Generalized derivative expansion and one-loop corrections to the vacuum energy of static background fields

Lai-Him Chan
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001

(Received 20 December 1996)

The derivative expansion of the effective action for field theories with spontaneous symmetry breaking has a variable expansion parameter which may become locally infinite. To circumvent this difficulty, I propose an alternative expansion series in which the series expansion is simultaneously developed in order of the number of derivatives of the field and in powers of the deviation of the field from its ground-state value. As an example, I have applied this new method to calculate the quantum correction to the energy of the (1+1)-dimensional soliton in various models which have been well investigated previously. The expansion series are calculated using MATHEMATICA to 14 terms. For the models in which exact solutions have been found, such as the sine-Gordon soliton, ϕ^4 soliton, and ϕ^4 soliton with a fermion loop, the new improved series can be recognized as well-known analytically summable series. The complete results of exact solutions are recovered. More importantly, for the cases where exact solutions may not be available, Padé approximants or the Borel summation can be used as an efficient method to provide an excellent approximation, in contrast with cumbersome numerical calculations. The Christ-Lee soliton and the ϕ^4 bag are used to illustrate this approximation. We also derive a compact hybrid formula in closed form to estimate the quantum correction to the static energy of the (1+1)-dimensional field. This new method can be easily extended to higher dimensions as well as other important applications such as vacuum tunneling, Skyrmion physics, etc. [S0556-2821(97)06210-3]

PACS number(s): 11.10.Lm, 11.15.Kc

I. INTRODUCTION

In local quantum field theory, short-distance effects come primarily from heavy particles, confining fields, and the higher frequency parts of the quantum fluctuation of the light particles. In the low-energy limit, it may be more convenient to eliminate those degrees of freedom not directly observable and incorporate their effects into an effective action of the observable low-energy fields. The expansion of this necessarily nonlocal effective action into an infinite series of local action in terms of the number of derivatives of the local fields is known as the derivative expansion of the effective action [1–8,15,16].

In the 1970s, the leading term of the derivative expansion, the effective potential, was used to understand the dynamics of spontaneous symmetry breaking in local-field theories. In the 1980s, with increasing interest toward effective theories with higher derivative terms, such as topological anomalies, Skyrmion physics, the strongly interacting Higgs sector of the standard model, etc., important progress was made toward an efficient derivation of the derivative expansion of the effective action by integrating out the nonobservable quantum degrees of freedom associated with short-distant dynamics [1–4].

However, as a functional series with no detectable pattern of regularity, the derivative expansion is very complex with no obvious parameter of expansion. The convergence of the series has not been thoroughly investigated. In spite of its many possible applications, use of the derivative expansion is severely restricted because the series does not converge fast enough or diverge in interesting cases. Important examples are vacuum energy in the presence of background fields and quantum correction to the soliton energy. In particular in the case of a scalar field with spontaneous symmetry breaking, the coefficients of the derivative expansion may be infinite term by term [9–13,17,18].

In this paper, we propose that, in the case of spontaneous symmetry breaking, the conventional derivative expansion of the effective action should be replaced by a new improved expansion series which incorporates the derivative expansion with additional expansion described below. To avoid unnecessary complications, we shall not include internal symmetry in this paper.

In most of the local quantum field theories, the calculation of the one-loop effective action in D dimensions can be effectively reduced to the calculation of the derivative expansion of the action [3],

$$\Gamma[u]=\frac{i}{2}\mathrm{Tr}\ln G(u)^{-1},$$

where G, in coordinate representation, is given by

$$G^{-1}(x,y)=[\partial_x^2+u(x)]\delta^D(x-y) = \int\frac{d^Dp}{(2\pi)^D}[\partial_x^2+u(x)]e^{ip\cdot(x-y)}. \tag{1}$$

The derivative expansion is a functional series of $u(x)$. Higher power derivative terms must be balanced by an appropriate inverse power of $u(x)$. In order for the derivative expansion to converge, it is necessary for the function $u(x)$ to be a slowly varying function for all x such that higher derivatives become smaller and smaller. More importantly, $u(x)$ cannot pass through zero to become negative. The latter case indeed occurs in field theories with spontaneous symmetry breaking. Associated with the spontaneous symmetry breaking is the asymmetric vacuum (ground state) such that

$$\lim_{x\to\infty} u(x) = M^2 > 0. \tag{2}$$

If $u(x)$ is a slow-varying function for all x, the finite difference $U(x) = u(x) - M^2$ should also be small.

Therefore, we propose that *a consistent expansion in this case should be a double expansion series expanding simultaneously in powers of the derivative as well as in powers of* $U(x)$. The expansion parameter can now be readily identified. The large cancellations between the terms in the double expansion render the expansion a much better-behaved series so that an exact or approximated analytic continuation of the series is possible, if the series is not already convergent. As a side benefit of this double expansion, the expansion is much simpler to carry out. This improved expansion not only circumvents some outstanding difficulties concerning the convergence of the expansion series when applied to theories of spontaneously broken symmetry but also leads to a well-defined path of analytic continuation of the series.

The simplification of our formulation of the derivative expansion is based on a hybrid approach that the effective action is kept in a configuration representation while the loop integration is performed conveniently in the momentum representation. The calculation of the effective action is essentially the calculation of the sum of vacuum graphs of a particle propagating in the potential of an external background field. The generalized derivative expansion is essentially a high-loop-energy expansion series of the effective action. The analytic properties of the loop-energy integrand in the complex energy plane is well known from study of the potential scattering and they can be used as additional constraints for the analytic continuation of the generalized derivative expansion series by the Padé approximation. In addition, we propose that the dominant contributions from the low-lying poles can be conveniently isolated such that the convergence of the analytic continuation of the remaining generalized derivative expansion series shows orders of magnitude improvement. The recognition of these properties in the loop-energy space provides a hitherto completely unexploited powerful tool which can turn the derivative expansion into a genuinely useful method of computation.

We have carried out the complete implementation of this approach for static background fields and have successfully calculated the energy for a large collection of models of (1+1)-dimensional solitons. The calculations, which include up to 22-derivative terms in the expansion, using the Padé approximant of the analytic continuation, were performed using the MATHEMATICA program [14]. The capability to do most parts of the calculations analytically and interactively is most important for the successful implementation of this program both in understanding the physics and in the improvement of the approximation. This new method opens up the possibility of asking some more fundamental questions, such as whether one can define the effective potential more meaningfully using some test spatially varying background fields, in the region when the conventional effective potential is not well defined for a constant background field.

In this paper, in order to illustrate the spirit and the power of this new development, we apply this method to the calculation of the one-loop quantum correction to the (1+1)-dimensional (with no internal symmetry) static soliton in various models, in which extensive works on exact solutions and numerical solutions have been carried out [9–13,17,18]. For the models in which exact solutions have been found, such as the sine-Gordon soliton, the ϕ^4 soliton, and the fermion-loop contribution to the ϕ^4 soliton, we are able to deduce a unique pattern of the improved expansion series and, therefore, succeed in resumming the series to obtain the exact results. More importantly, in the cases where exact solutions may not be available, well-known analytic continuation techniques such as the Padé approximant or the Borel summation can be used as a very efficient method to provide an excellent approximation, in contrast with relying on the cumbersome numerical calculations. The fermion-loop contribution to the ϕ^4 soliton mass with the exact solution is used to test the approximated analytic continuation method. Then, we apply the method to the Christ-Lee soliton and the ϕ^4 bag.

Finally, we derive a useful hybrid formula to estimate the quantum correction to the vacuum energy of the static field using the generalized derivative expansion series up to only the two-derivative terms and the low-lying pole energies.

II. AN IMPROVED DERIVATIVE EXPANSION

Following from Eq. (1) and commuting the exponential from the left to the right of the differential operator, we have [3]

$$\Gamma[u] = \frac{i}{2}\int d^Dx \frac{d^Dp}{(4\pi)^D} \ln[(\partial + ip)^2 + u(x)]$$
$$= \int d^Dx \frac{d^Dp}{(4\pi)^D} \ln[-p^2 - i\epsilon + u(x) + 2ip\cdot\partial + \partial^2]. \tag{3}$$

The conventional derivative expansion can be obtained by expanding the logarithmic function in power series of the operator $(2ip\cdot\partial + \partial^2)$. Special care must be taken for the noncommutivity between the two operators $(2ip\cdot\partial + \partial^2)$ and $u(x)$. The p integration can be performed to give

$$\Gamma[u] = \frac{1}{2(4\pi)^{D/2}}\int d^Dx\bigg[\Gamma\left(-\frac{D}{2}\right)u^{D/2} + \frac{1}{12}\Gamma\left(3-\frac{D}{2}\right)u^{-3+D/2}(\partial_\mu u)^2 + \frac{1}{288}\Gamma\left(6-\frac{D}{2}\right)u^{-6+D/2}(\partial_\mu u)^4 + \frac{1}{120}\Gamma\left(4-\frac{D}{2}\right)u^{-4+D/2}(\partial_\alpha\partial_\beta u)^2 - \frac{1}{72}\Gamma\left(5-\frac{D}{2}\right)u^{-5+D/2}\partial^\alpha\partial^\beta u\,\partial_\alpha u\,\partial_\beta u\bigg] + O(\partial^6). \tag{4}$$

This expansion has been calculated to $O(\partial^{10})$. It is clear that if the function $u(x)$ crosses from a positive value to a negative value through the point zero, the expansion series would definitely be invalid. It is also not clear how to analytically continue this functional series.

$u(x)$ can cross zero in a number of contexts. The most interesting case is the spontaneously symmetry-breaking case in which M^2 in Eq. (2) is positive and nonzero. The appropriate expansion should be a double expansion which includes the derivative expansion and the Taylor series expansion of $u(x)$ about $u(x)=M^2$. In principle, the double expansion can be achieved by further expanding Eq. (4) in Taylor series of $U(x)=u(x)-M^2$ to give

$$\begin{aligned}\frac{i}{2}\mathrm{Trln}G^{-1}=&\frac{1}{2(4\pi)^{D/2}}\int d^Dx\left\{\Gamma\left(-\frac{D}{2}\right)M^D-\Gamma\left(1-\frac{D}{2}\right)\right.\\&\times M^{-2+D}U+\frac{1}{2}\Gamma\left(2-\frac{D}{2}\right)M^{-4+D}U^2\\&+\frac{1}{12}\Gamma\left(3-\frac{D}{2}\right)M^{-6+D}[-2U^3+(\partial_\mu U)^2]\\&+\frac{1}{120}\Gamma\left(4-\frac{D}{2}\right)M^{-8+D}[-5U^4-10U(\partial_\mu U)^2\\&\left.+(\partial_\alpha\partial_\beta U)^2]+O(M^{-10+D})\right\}. \qquad (5)\end{aligned}$$

The expansion parameter can now be identified to be $1/M^2$. The first few terms can be used to isolate the divergent terms to be used for renormalization.

However, it is more convenient as well as more correct to develop the double expansion directly from Eq. (3). It is much easier to keep track of the higher order terms and the analytic property in momentum space is much better understood. We can rewrite Eq. (3),

$$\begin{aligned}\Gamma[U]=&\frac{i}{2}\mathrm{Trln}G^{-1}\\=&\frac{i}{2}\int\frac{d^Dx}{(4\pi)^D}d^Dp\ln\{-p^2-i\epsilon+M^2\\&+[U(x)+2ip\cdot\partial+\partial^2]\}, \qquad (6)\end{aligned}$$

and expand it as a power series in $[U(x)+2ip\cdot\partial+\partial^2]$. Since we are expanding the only operator in the expression, there is no complication because of noncommutativity.

III. QUANTUM CORRECTIONS TO THE VACUUM ENERGY OF STATIC BACKGROUND FIELDS

In this paper we shall consider exclusively the time-independent background field $U(\phi(\vec{x}))$. We decompose the momentum vector $p_\mu=(\omega,\vec{p})$ and effectively $\partial_\mu\rightarrow(0,\vec{\nabla})$. In that case we can factor out the constant time integration,

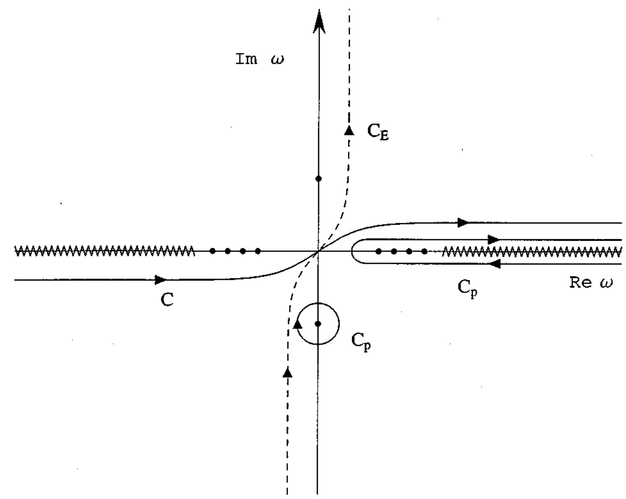

FIG. 1. Singularities and contour paths in the complex plane.

$$\Gamma_i[U]=-\left(\int dt\right)E[U], \qquad (7)$$

and identify the one-loop quantum correction to the field energy for a static field configuration:

$$E[U]=-\frac{i}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathrm{Trln}[-\omega^2-i\epsilon+M^2-\vec{\nabla}^2+U(\phi(x))]. \qquad (8)$$

This integral is in general divergent and requires subtractions to render it finite. With this subtraction understood and to be defined later, we can perform integration by parts for the ω integration:

$$\begin{aligned}E[U]_{\mathrm{sub}}=&-i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2\mathrm{Tr}\\&\times\left[\frac{1}{-\omega^2-i\epsilon+M^2-\nabla^2+U(\phi(x))}\right]_{\mathrm{sub}}. \qquad (9)\end{aligned}$$

Throughout this paper we will refer energy (E) or soliton energy to mean the one-loop quantum correction to the vacuum energy of the static background field, unless it is otherwise specified, such as classical energy ($E_{\mathrm{classical}}$). The trace may be evaluated by solving the quantum-mechanical problem of the Hamiltonian $H=-\nabla^2+U(\phi(x))$.

