

Black hole entropy in the Chern-Simons formulation of (2+1)-dimensional gravity

Máximo Bañados

*Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago, Chile
and Departamento de Física, Universidad de Santiago, Casilla 307, Correo 2, Santiago, Chile*

Andrés Gomberoff

*Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago, Chile
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We examine Carlip's derivation of the 2+1 Minkowskian black hole entropy. A simplified derivation of the boundary action, valid for any value of the level k , is given. [S0556-2821(97)05308-3]

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I. INTRODUCTION

In the last two years there has been major progress in the understanding of the quantum mechanics of black holes. On the one hand, Carlip [1] has given a statistical description for the entropy of the (2+1)-dimensional black hole [2]. More recently, string theory has also provided a statistical description of the black hole entropy for extremal and near extremal black holes [3]. Despite the success of these new formulations, much work remains to be done. In fact, Carlip's approach relies heavily on the Chern-Simons formulation of (2+1)-dimensional gravity and, therefore, its generalization to higher dimensions is not an easy task [4]. The formulation given in [3], on the other hand, can be implemented in various dimensions but only for extremal and near-extremal black holes. The real four-dimensional nonextremal black hole still seems far from being completely understood.

In this paper we address some issues concerning Carlip's derivation for the entropy of the Minkowskian 2+1 black hole. It was shown in [1] that the degeneracy of boundary degrees of freedom of (2+1)-dimensional gravity gives the correct value for the black hole entropy. However, the explicit form of the boundary action was not written in [1] because it involved a complicated Jacobian. It was argued instead that in the limit $k \rightarrow \infty$, the boundary degrees of freedom should be described by Kac-Moody currents subject to the constraint $L_0 = 0$ (this constraint was imposed because L_0 generates a gauge symmetry at the boundary). Here, we shall prove that the Kac-Moody currents are indeed the relevant degrees of freedom for any value of k , and the constraint $L_0 = 0$ is also necessary to ensure differentiability of the three-dimensional action. We also find the explicit formula for the Wess-Zumino-Witten (WZW) action that gives rise to the boundary degrees of freedom for any value of k . Our analysis is simple and relies only on some general considerations of Chern-Simons theory formulated in a manifold with a boundary. However, the quantization of the resulting boundary theory (which is classically well defined for all values of k) will be possible only in the limit $k \rightarrow \infty$. The reason is that the WZW action for the group $SO(2,1)$ is not completely understood. In particular, we do not know how to count states in the full non-Abelian theory.

Carlip's analysis has two main ingredients. First, it is assumed that the entropy can be associated to a field theory

lying at the horizon. This assumption has been extensively discussed in the last few years by Carlip himself [5] and by others [6,7]. One can further justify it by resorting to the zeroth law of black hole mechanics which states that the surface gravity κ is constant over the horizon. Therefore, the thermodynamic object is the horizon and it is thus natural to look for microscopic states defined on that surface. Second, the horizon is assumed to rotate with a rigid angular velocity and that parameter, which only depends on time, is varied in the boundary action principle. Given these two assumptions, the rest is done by the dynamics of (2+1)-dimensional gravity. It only remains to set appropriate boundary conditions to ensure the existence of a black hole, find the boundary action, and quantize it.¹

In this paper we shall mainly be concerned with the issue of imposing the correct boundary conditions and finding the boundary action; we shall not attempt to clarify or further analyze the two assumptions described above. As we shall see, the method followed here to find the boundary action is remarkably simple and may be, in principle, applicable to 3+1 dimensions.

For notational simplicity and to gain some generality, we shall start by analyzing the problem of boundary conditions in Chern-Simons theory for a general Lie group G . Once the general case is understood, the application to (2+1)-dimensional gravity will be straightforward.

