Kantowski-Sachs string cosmologies

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We present new exact solutions of the low-energy-effective-action string equations with both dilaton ϕ and axion H fields nonzero. The background universe is Kantowski-Sachs-type. We consider the possibility of a pseudoscalar axion field h $[H=e^{\phi}(dh)^*]$ that can be either time or space dependent. The case of a time-dependent h reduces to that of a stiff perfect-fluid cosmology. For a space-dependent h there is just one nonzero time-space-space component of the axion field H, and this corresponds to a distinguished direction in space which prevents the models from isotropizing. Also, in the latter case, both the axion field H and its tensor potential B $(H=dB)$ are dependent on time and space, yet the energy-momentum tensor remains time dependent as required by the homogeneity of the cosmological model. [S0556-2821(97)02302-3]

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I. INTRODUCTION

The motion of the bosonic string in background fields is governed by the action for the nonlinear σ model [1,2]. Mueller [3] has solved both zeroth and first-order inverse string tension α' equations for multidimensional Bianchi I cosmological model without the antisymmetric axion field. The isotropic cosmological backgrounds without axion have been extensively studied by Gasperini and Veneziano [4–7], and special attention has been paid to the pre-Big-Bang solutions in relation to scale-factor duality. The homogeneous axion-dilaton cosmology gives rise to the question [9] of whether the antisymmetric three-index axion field strength $H_{\alpha\beta\gamma}$, $(\alpha,\beta,\gamma=0,1,2,3)$, or its antisymmetric two-index tensor potential $B_{\beta\gamma}$ $(H_{\alpha\beta\gamma}=6\partial_{[\alpha}B_{\beta\gamma]})$ should be homogeneous. By analogy with the Einstein-Maxwell equations, most investigators have considered an homogeneous (time-dependent) axion field strength H [8–12], but Copeland *et al.* [9] assumed that it was the potential B which should be homogeneous. Actually, because of the antisymmetry of the axion field, one has either of the two possibilities: the components H_{oij}, $(i,j=1,2,3)$, vanish if the potential B_{ij} is space dependent and the components H_{ijk} vanish if the potential B_{ij} is time dependent. Copeland *et al.* [9] framed the problem in terms of the dual pseudoscalar axion field h $[H=e^{\phi}(dh)^*]$, (where * is the spacetime dual) which was taken to be time or space dependent, respectively]. They concluded that for time-dependent antisymmetric tensor potential B_{ij} there exists just one nonzero component of the axion field, $H_{0ij}, i,j=1,2,3$, and this gives rise to Bianchi I universes which cannot isotropize at late times. Similarly, Barrow and Kunze [13] have classified the degrees of freedom available to the antisymmetric field strength H in all Bianchi type spacetimes, assuming time dependence of the tensor potential B as well in the orthonormal-frame formalism. Other papers about the evolution of the axion field have also appeared [14,15].

In this paper we will consider a spatially homogeneous background spacetime of Kantowski-Sachs-type. This is the only spatially homogeneous universe that is not included in the Bianchi classification. It falls outside this classification of models with three-dimensional isometry groups because it possesses a four-dimensional group of motions with no simply transitive three-dimensional subgroup. We consider both time-dependent and space-dependent pseudoscalar axion field h (cf. the Appendix). In the former case we have effectively another scalar field (or equivalently, a stiff perfect-fluid) cosmology. In the latter case we produce a model which evolves like that given by Copeland *et al.* [9] for Bianchi I models.

The field equations will describe three different anisotropic 3+1 dimensional spacetimes. Those with zero and negative curvature are just axisymmetric Bianchi type I and III universes. The positive curvature models constitute the Kantowski-Sachs models (first found by Kompanyeets and Chernov [16]). They are closed anisotropic universes. In the special case where they become isotropic, they reduce to the closed Friedmann-Robertson-Walker universes.

We give solutions for models with all curvatures. For a time-dependent pseudoscalar axion field h, we give a parametric solution of the low-energy-effective-action equations for the system containing both dilaton and axion in Sec. II. We also give an explicit special solution in terms of the cosmic time when the axion field is absent in Sec. III. Following the discussion of Copeland *et al.* [9] in Sec. IV, we examine whether it is possible to employ a time-independent pseudoscalar axion field h in Kantowski-Sachs geometries and, if so, which of its components are allowed to be nonzero. In the Appendix we also discuss some relations between our work and these earlier studies.

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II. LOW-ENERGY-EFFECTIVE-ACTION EQUATIONS AND SOLUTIONS WITH TIME-DEPENDENT PSEUDOSCALAR AXION FIELD

The low-energy-effective-action field equations are given by [12]

$$R^{\nu}_{\mu}+\nabla_{\mu}\nabla^{\nu}\phi-\tfrac{1}{4}H_{\mu\alpha\beta}H^{\nu\alpha\beta}=0, \tag{2.1}$$

$$R-\nabla_{\mu}\phi\nabla^{\mu}\phi+2\nabla_{\mu}\nabla^{\mu}\phi-\tfrac{1}{12}H_{\mu\nu\beta}H^{\mu\nu\beta}=0, \tag{2.2}$$

$$\partial_{\mu}(e^{-\phi}\sqrt{-g}H^{\mu\nu\alpha})=0, \tag{2.3}$$

where ϕ is the dilaton field, $H_{\mu\nu\beta}=6\partial_{[\mu}B_{\nu\beta]}$ is the strength of the antisymmetric tensor field $B_{\mu\nu}=-B_{\nu\mu}$ is its antisymmetric tensor potential. We choose the metric of spacetime to be of Kantowski-Sachs form, with [17]

$$ds^2=dt^2-X^2(t)dr^2-Y^2(t)d\Omega_k^2, \tag{2.4}$$

where the angular metric is

$$d\Omega_k^2=d\theta^2+S^2(\theta)d\psi^2,$$

$$S(\theta)=\begin{cases}\sin\theta & \text{for } k=+1,\\ \theta & \text{for } k=0,\\ \sinh\theta & \text{for } k=-1,\end{cases} \tag{2.5}$$

and X and Y are the expansion scale factors. We shall consider models of all three curvatures in the same analysis. Strictly, only the $k=+1$ models fall outside the Bianchi classification, but we shall refer to them all as Kantowski-Sachs models for simplicity. The nonzero Ricci tensor components are

$$-R^0_0=\frac{\ddot{X}}{X}+2\frac{\ddot{Y}}{Y}, \tag{2.6}$$

$$-R^1_1=\frac{\ddot{X}}{X}+2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}, \tag{2.7}$$

$$-R^2_2=-R^3_3=\frac{k+\dot{Y}^2}{Y^2}+\frac{\ddot{Y}}{Y}+\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}, \tag{2.8}$$

and the scalar curvature is

$$-R=2\frac{\ddot{X}}{X}+4\frac{\ddot{Y}}{Y}+2\frac{k+\dot{Y}^2}{Y^2}+4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}. \tag{2.9}$$