In Fig. 1, we plot the singularities of the trace function and various contour paths of ω integration in the complex ω plane. The branch cuts extend from $\pm M$ to $\pm\infty$ along the real axes. Between $-M$ and M are the discrete bound state poles at ω_i. Along the imaginary axes are poles $\omega_j^2<0$. The path C is the original contour. With an appropriate subtraction, the contour C can be deformed to the path C_E along the imaginary axes. This path of integration is particularly useful to evaluate the integral when there is no pole on the imaginary axes. In this case the integral is dominated by the lowest-lying bound state pole on the real axes.

This contour may also be deformed to the contour C_p consisting of a set of contours encircling clockwise around poles at the negative imaginary axes and a contour path wrapping around singularities on the positive real axes. The ω integration along the real axes becomes the integration of the discontinuity (imaginary part) of the integrand in the physical region [19]:

$$E[U]=-\frac{i}{2}\sum_j |\omega_j|+2i\int_0^\infty \frac{d\omega}{2\pi}\omega^2 \mathrm{ImTr}\times\left[\frac{1}{-\omega^2-i\epsilon+M^2-\nabla^2+U(\phi(x))}\right]_{\mathrm{sub}}$$

$$=-\frac{i}{2}\sum_j |\omega_j|+\sum_{0<\omega_i<M}\tfrac{1}{2}(\omega_i-M)+\int_M^\infty \frac{d\omega}{2\pi}2\omega^2\frac{d}{d\omega^2}\sum_k \delta_k(\omega^2)\bigg|_{\mathrm{sub}}, \tag{10}$$

where δ_k is the phase shift for the k channel.

While the eigenvalues of the Hamiltonian are relatively simple to calculate numerically, the numerical evaluation of the phase shift for a continuous distribution of ω^2 can consume an enormous amount of computer time, especially for large ω^2 where numerical accuracy suffers because of large cancellations. A hybrid method has been suggested in which the ω integration is performed using the conventional approach from the threshold M up to a certain value Λ and then from Λ to ∞ using the first two terms of the derivative expansion [13]. The convergence of the derivative expansion series may be rather poor for some background fields. It is also not clear how to choose the artificial division Λ to optimize the calculation.

In the physical region, the conventional derivative expansion is equivalent to the WKB approximation which is a high-energy expansion. Since the integrand in Eq. (10) is not an analytic function of ω, it cannot be analytically continued to the complex ω plane.

In the case of absence of any pole on the imaginary axes, a more appropriate contour path in the complex ω plane for the ω integration would be the contour C_E ($\omega=i\omega_E$) along the imaginary axes far away from any singularity. Equation (9) becomes

$$E[U]=-\int_{-\infty}^\infty \frac{d\omega_E}{2\pi}\omega_E^2 \mathrm{Tr}\left[\frac{1}{\omega_E^2-i\epsilon+M^2-\nabla^2+U(\phi(x))}\right]_{\mathrm{sub}}. \tag{11}$$

In this paper the final evaluation of ω integrations will be performed using this path (C_E) while the physical variable ω will be used for intermediate expressions.

In this paper we shall show that our generalized derivative expansion can provide a powerful method to evaluate the expression in Eq. (9). This method can equally be adopted for other quantum-mechanical problems. The trace in Eq. (9) can be evaluated [3]:

$$E[U]=-i\int_{-\infty}^\infty \frac{d\omega}{2\pi}\omega^2\left[\int\frac{d\vec{x}d\vec{p}}{(2\pi)^{D-1}}\times\frac{1}{-\omega^2-i\epsilon+\vec{p}^2+M^2+[U(\vec{x})-2i\vec{p}\cdot\vec{\nabla}-\vec{\nabla}^2]}\right]_{\mathrm{sub}}. \tag{12}$$

After expanding the integrand in power of $[U(\vec{x})-2i\vec{p}\cdot\vec{\nabla}-\vec{\nabla}^2]/(-\omega^2-i\epsilon+\vec{p}^2+M^2)$ and integrating over $\vec{p}$, we obtain

$$E[U]=i\int_{-\infty}^\infty \frac{dw}{2\pi}\omega^2 z^{3/2}A(z), \tag{13}$$

$$A(z)=-z^{-3/2}\mathrm{Tr}\left[\frac{1}{-\omega^2-i\epsilon+M^2-\nabla^2+U(\phi(x))}\right]_{\mathrm{sub}}$$

$$=-\frac{1}{4\pi^{[(D-1)/2]}}\int_{-\infty}^\infty d\vec{x}\Bigg\{-\Gamma\left(\frac{5-D}{2}\right)Uz^{1-D/2}+\frac{1}{2}\Gamma\left(\frac{7-D}{2}\right)U^2z^{2-D/2}-\frac{1}{12}\Gamma\left(\frac{9-D}{2}\right)[2U^3+U_i^2]z^{3-D/2}$$
$$+\frac{1}{120}\Gamma\left(\frac{11-D}{2}\right)[5U^4+10UU_i^2+U_{ij}^2]z^{4-D/2}-\frac{1}{5040}\Gamma\left(\frac{13-D}{2}\right)[42U^5+210U^2U_i^2+70U_iU_jU_{ij}$$
$$+42UU_{ij}^2+3U_{ijk}^2]z^{5-D/2}+\frac{1}{30240}\Gamma\left(\frac{15-D}{2}\right)[42U^6+420U^3U_i^2+420UU_iU_jU_{ij}+126U^2U_{ij}^2+18UU_{ijk}^2$$
$$+105U_i^2U_j^2+22U_{ik}U_{kj}U_{ji}+84U_iU_{jk}U_{ijk}+U_{ijkl}^2]z^{6-D/2}+O(z^{7-D/2})\Bigg\}\Bigg|_{\mathrm{sub}}, \tag{14}$$

where $U_{i_1\cdots i_n}=\vec{\nabla}_{i_1}\cdots\vec{\nabla}_{i_n}U$ and $A(z)$ is expanded in power series of

$$z=\frac{1}{-\omega^2-i\epsilon+M^2}=\frac{1}{\omega_E^2-i\epsilon+M^2}. \quad (15)$$

The coefficient of the z^n term contains terms up to and including one term with $2n-4+D$ derivatives. Integration of x by part has been performed repeatedly on this coefficient to arrive at this unique representation such that there is no more than $n-2+D/2$ derivatives on each U and there is no contraction between derivative components on the same U.

In the region of integration, U is mostly negative. There is a large cancellation among these terms. As the power of z increases, the number of terms and, therefore, the degree of cancellation also increase accordingly. For the examples of (1+1)-dimensional soliton in this paper, the degree of cancellation can be as big as an extra order of magnitude for each higher order in z.

For reference purposes, we find it convenient to characterize the order of the power series by the maximum number of derivatives of a single term in its coefficient rather than the power of z. There should has been a leading term $\Gamma[(3-D)/2]z^{-D/2}$ in the power series $A(z)$, which is independent of U. It is, therefore, divergent even without the ω integration. This term has been dropped without any physical consequence. For terms z^n with $n<1$, the ω integration is divergent. Therefore, subtractions must be carried out to remove such terms. If the same functional forms of these terms appear in the original Lagrangian, the field theory is renormalizable and the subtractions would come from the corresponding counterterms introduced into the original Lagrangian for the purpose of renormalization. Our renormalization prescription is to truncate divergent terms in the beginning of the generalized derivative expansion series and start the series with the leading term z for any dimension D. In this paper all relevant expressions include proper subtraction unless explicitly indicated otherwise.

The exact function $A(z)$ is an analytic function of z or ω, and has singularities of poles and cuts slightly displaced from the real axis. The series in this present form is suitable for analytic continuation. The analytic continuation of the power series in Eq. (14) should yield

$$A(z(\omega))=\sum_{\omega_i^2<M^2}\frac{r_i}{-w^2-i\epsilon+\omega_i^2}+F(\omega)$$

$$=\sum_{\omega_i^2<M^2}\frac{r_i z(\omega)}{1+(\omega_i^2-M^2)z(\omega)}+F(z(\omega)), \quad (16)$$

where $F(\omega)$ is regular everywhere except possibly on the real axes of $\omega>M$ and $F(\omega)$ goes to zero faster than $1/\omega^2$ as $\omega\to\infty$. In order to be consistent with the result from Eq. (10), the coefficients r_i or, equivalently, the residues of the poles, are completely determined:

$$r_i=-(-\omega_i^2+M^2)^{3/2}. \quad (17)$$

In the expression for $\mathcal{E}(\omega)$, the bound state poles are multiplied by a ω-dependent factor $z^{3/2}$ which, in odd space dimension, has a branch cut above the threshold M. The branch cut would represent the continuum contribution. It is not possible to separate from this term the pole contribution and the cut contribution analytically.

We define the contribution of each pole $\omega_i<M^2$ to $A(z)$ to be

$$a(\omega^2;\omega_i^2,M^2)=-\left(\frac{(-\omega_i^2+M^2)^{3/2}}{-\omega^2-i\epsilon+\omega_i^2}\right). \quad (18)$$

For $M^2>\omega_i^2>0$, its contribution to the soliton mass is

$$E_p(\omega_i^2,M^2)=-\frac{1}{\pi}\sqrt{M^2-\omega_i^2}\left(1-\sqrt{\frac{\omega_i^2}{M^2-\omega_i^2}}\times\arctan\sqrt{\frac{M^2-\omega_i^2}{\omega_i^2}}\right). \quad (19)$$

The ω integration has been performed with contour C_E along the imaginary axes. It is interesting to point out that the discrete bound state pole contribution to the soliton mass E_B after the subtraction is independent of the dimension D.

Equation (19) can be analytically continued to $\omega_i^2<0$:

$$E_p(\omega_i^2,M^2)=-\frac{i}{2}|\omega_i|-\frac{1}{\pi}\sqrt{M^2+|\omega_i|^2}\times\left(1-\frac{1}{2}\sqrt{\frac{|\omega_i|^2}{M^2+|\omega_i|^2}}\times\ln\frac{1+\sqrt{\frac{|\omega_i|^2}{M^2+|\omega_i|^2}}}{1-\sqrt{\frac{|\omega_i|^2}{M^2+|\omega_i|^2}}}\right). \quad (20)$$

At $\omega_i=0$, $E_p(0,M^2)=-M/\pi$.

To remove a pole at ω_i and its influence, from $A(z(\omega))$ analytically, it is necessary to subtract not just the simple pole but the entire expression. The extraction procedure is defined by the subtraction

$$\widetilde{A}(z(\omega),-\{\omega_1,\ldots,\omega_n\})=A(z(\omega))-\sum_{i=1}^{n}a(\omega^2;\omega_i^2,M^2), \quad (21)$$

where $\widetilde{A}(z(\omega),-\{\omega_1,\ldots,\omega_n\})$ is an analytic function of ω and has the identical singularities as $A(z(\omega))$ except that the poles at $\{\omega_1,\ldots,\omega_n\}$ have been completely removed. We express

$$\widetilde{E}(-\{\omega_1,\ldots,\omega_n\})=E-\sum_{i=1}^{n}E_p(\omega_i,M^2) \quad (22)$$

$$=i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2 z^{3/2}\widetilde{A}(z(\omega),-\{\omega_1,\ldots,\omega_n\}). \quad (23)$$

We can now expand $a[\omega^2(z);\omega_i^2,M^2]$ in a power series in z:

$$a[\omega^2(z);\omega_i^2,M^2]=-(-\omega_i^2+M^2)^{3/2}\frac{z}{1-z(M^2-\omega_i^2)}$$
$$=-(-\omega_i^2+M^2)^{3/2}[z+z^2(M^2-\omega_i^2)+z^3(M^2-\omega_i^2)^2+\cdots], \tag{24}$$

and combine this expansion with the expansion $A(z)$ in Eq. (14) to give a new power series $\widetilde{A}(z,-\{\omega_1,\ldots,\omega_n\})$. We shall refer to this process of removing the pole contribution analytically as pole extraction and to the new power series $\widetilde{A}(z,-\{\omega_1,\ldots,\omega_n\})$ as remainder series.

We would like to evaluate the ω integration with the contour path along the imaginary axes such that the path would be far away from the singularities. The integral is dominated by the nearest singularity, in this case, lowest-lying poles which also control the convergence of the generalized derivative expansion series. If we can extract the low-lying poles in a manner described above, we would have replaced the dominant pole contributions by the known analytic expression, Eq. (19), and at the same time we would have weakened the continuum (cut) contributions substantially. The remaining singularities would be much farther away from the integration path and the remainder series would be much more convergent.

However, the location of the low-lying bound states may not be given explicitly; we may choose to proceed in one of the following ways or some combinations of them.

(1) For a soliton background field, spatial translational invariance implies the existence of zero frequency ($\omega_i=0$) pole(s) whose contribution to the soliton mass can be extracted easily. Even though the zero-frequency pole is superficially removed by the overall factor of ω^2 in the integrand of Eq. (12), there is known continuum contribution associated with this mode which has to be removed by our extraction procedure.