II. CLASSICAL CHERN-SIMONS THEORY ON A MANIFOLD WITH A BOUNDARY

A. The action

In this section we introduce some general aspects of Chern-Simons theory on a manifold with a boundary. We consider a Chern-Simons action formulated on a manifold M with the topology $\Sigma \times \mathcal{R}$ and the "spacelike" surface Σ has the topology of an annulus. The manifold M has thus two disconnected "timelike" boundaries given by

$$B_+ = \partial \Sigma_+ \times \mathcal{R}, \quad B_- = \partial \Sigma_- \times \mathcal{R}, \quad (1)$$

¹As stressed in [1], because of the noncompact nature of the symmetry group and the lack of a full diffeomorphism invariance, the resulting Hilbert space has states with negative norm.

where $\partial\Sigma_+$ and $\partial\Sigma_\infty$ are the boundaries of Σ . Since both $\partial\Sigma_+$ and $\partial\Sigma_\infty$ are topologically circles, B_+ and B_∞ are cylinders.

An important difference between the inner (B_+) and outer (B_∞) boundaries is that B_∞ is located at an infinite distance while B_+ is located at a fixed finite distance. As it has been proved in [8], the asymptotic group (the group of transformations that leave the asymptotic conditions invariant) at B_∞ has a classical central charge. This central charge is absent at the inner boundary because B_+ is located at a finite distance and, therefore, diffeomorphisms normal to the boundary, responsible for the central charge, are not accepted [9].

The Chern-Simons (CS) action is given by

$$I_{\text{CS}} = kW[A] + B, \quad (2)$$

where

$$W[A] = \frac{1}{4\pi} \int_M \text{Tr}(AdA + \frac{2}{3}A^3) \quad (3)$$

is the Chern-Simons functional, and B is a boundary term. Its variation gives rise to the equations of motion $F=0$, where $F=dA+A\wedge A$ is the Yang-Mills curvature two-form. These equations can be split in the convenient (2+1)-form

$$\dot{A}_i^a = D_i A_t^a, \quad (4)$$

$$F_{ij}^a = 0, \quad (5)$$

showing that the time evolution is generated by a gauge transformation with parameter A_t^a . Equation (5) is a constraint over the initial conditions. Here, we have denoted by $x^0=t$ the coordinate running along $\mathfrak{R}$, and x^i are local coordinates on Σ .

An important point to ensure the validity of the above equations is the cancellation of the boundary term

$$-\frac{k}{4\pi} \int_{\partial M} \text{Tr} A \wedge \delta A + \delta B = 0, \quad (6)$$

which appears when Eq. (2) is varied. As usual, at the initial and final boundaries, Eq. (6) is canceled by imposing $\delta A=0$ and $B=0$. However, in our present case there are two other timelike boundaries, namely, B_+ and B_∞ . The treatment of the outer boundary (B_∞) is standard and we shall not repeat it here. The interested reader can consult [8,10,11] for the case of gravity and [12,9] for the general case. We will concentrate on the inner boundary which in the next section will be associated to the black hole horizon.

B. Boundary conditions

Let φ be an angular coordinate running along $\partial\Sigma_+$ and $x^0=t$, then the boundary term (6) at the inner boundary reads

$$-\frac{k}{4\pi} \int_{B_+} dt d\varphi \text{Tr}(A_t \delta A_\varphi - A_\varphi \delta A_t) + \delta B = 0. \quad (7)$$

A simple way to cancel Eq. (7) is by imposing the boundary condition $A_t=0$ and $B=0$. The group of gauge transforma-

tions leaving these boundary conditions invariant are those whose parameters do not depend on time. These transformations are global symmetries and are generated by Kac-Moody currents [13,14]. A second possibility to ensure the vanishing of Eq. (7) is to set A_t^a equal to a fixed given value, i.e., $\delta A_t^a=0$ at B_+ . We then set $B=(k/4\pi) \int \text{Tr}(A_t A_\varphi)$, producing an action which has well-defined variations. The residual group in this case is given by the set of parameters λ^a satisfying $D_0 \lambda^a = \dot{\lambda}^a + [A_t, \lambda]^a = 0$. Thus, in this case the parameters can depend on time but their dependence is not arbitrary because A_t is fixed. Again, these transformations are generated by Kac-Moody currents and are global transformations.