Since the metric is spatially homogeneous, the dilaton field can only depend on time and we have

$$\nabla_{\mu}\nabla^{\nu}\phi=\phi^{,\nu}_{,\mu}+\Gamma^{\nu}_{\mu\rho}\phi^{,\rho}, \tag{2.10}$$

so

$$\nabla_0\nabla^0\phi=\ddot{\phi}, \tag{2.11}$$

$$\nabla_1\nabla^1\phi=\frac{\dot{X}}{X}\dot{\phi}, \tag{2.12}$$

$$\nabla_2\nabla^2\phi=\nabla_3\nabla^3\phi=\frac{\dot{Y}}{Y}\dot{\phi}, \tag{2.13}$$

$$\nabla_0\phi\nabla^0\phi=\dot{\phi}^2. \tag{2.14}$$

For the torsion field, $H_{\alpha\beta\gamma}$, we assume the simple ansatz, similar to that of Batakis and Kehagias [10] and Mimóso and Wands [11], that $H_{\alpha\beta\gamma}$ takes the form

$$H_{123}=AS(\theta), \tag{2.15}$$

where $A=$const and the rest of the components are taken to be zero. This ansatz corresponds to a space-dependent antisymmetric potential $B_{\mu\nu}=B_{\mu\nu}(r,\theta,\psi)$ or to a time-dependent pseudoscalar axion field h [cf. the Appendix—Eq. (A5)]. It is interesting to note that the antisymmetric tensor potential components for Eq. (2.15) are given by $B_{12}=A\psi\sin\theta$, $B_{23}=Ar\sin\theta$, $B_{31}=-A\cos\theta$ for $k=+1$; $B_{12}=A\psi\sinh\theta$, $B_{23}=Ar\sinh\theta$, $B_{31}=A\cosh\theta$ for $k=-1$ and $B_{12}=A\psi, B_{23}=Ar, B_{31}=A\theta$ for $k=0$. With the choice (2.15) the field equations (2.1) become

$$\frac{\ddot{X}}{X}+2\frac{\ddot{Y}}{Y}-\ddot{\phi}=0, \tag{2.16}$$

$$\frac{\ddot{X}}{X}+2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}-\dot{\phi}\frac{\dot{X}}{X}-\frac{1}{2}\frac{A^2}{X^2Y^4}=0, \tag{2.17}$$

$$\frac{k+\dot{Y}^2}{Y^2}+\frac{\ddot{Y}}{Y}+\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}-\dot{\phi}\frac{\dot{Y}}{Y}-\frac{1}{2}\frac{A^2}{X^2Y^4}=0. \tag{2.18}$$

The sum of Eqs. (2.16) and (2.17) added to twice Eq. (2.18) gives

$$2\frac{\ddot{X}}{X}+4\frac{\ddot{Y}}{Y}+2\frac{k+\dot{Y}^2}{Y^2}+4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}-\ddot{\phi}-\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)\dot{\phi}-\frac{3}{2}\frac{A^2}{X^2Y^4}=0. \tag{2.19}$$

The field equation (2.2) reads

$$-2\frac{\ddot{X}}{X}-4\frac{\ddot{Y}}{Y}-2\frac{k+\dot{Y}^2}{Y^2}-4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}+2\ddot{\phi}-\dot{\phi}^2+2\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)\dot{\phi}+\frac{1}{2}\frac{A^2}{X^2Y^4}=0, \tag{2.20}$$

so from the sum of Eqs. (2.19) and (2.20) we have

$$\ddot{\phi}-\dot{\phi}^2+\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)\dot{\phi}-\frac{A^2}{X^2Y^4}=0. \tag{2.21}$$

At this stage we introduce a new time coordinate τ via relation [which is proportional to the scalar field χ Eq. (91) of [11]]

$$dt=XY^2e^{-\phi}d\tau, \tag{2.22}$$

and then Eq. (2.21) becomes

$$\phi_{,\tau\tau}-A^2e^{-2\phi}=0, \tag{2.23}$$

which solves as [cf. also Eq. (100) of [11]]

$$e^{\phi}=\cosh\alpha\tau+\sqrt{1-(A^2/\alpha^2)}\sinh\alpha\tau, \tag{2.24}$$

with α constant ($\alpha^2>A^2$). If we turn off the $H_{\mu\nu\rho}$ field, the solution of (2.24) is simply

$$\phi(\tau)=\alpha\tau+\gamma, \tag{2.25}$$

with γ a constant, which we may set to zero without loss of generality. A useful relation, implied by Eq. (2.24), is

$$\phi_{,\tau\tau}+\phi_{,\tau}^2=\alpha^2. \tag{2.26}$$

Using the time coordinate (2.22), equations (2.16)–(2.18) become

$$\left(\frac{X_{,\tau}}{X}\right)_{,\tau}+2\left(\frac{Y_{,\tau}}{Y}\right)_{,\tau}-2\frac{Y_{,\tau}}{Y}\left(\frac{Y_{,\tau}}{Y}+2\frac{X_{,\tau}}{X}\right)+2\phi_{,\tau}\left(\frac{X_{,\tau}}{X}+2\frac{Y_{,\tau}}{Y}\right)-\phi_{,\tau\tau}-\phi_{,\tau}^2=0, \tag{2.27}$$

$$\left(\frac{X_{,\tau}}{X}\right)_{,\tau}-\frac{1}{2}\phi_{,\tau\tau}=0, \tag{2.28}$$

$$\left(\frac{Y_{,\tau}}{Y}\right)_{,\tau}-\frac{1}{2}\phi_{,\tau\tau}+kX^2Y^2e^{-2\phi}=0. \tag{2.29}$$