(2) The remainder power series $\widetilde{A}(z)$ may be analytically continued by Padé approximants. Let $A_{n,m}$ be the $\{n,m\}$ Padé approximant of $\widetilde{A}(z)$. $A_{n,m}$ is a ratio of two polynomials of z of order n and m and its power series expansion matches that of the power series $A(z)$ for the first $n+m$ terms. Since there is an overall factor of z in $A(z)$, we shall take the Padé sequence $A_{n,n}$. To calculate $A_{n,n}$ requires the series $A(z)$ calculated up to the z^{2n} term or up to terms containing $4n-4+D$ derivatives. We rewrite $A_{n,n}$ as a ratio of two polynomials in ω^2. After factorizing the denominator and performing partial fractions, we can then express the result as

$$A_{n,n}=-\sum_{i=1}^{k}\frac{b_{n,i}(-\alpha_{n,i}^2+M^2)^{3/2}}{-\omega^2-i\epsilon+\alpha_{n,i}^2}-\sum_{i=k}^{n}\frac{c_{n,i}M(-\alpha_{n,i}^2+M^2)}{-\omega^2-i\epsilon+\alpha_{n,i}^2}, \tag{25}$$

where $\alpha_{n,i}$ is arranged in order such that $\alpha_{n,i}^2<\alpha_{n,j}^2$ for $i<j$, $\alpha_{n,i}^2<M^2$ for $i<k$, and $\alpha_{n,i}>M$ for $i>k$. The poles at $\alpha_i^2<M^2$ are physical poles whereas the poles at $\alpha_i^2>M^2$ are there to approximate the additional continuum contributions. As n increases, $\alpha_{n,i}^2<M^2$ converges to the correct eigenvalues ω_i^2 from above while $\alpha_{n,i}>M^2$ increases in number and becomes more dense. Equations (16) and (17) imply that for an exact solution $b_{n,i}$ must be 1 or a positive integer which represents the multiplicity of the degenerate eigenstates with eigenvalue ω_i^2. For the Padé approximant, $b_{n,i}>1$ approaches 1 or integer values from above. The deviation of the value of $b_{n,i}$ from 1 can be used as a measure of the deviation of $\alpha_{n,i}^2$ from the correct value ω_i^2. In principle, it is possible to exploit the constraint of the b's exactly or iteratively for better approximation of the α's. We have developed some preliminary methods which show great improvement over the standard Padé approximation. Our goal is to calculate the soliton mass here. With perhaps some exceptions, there is no need to go beyond the present approach. In this approximation, the quantum correction to the soliton mass is given by

$$E_n=-\sum_{i=1}^{k}b_{n,i}E_p(\alpha_{n,i}^2,M^2)-\sum_{i=k}^{n}c_{n,i}E_c(\alpha_{n,i}^2,M^2), \tag{26}$$

where

$$E_c(\alpha^2,M^2)=-\frac{M}{\pi}\left[1-\frac{1}{2\sqrt{1-M^2/\alpha^2}}\times\ln\left(\frac{1+\sqrt{1-\frac{M^2}{\alpha^2}}}{1-\sqrt{1-\frac{M^2}{\alpha^2}}}\right)\right] \tag{27}$$

is the analytic continuation of the function $(1/\sqrt{1-M^2/\alpha^2})E_p(\alpha^2,M^2)$ from $\alpha^2<M^2$ to $\alpha^2>M^2$.

(3) It is relatively simple to calculate numerically the low-lying ω_i^2 for a given potential $U(\phi(x))$. Extracting just one lowest-lying pole would improve the convergence of the Padé approximant greatly, which means that fewer higher derivative terms will be needed. In general, we would like to extract all poles along the imaginary axes and those along the real axes with $\omega_i^2\ll M^2$, such that the integration path can be rotated to C_E, the imaginary axes, and away from any singularity.

In the following section we shall use this result to calculate the quantum correction to the soliton energy.

IV. SCALAR FIELD

The Lagrangian density for a scalar field coupled to a fermion is

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-V(\phi)+\overline{\psi}[i\partial\!\!\!/-V_F(\phi)]\psi, \tag{28}$$

where $V(\phi)$ satisfies reflection symmetry $V(-\phi)=V(\phi)$ so that the Lagrangian is reflection symmetric with respect to ϕ. Spontaneous symmetry breaking occurs when there are more than two degenerate absolute minima $\phi=\pm\phi_0$ such that $V'(\pm\phi_0)=0$ and $V''(\pm\phi_0)=M^2>0$. We also choose to normalize $V(\pm\phi_0)=0$. For a finite energy field, $\phi(x)$ must satisfy the asymptotic conditions that as $|x|\to\infty$:

$$\begin{aligned} |\phi(x)| &\to \phi_0, \\ V(\phi(x)) &\to 0, \\ V'(\phi(x)) &\to 0, \\ V''(\phi(x)) &\to M^2. \end{aligned} \tag{29}$$

In the one-loop order, the effective action is given by

$$\Gamma[\phi]=\Gamma_0[\phi]+\Gamma_B[\phi]+\Gamma_F[\phi]=\int dx\mathcal{L}_{\text{eff}}, \tag{30}$$

where

$$\Gamma_0=\int dx\mathcal{L}, \tag{31}$$

$$\Gamma_B=\frac{i}{2}\text{Trln}[\partial^2+V''(\phi(x))]|_{\text{sub}}, \tag{32}$$

$$\Gamma_F=-i\text{Trln}[-(i\partial\!\!\!/-g\phi)]|_{\text{sub}}. \tag{33}$$

The derivative expansion of Γ_B in Eq. (32) is given by Eq. (4) with $u(x)=V''(\phi(x))$. Since $V'(\pm\phi_0)=0$, unless $V'(\phi)$ is identically zero for the whole range $-\phi_0<\phi<\phi_0, V'(\phi)$ must have at least a maximum or minimum and, therefore, $V''(\phi)$ must has at least one zero in this range. From $V''(\pm\phi_0)=M_B^2>0$ it follows that $V''(\phi)$ must have at least two zeros in this range, which means that $u(x)=V''(\phi)$ must go through zero to a negative value and back to a positive value at least once. In that case the derivative expansion, Eq. (4), may not be appropriate. The proper expansion should be Eq. (14) with

$$U(\phi(x))=V''(\phi(x))-M_B^2, \tag{34}$$

where $V''(\phi)=(d^2/d\phi^2)V(\phi(x))$.

In the (1+1)-dimensional case and for the static field configuration $\phi(\vec{x})$, we have calculated the series up to the z^{16} term using MATHEMATICA [14]. For the meson loop, we obtain

$$E_B[U]=i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathcal{E}_B(\omega)=i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2 z^{3/2}A_B(z), \tag{35}$$

$$\begin{aligned} A_B(z)=-\frac{1}{2}\int_{-\infty}^{\infty}dx\bigg\{&\frac{3}{8}zU^2-\frac{5}{32}z^2[2U^3+(U')^2] \\ &+\frac{7}{128}z^3[5U^4+10U(U')^2+(U'')^2]-\frac{9}{512}z^4[14U^5 \\ &+70U^2(U')^2+14U(U'')^2+(U''')^2]+\frac{11}{2048}z^5[42U^6 \\ &+420U^3(U')^2-35(U')^4+126U^2(U'')^2-20(U'')^3 \\ &+18U(U''')^2+(U'''')^2]+\cdots+O(z^{15})\bigg\}, \end{aligned} \tag{36}$$

where $U'(x)=(d/dx)U(\phi(x))$, $U''(x)=(d^2/dx^2)U(\phi(x))$, ... A more complete expression of $A_B(z)$ up to z^{13} is given in the Appendix.

For the fermion loop, we obtain

$$\begin{aligned} E_F[U]&=-\frac{1}{2}\{E_B[W+V_F'(x)]+E_B[W-V_F'(x)]\} \\ &=i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2 z^{3/2}A_F(z), \end{aligned}$$

$$\begin{aligned} A_F(z)=\frac{1}{2}\int_{-\infty}^{\infty}dx\bigg\{&\frac{3}{8}z[W^2+(V_F')^2]-\frac{5}{32}z^2[2W^3+(W')^2 \\ &+6W(V_F')^2+(V_F'')^2]+\frac{7}{128}z^4[5W^3+10W(W')^2 \\ &+(W'')^2+30W^2(V_F')^2+10W(V_F'')^2+20V_F'V_F''W' \\ &+5(V_F')^4+(V_F''')^2]+\cdots+O(z^{14})\bigg\} \\ =\frac{1}{2}\int_{-\infty}^{\infty}dx\bigg\{&\frac{3}{8}z[W^2+(V_F')^2]-\frac{5}{32}z^2[2W^3 \\ &+4V_F^2(V_F')^2+6W(V_F')^2+(V_F'')^2]+\frac{7}{128}z^3[5W^4 \\ &+30W^2(V_F')^2+10W(V_F'')^2+40W(V_F)^2(V_F')^2 \\ &-7(V_F')^4+4V_F^2(V_F'')^2+(V_F''')^2]+\cdots+O(z^{14})\bigg\}, \end{aligned} \tag{37}$$

where we have defined $W=W(\phi(x))=V_F(\phi(x))^2-M_F^2$ with $M_F^2=V_F(\phi_0)^2$, $z=1(-\omega^2-i\epsilon+M_F^2)$, and $V_F(x)=V_F(\phi(x))$.

In this paper we shall examine the renormalizable models in Table I. The infinite subtraction term linear in U has been absorbed into the renormalization of V.

V. ONE-LOOP CORRECTIONS TO SOLITONS IN (1+1)-DIMENSIONAL THEORIES

The (1+1)-dimensional soliton solution $\phi_s(x)$ is the solution of the differential equation

$$\frac{d^2}{dx^2}\phi_s(x)=\frac{d}{d\phi_s(x)}V(\phi_s(x)), \tag{38}$$

which can be integrated by an integrating factor $(d/dx)\phi_s(x)$:

$$\left(\frac{d}{dx}\phi_s(x)\right)^2=2V(\phi_s(x)). \tag{39}$$

This equation can be further integrated with the boundary condition that $\phi_s(x)\to\phi_0$ as $|x|\to\infty$ and ϕ_0 is defined by Eq. (29), we get

TABLE I. Models of the (1+1)-dimensional soliton.

Model	$V(\phi)$	ϕ_0	M^2	$U(\phi)$	$E_{\text{classical}}$
sine-Gordon	$\frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)+1\right]$	$\frac{m\pi}{\sqrt{\lambda}}$	m^2	$-m^2\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)+1\right]$	$8\frac{m^3}{\lambda}$
ϕ^4	$\frac{1}{8}\lambda\left(\phi^2-\frac{m^2}{\lambda}\right)^2$	$\frac{m}{\sqrt{\lambda}}$	m^2	$\frac{3}{2}\lambda\left(\phi^2-\frac{m^2}{\lambda}\right)$	$\frac{2}{3}\frac{m^3}{\lambda}$
Christ-Lee	$\frac{1}{8}\frac{\lambda}{(1+c^2)}\left(\phi^2-\frac{m^2}{\lambda}\right)^2 \times\left(1+c^2\frac{\lambda}{m^2}\phi^2\right)$	$\frac{m}{\sqrt{\lambda}}$	m^2	$\frac{3}{4}\frac{\lambda}{(1+c^2)}\left(\phi^2-\frac{m^2}{\lambda}\right)\times\left(2+c^2+5c^2\frac{\lambda}{m^2}\phi^2\right)$	$\frac{m^3}{\lambda}\frac{1}{8c^3\sqrt{1+c^2}}[c\sqrt{1+c^2}(2c^2-1)+(1+4c^2)\ln(1+\sqrt{1+c^2})]$
Fermion Yukawa Coupling	$g\phi$	ϕ_0	$g^2\phi_0^2$	$g^2[\phi(x)^2-\phi_0^2]\pm g\phi'(x)$	

$$x-x_0=\int_0^{\phi_s}\frac{d\phi}{\sqrt{2V(\phi)}},\tag{40}$$

where x_0 is defined as the center of the soliton such that $\phi_s(x_0)=0$ and is set equal to zero in this paper with no loss of generality. Inverting this equation yields the soliton profile $\phi_s(x)$. However, it is much simpler to compute the classical energy of the soliton and its quantum correction directly using Eq. (39) rather than the solution $\phi_s(x)$. The classical energy is given by

$$E_{\text{classical}}=\int_{-\infty}^{\infty}dxV(\phi_s(x))=\int_{-\phi_0}^{\phi_0}d\phi_s\frac{2}{\left|\frac{d\phi_s}{dx}\right|}V(\phi_s)$$
$$=2\sqrt{2}\int_0^{\phi_0}d\phi\sqrt{V(\phi)},\tag{41}$$

and the corresponding values for various models are given in Table I.

The normal modes $\eta_i(x)$ of fluctuations around $\phi_s(x)$ satisfy the eigenvalue equation

$$\left[-\frac{d^2}{dx^2}+V''(\phi_s(x))\right]\eta_i(x)=\omega_i^2\eta_i(x),\tag{42}$$

with the boundary condition $\eta(x)\to 0$ as $|x|\to\infty$. The ground state $\eta_0(x)=d\phi_s/dx$ is the zero-energy (translational) mode ($\omega_0=0$) obtained directly from differentiating Eq. (38). The spectrum contains possible discrete eigenvalues of positive ω_i^2 and a real continuous spectrum above the threshold M^2.

The solution of this eigenvalue problem can be used to evaluate the trace in Eq. (9):

$$\mathrm{Tr}\left.\frac{1}{-\omega^2-i\epsilon-\frac{d^2}{dx^2}+V''(\phi_s(x))}\right|_{\text{sub}}.$$

However, the technical problems of solving these differential equations and evaluating the trace are highly nontrivial. Although exact solutions of the sine-Gordon and the ϕ^4 solitons have been found, there is no simple method of approximation to evaluate the quantum correction to the soliton energy efficiently without heavily relying on tedious and CPU time-consuming numerical computation. Derivative expansion can be combined with numerical calculation to reduce the computation time substantially [13]. However, it cannot replace the numerical calculation entirely because of the deficiency of the derivative expansion in the presence of the spontaneous breaking of symmetry. This difficulty can be completely removed by the new improved expansion proposed in this paper.

In this new approach, there is no need for solving any differential equation. Instead, we use Eq. (39) to transform the integration in Eq. (36) from x to ϕ_s. The spatial derivatives of $U(\phi_s(x))$ can be expressed in terms of function only of ϕ_s. In this section, we deal exclusively with the soliton solution as the background field and we shall suppress the subscript s:

$$U'(\phi(x))\to\sqrt{2}\sqrt{V(\phi)}V^{(3)}(\phi),$$

$$U''(\phi(x))\to V'(\phi)V^{(3)}(\phi)+2V(\phi)V^{(4)}(\phi),$$

$$U^{(3)}(\phi(x))\to\sqrt{2}\sqrt{V(\phi)}[V''(\phi)V^{(3)}(\phi)+3V'(\phi)V^{(4)}(\phi)+2V(\phi)V^{(5)}(\phi)].$$

Equation (36) can be rewritten in terms of the integration of ϕ:

$$A_B(\omega)=-\int_0^{\phi_0}\frac{d\phi}{\sqrt{2V(\phi)}}\Bigg\{\frac{3}{8}zU(\phi)^2-\frac{5}{16}z^2\{U(\phi)^3+V(\phi)\times[V^{(3)}(\phi)]^2\}+\frac{7}{128}z^3\{5U(\phi)^4+20U(\phi)^2V(\phi)\times[V^{(3)}(\phi)]^2+[V'(\phi)V^{(3)}(\phi)+2V(\phi)V^{(4)}(\phi)]^2\}+\cdots+O(z^{14})\Bigg\}.\tag{43}$$

With the functions of ϕ given for various models in Table I, the integration of ϕ can be carried out analytically.