In our application to black hole physics, we will need a different set of boundary conditions. Consider the case on which the surface $\partial\Sigma_+$ (which is topologically a circle) rotates with angular velocity $w(t)$. Since the time evolution is generated by a gauge transformation with parameter A_t^a [see Eq. (4)], the appropriate boundary condition is

$$A_t = w(t) A_\varphi \quad (8)$$

because, in Chern-Simons theory, a displacement in φ with parameter $w(t)$ is equivalent to a gauge transformation with parameter $w(t) A_\varphi^a$ [15].

Having chosen the boundary conditions, we now have to address two remaining things: Whether the boundary conditions (8) are enough to ensure the differentiability of the action and what is the set of gauge transformations that leave Eq. (8) invariant. These two issues are connected.

Under Eq. (8) the boundary term (7) reduces to

$$\frac{k}{4\pi} \int_{B_+} dt d\varphi \text{Tr}(A_\varphi^2) \delta w(t) + \delta B = 0. \quad (9)$$

To ensure the vanishing of this boundary term we have two possibilities. One could impose $\delta w(t)=0$ and $B=0$. In this case, the surface rotates with a given, fixed, angular velocity. A second possibility, which will be the relevant boundary condition for the black hole, is to vary with respect to $w(t)$. This implies that the coefficient of $\delta w(t)$ in Eq. (9) must vanish, which in turn ensures the differentiability of the action (with $B=0$).

Indeed, if $w(t)$ is varied, there exists a *gauge* symmetry at the boundary whose generator is the coefficient of $\delta w(t)$ in Eq. (9). This can be seen as follows. We look at the most general set of gauge transformations $\delta A_\mu^a = -D_\mu \lambda^a$ leaving Eq. (8) invariant. This group will be called “the boundary group” at B_+ . One finds the condition over λ^a :

$$\dot{\lambda}^a = -\delta w(t) A_\varphi^a + w(t) \partial_\varphi \lambda^a. \quad (10)$$

Note that, since $w(t)$ is not fixed, we have allowed for transformations with $\delta w \neq 0$.

The boundary group has two pieces. First, for those transformations with $\delta w(t)=0$, one finds that the time derivative of λ^a is completely determined by Eq. (10). These are global symmetries and are generated by Kac-Moody currents. A different solution to Eq. (10) is provided by

$$\delta w(t) = -\dot{\epsilon}(t), \quad \lambda^a = \epsilon(t) A_\varphi^a, \quad (11)$$

where $\epsilon(t)$ is an *arbitrary* function of time and A_φ satisfies its equation of motion. This is a gauge symmetry because it contains an arbitrary function of time. The transformation (11) corresponds to rigid (φ -independent) time-dependent rotations of the surface $\partial\Sigma_+$ [1]. The generator of these rotations is the zero mode of $g_{ab}A_\varphi^a A_\varphi^b$ which should then vanish because its associated transformation is a gauge symmetry. Going back to Eq. (9), we see that the vanishing of

$$L_0 \equiv \frac{k}{2} g_{ab} A_\varphi^a A_\varphi^b \Big|_{\text{zero mode}} = 0 \quad (12)$$

also ensures the differentiability of the action (with $B=0$). In summary, the group of transformations that leave the boundary condition (8) invariant is given by the semidirect product of the Kac-Moody symmetry times the (time-dependent) rigid translations along φ . Note that L_0 is the zero-mode Virasoro operator of the theory. [Only the zero-mode Virasoro constraint appears because $w(t)$ does not depend on φ .]

C. The induced theory at the boundary

Having chosen the boundary conditions, we can now study the induced theory at the boundary. As it is well known, Chern-Simons theory in 2+1 dimensions does not possess local degrees of freedom² so fixing the gauge will leave us only with some global degrees of freedom. These global degrees of freedom can be of two types. On the one hand, there may be nontrivial holonomies. This is certainly our case because the spatial manifold has the topology of an annulus. Another set of degrees of freedom are the boundary values of the gauge field which cannot be set equal to zero by an allowed gauge transformation. The number of these states is infinite and for a fixed value of the black hole area, Carlip has shown that their degeneracy gives rise to the correct value for the 2+1 black hole entropy [1].

