

# Fluctuations of the vacuum energy density of quantum fields in curved spacetime via generalized $\zeta$ functions

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For quantum fields on a curved spacetime with a Euclidean section, we derive a general expression for the stress-energy tensor two-point function in terms of the effective action. The renormalized two-point function is given in terms of the second variation of the Mellin transform of the trace of the heat kernel for the quantum fields. For systems for which a spectral decomposition of the wave operator is possible, we give an exact expression for this two-point function. Explicit examples of the variance to the mean ratio  $\Delta' = (\langle \rho^2 \rangle - \langle \rho \rangle^2) / \langle \rho \rangle^2$  of the vacuum energy density  $\rho$  of a massless scalar field are computed for the spatial topologies of  $R^d \times S^1$  and  $S^3$ , with results of  $\Delta'(R^d \times S^1) = (d+1)(d+2)/2$ , and  $\Delta'(S^3) = 111$ , respectively. The large variance signifies the importance of quantum fluctuations and has important implications for the validity of semiclassical gravity theories at sub-Planckian scales. The method presented here can facilitate the calculation of stress-energy fluctuations for quantum fields useful for the analysis of fluctuation effects and critical phenomena in problems ranging from atom optics and mesoscopic physics to early universe and black hole physics. [S0556-2821(97)02410-7]

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## I. INTRODUCTION

Many important physical processes involve vacuum fluctuations of quantum fields. Famous examples are the Casimir effect [1] and Lamb shift. These effects shed light on some basic issues of quantum field theory, and guide the beginning probes and queries into establishing a viable theory of quantum fields, in curved or topologically nontrivial spacetimes (see, e.g., [2]). Basic issues such as the ambiguity in the definition of vacuum states, particles, energy, and the regularization of the energy-momentum tensor occupied the central attention of investigators in the 1970s. Amongst the many formalisms developed, the  $\zeta$ -function [3,4] and point-separation [5,6] regularizations are particularly relevant to our problem. In the 1980s, two directions were noteworthy in this field: the back reaction of particle creation on the dynamics of the Universe [7] and the fate of black hole collapse [8], and the study of symmetry breaking and critical phenomena via interacting field theory in curved spacetime (see, e.g., [9]). These studies are needed for understanding the dynamics of the inflationary universe, such as spinodal decomposition and nucleation, and from it, astrophysical consequences such as entropy generation and galaxy formation. Since particle creation stems from amplification of vacuum fluctuations, and galaxy formation have them as seeds, both directions share a common need to understand better the properties of vacuum fluctuations of quantum fields and how they affect the dynamics of spacetime in the early universe and state of the matter at the observable classical domains or epochs. In the 1990s, the theoretical underpinning of this issue was taken up afresh in the work of Ford and co-workers [10,11], who investigated amongst other problems the effect of vacuum fluctuations in a quantum field on the causal structure of a quantum field theory, and in the work of Hu and co-workers [12,13], who introduced nonequilibrium statistical mechanical concepts and techniques for the study

of noise and fluctuations associated with a quantum field, with applications to back reaction problems in semiclassical gravity [14,15] and structure formation problems in inflationary cosmology [16,17].

In this paper, we aim to set up the basic framework for the calculation of the fluctuations of energy and momentum for quantum fields in curved spacetimes, focusing on the generalization of the zeta-function regularization method [18]. Elegant and powerful, its application is unfortunately limited to spacetimes which admit an Euclidean section. But this already includes many cases of physical importance, such as the de Sitter universe (in the  $S^4$  coordination), Kaluza-Klein theory ( $M^4 \times B^D$ ), and finite temperature theory (in the imaginary-time formulation  $R^3 \times S^1$ ). Later we intend to connect it with the point-separation method, which, as one of us has anticipated, actually contains untapped useful information about the statistical mechanical (or kinetic theory) properties of systems of particles interacting with fields in a curved spacetime. The result obtained here is also useful for the calculation of fluctuations of Casimir energy relevant in atom optics [19], mesoscopic physics [20], and correlations of quantum fields [21,22] relevant to black hole fluctuations [23–25] and possible Planck scale phase transitions [26].