The equations (2.28) and (2.29) can be rewritten as

$$(\ln X^2e^{-\phi})_{,\tau\tau}=0, \tag{2.30}$$

$$(\ln Y^2e^{-\phi})_{,\tau\tau}+2kX^2Y^2e^{-2\phi}=0. \tag{2.31}$$

From, Eq. (2.20), we see that the constraint equation (2.16) can be rewritten as

$$\frac{k+\dot{Y}^2}{Y^2}+2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}+\frac{1}{2}\dot{\phi}^2-\dot{\phi}\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)-\frac{1}{4}\frac{A^2}{X^2Y^4}=0, \tag{2.32}$$

so, in terms of the time coordinate (2.22),

$$\tfrac{1}{2}(\ln X^2e^{-\phi})_{,\tau}(\ln Y^2e^{-\phi})_{,\tau}+(\ln Y)_{,\tau}(\ln Y-\phi)_{,\tau}+kX^2Y^2e^{-2\phi}=\tfrac{1}{4}A^2e^{-2\phi}. \tag{2.33}$$

The solution of Eq. (2.30) is

$$X^2e^{-\phi}=X_0e^{p\tau}, \tag{2.34}$$

with X_0 and p constants. Then, from Eqs. (2.24) and (2.34), for $A\neq 0$, we have

$$X(\tau)=\sqrt{X_0}e^{(1/2)p\tau}\sqrt{\cosh\alpha\tau+\sqrt{1-(A^2/\alpha^2)}\sinh\alpha\tau}, \tag{2.35}$$

or, for $A=0$,

$$X(\tau)=\sqrt{X_0}e^{1/2(p+\alpha)\tau}. \tag{2.36}$$

The solution of Eq. (2.31) for the scale factor Y is given by

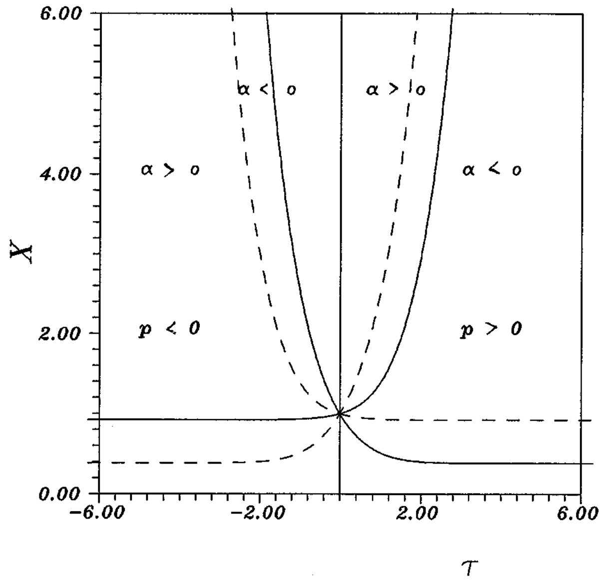

FIG. 1. The plots of the scale factor X [Eq. (2.35)] in terms of the parametric time τ defined by Eq. (2.22). The plots do not depend on the spatial curvature index k. Different shapes of the plots depend on the values of the constants $\alpha=\pm 1$ and $p=\pm 1$.

$$Y(\tau)=\frac{1}{\sqrt{X_0}}e^{-(1/2)p\tau}\times\sqrt{\cosh\alpha\tau+\sqrt{1-(A^2/\alpha^2)}\sinh\alpha\tau}\sqrt{M(\tau)}, \tag{2.37}$$

where $M(\tau)=X^2Y^2\exp(-2\phi)$ satisfies

$$(\ln M)_{,\tau\tau}+2kM=0, \tag{2.38}$$

hence,

$$\frac{1}{\sqrt{M(\tau)}}=\cosh\beta\tau+\sqrt{1-(k/\beta^2)}\sinh\beta\tau, \tag{2.39}$$

and $\beta^2>k$. The constraint equation (2.33) may be now rewritten in terms of M as

$$\frac{1}{4}\left(\frac{M_{,\tau}}{M}\right)^2+kM=\frac{1}{4}(\alpha^2+p^2), \tag{2.40}$$

which gives the condition

$$\beta^2=\tfrac{1}{4}(\alpha^2+p^2). \tag{2.41}$$

The solutions for the two scale factors $\tilde{a}_1$ and $\tilde{a}_2$ in the Einstein frame, which relate via conformal transformation (3.18)–(3.20) to our X and Y, in the string Einstein frame are given in the paper by Mimóso and Wands [11].

Although τ is a parametric time related to the cosmic time by Eq. (2.22), we give some plots of the scale factors X and Y [Eqs. (2.35) and (2.37)] and the dilaton ϕ [Eq. (2.24)] in Figs. 1–3. Different plots are given for different values of the constants α and p, which reflects the string duality symmetry here. As for Y we give just the plot for curvature index $k=+1$. Note that the value of the constant β is constrained by Eq. (2.41) and that α and p do not have to be positive.

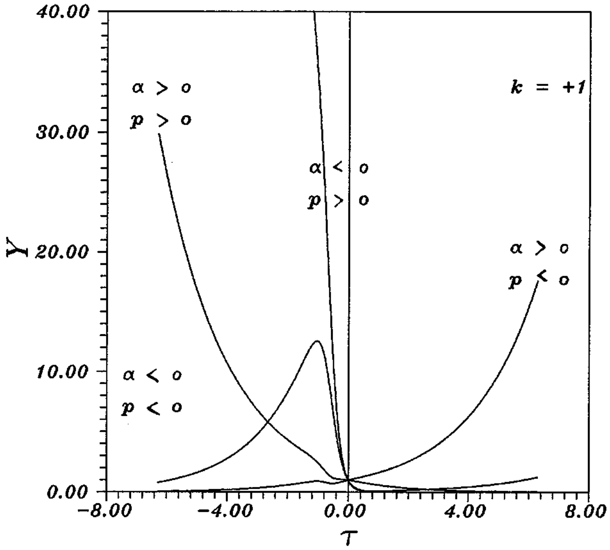

FIG. 2. The plots of the scale factor Y [Eq. (2.37)] in terms of the parametric time τ for spatial curvature index $k=+1$. The plots depend just on the two constants $\alpha=\pm 5$ and $p=\pm 1$ since the third constant β is constrained by Eq. (2.41).