Let us thus fix the gauge in order to isolate the boundary degrees of freedom. As it is well known, the theory at the boundary is described by a WZW model [13,14]. However, it is instructive to obtain it directly from the equations of motion projected to the boundary. An appropriate [17] gauge-fixing condition is³

$$A_r^a = 0. \quad (13)$$

This gauge-fixing condition, together with the constraint (5), simply implies that the tangential component of the connection A_φ^a does not depend on the radial component. Thus, hereafter, we define

$$A_\varphi^a(t, r, \varphi) = A^a(t, \varphi). \quad (14)$$

²It has been proved in [16] that this property is not carried over to higher-dimensional Chern-Simons theories. For $D > 3$, the gauge symmetries are not enough to kill all the degrees of freedom and local excitations do exist.

³Here, we fix the gauge in the interior. The residual gauge freedom of the boundary conditions (11) is not fixed by this gauge condition.

Equation (4), on the other hand, contains the dynamical information. The radial component, together with the gauge condition (13), allows the Lagrange multiplier A_t^a to be solved. We find that A_t^a does not depend on r , which is also consistent with the boundary condition (8). The angular component of Eq. (4) gives the dynamics of A_φ^a . Projecting to the boundary and using Eq. (8), it reads

$$\frac{d}{dt} A^a = w(t) \partial_\varphi A^a. \quad (15)$$

This equation, together with the constraint (12), defines the dynamics at the boundary. The values of A^a at B_+ cannot be set equal to zero by an allowed gauge transformation.

Equation (15) has the symmetries of the boundary conditions. Indeed, Eq. (15) is invariant under the gauge transformation

$$\delta A^a = \epsilon(t) \partial_\varphi A^a, \quad \delta w(t) = -\dot{\epsilon}(t), \quad (16)$$

where $\epsilon(t)$ is an arbitrary function of t . As stressed above, the generator of this gauge transformation is the zero-mode Virasoro constraint $L_0 = 0$, defined in Eq. (12). Equation (15) has also the Kac-Moody global symmetry given by the transformation

$$\delta A^a = \partial_\varphi \lambda^a + [\lambda, A]^a, \quad \delta w(t) = 0, \quad (17)$$

where λ satisfies the equation $\dot{\lambda} = w \partial_\varphi \lambda$ [see Eq. (10)] but is otherwise arbitrary. Finally, Eq. (15) has also a global symmetry given by the translation

$$A^a \rightarrow A^a + \alpha^a, \quad (18)$$

where α^a is a constant Lie-algebra-valued element. The conserved quantities associated with this symmetry are the zero modes of $A^a(\varphi)$ as can be directly verified from the Eq. (15).

Equation (15) is already in the Hamiltonian form. We define the (noncanonical) Poisson brackets

$$[A^a(\varphi), A^b(\varphi')] = \frac{2\pi}{k} [f_c^{ab} A^c(\varphi) + k g^{ab} \partial_\varphi] \delta(\varphi, \varphi'), \quad (19)$$

and it is straightforward to check that Eq. (15) can be written in the form

$$\frac{dA(\varphi)^a}{dt} = [A^a(\varphi), H], \quad (20)$$

where the Hamiltonian is

$$H = \frac{k}{4\pi} \int d\varphi w(t) A^2. \quad (21)$$

The symmetries of Eq. (15) can also be written in the Hamiltonian form. The generator of the gauge symmetry (16) is the Hamiltonian itself, while the generator of the Kac-Moody symmetry (17) is $K(\lambda) = \int \lambda^a A^b g_{ab}$. Note that K is a conserved quantity, and thus a symmetry only when λ^a belongs to the boundary group, that is, it satisfies $\dot{\lambda}^a = w \partial_\varphi \lambda^a$.