The goal of this paper is to develop a regularized expression for the stress-energy tensor two-point function of a quantum field in a curved spacetime. This is done by first expressing the two-point function solely in terms of the quantum field effective action and its variations with respect to the metric. The effective action is derived by using the trace of the heat kernel that corresponds to the field action. Since the effective action can be expressed as a function of the Mellin transform of the heat kernel, the needed variations of the effective action can be evaluated by considering the variations of the heat kernel. The variations of the heat kernel can be related to variations of an operator acting on the heat kernel in the varied heat equation.

In Sec. II we relate the stress-energy two-point function to the second metric variation of the effective action. In Sec. III we show how to express this second variation in terms of the generalized zeta function for the system. Section IV presents the specialization of these expressions to the Klein-Gordon scalar field. In particular, we derive the expression for the variance of the energy density, and discuss the need for regularization. In Secs. V and VI we explicitly compute the energy density variance for a flat spacetime with one periodic dimension and for the Einstein universe, respectively. We give a brief discussion of our results in Sec. VII.

## II. STRESS ENERGY TWO-POINT FUNCTION

We start by considering the generating functional (or partition function) of a scalar field  $\phi$  in the Euclidean section  $\Sigma$  of a spacetime manifold  $\mathcal{M}$ ,

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \langle 0, \text{out} | 0, \text{in} \rangle \quad (2.1)$$

and its functional derivatives with respect to the metric:

$$\begin{aligned} \frac{\delta Z}{\delta g^{ab}(x)} &= - \int \mathcal{D}\phi \frac{\delta S}{\delta g^{ab}(x)} e^{-S} = - \left\langle 0, \text{out} \left| \frac{\delta S}{\delta g^{ab}(x)} \right| 0, \text{in} \right\rangle, \\ \frac{\delta^2 Z}{\delta g^{ab}(x) \delta g^{cd}(y)} &= \int \mathcal{D}\phi \left[ \frac{\delta S}{\delta g^{ab}(x)} \frac{\delta S}{\delta g^{cd}(y)} - \frac{\delta^2 S}{\delta g^{ab}(x) \delta g^{cd}(y)} \right] e^{-S} \\ &= \left\langle 0, \text{out} \left| \frac{\delta S}{\delta g^{ab}(x)} \frac{\delta S}{\delta g^{cd}(y)} \right| 0, \text{in} \right\rangle - \left\langle 0, \text{out} \left| \frac{\delta^2 S}{\delta g^{ab}(x) \delta g^{cd}(y)} \right| 0, \text{in} \right\rangle \end{aligned} \quad (2.2)$$

where  $|0, \text{in}, \text{out}\rangle$  are the vacua defined at the in and out states. In terms of the effective action  $W = \ln Z$ , this becomes

$$\frac{\delta^2 W}{\delta g^{ab}(x) \delta g^{cd}(y)} = \frac{1}{Z} \frac{\delta^2 Z}{\delta g^{ab}(x) \delta g^{cd}(y)} - \frac{\delta W}{\delta g^{ab}(x)} \frac{\delta W}{\delta g^{cd}(y)}. \quad (2.3)$$

The expectation value of the quantum stress-energy tensor is given by

$$\langle T_{ab} \rangle = \frac{\langle 0, \text{out} | T_{ab} | 0, \text{in} \rangle}{\langle 0, \text{out} | 0, \text{in} \rangle} = - \frac{2}{\sqrt{g(x)}} \frac{\langle 0, \text{out} | \delta S / \delta g^{ab}(x) | 0, \text{in} \rangle}{\langle 0, \text{out} | 0, \text{in} \rangle} = - \frac{2}{\sqrt{g(x)}} \frac{\delta W}{\delta g^{ab}(x)}. \quad (2.4)$$

In analogue with this, we define the correlation function for the stress-energy tensor as

$$\begin{aligned} \langle T_{ab}(x) T_{cd}(y) \rangle &= \frac{\langle 0, \text{out} | T_{ab}(x) T_{cd}(y) | 0, \text{in} \rangle}{\langle 0, \text{out} | 0, \text{in} \rangle} = \frac{4}{\sqrt{g(x)g(y)}} \frac{1}{Z} \left\langle 0, \text{out} \left| \frac{\delta S}{\delta g^{ab}(x)} \frac{\delta S}{\delta g^{cd}(y)} \right| 0, \text{in} \right\rangle \\ &= \frac{4}{\sqrt{g(x)g(y)}} \left[ \frac{\delta^2 W}{\delta g^{ab}(x) \delta g^{cd}(y)} + \frac{\delta W}{\delta g^{ab}(x)} \frac{\delta W}{\delta g^{cd}(y)} + \left\langle \frac{\delta^2 S}{\delta g^{ab}(x) \delta g^{cd}(y)} \right\rangle \right]. \end{aligned} \quad (2.5)$$