Also note that despite X and ϕ, the plots of $Y(\tau)$ are time asymmetric in τ. This refers to the string duality which is more sophisticated for homogeneous models as it has been shown for Bianchi I models in [9].

III. DEPARAMETRIZED SOLUTIONS FOR VANISHING AXION FIELD

From the results of Sec. II, we see that without the axion field ($A=0$) the solutions for ϕ, X, and Y reduce to

$$\phi(\tau)=\alpha\tau, \tag{3.1}$$

$$X(\tau)=\sqrt{X_0}e^{(1/2)(\alpha+p)\tau}, \tag{3.2}$$

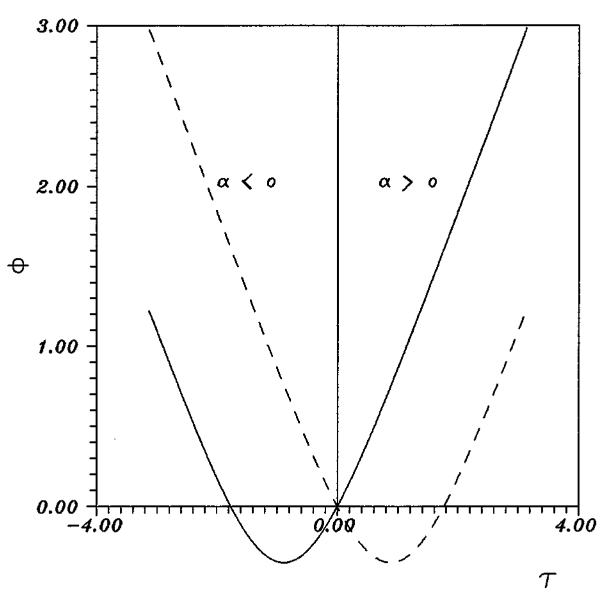

FIG. 3. The plots of the dilaton ϕ [Eq. (2.24)] in terms of the parametric time τ for different values of $\alpha=\pm 1$.

$$Y(\tau)=\frac{1}{\sqrt{X_0}}e^{(1/2)(\alpha-p)\tau}\widetilde{M}, \tag{3.3}$$

where $\widetilde{M}(\tau)$, the solution of Eq. (2.38), is given by

$$\widetilde{M}=\begin{cases}\beta\cosh^{-1}(\beta\tau+\delta) & \text{for } k=+1,\\ \exp(\beta\tau+\delta) & \text{for } k=0,\\ \beta\sinh^{-1}(\beta\tau+\delta) & \text{for } k=-1,\end{cases} \tag{3.4}$$

where δ is constant, with the constraint given by Eq. (2.40).

From Eq. (2.22) and (3.1)–(3.3), we find that the time parameter in the string frame is

$$t(\tau)=\frac{1}{\sqrt{X_0}}\int e^{-(1/2)(\alpha-p)\tau}\widetilde{M}^2 d\tau. \tag{3.5}$$

Hence, this relation is integrable for $k\neq 0$, provided $\alpha=p$ [that is, from Eq. (2.41), if $\beta^2=\alpha^2/2$]. In this case we have

$$t(\tau)=\frac{\alpha^2}{2\sqrt{X_0}}\begin{cases}\pm\dfrac{\sqrt{2}}{\alpha}\tanh\left(\pm\dfrac{\alpha}{\sqrt{2}}\tau+\delta\right), & k=+1,\\ \mp\dfrac{\sqrt{2}}{\alpha}\coth\left(\mp\dfrac{\alpha}{\sqrt{2}}\tau+\delta\right), & k=-1.\end{cases} \tag{3.6}$$

For $k=0$ it is always integratable and gives

$$t(\tau)=\frac{1}{X_0 s}\exp(-s\tau-2\delta), \tag{3.7}$$

where

$$s=-\tfrac{1}{2}(\alpha-p+4\beta). \tag{3.8}$$

After deparametrization, Eqs. (3.1)–(3.3) provide a simple solution of Eqs. (2.16)–(2.18) for $A=0$ and $\dot{\phi}=\dot{X}/X$. When $k\neq 0$, it is given by

$$X(t)=\left(k\frac{\alpha/\sqrt{2}-t}{\alpha/\sqrt{2}+t}\right)^{1/\sqrt{2}}, \tag{3.9}$$

$$Y(t)=\sqrt{k(\alpha^2/2-t^2)}, \tag{3.10}$$

$$\phi(t)=\ln\left(k\frac{\alpha/\sqrt{2}-t}{\alpha/\sqrt{2}+t}\right)^{1/\sqrt{2}}, \tag{3.11}$$

where the time coordinate has the ranges

$$0\leqslant t\leqslant\frac{\alpha^2}{2}\quad\text{for}\quad k=+1, \tag{3.12}$$

$$t\geqslant\frac{\alpha^2}{2}\quad\text{for}\quad k=-1. \tag{3.13}$$

The volume expansion is given by

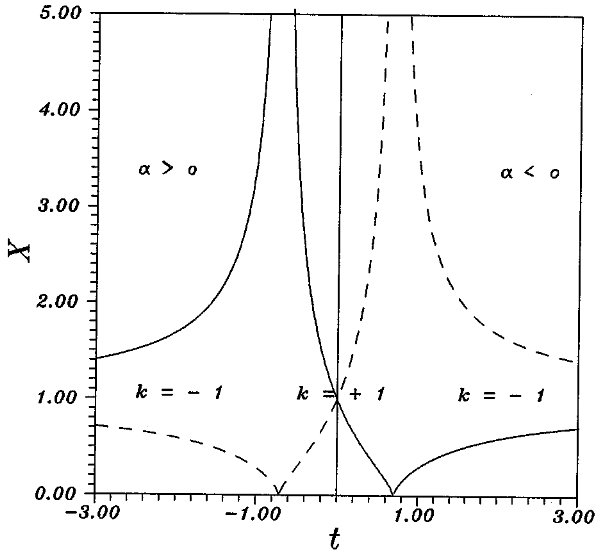

FIG. 4. The plot of the scale factor $X(t)$ for an exact Kantowski-Sachs low-energy-effective-action model (3.9). We take $\alpha = \pm 1$ and the ranges of t are given by Eqs. (3.12) and (3.13).