We can now make contact with the well-known fact that the dynamics at the boundary of a Chern-Simons theory is described by a WZW model [13,14]. Making the usual change of variables $A = U^{-1}dU$, the above equations of motion can be derived from the 1+1 action

$$I = I_{\text{WZW}}(U) + \frac{1}{2\pi} \int dt d\varphi w(t) L_0. \quad (22)$$

Note that $w(t)$ enters in the action as a Lagrange multiplier. This action can thus be interpreted as a constrained WZW model in which the variation of I with respect to $w(t)$ imposes the constraint $L_0 = 0$ among the Kac-Moody fields.

The reader may notice that we have somehow rederived the well-known relation between the WZW action and Chern-Simons theory. We have chosen not to start with the WZW action from the very beginning to stress the fact that, in principle, the method followed here could be applied to (3+1)-dimensional gravity. The boundary theory can be found solely from the boundary conditions and the equations of motion. The real problem is the quantization of the resulting theory. The simplicity of the 2+1 theory relies in the fact that the quantization of a WZW model is well understood for compact groups and that there are no bulk degrees of freedom. This allowed us to isolate the boundary degrees of freedom in a simple way.

The quantization of the above action is straightforward. The canonical commutation relations (19) can be promoted to quantum commutators without any trouble. The Hamiltonian H is more delicate because it has to be regularized. Fortunately, this problem has been extensively studied in the literature. The correct quantum Virasoro operator is

$$L_0 = \frac{1}{2k + \hbar q} \left(T_0^2 + \sum_{n=1}^{\infty} T_{-n}^a T_n^b g_{ab} \right), \quad (23)$$

where q is the second Casimir in the adjoint representation and the T_n^a 's are the Fourier components of $A^a(\varphi)$,

$$A^a(\varphi) = \frac{1}{k} \sum_n T_n^a e^{in\varphi}. \quad (24)$$

The commutator between the Fourier components T_n^a is

$$[T_n^a, T_m^b] = i f_c^{ab} T_{n+m}^c + k m g^{ab} \delta_{n+m} \quad (25)$$

and the normal-ordered Virasoro operator satisfies the commutation relation

$$[L_0, T_n^a] = -n T_n^a. \quad (26)$$

This commutation relation implies that L_0 has the form $L_0 = C/(2k + \hbar q) + N$ where

$$C = g_{ab} T_0^a T_0^b \quad (27)$$

and N is the number operator.

III. (2+1)-DIMENSIONAL GRAVITY AND BLACK HOLE ENTROPY

In this section we shall apply the results of the last section to the special case of (2+1)-dimensional gravity. As we shall see, this leads directly to Carlip's formulation of the 2+1 black hole entropy [1]. The Chern-Simons formulation of (2+1)-dimensional gravity consists of the sum of two copies of the Chern-Simons action for the group $\text{SO}(2,1)$ [18,15],

$$I = kW[A] - kW[\tilde{A}] + B, \quad (28)$$

where the Chern-Simons functional W was defined in Eq. (3). The connections are related to the triad and spin connection through

$$A_\mu^a = \omega_\mu^a + \frac{e_\mu^a}{l}, \quad \tilde{A}_\mu^a = \omega_\mu^a - \frac{e_\mu^a}{l}, \quad (29)$$

where A^a and $\tilde{A}^a$ are both $\text{SO}(2,1)$ connections and l is a parameter with dimensions of length. The Chern-Simons coupling constant is related to Newton's constant by

$$k = \frac{l}{8G}. \quad (30)$$

In order to agree with the conventions followed in [2], we use units in which $G = 1/8$ and hence $k = l$.

A. Boundary conditions: A 1+1 generally covariant theory

Consider the Chern-Simons action for the group $\text{SO}(2,1) \times \text{SO}(2,1)$. We apply the boundary condition (8) to each $\text{SO}(2,1)$ copy; thus, we impose

$$A_0^a = w A_\varphi^a, \quad \tilde{A}_0^a = \tilde{w} \tilde{A}_\varphi^a. \quad (31)$$

The boundary term coming from the variation of the Chern-Simons action is

$$\frac{k}{4\pi} \int (\delta w A^2 - \delta \tilde{w} \tilde{A}^2) + \delta B. \quad (32)$$