Because of translational invariance, there is a zero-frequency mode $\omega_0=0$ for Eq. (42), with the eigenfunction $\eta_0(x)=(d/dx)\phi_s(x)$. We can obtain a much improved power series by extracting out this zero frequency pole from Eq. (43). Using Eqs. (18)–(24) and noting that $E_B(0,M^2)=-M/\pi$, we obtain

$$E_{\text{soliton}}=E_p(0,M^2)+i\int\frac{d\omega}{2\pi}\omega^2 z^{3/2}\widetilde{A}_B[z(\omega),-\{0\}]$$
$$=-\frac{M}{\pi}+i\int\frac{d\omega}{2\pi}\omega^2 z^{3/2}[A_B(z(\omega))+zM^3\times(1+zM^2+z^2M^4+\cdots)]. \qquad (44)$$

If the path of ω integration is taken along the imaginary axes, (C_E), then

$$0<M^2z=\frac{M^2}{\omega_E^2+M^2}<1. \qquad (45)$$

This series and the series in the following examples are convergent series.

A. The sine-Gordon soliton

We can use $V(\phi)$ and $U(\phi)$ for the sine-Gordon soliton in Table I to evaluate Eq. (43). After the ϕ integration, we find

$$A_B^{\text{sine-Gordon}}(z)=-m^3z[1+m^2z+m^4z^2+\cdots+m^{24}z^{13}+O(z^{14})]$$
$$=-\frac{m^3z}{1-m^2z}=\frac{m^3}{\omega^2}=a(\omega^2;0,m^2). \qquad (46)$$

The $\{1,1\}$ Padé approximant using the first two terms of the series up to only the two-derivative contribution has already converged to the exact solution and generates the required zero-frequency bound state of the translational invariance. Higher order Padé approximant $\{n,n\}$ with $n>1$ gives the same result. Alternatively, we can apply Eq. (44) directly to calculate the static energy. Since

$$\widetilde{A}_B^{\text{sine-Gordon}}(z,-\{0\})=0+O(z^{14}),$$

we obtain the correct energy with no effort

$$E_B^{\text{sine-Gordon}}=-\frac{m}{\pi}.$$

B. The ϕ^4 soliton

Similarly, for the ϕ^4 soliton, we obtain

$$A_B^{\phi^4}(z)=-m^3z\left[\frac{9}{8}+\frac{33}{32}m^2z+\frac{129}{128}m^4z^2+\frac{513}{512}m^6z^3+\cdots+O(z^{13})\right]$$
$$=-\frac{m^3z}{1-m^2z}-\frac{1}{8}\frac{m^3z}{1-\frac{1}{4}m^2z}$$
$$=\frac{m^3}{\omega^2}+\frac{1}{8}\frac{m^3}{\omega^2-\frac{3}{4}m^2}$$
$$=a(\omega^2;0,m^2)+a(\omega^2;\tfrac{3}{4}m^2,m^2). \qquad (47)$$

The summation of this series is less obvious because it requires $\{2,2\}$ with six-derivative or higher order Padé approximant to converge to the exact sum. By extracting the zero-frequency bound state pole, we have

$$\widetilde{A}_B^{\phi^4}(z)=-\frac{1}{8}m^3\left[1+\frac{m^2z}{4}+\left(\frac{m^2z}{4}\right)^2+\left(\frac{m^2z}{4}\right)^3+\cdots+\left(\frac{m^2z}{4}\right)^{12}+O(z^{13})\right]=\frac{1}{8}\frac{m^3}{\omega^2-\frac{3}{4}m^2}$$
$$=a(\omega^2;\tfrac{3}{4}m^2,m^2). \qquad (48)$$

This convergent geometric series can be summed by $\{1,1\}$ Padé approximant and

$$\widetilde{E}_B^{\phi^4}=-\frac{m}{\pi}+E_p(\tfrac{3}{4}m^2,m^2)=m\left[-\frac{1}{\pi}+\left(\frac{1}{4\sqrt{3}}-\frac{1}{2\pi}\right)\right]$$
$$=m\left(\frac{1}{4\sqrt{3}}-\frac{3}{2\pi}\right).$$

Thus, we have recovered again the exact solutions for the ϕ^4 solitons [19] using terms only up to the two derivative.

C. Fermion-loop contribution to the ϕ^4 soliton with Yukawa coupling: Exact

The Yukawa coupling is defined by

$$V_F(\phi(x))=g\phi(x), \qquad (49)$$

and from this one deduces the fermion mass

$$M_F=g\phi_0=g\frac{m}{\sqrt{\lambda}}. \qquad (50)$$

It is convenient to define the dimensionless variables

$$G=\frac{2M_F}{m}=\frac{2g}{\sqrt{\lambda}}, \qquad (51)$$

$$\nu = i\nu_E = \frac{2\omega}{m} = \sqrt{-\frac{1}{Z} + G^2}, \tag{52}$$

$$Z = \frac{1}{4}m^2 z = \frac{1}{4}\,\frac{m^2}{-\omega^2 - i\epsilon + \frac{1}{4}m^2G^2}$$
$$= \frac{1}{-\nu^2 - i\epsilon + G^2} = \frac{1}{\nu_E^2 + G^2}. \tag{53}$$

The fermion-loop contribution to the ϕ^4 soliton mass is

$$E_F(G) = i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathcal{E}_F(\omega) = i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega^2 z^{3/2}A_F(z(\omega),G)$$
$$= i\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\frac{m}{2}\mathcal{E}_F(\nu), \tag{54}$$

where

$$A_F(Z,G) = \frac{m}{2}\{\tfrac{1}{2}G^2(G^2+1)Z + \tfrac{1}{6}G^2(2G^4+5G^2-1)Z^2$$
$$+ \tfrac{1}{12}G^2(3G^6+14G^4-7G^2+2)Z^3$$
$$+ \tfrac{1}{10}G^2(2G^8+15G^6-14G^4+10G^2-3)Z^4$$
$$+ \cdots + O(Z^{15})\}. \tag{55}$$

It is clear that $A_F(z,0)=0$ and $E_F(0)=0$. Following Eqs. (18)–(24) and folding in the extra factor of -1 for the fermion loop, we extract out the zero-frequency mode:

$$E_F(G) = \frac{Gm}{2\pi} + i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega^2 z^{3/2}\widetilde{A}_F(z(\omega),G). \tag{56}$$

The first term comes from $-E_p(0,M_F^2=\frac{1}{4}m^2G^2)$ and

$$\widetilde{A}_F(Z,G) = A_F(Z,G) + a(\omega,0,\tfrac{1}{4}m^2G^2)$$
$$= A_F(z,G) - \tfrac{1}{2}mG^3Z(1+G^2Z+G^4Z^2+\cdots)$$
$$= \tfrac{1}{2}m\{\tfrac{1}{2}G^2(G-1)^2Z + \tfrac{1}{6}G^2(G-1)^2(2G^2-2G-1)Z^2 + \tfrac{1}{12}G^2(G-1)^2(3G^4-6G^3-G^2+4G+2)Z^3$$
$$+ \tfrac{1}{10}G^2(G-1)^2(G^2-G-1)(2G^4-4G^3-G^2+3G+3)Z^4 + \cdots + O(Z^{15})\}. \tag{57}$$

When G is an integer, this series can be rewritten and analytically continued:

$$\widetilde{A}_F(Z,G) = m\sum_{n=1}^{G-1}[n^3Z + n^5Z^2 + n^7Z^3 + \cdots + n^{29}Z^{14} + O(Z^{15})] = m\sum_{n=1}^{G-1}\frac{n^3Z}{1-n^2Z} = 2\sum_{n=1}^{G-1}\frac{\frac{1}{8}n^3m^3}{-\omega^2 - i\epsilon + \frac{1}{4}m^2G^2 - \frac{1}{4}n^2m^2}$$
$$= 2\sum_{n=1}^{G-1}\frac{(M_F^2-\omega_n^2)^{3/2}}{-\omega^2 - i\epsilon + \omega_n^2} = -2\sum_{n=1}^{G-1}a(\omega^2,\omega_n^2,\tfrac{1}{4}m^2G^2), \tag{58}$$

where $\omega_n^2 = \frac{1}{4}m^2(G^2-n^2)$ for $n=1,\ldots,G-1$, are the $G-1$ doubly degenerate ($n=\pm|n|$) bound state poles and $M_F^2 = \frac{1}{4}m^2G^2$ is the threshold. Therefore, for integer values of G, we recover the exact solution [20]

$$A_F(\omega,G) = -a(\omega,0,\tfrac{1}{4}m^2G^2) - 2\sum_{n=1}^{G-1}a[\omega,\tfrac{1}{4}m^2(G^2-n^2),\tfrac{1}{4}m^2G^2], \tag{59}$$

and

$$E_F(G) = -E_p(0,\tfrac{1}{4}m^2G^2) - 2\sum_{n=1}^{G-1}E_p[\tfrac{1}{4}m^2(G^2-n^2),\tfrac{1}{4}m^2G^2] = \frac{m}{2\pi}G + \frac{m}{\pi}\sum_{n=1}^{G-1}\left(n - \sqrt{G^2-n^2}\arctan\sqrt{\frac{n^2}{G^2-n^2}}\right)$$
$$= \frac{m}{2\pi}\left(G^2 - 2\sum_{n=1}^{G-1}\sqrt{G^2-n^2}\arctan\sqrt{\frac{n^2}{G^2-n^2}}\right). \tag{60}$$

The analytic continuation of Eq. (58) from integer G to noninteger value can obtained by rewriting the expression

$$\widetilde{A}_F(Z,G) = m\sum_{n=1}^{G-1}\frac{n^3Z}{1-n^2Z} = \frac{m}{2Z}\sum_{n=1}^{G-1}\left[\left(\frac{1}{Z^{-1/2}-n} - \frac{1}{Z^{-1/2}+n}\right) - G(G-1)Z\right] = \frac{m}{2Z}\left[\frac{d}{dZ^{-1/2}}\ln\prod_{n=1}^{G-1}\frac{Z^{-1/2}-n}{Z^{-1/2}+n} - G(G-1)Z\right]$$
$$= \frac{m}{2Z}\left[\frac{d}{dZ^{-1/2}}\ln\frac{\Gamma(1+Z^{-1/2})^2}{\Gamma(1+Z^{-1/2}+G)\Gamma(1+Z^{-1/2}-G)} + \frac{1}{Z^{-1/2}+G} - Z^{\frac{1}{2}} - G(G-1)Z\right]. \tag{61}$$

The final result for the fermion-loop contribution to the ϕ^4 soliton mass for any real value of G is given by

$$E_F^{\phi^4}=\frac{m}{2\pi}\int_0^\infty d\nu_E\left\{\frac{\nu_E^2}{\sqrt{\nu_E^2+G^2}}\left[2\psi(1+\sqrt{\nu_E^2+G^2})-\psi(1+\sqrt{\nu_E^2+G^2}+G)-\psi(1+\sqrt{\nu_E^2+G^2}-G)-\frac{G^2}{\nu_E^2+G^2}\right]+\frac{G^2}{\nu_E^2+G^2}\right\} \tag{62}$$

$$=\frac{m}{2\pi}\int_0^\infty d\nu_E\left(\frac{G^2}{\nu_E^2+G^2}-\frac{G^2}{\sqrt{\nu_E^2+G^2}}-\ln\frac{\Gamma(1+\sqrt{\nu_E^2+G^2})^2}{\Gamma(1+\sqrt{\nu_E^2+G^2}+G)\Gamma(1+\sqrt{\nu_E^2+G^2}-G)}\right), \tag{63}$$

where $\psi(x)=(d/dx)\ln\Gamma(x)$ is the polygamma function. Integration by part is performed in the last equality using the derivative from the definition of ψ.

The integrand is an analytic function of ν. Our expression appears to have a much simpler form than that of the previous exact solution but is otherwise completely equivalent [25]. The arguments in the γ function and the polygamma function are always real and greater than 1. In principle, the integral in Eq. (62) can be evaluated numerically. However, for large G and large ν_E, there exists a many order of magnitude cancellation between terms in the integrand which renders the numerical integration either inaccurate or time consuming. In that case it would be useful to replace the integrand by its asymptotic expansion, such as the expansion equations (54) and (55) for the integrand in Eq. (62). The loss of accuracy from the high-frequency contribution has been a problem in any numerical method in this type of calculation. Derivative expansions have been found to be extremely useful to supplement numerical methods, even in the case that the complete solution has been reduced to a quadrature form. For $G>20$, the numerical evaluation of the integrand in Eq. (62) has a large uncertainty for large ν_E. It becomes more practical to evaluate the numerical integration by breaking up the integration into a sum of two integrations: use Eq. (62) to integrate ν_E from 0 to $\overline{\nu_E}$ and use the expansion equations (54) and (55) to integrate ν_E from $\overline{\nu_E}$ to ∞. The choice of $\overline{\nu_E}$ depends on how many terms in the expansion equations (54) and (55) are being included.