$$V(t) = XY^2 = \left[k\left(\frac{\alpha}{\sqrt{2}} - t \right) \right]^{(\sqrt{2}+1)/\sqrt{2}} \left(\frac{\alpha}{\sqrt{2}} + t \right)^{(\sqrt{2}-1)/\sqrt{2}}, \tag{3.14}$$

and its evolution is qualitatively the same as that of the scale factor $Y(t)$.

For $k=0$ we have

$$X(t) = \sqrt{X_0}(X_0 st)^{(\alpha+p)/(\alpha-p+4\beta)}, \tag{3.15}$$

$$Y(t) = \frac{1}{X_0}(X_0 st)^{(\alpha-p+2\beta)/(\alpha-p+4\beta)}, \tag{3.16}$$

$$\phi(t) = \frac{2\alpha}{\alpha - p + 4\beta}\ln(X_0 st). \tag{3.17}$$

The plots of $X(t), Y(t)$ and $\phi(t)$, for $k = \pm 1$ are given in Figs. 4–6. It is interesting to note that the scale factors and the dilaton remain the same after the change $\alpha \to -\alpha$ and $t \to -t$ which refers to the string duality symmetry here [4].

For $k = +1$, the universe starts at a cigar singularity with $X = \infty, Y = 0$, and terminates at a point singularity with $X = Y = 0$ [18]. For $k = -1$ the universe either starts at $t = \alpha^2/2$ with a point singularity with the ensuing volume expansion going to infinity (with asymptotic value of $X = 1$ for $t \to \infty$), or it starts with infinite volume (with X taken to be equal to one at minus infinity) and collapses to a cigar singularity.

In order to develop these solutions in the Einstein frame, we need to change the scale factors X and Y, together with the time coordinate, to

$$\tilde{X} = e^{-\phi/2}X, \tag{3.18}$$

$$\tilde{Y} = e^{-\phi/2}Y, \tag{3.19}$$

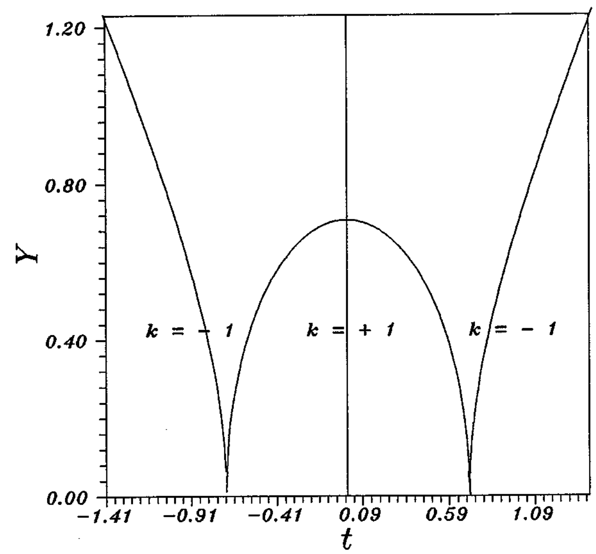

FIG. 5. The plot of the scale factor $Y(t)$ for an exact Kantowski-Sachs low-energy-effective-action model (3.10) $(\alpha = \pm 1)$. The qualitative volume evolution (3.14) has behavior of the same form.

$$d\tilde{t} = e^{-\phi 2}dt = \left(k\frac{\alpha/\sqrt{2} - t}{\alpha/\sqrt{2} + t} \right)^{1/2\sqrt{2}} dt. \tag{3.20}$$

The calculations for the $k=0$ (Bianchi I) case have already been given in [9].

IV. SOLUTIONS WITH TIME-INDEPENDENT PSEUDOSCALAR AXION FIELD

In this section we consider the space-dependent pseudoscalar axion field h. This requirement, however, does not strictly correspond to the condition that the antisymmetric tensor field strength $H_{\mu\nu\alpha}$ and its antisymmetric potential

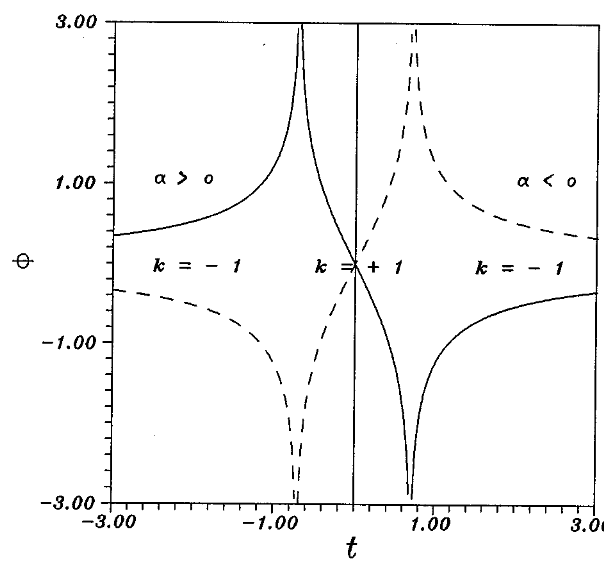

FIG. 6. The plot of the dilaton field $\phi(t)$ for an exact Kantowski-Sachs low-energy-effective-action model (3.11) $(\alpha = \pm 1)$.