We shall shortly impose some conditions over the functions w and $\tilde{w}$. However, it is convenient to keep them as arbitrary functions in order to clarify their geometrical meaning. If w and $\tilde{w}$ are arbitrary functions of time, we get at the boundary the two Virasoro constraint equations

$$L = (k/2) A^2 = 0, \quad \tilde{L} = (-k/2) \tilde{A}^2 = 0, \quad (33)$$

ensuring the vanishing of the boundary term (32), with $B = 0$. It is a standard result that if L and $\tilde{L}$ satisfy the Virasoro algebra, then the combinations $H = L - \tilde{L}$ and $H_\varphi = L + \tilde{L}$ satisfy the Dirac 1+1 deformation algebra. This means that the induced theory at the boundary is diffeomorphism invariant. H represents the generator of timelike deformations (conveniently densitized) and H_φ is the generator of diffeomorphisms along φ . The induced theory is then given by the two copies of the $\text{SO}(2,1)$ Kac-Moody currents subject to the constraints equations (33) or, equivalently, $H = 0$ and $H_\varphi = 0$. The boundary action can thus be written as

$$I = I_{\text{WZW}}(U) - I_{\text{WZW}}(\tilde{U}) + \int dt d\varphi (N^\perp H + N^\varphi H_\varphi), \quad (34)$$

with $N^\perp = (w - \tilde{w})/2$ and $N^\varphi = (w + \tilde{w})/2$. The theory described by the action (34), which can be understood as a non-Abelian string theory in six dimensions, is certainly interesting in its own right. [Unitary representations for (one copy of) the above action have been found in [19].] However, in our application to black hole physics we shall make some simplifications and consider only a special case. First, we shall impose that, at the horizon, the lapse function $N^\perp$ vanishes:

$$N^\perp = 0. \quad (35)$$

This condition is quite natural for a black hole. Indeed, at the horizon (in these coordinates), the lapse $N^\perp$ vanishes on shell. Second, since H_φ is the generator of diffeomorphisms along φ , N^φ represents the angular velocity of the horizon. We use the same condition as in the last section, $\partial_\varphi N^\varphi = 0$, and define

$$N^\varphi \equiv w(t). \quad (36)$$

Under conditions (35) and (36), not all the equations (33) are imposed at the boundary. Actually, only one of them is imposed, namely, the zero mode (total) Virasoro operator

$$L_0 + \tilde{L}_0 = \frac{k}{2} (A^2 - \tilde{A}^2) \Big|_{\text{zero mode}} = 0. \quad (37)$$

Since $N^\perp$ is fixed by Eq. (35) and the nonzero modes of N^φ are fixed by Eq. (36), the other modes of Eqs. (33) are not imposed. The boundary action appropriate to the boundary conditions (36) and (35) is a modification of Eq. (34):

$$I = I_{\text{WZW}}(U) - I_{\text{WZW}}(\tilde{U}) + \frac{1}{2\pi} \int dt d\varphi w(t) (L_0 + \tilde{L}_0). \quad (38)$$

This is Carlip's boundary action and its quantization gives rise to the $2+1$ black hole entropy.

Before going to the quantization of this action, let us clarify some of the differences between Eqs. (34) and (38). In Eq. (34), there are two constraints per point which are a consequence of the arbitrariness of the Lagrange multipliers $N^\perp$ and N^φ . In Eq. (38), on the other hand, the Lagrange multipliers are severely restricted by Eqs. (36) and (35), hence there is only one constraint, $L_0 + \tilde{L}_0 = 0$. Furthermore, from Eq. (32) we see that $N^\perp$ and $L_0 - \tilde{L}_0$ are conjugate pairs (in a radial quantization), thus fixing $N^\perp = 0$ implies that $L_0 - \tilde{L}_0$ is undetermined. This will have an important consequence in the next section.

B. Quantization and counting of states

The quantization of this system is implemented with the quantum version of Eq. (37):

$$(L_0 + \tilde{L}_0) |\psi\rangle = 0 \quad (39)$$

plus the condition that the eigenvalues of $L_0 - \tilde{L}_0$ are undetermined.