Equation (62) may also be derived directly from Eqs. (54) and (55) by Borel summation. To the order indicated in Eq. (55), the coefficients of this series can be expressed in terms of the Bernoulli polynomials $B_n(x)$:

$$A_F(\nu)=\frac{m}{2Z}\sum_{n=2}^{\infty}\frac{1}{2n}Z^n\{B_{2n}(G)+B_{2n}(-G)-2B_{2n}\}$$
$$=\frac{m}{2Z}\left[\left(\sum_{n=1}^{\infty}\frac{1}{n}Z^{n/2}\{B_n(G)+B_n(-G)-2B_n\}-ZG^2\right)-\frac{G^2}{\nu^2}\sqrt{Z}\right]. \tag{64}$$

The Bernoulli polynomials are given by

$$B_n(x)=\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}B_m x^{n-m}$$

and B_n are the Bernoulli numbers. Using the generating functional of the Bernoulli polynomials

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},$$

we can analytically continue the series A_F in Eq. (64) by Borel summation:

$$A_F(\nu)=\frac{m}{2Z}\left(\int_0^\infty ds e^{-s}\sqrt{Z}\left[\frac{e^{Gs\sqrt{Z}}+e^{-Gs\sqrt{Z}}-2}{e^{s\sqrt{Z}}-1}\right]+ZG^2+\frac{G^2}{\nu^2}\sqrt{Z}\right) \tag{65}$$

$$=\frac{m}{2Z}\left[\int_0^\infty dt\left(\frac{e^{-(1+Z^{-1/2}+G)t}+e^{-(1+Z^{-1/2}-G)t}-2e^{-(1+Z^{-1/2})t}}{1-e^{-t}}\right)+ZG^2+\frac{G^2}{\nu^2}\sqrt{Z}\right], \tag{66}$$

where we have changed the integration variable to $t=s\sqrt{Z}$. The last integral can be expressed in terms of the polygamma function defined by

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\int_0^\infty dt\left[\frac{e^{-t}}{t}-\frac{e^{-xt}}{1-e^{-t}}\right]. \tag{67}$$

We obtain

$$E_F^{\phi^4}=\frac{m}{4\pi}i\int_{-\infty}^{\infty}d\nu\sqrt{Z}\{\nu^2[2\psi(1+Z^{-1/2})-\psi(1+Z^{-1/2}+G)-\psi(1+Z^{-1/2}-G)-ZG^2]-G^2\sqrt{Z}\}, \tag{68}$$

which becomes Eq. (62) after rotating the path of integration in the complex ν plane by 90°, $\nu=i\nu_E$.

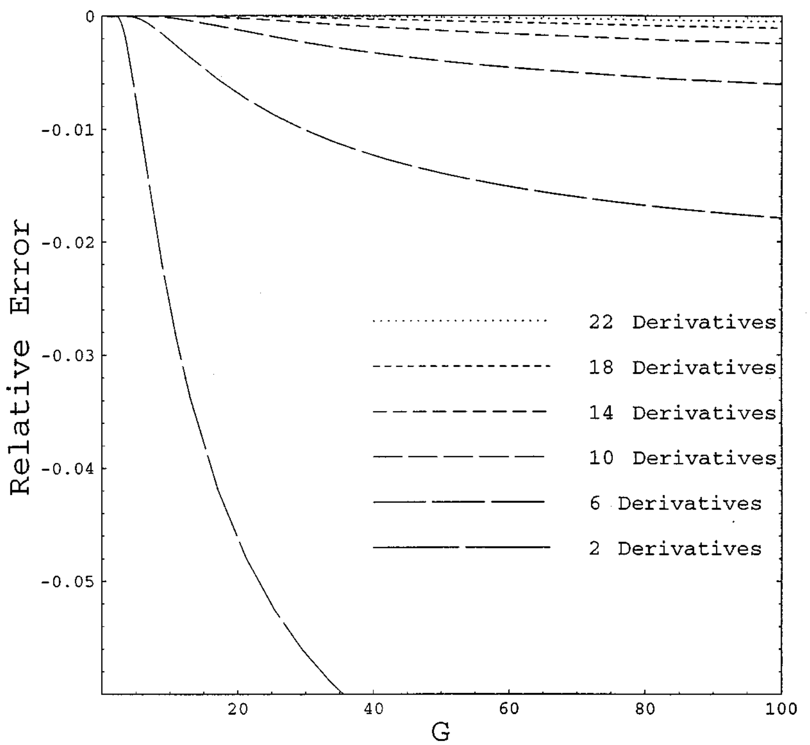

FIG. 2. Relative error of the fermion-loop correction to the ϕ^4 soliton mass.

D. Fermion-loop contribution to the ϕ^4 soliton with Yukawa coupling: Approximation

For the examples in the previous sections, the expansion series are relatively simple that we are able to deduce the unique pattern of the infinite series by calculating more than sufficient but still a finite number of terms. Perhaps, this simplicity is related to the fact that the potentials are nonreflective and the problems are exactly solvable. However, if the series is not recognizable, we may have to rely on a sequence of Padé approximants to approximate the analytic continuation of the generalized derivative expansion. In order to evaluate the reliability of this approximation, we shall apply the scheme outlined in Sec. III to the fermion-loop contribution to the ϕ^4 soliton mass.

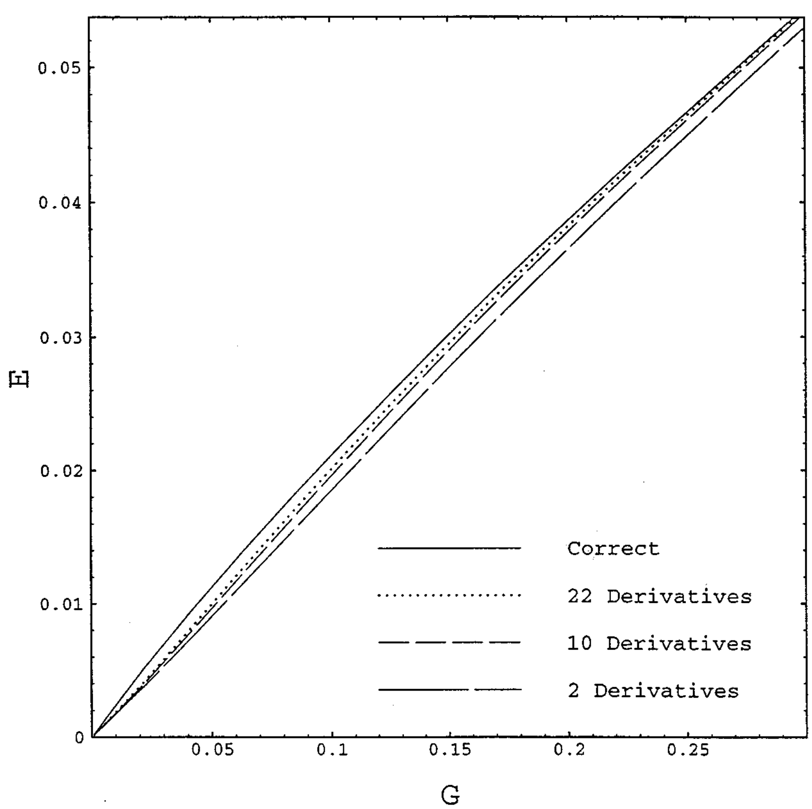

FIG. 3. Fermion-loop correction to the ϕ^4 soliton mass ($G \ll 1$).

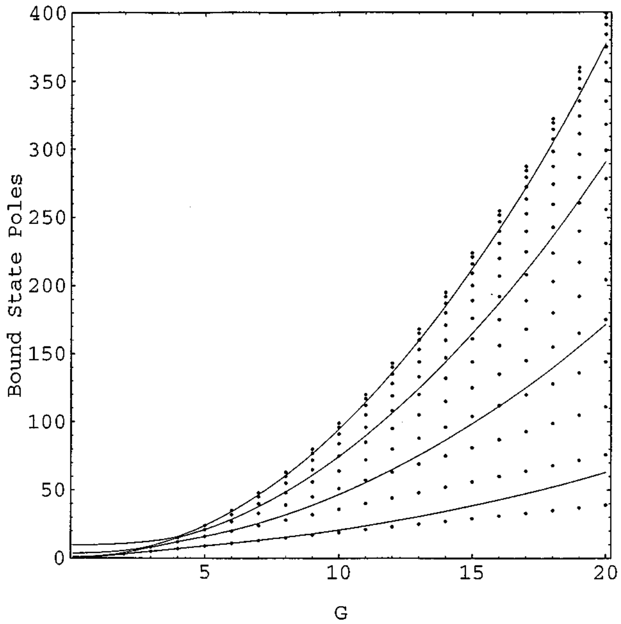

FIG. 4. Fermion-loop bound states energy in unit of boson mass m.

We calculate the Padé approximant $A_{n,n}$ for the expression in Eq. (57) and follow the general procedure as in Eqs. (18)–(24) and the description in that section. The resulting $E_F(G)$ is compared with the correct values calculated using Eq. (62). In Fig. 2, we plotted the relative error

$$\frac{E_F(G)-E_F(G)_{\rm correct}}{E_F(G)_{\rm correct}}$$

as a function of G for various n. Instead of n, we label the curves by the corresponding $4n-2$ derivatives which is the highest number of derivatives for a single term required to construct the Padé approximant $A_{n,n}$. The convergence of the Padé sequence is excellent for a wide range of G, although it may require more terms as G becomes very large. When $G\to 0$, the fermion mass $M_F\to 0$. The Padé sequence is not expected to be uniformly convergent at $G=0$. In Fig. 3 we plot $E_F(G)$ for $G\ll 1$. The convergence becomes worse as G decreases, but the deviations from the correct value are small in absolute terms. There is no reason to expect the Padé approximant to be a good approximation for analytic continuation in this region.

We shall use $A_{4,4}$ to illustrate more details of the approximation. Since $A_{4,4}$ is a ratio of two polynomials both of degree 4, it can only have four poles in the ω plane. However, the correct number of poles should be $G-1$. $A_{4,4}$ would give the correct pole positions and the correct E_F for integer $G\leqslant 5$. For $G>5$, the discrepancy between the correct number of poles and the number four given by $A_{4,4}$, increases as G increases. In Fig. 4, we plot the correct position of the poles for integer values of G (solid circles), and the trajectories of the four poles from $A_{4,4}$. The four poles are distributed optimally to produce a well-approximated value for the correct function at the imaginary axes, the integration path. The lowest trajectory is always above the correct lowest-lying pole.

E. Christ-Lee soliton

The potential for the Christ-Lee model is given by [21]

$$V(\phi)=\frac{1}{8}\frac{\lambda}{(1+c^2)}\left(\phi^2-\frac{m^2}{\lambda}\right)^2\left(1+c^2\frac{\lambda}{m^2}\phi^2\right)$$
$$\underset{c->0}{\to}\frac{1}{8}\lambda\left(\phi^2-\frac{m^2}{\lambda}\right)^2. \tag{69}$$

Since we can always suppress the dependence on m and λ by rescaling $x\to x/m$ and $\phi\to m/\sqrt{\lambda}$, we can set $m=1$ and $\lambda=1$, or simply drop the factor of m or λ without loss of generality; however, for clarity, with the exception of plotting graphs, we shall keep the overall factor of m for dimensional purposes and set only $m^2/\lambda=1$. In Figs. 5 and 6, we plot the potential $V(\phi)$ and the corresponding $U(\phi)$ as functions of the field ϕ for various values of c in Figs. 5 and 6. The potential is normalized such that it becomes the ϕ^4 model for $c=0$. As $c\to\infty$, the potential at $\phi=0$ approaches zero and the ground states become triply degenerate. The soliton with the topological boundary condition $\phi(\pm\infty)\to\pm 1$ breaks up into two independent solitons, one with the boundary condition $[\phi(-\infty)\to -1,\phi(\infty)\to 0]$ and the other with the boundary condition $[\phi(-\infty)\to 0,\phi(\infty)\to 1]$.

For the potential (69), Eq. (39) can be integrated analytically to yield the soliton configuration

$$\phi(x)=\frac{m}{\sqrt{\lambda}}\frac{\sinh\left(\frac{mx}{2}\right)}{\sqrt{c^2+\cosh\left(\frac{mx}{2}\right)^2}}, \tag{70}$$

which is plotted in Fig. 7. The potentials $V(\phi_s(x))$ and $U(\phi_s(x))$ for the soliton background field are plotted as functions of x in Figs. 8 and 9. It is clear that for small c, the kink is located at $x=0$. As c increases, the single kink gradually separates into two kinks [22],

$$\phi(x)\approx\phi^+_{c->\infty}(x)-\phi^-_{c->\infty}(x), \tag{71}$$

$$\phi^{\pm}_{c\to\infty}(x)=\sqrt{\frac{1}{1+e^{\mp x+\ln(1+4c^2)}}}=\sqrt{\frac{1}{1+(1+4c^2)e^{\mp x}}} \tag{72}$$

which are soliton solutions of Eq. (39) with $V(\phi)$ from Eq. (69) at $c\to\infty$ and boundary conditions

$$\lim_{x\to\pm\infty}\phi^{\pm}_{c\to\infty}(x)\to 1 \quad\text{and}\quad \lim_{x\to\mp\infty}\phi^{\pm}_{c\to\infty}(x)\to 0. \tag{73}$$

It is interesting to point out that Eq. (71) is not only an excellent approximation to Eq. (70) but also a respectable approximation for small c down to $c=0$.

A closed form for the one meson-loop contribution to the soliton mass, $E_M^{\text{Christ-Lee}}$, has not been found. We calculate the expansion equation (43):

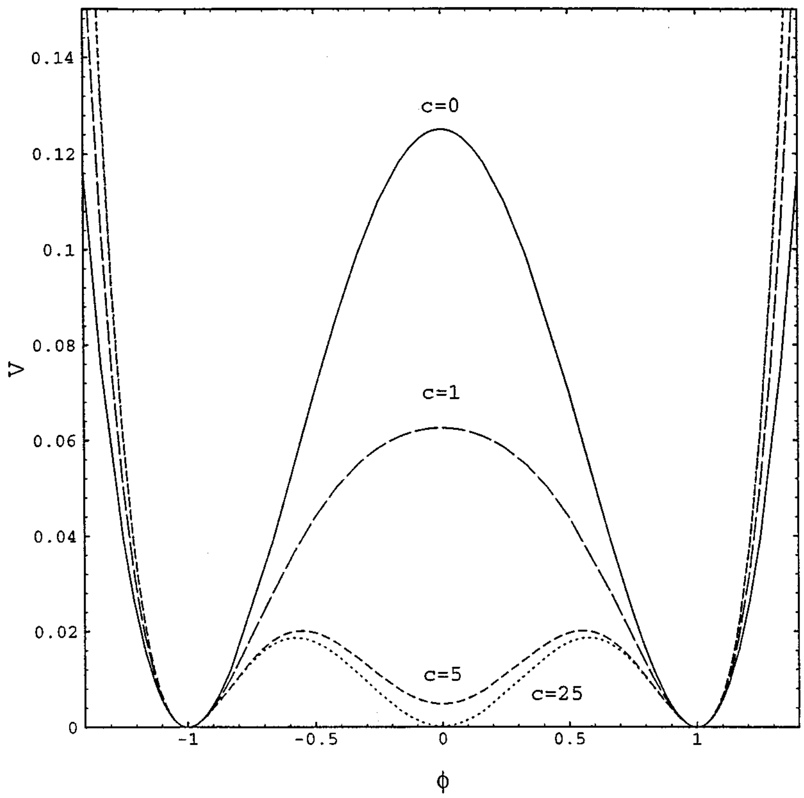

FIG. 5. Christ-Lee model potential V as a function of ϕ.