$B_{\mu\nu}$ are space dependent as was the case in the Bianchi I calculations of Copeland *et al.* [9] and the Bianchi-type universes studied by Barrow and Kunze [13]. If we have $B_{\mu\nu}=B_{\mu\nu}(t)$, then because of the antisymmetry

$$H_{ijk}=0 \quad \text{and} \quad H_{oij}\neq 0. \tag{4.1}$$

As we will show later, this is not the case in our formulation and it arises from the fact that we do not use orthonormal frames. If we consider a space-dependent pseudoscalar axion field, $h=h(r,\theta,\psi)$, then relation (A1) of the Appendix requires $H^{123}=\epsilon^{1230}e^{\phi}\partial_0 h=0$ and only the H_{0ij} components of the axion field H can be nonzero (a similar situation to that considered in Refs. [9,13]).

There are some conditions which must be satisfied if the low-energy-effective-action equations (2.1)–(2.3) are to admit the axion field into the Kantowski-Sachs geometry. One is that the off-diagonal components of $H_{\mu\lambda\sigma}H^{\nu\lambda\sigma}$ in Eq. (2.2) should vanish and we have the condition

$$g^{jj}g^{mm}H_{iom}H_{jom}=0 \quad (i\neq j,\text{no sum}), \tag{4.2}$$

which means that only one of H_{012}, H_{023}, or H_{013} may be nonzero.

Suppose we choose $H_{012}\neq 0$. From the equation of motion (2.3), we obtain

$$H^{012}=\frac{Be^{\phi}}{XY^2S(\theta)}, \tag{4.3}$$

with B constant, and $S(\theta)$ given by Eq. (2.5), so

$$H_{012}=-\frac{BXe^{\phi}}{S(\theta)}. \tag{4.4}$$

One can easily check that the integrability condition is fulfilled, since $\partial_3 H_{012}=0$. However, Eq. (2.1) gives

$$H^2\equiv H_{\mu\nu\lambda}H^{\mu\nu\lambda}=-6B^2\frac{e^{2\phi}Y^2}{S^2(\theta)} \tag{4.5}$$

and there is explicit dependence on the spatial coordinate θ, which means that the ansatz $H_{012}\neq 0$ is inconsistent with the geometries under consideration.

The next possibility is $H_{013}\neq 0$. This means that

$$H^{013}=\frac{Ce^{\phi}}{XY^2S(\theta)} \tag{4.6}$$

and

$$H_{013}=Ce^{\phi}XS(\theta) \tag{4.7}$$

with C constant. Then

$$H^2=\frac{C^2e^{2\phi}}{Y^2}, \tag{4.8}$$

which does depend just on time. However, in this case we see that the integrability condition $\partial_2 H_{013}=0$ is not fulfilled and so the choice $H_{013}\neq 0$ is also impossible.

The last possibility is given by

$$H_{023}=De^{\phi}\frac{Y^2}{X}S(\theta)=Y^4S^2(\theta)H^{023}, \tag{4.9}$$

where D is constant. This time-integrability condition $\partial_1 H_{023}=0$ is fulfilled and

$$H^2=6D^2\frac{e^{2\phi}}{X^2}, \tag{4.10}$$

which depends only on the time coordinate, as required. Thus, $H_{023}\neq 0$ provides the only consistent choice. This naturally gives the space dependence of the tensor potential B because $H_{023}=\partial_0 B_{23}$ (due to the gauge transformation, we can eliminate all the components B_{0i} [9]), so $B_{23}=DS(\theta)\int Y^2X^{-1}e^{\phi}dt$ and $B_{12}=B_{13}=0$. This also shows that H_{123} component of the axion field must vanish [$H_{312}=\partial_3 B_{12}=0,\ H_{231}=\partial_2 B_{13}=0,\ H_{123}=\partial_1 B_{23}(t,\theta)=0$]. The final conclusion is that our ansatz (4.9) is correct.

Using this choice, the field equations (2.1)–(2.2) become

$$\frac{\ddot X}{X}+2\frac{\ddot Y}{Y}-\ddot\phi=-\frac{D^2e^{2\phi}}{2X^2}, \tag{4.11}$$

$$\frac{\ddot X}{X}+2\frac{\dot X}{X}\frac{\dot Y}{Y}-\dot\phi\frac{\dot X}{X}=0, \tag{4.12}$$

$$\frac{k+\dot Y^2}{Y^2}+\frac{\ddot Y}{Y}+\frac{\dot X}{X}\frac{\dot Y}{Y}-\dot\phi\frac{\dot Y}{Y}=-\frac{D^2e^{2\phi}}{2X^2}. \tag{4.13}$$

The sum of Eq. (4.11) and (4.12) together with twice Eq. (4.13) gives

$$2\frac{\ddot X}{X}+4\frac{\ddot Y}{Y}+2\frac{k+\dot Y^2}{Y^2}+4\frac{\dot X}{X}\frac{\dot Y}{Y}-\ddot\phi-\left(\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right)\dot\phi-\frac{3}{2}\frac{D^2e^{2\phi}}{X^2}=0. \tag{4.14}$$

But Eq. (2.2) is

$$-2\frac{\ddot X}{X}-4\frac{\ddot Y}{Y}-2\frac{k+\dot Y^2}{Y^2}-4\frac{\dot X}{X}\frac{\dot Y}{Y}+2\ddot\phi-\dot\phi^2+2\left(\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right)\dot\phi-\frac{1}{2}\frac{D^2e^{2\phi}}{X^2}=0, \tag{4.15}$$

so from the sum of Eqs. (4.14) and (4.15), we have

$$\ddot\phi-\dot\phi^2+\left(\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right)\dot\phi-\frac{D^2e^{2\phi}}{X^2}=0. \tag{4.16}$$