The states of the theory are then defined by representations of the two Kac-Moody algebras $\{T_n^a, \tilde{T}_n^a\}$ subject to the constraint (39). It is standard to consider only highest weight representations which are determined by a representation of the subalgebra $\{T_0^a, \tilde{T}_0^a\}$ [the two copies of $\text{SO}(2,1)$], which acts as vacuum state, and the value of the central charge k . We shall parametrize the Casimir operators C and $\tilde{C}$ in the form

$$\begin{aligned} C &= 2 \eta_{ab} T_0^a T_0^b = 2(r_+ - r_-)^2, \\ \tilde{C} &= 2 \eta_{ab} \tilde{T}_0^a \tilde{T}_0^b = 2(r_+ + r_-)^2. \end{aligned} \quad (40)$$

The parameters r_+ and r_- can be identified, on shell, with the outer and inner horizons of the black hole solution [2]. They are also related with the $\text{SO}(2,1) \times \text{SO}(2,1)$ holonomy existing in the black hole topology [20]. Of course, the area of the outer horizon is equal to $2\pi r_+$. Note that r_+ and r_- are also related to the mass and angular momentum of the solution through $M = (r_+^2 + r_-^2)/l^2$ and $J = (2r_+ r_-)/l$. However, mass and angular momentum are concepts defined at infinity while r_+ and r_- depend only on the topology.

Using Eq. (40) and the normal-ordered expression for the Virasoro operators (23), the constraint equation (39) reads [1]

$$L_0 + \tilde{L}_0 = \hbar(N + \tilde{N}) + Q^2 - \frac{4r_+^2}{\hbar} = 0 \quad (41)$$

and the combination $L_0 - \tilde{L}_0$ reads

$$L_0 - \tilde{L}_0 = \hbar(N - \tilde{N}) + \frac{\hbar Q^2}{2k} + \frac{2r_-^2}{k} \equiv H. \quad (42)$$

Here, N and $\tilde{N}$ are number operators for each affine $\text{SO}(2,1)$ algebra, and Q^2 is a shorthand for

$$Q^2 = \frac{4\hbar}{4k^2 - \hbar^2} \left(\frac{2kr_+}{\hbar} - r_- \right)^2. \quad (43)$$

We showed at the end of last section that the operator H is canonically conjugate to the lapse function $N^\perp$ and since $N^\perp$ at the horizon has been set equal to zero, the eigenvalues of H are undetermined.

We now count states with a fixed value of r_+ . In the limit in which the number operators N and $\tilde{N}$ are large, the difference $N - \tilde{N}$ approaches to zero. Since r_+ is fixed and H is undetermined, Eq. (42) implies that r_- is undetermined. Equation (41), on the other hand, expresses the number operator $N + \tilde{N}$ in terms of r_+ and r_- . Since r_- is undetermined, we have to sum over its possible values. In the thermodynamical limit, the largest contribution to the number operator comes from $Q = 0$ ($r_- = 2kr_+/\hbar$) and one obtains $N = (2r_+/\hbar)^2$. As shown in [1] the logarithm of the degeneracy of states produces an entropy given by

$$S = \frac{2\pi r_+}{4\hbar}, \quad (44)$$

which coincides exactly with the Bekenstein-Hawking value for the 2+1 black hole entropy.

In this calculation there is one point that deserves special attention. The boundary theory was defined for any value of the level k . However, the calculation of the entropy makes use of the limit $k \rightarrow \infty$. This limit is necessary because the $SL(2, \mathfrak{K})$ WZW model is not completely understood (although unitary representations have been found in [19]). It is rather odd that at the very end we need to use that limit since we do not know how to count states in the full non-Abelian theory. A striking feature of this calculation is the fact that the non-Abelian nature of the theory does play a central role anyway. Indeed, the result (44) depends crucially in the shift of the coupling constant $k \rightarrow k + \hbar q/2$ induced by the non-Abelian Sugawara construction [see Eq. (23)]. Had we taken

the limit $k \rightarrow \infty$ at the very beginning, we would not have obtained the right value for the black hole entropy [1].

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