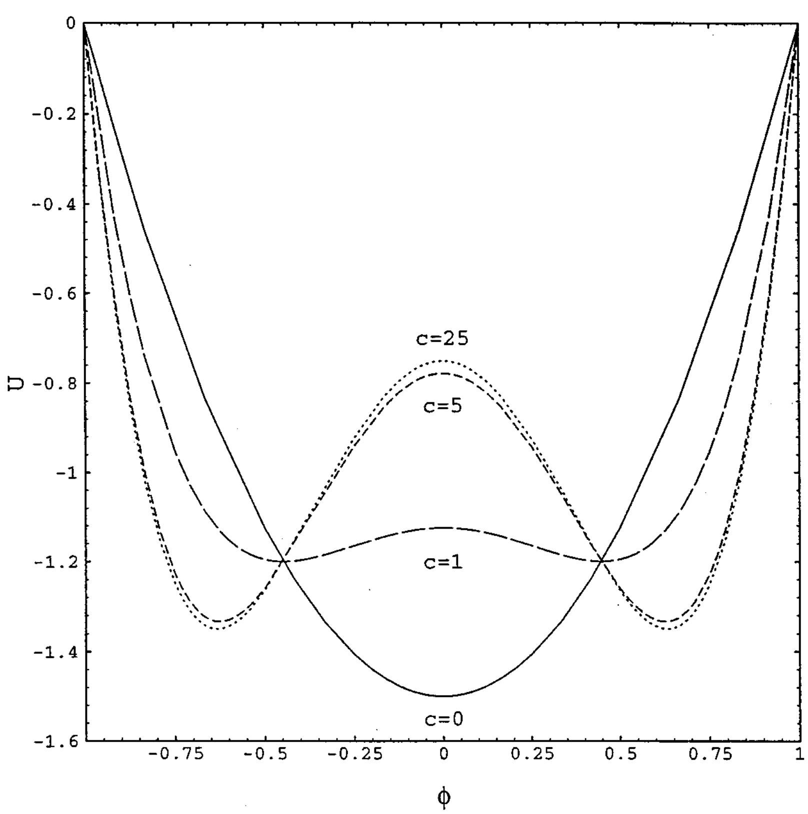

FIG. 6. Christ-Lee model potential U as a function of ϕ.

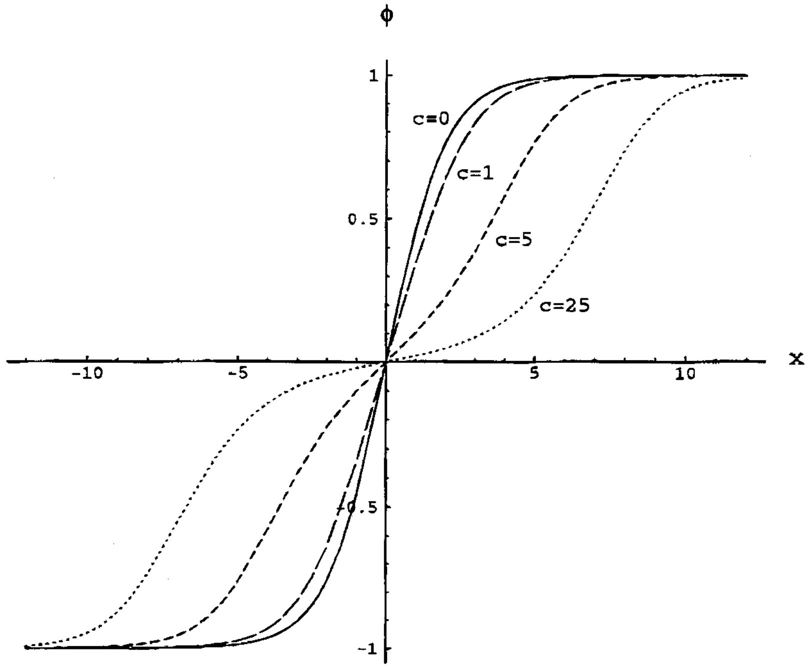

FIG. 7. Christ-Lee soliton configuration.

$$A_B^{\text{Christ-Lee}}(z) = -m\frac{3}{8}\left\{ 3\left(\frac{zm^2}{2^7c^2(1+c^2)}\right)\left[-(1-2c^2)(111+98c^2)+3(37+26c^2-8c^4+16c^6)\left(\frac{\text{arcsinh}(c)}{c\sqrt{1+c^2}}\right)\right] + 5\left(\frac{zm^2}{2^7c^2(1+c^2)}\right)^2\left[(10479-848c^2-10900c^4+17248c^6+15264c^8) + 3(-3493-2046c^2+7136c^4+5056c^6-384c^8+768c^{10})\left(\frac{\text{arcsinh}(c)}{c\sqrt{1+c^2}}\right)\right] + \cdots + O(z^{14})\right\}. \quad (74)$$

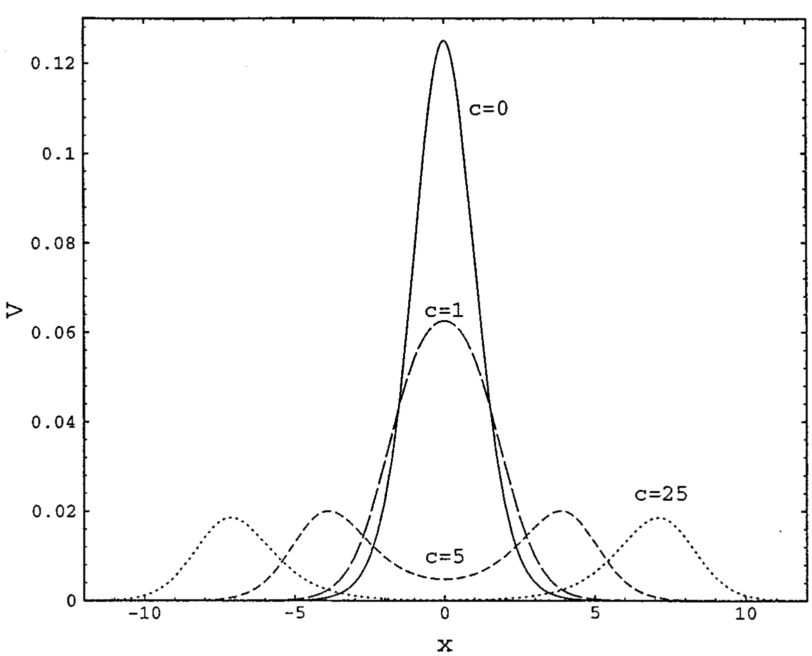

FIG. 8. Christ-Lee model potential V as a function of x.

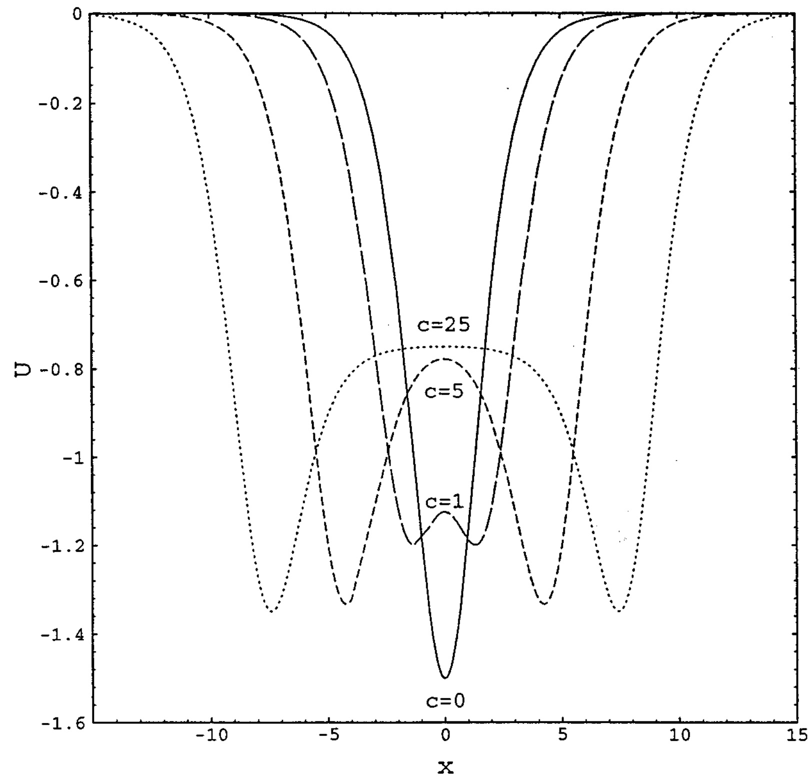

FIG. 9. Christ-Lee model potential U as a function of x.

This series appears to be far more complex than the examples encountered previously and does not seem to be summable. If we proceed to continue analytically this series by a sequence of Padé approximants, we can obtain reasonably good convergence for $c<6$, as shown in Fig. 10. However, if we first extract the zero-frequency modes from $A_B^{\text{Christ-Lee}}(z)$, the convergence of the series $\widetilde{A}_B^{\text{Christ-Lee}}$ is much improved. In the limit that $c\to 0$, $\widetilde{A}_B^{\text{Christ-Lee}}$ reduces to Eq. (47) for all 14 terms as it should. In Fig. 11, we plot $E_M^{\text{Christ-Lee}}$ as a function of c for the sequence of Padé approximations. The convergence is excellent for a wide range of c, except for very large values of c where the ground states become almost triply degenerate, which means that there is another pole at ω_1^2 very close to zero.

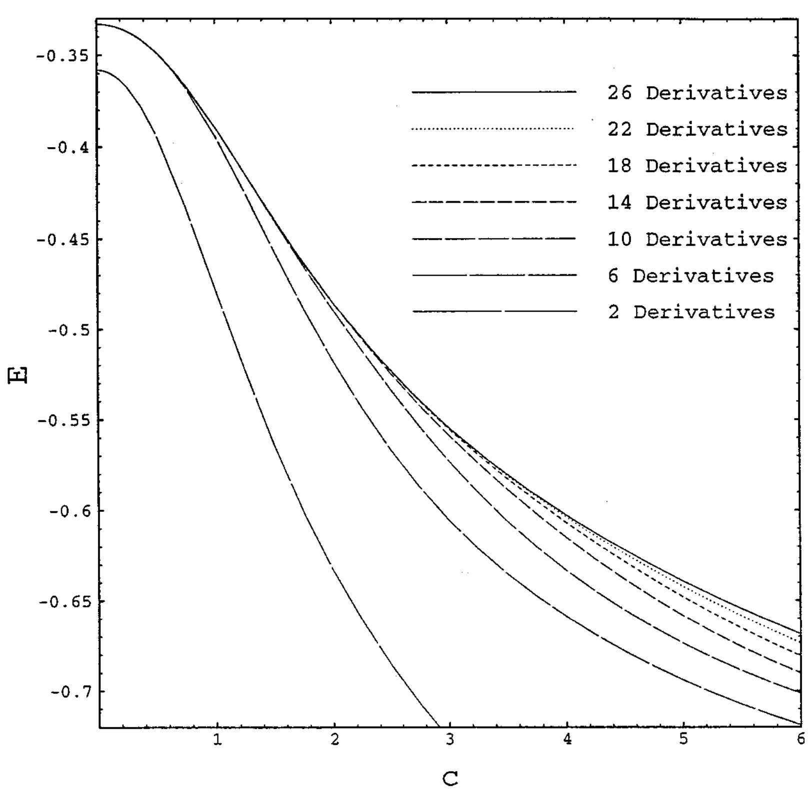

FIG. 10. Boson-loop correction to the Christ-Lee soliton mass.

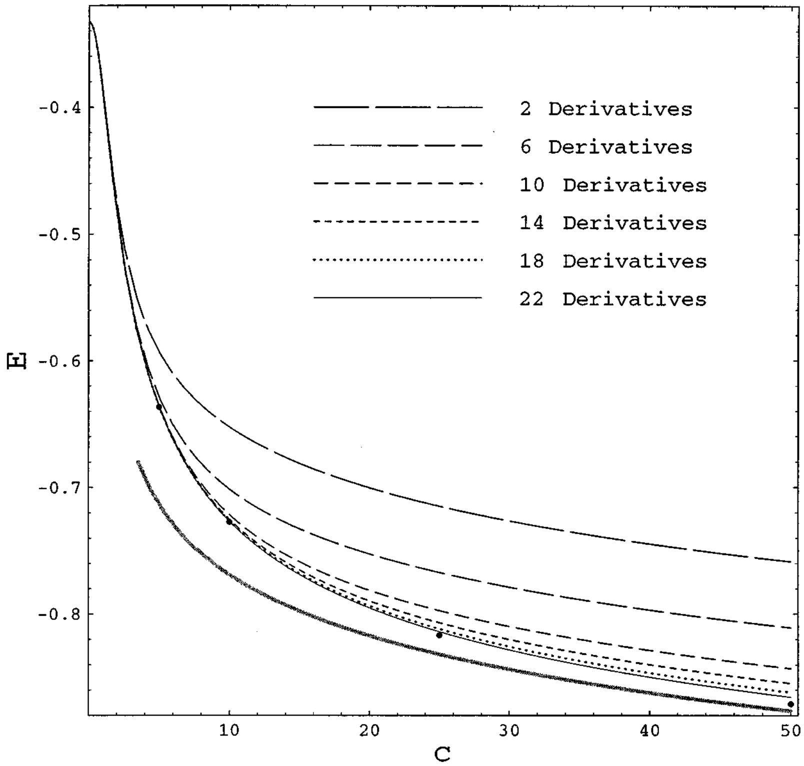

FIG. 11. Boson-loop correction to the Christ-Lee soliton mass with pole extraction.