Using the time coordinate (2.22), Eq. (4.16) becomes

$$\phi_{,\tau\tau}+D^2Y^4=0. \tag{4.17}$$

Equations (4.11)–(4.13) now become

$$\left(\frac{X_{,\tau}}{X}\right)_{,\tau}+2\left(\frac{Y_{,\tau}}{Y}\right)_{,\tau}-2\frac{Y_{,\tau}}{Y}\left(\frac{Y_{,\tau}}{Y}+2\frac{X_{,\tau}}{X}\right)+2\phi_{,\tau}\left(\frac{X_{,\tau}}{X}+2\frac{Y_{,\tau}}{Y}\right)$$
$$-\frac{3}{2}\phi_{,\tau\tau}-\phi_{,\tau}^{2}=0, \tag{4.18}$$

$$\left(\frac{X_{,\tau}}{X}\right)_{,\tau}=0, \tag{4.19}$$

$$\left(\frac{Y_{,\tau}}{Y}\right)_{,\tau}-\frac{1}{2}\phi_{,\tau\tau}+kX^2Y^2e^{-2\phi}=0, \tag{4.20}$$

and Eqs. (4.19) and (4.20) can be rewritten as

$$(\ln X)_{,\tau\tau}=0, \tag{4.21}$$

$$(\ln Y^2e^{-\phi})_{,\tau\tau}+2kX^2Y^2e^{-2\phi}=0. \tag{4.22}$$

The solution of Eq. (4.21) is

$$X(\tau)=\exp(r\tau+s) \tag{4.23}$$

with r and s constants. For $k=0$ it is also possible to solve Eq. (4.22) to obtain

$$Y(\tau)=\exp\{\tfrac{1}{2}[\phi(\tau)+m\tau+n]\} \tag{4.24}$$

with m and n constants. Using Eqs. (4.23)–(4.24) we may solve Eqs. (4.17)–(4.20) for Y and ϕ to give

$$Y(\tau)=2^{1/4}\sqrt{\frac{b}{D}}(\cosh b\sqrt{2}\tau)^{-1/2}, \tag{4.25}$$

$$\phi(\tau)=\phi_0-m\tau-\ln\cosh b\sqrt{2}\tau \tag{4.26}$$

with ϕ_0 constant, and

$$b^2=m(m+2r). \tag{4.27}$$

Using Eq. (2.22) we have

$$\tau(t)=A+\ln t^{1/(r+m)}, \tag{4.28}$$

where

$$A=\frac{1}{r+m}\left(\ln\frac{D(r+m)}{b\sqrt{2}}-s-\phi_0\right). \tag{4.29}$$

Finally, the solution of Eqs. (4.17)–(4.20) in terms of the cosmic time t in the string frame is given by

$$X(t)=t^{r/(r+m)}, \tag{4.30}$$

$$Y(t)=2^{1/4}\sqrt{\frac{b}{D}}\left(t_0t^{w}+\frac{1}{t_0}t^{-w}\right)^{-1/2}, \tag{4.31}$$

$$e^{\phi(t)}=e^{-ma}t^{-m/(r+m)}\left(t_0t^{w}+\frac{1}{t_0}t^{-w}\right)^{-1}, \tag{4.32}$$

where $t_0=\exp b\sqrt{2}A$ and $w=b\sqrt{2}/(r+m)$ are constants.

This is an axisymmetric (LRS) subcase of the Bianchi type I axion-dilaton solution first given by Copeland *et al.* [9] [compare their Eqs. (2.45)–(2.48) for $a_1=a_2$]. The axisymmetric limit in [9] is given by taking their constant r to be equal to zero in their Eqs. (2.46) and (2.47) or $C_1=C_2$ in their Eq. (2.36).

Some special solutions for the $k\neq0$ cases can be given by solving Eqs. 4.17 for Y; that is, by taking

$$Y^2=\frac{1}{D}(-\phi_{,\tau\tau})^{1/2}, \tag{4.33}$$

and substituting into Eq. (4.20), to obtain

$$\{\ln[(-\phi_{,\tau\tau})^{1/2}D^{-1}e^{-\phi}]\}_{,\tau\tau}+2kD^{-1}$$
$$(-\phi_{,\tau\tau})^{1/2}e^{-2\phi(\tau)+2r\tau+2s}=0. \tag{4.34}$$

We now seek solutions of the form

$$\phi(\tau)=\alpha^2\ln\tau+\beta\tau+\epsilon, \tag{4.35}$$

with α^2,β, and ϵ constants. From the field equations (4.17)–(4.20), we have

$$X(\tau)=\exp(r\tau+s), \tag{4.36}$$

$$\phi(\tau)=\tfrac{1}{2}\ln\tau+r\tau+\epsilon, \tag{4.37}$$

$$Y(\tau)=\pm2^{-1/4}D^{-1/2}\tau^{-1/2}, \tag{4.38}$$

with a constraint

$$e^{2(\epsilon-s)}=\pm\frac{4k}{3D\sqrt{2}}, \tag{4.39}$$

which, because $D>0$, means we take the plus sign for $k=+1$ and the minus sign for $k=-1$ universes. Using Eq. (2.22) and Eqs. (4.36)–(4.38), we write

$$t(\tau)=\mp D^{-1}\sqrt{2}e^{s-\epsilon}\tau^{-1/2}. \tag{4.40}$$

After deparametrization, our solutions (4.36)–(4.38) give

$$X(t)\propto\exp\left(\frac{r}{t^2}+s\right), \tag{4.41}$$

$$Y(t)\propto t, \tag{4.42}$$

$$\phi(t)\propto\ln t+\frac{\text{const}}{t^2}. \tag{4.43}$$

V. DISCUSSION

In this paper we have considered the low-energy-effective-action string equations for a Kantowski-Sachs background spacetime. We have included the full bosonic spectrum of fields, with the graviton $g_{\mu\nu}$, dilaton ϕ, and the axion H. We consider the two forms of ansatz for the axion. In terms of the pseudoscalar axion field h, they correspond to it depending on either the time or space coordinates alone.

For the time-dependent case, $h=h(t)$, we found an exact parametric solution of the field equations given in Sec. II. In such a case the axion field behaves effectively as a stiff perfect fluid (pressure=density) distributed homogeneously over space. These solutions were also discussed in the context of the scalar-tensor cosmologies by using slightly different parametrization in [11].

We also find that, for vanishing axion $H=0$, there is a deparametrized exact solution for $\dot{\phi}=\dot{X}/X$. We discuss this solution in Sec. III. It appears to be the most interesting Kantowski-Sachs solution in which to study the duality problem, which we shall address to a separate paper.