If ω_1^2 is known, we can perform another extraction to improve the convergence at large c. In Table II we list the numerical value of ω_1^2 calculated from Eq. (42) and the sequence of the one-loop corrections to the soliton mass for various values of c. The convergence is drastically improved. These results are summarized by four black dots in Fig. 11 representing almost the exact values. The results at the two-derivative level are within a one-half percent of the final values. This is very important because in some cases it may not be practical to compute the higher derivative contributions.

As c approaches infinity, the two kinks in Eq. (71) become well separated. The interaction between them becomes weaker and weaker. The two kinks eventually become two independent solitons and have two distinct translational modes. As $c\to\infty$, $\omega_1\to 0$ and we have a double degeneracy at $\omega=0$. Therefore, for large values of c, a much improved approximation can be achieved by extraction of two $\omega=0$ modes instead of one. Even at the two-derivative level, this approximation shows a drastic improvement over the single zero-mode extraction approximation for large c. This is shown by a broad grey curve in Fig. 11. The transition between the two approximations is approximately at $c\approx 6$.

VI. THE ϕ^4 BAG MODEL

It is clear that the convergence of the expansion series depends critically on the field configuration. The soliton field configurations are perhaps the most smooth field configurations. We have shown that the generalized derivative expansion indeed works extremely well for the solitons. It is necessary to test this improved expansion for nonsoliton configuration.

For this purpose, we choose the ϕ^4 model with the set of parameters specified by $\lambda=m^2=2$ such that $\phi_0=1$. A simple field configuration [9,13]

$$\Phi(x)=1-\frac{\phi_b}{1+\exp\left(\dfrac{x^2-R^2}{T^2}\right)} \tag{75}$$

has been used to study the dependence of this convergence on the shape of the field configuration. The field $\Phi(x)$ is reflection symmetric $\Phi(x)=\Phi(-x)$. It varies from $\Phi(0)=1-\phi_b[1+\exp(-R^2/T^2)]$ at the center to $\Phi(\pm\infty)=1$ monotonically with the steepest increase at the range $x\approx R$. We further specify $R=1$, $T=0.5$ and a set of values for ϕ_b=0.2, 0.4, 0.6, and 0.8.

TABLE II. Boson-loop energy for the Christ-Lee soliton with pole extraction.

c	ω_1^2	2 Der	6 Der	10 Der	14 Der	18 Der	22 Der
5.	0.0369180	−0.63566	−0.63667	−0.63669	−0.63669	−0.63669	−0.63669
10.	0.0097447	−0.72505	−0.72709	−0.72716	−0.72717	−0.72717	−0.72717
25.	0.0015912	−0.81276	−0.81602	−0.81616	−0.81618	−0.81618	−0.81618
50.	0.0003993	−0.86661	−0.87071	−0.87090	−0.87092	−0.87092	−0.87092

TABLE III. Boson-loop energy for the ϕ^4 bag.

ϕ_b	6 Der	10 Der	14 Der	18 Der	22 Der	Wasson [13]
0.2	−0.020968	−0.021126	−0.021193	−0.021227	−0.021247	−0.021
0.4	−0.080926	−0.081955	−0.082424	−0.082680	−0.082836	−0.083
0.6	−0.173806	−0.178064	−0.180280	−0.181622	−0.182511	−0.188
0.8	−0.286022	−0.301124	−0.310733	−0.317608	−0.322873	−0.354

In this case there is no soliton equation and no zero-frequency mode to extract. We have to perform the integration in Eq. (36) numerically to obtain the generalized derivative expansion $A^{\phi^4}_{\rm bag}$ and then proceed with the analytic continuation by the Padé approximants sequence. The results are summarized in Table III together with the numerical values calculated by Wasson and Koonin [13]. The convergence is slow for large ϕ because the energy of the lowest-lying pole, which gives the dominant contribution to the energy integral, becomes very close to zero.

It is possible to improve the convergence greatly if we are willing to calculate numerically the energy of the first bound state pole and extract its contribution from the generalized derivative expansion before the analytic continuation by the Padé approximants. Table IV shows that the convergence in this case is extremely rapid and very few derivative terms are needed to achieve convergence.

We have pointed out in Sec. III that the Padé approximant $A_{n,n}$ does not satisfy the constraint on the residues of the bound state poles, $b_{n,i}=1$, in Eq. (25). Both $b_{n,i}$ and the pole positions (eigenvalues) $\alpha^2_{n,i}$ [in Eq. (25)] approach to their correct values from above. This information can be used to improve the estimate of the bound state energy eigenvalues substantially through some iteration scheme or fitting parameters in analogy to the variation method.

VII. HYBRID FORMULA

In the previous section we have demonstrated that our generalized derivative expansion has indeed provided a convergent method to calculate the quantum correction to the vacuum energy of a static background field in the presence of spontaneous symmetry breaking. If we can extract all poles along the imaginary axes and along the real axes with $|\omega_i|\leqslant M$, we can obtain a good estimate of the quantum correction to the vacuum energy with two terms of the series up to only the two-derivative terms and the result can be expressed in a compact formula in closed form. Thus, the problem is reduced to the calculation of the positions of the lowest-lying poles and the generalized derivative expansion up and including the second derivative term.

From Eqs. (21) and (24), we have

$$\begin{aligned}\tilde{A}(z(\omega),-\{\omega_1,\ldots,\omega_k\}) &= A(z(\omega))-\sum_{\omega_i^2<\omega_N^2} a(\omega^2;\omega_i^2,M^2)\\ &= z\sum_{n=0}^{\infty}\left[A_n-\sum_{\omega_i^2<\omega_N^2}(M^2-\omega_i^2)^{n+3/2}\right]z^n\\ &= z\sum_{n=0}^{\infty}\tilde{A}_n z^n, \end{aligned}\tag{76}$$

where ω_N is to be chosen such that $0<\omega_N^2\leqslant M^2$. Applying the $\{1,1\}$ Padé approximant to the two terms of the series, we obtain

$$\begin{aligned}\tilde{A}(z(\omega),-\{\omega_1,\ldots,\omega_k\}) &= \frac{\tilde{A}_0 z}{1-\dfrac{\tilde{A}_1}{\tilde{A}_0}z}\\ &= \frac{\tilde{A}_0}{(M^2-\omega_c^2)^{3/2}}a(\omega^2,\omega_c^2,M^2),\end{aligned}\tag{77}$$

where

$$\omega_c^2 = M^2-\frac{\tilde{A}_1}{\tilde{A}_0}.\tag{78}$$

The final formula for the vacuum energy of a static background field is, for $\omega_c^2<M^2$,

TABLE IV. Boson-loop energy for the ϕ^4 bag with extraction of the lowest-lying pole.

ϕ_b	2 Der	6 Der	10 Der	14 Der	18 Der	22 Der	Wasson [13]
0.2	−0.021093	−0.02121	−0.021249	−0.021267	−0.021278	−0.021284	−0.021
0.4	−0.083087	−0.08317	−0.083188	−0.083195	−0.083198	−0.083199	−0.083
0.6	−0.185006	−0.18525	−0.185314	−0.185338	−0.185349	−0.185355	−0.188
0.8	−0.353119	−0.35376	−0.353954	−0.354038	−0.354083	−0.354111	−0.354

$$E[U(\phi(x))] = -\sum_{\omega_j^2<0}\left[\frac{i}{2}|\omega_j| + \frac{1}{\pi}\sqrt{M^2+|\omega_j|^2}\left(1-\frac{1}{2}\sqrt{\frac{|\omega_j|^2}{M^2+|\omega_j|^2}}\ln\frac{1+\sqrt{\frac{|\omega_j|^2}{M^2+|\omega_j|^2}}}{1-\sqrt{\frac{|\omega_j|^2}{M^2+|\omega_j|^2}}}\right)\right]$$

$$+\sum_{0<\omega_i^2<\omega_N^2}\frac{1}{\pi}\sqrt{M^2-\omega_i^2}\left(1-\sqrt{\frac{\omega_i^2}{M^2-\omega_i^2}}\arctan\sqrt{\frac{M^2-\omega_i^2}{\omega_i^2}}\right)$$

$$+\frac{\widetilde{A}_0}{M^2-\omega_c^2}\frac{1}{\pi}\begin{cases}1-\sqrt{\frac{\omega_c^2}{M^2-\omega_c^2}}\arctan\sqrt{\frac{M^2-\omega_c^2}{\omega_c^2}} & \text{if } 0<\omega_c^2<M^2,\\ 1-\frac{1}{2\sqrt{1-\frac{M^2}{\omega_c^2}}}\ln\left(\frac{1+\sqrt{1-\frac{M^2}{\omega_c^2}}}{1-\sqrt{1-\frac{M^2}{\omega_c^2}}}\right) & \text{if } \omega_c^2>M^2.\end{cases} \quad (79)$$

The summation should include a multiplicity factor if there is a degeneracy of a particular energy eigenvalue. The imaginary part of E corresponds to the decay width of the field configuration.

For a particular even space-time dimension field theory model $U(\phi)$ and a particular static field configuration $\phi(\vec{x})$, Eq. (79) provides a fair estimate of the quantum correction to the vacuum energy. The ω_i are the eigenvalues of the well-studied operator $-\nabla^2+M^2+U(\phi(\vec{x}))$ and their values can be computed analytically or numerically by standard methods. The accuracy of the estimate depends crucially on the number of ω_i extracted.

Because the number of required infinite subtraction terms is different, the definitions of $\widetilde{A}_0$ and $\widetilde{A}_1$ depend on the dimension D. For $1+1$ dimension they can be obtained from Eq. (36) or directly from Eq. (14):

$$\widetilde{A}_0 = -\frac{3}{16}\int_{-\infty}^{\infty}dx\,U(\phi(x))^2 - \sum_{\omega_i^2<\omega_N^2}(M^2-\omega_i^2)^{3/2},$$

$$\widetilde{A}_1 = \frac{5}{64}\int_{-\infty}^{\infty}dx[2U(\phi(x))^3+U'(\phi(x))^2] - \sum_{\omega_i^2<\omega_N^2}(M^2-\omega_i^2)^{5/2}. \quad (80)$$

Using Eqs. (78)–(80), one can easily reproduce all the results of the previous sections for the two-derivative approximation for various situations. If ω_N is chosen appropriately, this set of equations can provide a very direct method of calculation of one-loop corrections to the vacuum energy of static background fields.

For $1+3$ dimension, substituting Eq. (14) into Eq. (76), we obtain

$$\widetilde{A}_0 = \frac{1}{128\pi}\int_{-\infty}^{\infty}d^3x\{2U(\phi(\vec{x}))^3+[\vec{\nabla}U(\phi(\vec{x}))]^2\} - \sum_{\omega_i^2<\omega_N^2}(M^2-\omega_i^2)^{3/2},$$

$$\widetilde{A}_1 = -\frac{1}{512\pi}\int_{-\infty}^{\infty}d^3x\{5U(\phi(\vec{x}))^4 + 10U(\phi(\vec{x}))[\vec{\nabla}U(\phi(\vec{x}))]^2+[\nabla_i\nabla_jU(\phi(\vec{x}))]^2\} - \sum_{\omega_i^2<\omega_N^2}(M^2-\omega_i^2)^{5/2}. \quad (81)$$

Here we have demonstrated that our approach functions equally well in higher dimension in principle. Calculations of the eigenvalues and the integrations may be more involved but systematic and standard numerical methods are available. More accurate calculation can be carried out in a similar manner using higher order Padé approximation.

VIII. CONCLUSION

We have presented a generalized derivative expansion for the action of field theories with spontaneous symmetry breaking and a procedure of its analytic continuation. We have carried out the complete implementation of this procedure for the (1+1)-dimensional static field configurations and have successfully calculated the energy for various models of soliton, fermion loop, and bag. The results indicate that the improved derivative expansion is not only an elegant tool in theory but a viable technique for numerical computation. This approach may open up the possibility of asking some more fundamental questions, such as whether one can define the effective potential more meaningfully by using some spatially varying background field, in the region in which the conventional effective potential is not well defined for constant background field. Another interesting question is whether quantum corrections can stabilize or destabilize a classical soliton.

An immediate extension of the (1+1)-dimensional soliton calculation is the (1+3)-dimensional classical stationary background field with no internal symmetry, such as the configuration of the critical bubble. A nontrivial extension would be to generalize the present approach from the zero-temperature formulation to finite-temperature field theories. More interesting applications would have to include internal symmetry such as Skyrmion physics. As the dimension and internal degrees of freedom increase, the generalized derivative expansion unavoidably becomes more complex. However, the improved derivative expansion is actually easier to calculate than the conventional derivative expansion. The general procedure is very systematic and the underlying analytic continuation is basically the same. The capacity to do most parts of the calculations analytically and interactively is tremendously helpful in understanding the mathematics of the physical problems and in making progress toward a better solution by experimentation which is otherwise too time consuming to try.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy. The calculations and the graphs of this paper were obtained almost exclusively using MATHEMATICA, without which this work would have become impossible.

APPENDIX: MESON-LOOP DERIVATIVE EXPANSION

In this appendix, we give a more complete expression of Eq. (36) up to the z^{12} term:

$$A_B(z)=-\frac{1}{2}\int_{-\infty}^{\infty}dx\Big\{U+\frac{3}{8}zU^2-\frac{5}{32}z^2[2U^3+(U')^2]+\frac{7}{128}z^3[5U^4+10U(U')^2+(U'')^2]-\frac{9}{512}z^4$$

$$\times[14U^5+70U^2(U')^2+14U(U'')^2+(U''')^2]+\frac{11}{2048}z^5[42U^6+420U^3(U')^2-35(U')^4+126U^2(U'')^2-20(U'')^3$$

$$+18U(U''')^2+(U'''')^2]-\frac{13}{8192}z^6[132U^7+2310U^4U'^2-770UU'^4+924U^3U''^2-462U'^2U''^2-440UU''^3$$

$$+198U^2U^{(3)}(x)^2-110U''U^{(3)}(x)^2+22UU^{(4)^2}+U^{(5)^2}]+\frac{15}{32768}z^7[429U^8+12012U^5U'^2-10010U^2U'^4$$

$$+6006U^4U''^2-12012UU'^2U''^2-5720U^2U''^3+1001U''^4+1716U^3U^{(3)^2}-858U'^2U^{(3)^2}-2860UU''U^{(3)^2}$$

$$+286U^2U^{(4)^2}-182U''U^{(4)^2}+26UU^{(5)^2}+U^{(6)^2}]$$

$$-\frac{17}{131072}z^8[1430U^9+60060U^6U'^2-100100U^3U'^4+10010U'^6$$

$$+36036U^5U''^2-180180U^2U'^2U''^2-57200U^3U''^3+52624U'^2U''^3+30030UU''^4+12870U^4U^{(3)^2}$$

$$-25740UU'^2U^{(3)^2}-42900U^2U''U^{(3)^2}+16406U''^2U^{(3)^2}+5096U'U^{(3)^3}+2860U^3U^{(4)^2}-1430U'^2U^{(4)^2}$$

$$-5460UU''U^{(4)^2}+252U^{(4)^3}+390U^2U^{(5)^2}-280U''U^{(5)^2}+30UU^{(6)^2}+U^{(7)^2}]$$

$$+\frac{19}{524288}z^9[4862U^{10}+291720U^7U'^2-850850U^4U'^4+340340UU'^6+204204U^6U''^2-2042040U^3U'^2U''^2$$

$$+510510U'^4U''^2-486200U^4U''^3+1789216UU'^2U''^3+510510U^2U''^4-101660U''^5+87516U^5U^{(3)^2}$$

$$-437580U^2U'^2U^{(3)^2}-486200U^3U''U^{(3)^2}+418132U'^2U''U^{(3)^2}+557804UU''^2U^{(3)^2}+173264UU'U^{(3)^3}$$

$$-16099U^{(3)^4}+24310U^4U^{(4)^2}-48620UU'^2U^{(4)^2}-92820U^2U''U^{(4)^2}+38726U''^2U^{(4)^2}$$

$$+32368U'U^{(3)}U^{(4)^2}+8568UU^{(4)^3}+4420U^3U^{(5)^2}-2210U'^2U^{(5)^2}-9520UU''U^{(5)^2}+1428U^{(4)}U^{(5)^2}$$

$$+510U^2U^{(6)^2}-408U''U^{(6)^2}+34UU^{(7)^2}+U^{(8)^2}]$$

$$-\frac{7}{2097152}z^{10}[50388U^{11}+4157010U^8U'^2-19399380U^5U'^4+19399380U^2U'^6$$

$$+3325608U^7U''^2-58198140U^4U'^2U''^2+58198140UU'^4U''^2-11085360U^5U''^3+101985312U^2U'^2U''^3$$

$$+19399380U^3U''^4-30207606U'^2U''^4-11589240UU''^5+1662804U^6U^{(3)^2}-16628040U^3U'^2U^{(3)^2}$$
$$+4157010U'^4U^{(3)^2}-13856700U^4U''U^{(3)^2}+47667048UU'^2U''U^{(3)^2}+31794828U^2U''^2U^{(3)^2}$$
$$-10949700U''^3U^{(3)^2}+9876048U^2U'U^{(3)^3}-11795960U'U''U^{(3)^3}-1835286UU^{(3)^4}$$
$$+554268U^5U^{(4)^2}-2771340U^2U'^2U^{(4)^2}-3527160U^3U''U^{(4)^2}+2872116U'^2U''U^{(4)^2}+4414764UU''^2U^{(4)^2}$$
$$+3689952UU'U^{(3)}U^{(4)^2}-753882U^{(3)^2}U^{(4)^2}+488376U^2U^{(4)^3}-519384U''U^{(4)^3}+125970U^4U^{(5)^2}$$
$$-251940UU'^2U^{(5)^2}-542640U^2U''U^{(5)^2}+246126U''^2U^{(5)^2}+186048U'U^{(3)}U^{(5)^2}+162792UU^{(4)}U^{(5)^2}$$
$$+19380U^3U^{(6)^2}-9690U'^2U^{(6)^2}-46512UU''U^{(6)^2}+7524U^{(4)}U^{(6)^2}+1938U^2U^{(7)^2}-1710U''U^{(7)^2}+114UU^{(8)^2}$$
$$+3U^{(9)^2}]$$

$$+\frac{23}{25165824}z^{11}[176358U^{12}+19399380U^9U'^2-135795660U^6U'^4+271591320U^3U'^6-24249225U'^8$$
$$+17459442U^8U''(x)^2-488864376U^5U'^2U''^2+1222160940U^2U'^4U''^2-77597520U^6U''^3$$
$$+1427794368U^3U'^2U''^3-519903384U'^4U''^3+203693490U^4U''^4-1268719452UU'^2U''^4-243374040U^2U''^5$$
$$+54936486U''^6+9976824U^7U^{(3)^2}-174594420U^4U'^2U^{(3)^2}+174594420UU'^4U^{(3)^2}-116396280U^5U''U^{(3)^2}$$
$$+1001008008U^2U'^2U''U^{(3)^2}+445127592U^3U''^2U^{(3)^2}-624962364U'^2U''^2U^{(3)^2}-459887400UU''^3U^{(3)^2}$$
$$+138264672U^3U'U^{(3)^3}-89085984U'^3U^{(3)^3}-495430320UU'U''U^{(3)^3}-38541006U^2U^{(3)^4}+55604450U''U^{(3)^4}$$
$$+3879876U^6U^{(4)^2}-38798760U^3U'^2U^{(4)^2}+9699690U'^4U^{(4)^2}-37035180U^4U''U^{(4)^2}$$
$$+120628872UU'^2U''U^{(4)^2}+92710044U^2U''^2U^{(4)^2}-33414996U''^3U^{(4)^2}+77488992U^2U'U^{(3)}U^{(4)^2}$$
$$-97900008U'U''U^{(3)}U^{(4)^2}-31663044UU^{(3)^2}U^{(4)^2}+6837264U^3U^{(4)^3}-8488440U'^2U^{(4)^3}$$
$$-21814128UU''U^{(4)^3}+1219515U^{(4)^4}+1058148U^5U^{(5)^2}-5290740U^2U'^2U^{(5)^2}-7596960U^3U''U^{(5)^2}$$
$$+5914776U'^2U''U^{(5)^2}+10337292UU''^2U^{(5)^2}+7814016UU'U^{(3)}U^{(5)^2}-1743402U^{(3)^2}U^{(5)^2}$$
$$+3418632U^2U^{(4)}U^{(5)^2}-3842028U''U^{(4)}U^{(5)^2}-406296U'U^{(5)^3}+203490U^4U^{(6)^2}-406980UU'^2U^{(6)^2}$$
$$-976752U^2U''U^{(6)^2}+479598U''^2U^{(6)^2}+330372U'U^{(3)}U^{(6)^2}+316008UU^{(4)}U^{(6)^2}-10296U^{(6)^3}$$
$$+27132U^3U^{(7)^2}-13566U'^2U^{(7)^2}-71820UU''U^{(7)^2}+12474U^{(4)}U^{(7)^2}+2394U^2U^{(8)^2}-2310U''U^{(8)^2}$$
$$+126UU^{(9)^2}+3U^{(10)^2}]$$

$$-\frac{25z^{12}}{100663296}[624036U^{13}+89237148U^{10}U'^2-892371480U^7U'^4+3123300180U^4U'^6-1115464350UU'^8$$
$$+89237148U^9U''^2-3747960216U^6U'^2U''^2+18739801080U^3U'^4U''^2-3123300180U'^6U''^2$$
$$-509926560U^7U''^3+16419635232U^4U'^2U''^3-23915555664UU'^4U''^3+1873980108U^5U''^4$$
$$-29180547396U^2U'^2U''^4-3731735280U^3U''^5+8928350496U'^2U''^5+2527078356UU''^6+57366738U^8U^{(3)^2}$$
$$-1606268664U^5U'^2U^{(3)^2}+4015671660U^2U'^4U^{(3)^2}-892371480U^6U''U^{(3)^2}+15348789456U^3U'^2U''U^{(3)^2}$$
$$-5443466028U'^4U''U^{(3)^2}+5118967308U^4U''^2U^{(3)^2}-28748268744UU'^2U''^2U^{(3)^2}-10577410200U^2U''^3U^{(3)^2}$$
$$+3603139290U''^4U^{(3)^2}+1590043728U^4U'U^{(3)^3}-4097955264UU'^3U^{(3)^3}-11394897360U^2U'U''U^{(3)^3}$$
$$+8637846880U'U''^2U^{(3)^3}-590962092U^3U^{(3)^4}+1413396966U'^2U^{(3)^4}+2557804700UU''U^{(3)^4}$$
$$+25496328U^7U^{(4)^2}-446185740U^4U'^2U^{(4)^2}+446185740UU'^4U^{(4)^2}-340723656U^5U''U^{(4)^2}$$

$$
\begin{aligned}
&+2774464056U^2U'^2U''U^{(4)^2}+1421554008U^3U''^2U^{(4)^2}-1825661892U'^2U''^2U^{(4)^2}-1537089816UU''^3U^{(4)^2}\\
&+1188164544U^3U'U^{(3)}U^{(4)^2}-747060240U'^3U^{(3)}U^{(4)^2}-4503400368UU'U''U^{(3)}U^{(4)^2}\\
&-728250012U^2U^{(3)2}U^{(4)^2}+1088502324U''U^{(3)^2}U^{(4)^2}+78628536U^4U^{(4)^3}-390468240UU'^2U^{(4)^3}\\
&-501724944U^2U''U^{(4)^3}+341468304U''^2U^{(4)^3}+378511920U'U^{(3)}U^{(4)^3}+56097690UU^{(4)^4}+8112468U^6U^{(5)^2}\\
&-81124680U^3U'^2U^{(5)^2}+20281170U'^4U^{(5)^2}-87365040U^4U''U^{(5)^2}+272079696UU'^2U''U^{(5)^2}\\
&+237757716U^2U''^2U^{(5)^2}-90112896U''^3U^{(5)^2}+179722368U^2U'U^{(3)}U^{(5)^2}-239975928U'U''U^{(3)}U^{(5)^2}\\
&-80196492UU^{(3)^2}U^{(5)^2}+52419024U^3U^{(4)}U^{(5)^2}-61040160U'^2U^{(4)}U^{(5)^2}-176733288UU''U^{(4)}U^{(5)^2}\\
&+20441250U^{(4)2}U^{(5)^2}-18689616UU'U^{(5)^3}+9687876U^{(3)}U^{(5)^3}+1872108U^5U^{(6)^2}-9360540U^2U'^2U^{(6)^2}\\
&-14976864U^3U''U^{(6)^2}+11232648U'^2U''U^{(6)^2}+22061508UU''^2U^{(6)^2}+15197112UU'U^{(3)}U^{(6)^2}\\
&-3679770U^{(3)^2}U^{(6)^2}+7268184U^2U^{(4)}U^{(6)^2}-8637420U''U^{(4)}U^{(6)^2}-2541132U'U^{(5)}U^{(6)^2}-473616UU^{(6)^3}\\
&+312018U^4U^{(7)2}-624036UU'^2U^{(7)2}-1651860U^2U''U^{(7)^2}+874230U''^2U^{(7)^2}+552552U'U^{(3)}U^{(7)^2}\\
&+573804UU^{(4)}U^{(7)^2}-59202U^{(6)}U^{(7)^2}+36708U^3U^{(8)^2}-18354U'^2U^{(8)^2}-106260UU''U^{(8)^2}+19734U^{(4)}U^{(8)^2}\\
&+2898U^2U^{(9)^2}-3036U''U^{(9)^2}+138UU^{(10)^2}+3U^{(11)^2}]+\cdots\Bigg\},
\end{aligned}
$$

where $U^{(n)}=(d^n/dx^n)U(\phi(x))$. Integration by parts has been repeatedly used to reduce the number of derivatives on the function $U(\phi(x))$ in each term to a minimum possible value. The highest value of derivative for a single U for the z^n term is $n-2$. Such a representation, irreducible by further integration by parts, is unique.

We have computed the series up to the z^{14} term with 26 derivatives for the calculations in this paper. However, the remaining terms are too lengthy to present here. The calculation is performed using MATHEMATICA [14]. The CPU time increases by approximately a factor of 4 for computations of each higher order term.

[1] L.-H. Chan, Phys. Rev. Lett. **57**, 1199 (1986).
[2] L.-H. Chan, Phys. Rev. Lett. **55**, 21 (1985).
[3] L.-H. Chan, Phys. Rev. Lett. **54**, 1222 (1985); **56**, 404 (1985).
[4] L.-H. Chan, Phys. Rev. D **36**, 3755 (1987).
[5] C. M. Fraser, Z. Phys. C **28**, 101 (1985).
[6] I. J. R. Aitchison and C. M. Fraser, Phys. Lett. **146B**, 63 (1984); Phys. Rev. D **31**, 2605 (1985).
[7] O. Cheyette, Phys. Rev. Lett. **55**, 2394 (1985).
[8] M. K. Gaillard, Nucl. Phys. **B268**, 669 (1986).
[9] M. Li, R. Perry, and L. Wilets, Phys. Rev. D **36**, 596 (1987).
[10] R. Perry and M. Li, Mod. Phys. Lett. A **2**, 353 (1987).
[11] R. Perry and M. Li, Phys. Rev. D **37**, 1670 (1988).
[12] R. J. Perry, Phys. Lett. B **182**, 269 (1986).
[13] D. A. Wasson and S. E. Koonin, Phys. Rev. D **43**, 3400 (1991).
[14] Wolframe Research, Inc. MATHEMATICA Version 2.2 (1993) and Version 3.0 (1996).
[15] L.-H. Chan, in *Chiral Soliton*, edited by K. F. Liu (World Scientific, Singapore, 1987), Chap. 2.
[16] L.-H. Chan, Phys. Rev. D **36**, 3755 (1987).
[17] L.-H. Chan, Phys. Rev. D **38**, 3739 (1988).
[18] S. G. Naculich, Phys. Rev. D **46**, 5487 (1992).
[19] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10**, 4114 (1974); **10**, 4130 (1974).
[20] S. J. Chang and T. M. Yan, Phys. Rev. D **12**, 3225 (1975).
[21] N. H. Christ and T. D. Lee, Phys. Rev. D **12**, 1606 (1975).
[22] L.-H. Chan (in preparation).