For the spatially dependent case, $h=h(r,\theta,\psi)$, we find that there is only one possible form for the torsion field in spatially homogeneous closed universes of Kantowski-Sachs-type. Its 3-form strength can have just one nonzero component, H_{023}, which distributes the field along the two spatial directions on the two-sphere S^2. This component depends both on time and space and leads to space and time dependence of the only nonzero component of the tensor potential, $B_{23}=B_{23}(t,\theta)$. This is expected because we are working in coordinate frames rather than in orthonormal frames of Refs. [10,13]. In effect, there is an anisotropic stress in the universe. For such a dilaton-axion anisotropic cosmology, we write down the field equations and find some new solutions. In the zero-curvature case we recover the axisymmetric Bianchi I (LRS) solutions given in [9]. These results provide, in particular, a new type of closed universe in string cosmology.

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APPENDIX: THREE-INDEX AXION FIELD AND PSEUDOSCALAR AXION FIELD NOTATIONS

In this appendix we connect our notation for the three-index axion field H to that of [9] which uses the pseudoscalar axion field h. Following [9], we define

$$H^{\mu\nu\alpha}=e^{\phi}\epsilon^{\mu\nu\alpha\beta}h_{,\beta}. \tag{A1}$$

and

$$\epsilon^{\mu\nu\alpha\beta}=\frac{4!}{\sqrt{-g}}\delta^{\mu}_{[0}\delta^{\nu}_{1}\delta^{\alpha}_{2}\delta^{\beta}_{3]}. \tag{A2}$$

The equation of motion for the h field is obtained via the integrability conditions as

$$\nabla^{\mu}\nabla_{\mu}h+\nabla^{\mu}\phi\nabla_{\mu}h=0. \tag{A3}$$

Notice that for the antisymmetric tensor potential, $B_{\mu\nu}=B_{\mu\nu}(x)$, the h field can only depend on time, and Eq. (A3) reads

$$\ddot{h}+\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)\dot{h}+\dot{\phi}\dot{h}=0, \tag{A4}$$

which integrates to give

$$\dot{h}=-A\frac{e^{-\phi}}{XY^2}. \tag{A5}$$

So, from Eq. (A1), we have

$$H^{123}=-\frac{A}{X^2Y^4S(\theta)}, \tag{A6}$$

or

$$H_{123}=AS(\theta), \tag{A7}$$

as required by Eq. (2.5). With the H field chosen as above, the equation of motion (2.3) is satisfied. There is also a trivial solution of Eq. (A4), $\dot{h}=0$, but it corresponds to a constant torsion field.

If instead we assume that the h field cannot depend on time at all, and the equation of motion (A3) reads

$$\partial^{\mu}\partial_{\mu}h+\Gamma^{\mu}_{\rho\mu}\partial^{\rho}h=0. \tag{A8}$$

For the Kantowski-Sachs metric, Eq. (2.4), this reads

$$\frac{1}{X^2}\partial_1^2h+\frac{1}{Y^2}\partial_2^2h+\frac{1}{Y^2S^2(\theta)}\partial_3^2h+\frac{1}{Y^2}C(\theta)\partial_2h=0, \tag{A9}$$

where

$$C(\theta)=\begin{cases}\cot\theta & \text{for } k=+1,\\ 0 & \text{for } k=0,\\ \coth\theta & \text{for } k=-1,\end{cases} \tag{A10}$$

However, from the vanishing of the off-diagonal components of the energy-momentum tensor, which is the necessary condition to match homogeneous Kantowski-Sachs geometry with the axion field, we have

$$e^{2\phi}g^{jk}\partial_ih\partial_kh=0,\quad i\neq j(\text{no sum over } i \text{ and } j). \tag{A11}$$

This means that only one of the three ∂_ih may be nonzero. The solutions of the equation of motion (A9) which satisfy the condition (A11) are

$$\partial_1h=D=\text{const},\quad \partial_2h=\partial_3h=0, \tag{A12}$$

$$\partial_3h=B=\text{const},\quad \partial_1h=\partial_2h=0, \tag{A13}$$

$$\partial_2h=\frac{C}{S(\theta)},\quad \partial_1h=\partial_3h=0 \tag{A14}$$

with C constant and $S(\theta)$ given by Eq. (2.5). From Eq. (A1), we see that these correspond to the components of the axion field H given by Eq. (4.9) and (4.3) for Eqs. (A12) and (A13), respectively, while the situation for Eq. (A14) is more complicated. This is because from the definition (A1) we have, for Eq. (A14),

$$H^{013}=\frac{Ce^{\phi}}{XY^2S(\theta)}, \tag{A15}$$

which differs in the denominator from Eq. (4.6). With this choice, the axion equation of motion (2.3) reads

$$\partial_0\left(\frac{C}{S(\theta)}\right)=0 \tag{A16}$$

and is also fulfilled. However, we find

$$H^2=H_{013}H^{013}=\frac{e^{2\phi}}{Y^2S^2(\theta)}. \tag{A17}$$

This means that H depends on θ and the ansatz (A14) is inappropriate for a spatially homogeneous cosmology—a result which has been already discussed in Sec. IV. The only possible solution of Eq. (A9) that remains is $\partial_2 h=C=0$, which is a trivial solution with constant axion. This means that there is only one consistent choice of the components of the axion field H, and this is

$$H_{023}\neq 0. \tag{A18}$$

However, by the definition of axion field strength, this implies that

$$H_{023}=\partial_0 B_{23}\neq 0 \tag{A19}$$

and then, bearing in mind Eq. (4.9), it follows that the antisymmetric tensor field potential $B_{\mu\nu}$ must depend both on time and space coordinate, i.e.,

$$B_{\mu\nu}=B_{\mu\nu}(t,\theta) \tag{A20}$$

in a Kantowski-Sachs model. This is acceptable because the only quantity which should be homogeneous (depending only on time) for homogeneous geometry is the energy-momentum tensor expressible in terms of axion H or the pseudoscalar axion field h [cf. Eqs. (2.1), (2.2), and [9]] which is the case for our choice (A18). This difference appears here because we have worked in coordinate frames rather than in orthonormal frames of Refs. [10,13]. Copeland *et al.* [9] used coordinate frames in their calculations of Bianchi I model, but it did not really matter because the basis forms in type I are just $\sigma^1=dx^1, \sigma^2=dx^2, \sigma^3=dx^3$, and do not involve any spatial dependence. One could of course elaborate the problem in orthonormal frames [19] coming to the same conclusion as in the coordinate frames.

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