For any local action the last term will not contribute to the final expression for the stress-energy correlation function. For such an action, this expression will depend on  $x, y$  as a  $\delta$  function:  $\delta(x-y)$ . Thus it need not be considered when computing the correlation function for  $x \neq y$ . The autocorrelation is understood as resulting from this by taking the  $y \rightarrow x$  or coincidence limit. Recognizing the second term in the last line as a product of expectation values of the stress-energy tensor, we can define the bitensor

$$\begin{aligned} \Delta T_{abcd}^2(x, y) &\equiv \langle T_{ab}(x) T_{cd}(y) \rangle - \langle T_{ab}(x) \rangle \langle T_{cd}(y) \rangle \\ &= \frac{4}{\sqrt{g(x)g(y)}} \frac{\delta^2 W}{\delta g^{ab}(x) \delta g^{cd}(y)}. \end{aligned} \quad (2.6)$$

## III. SECOND VARIATION OF THE EFFECTIVE ACTION

The classical action of a scalar field  $\phi(x)$  is

$$S[\phi] = \frac{1}{2} \int d\mathbf{x} \phi(x) H \phi(x), \quad (3.1)$$

where  $H$  is a second-order elliptic operator. From the spectral decomposition of this operator,  $H = \sum_n \lambda_n |n\rangle \langle n|$ , where  $n$  denotes the collective indices of the spectrum, the effective action can be expressed as

$$W = - \frac{1}{2} \ln \det(H/\mu) = - \frac{1}{2} \text{Tr} \ln \frac{H}{\mu} = - \frac{1}{2} \sum_n \ln \frac{\lambda_n}{\mu}, \quad (3.2)$$

where we assume the zero modes of  $H$  have been projected out and  $\mu$  is a normalization constant with dimensions of mass squared. This expression for the effective action is only formal since  $H$  is not trace class. We regularize the effective action and the expressions derived from it via the  $\zeta$ -function method [3,4].

The  $\zeta$  function for this system is defined as

$$\zeta_H(s) = \text{Tr} e^{-s \ln H/\mu} = \mu^s \sum_n \lambda_n^{-s}. \quad (3.3)$$

For  $2s > \dim \mathcal{M} = d$ , this sum is convergent. Then an analytic continuation in  $s$  is found such that it includes a neighborhood of  $s=0$ . From this analytic continuation the regularized effective action is given as

$$W_R = \frac{1}{2} \left. \frac{d\zeta(s)}{ds} \right|_{s=0} = \frac{1}{2} \zeta'(0). \quad (3.4)$$

For positive  $H$  the definition of the  $\Gamma$  function yields

$$\zeta_H(s) = \frac{\mu^s}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} U_0(t) dt, \quad U_0(t) = e^{-tH}; \quad (3.5)$$

i.e., the  $\zeta$  function is given as the Mellin transform of the trace of the heat kernel  $U_0(t)$ . We know  $\text{Tr} U_0(t) \sim t^{-d/2}$  for  $t \rightarrow 0$ . Hence for Eq. (3.5) to be convergent, we have the condition  $s > d/2$ . This is most often the expression from which an analytic continuation is derived. In fact, one of the main points of the  $\zeta$  function idea is that formal expressions

such as  $\text{Tr} U_0(t)$  need be modified by the introduction of a factor  $t^\nu$ . Then, once the analytic continuation is found, one takes  $\nu=0$ .

We now consider the effect of two small metric perturbations  $\delta_1$  and  $\delta_2$  on the effective action. (They are assumed to be independent of the order with which they act.) The response of the effective action to these perturbations is

$$\begin{aligned} \delta_2 \delta_1 W_R[g] &= W_R[g + \delta_1 + \delta_2] + W_R[g] - W_R[g + \delta_1] \\ &\quad - W_R[g + \delta_2] \\ &= \frac{1}{2} \frac{d}{ds} [\delta_2 \delta_1 \zeta_H(s)]|_{s=0}, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \delta_2 \delta_1 \zeta_H(s) &= \frac{\mu^s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} \{ e^{-t(H + \delta_1 H + \delta_2 H)} \\ &\quad - e^{-t(H + \delta_1 H)} - e^{-t(H + \delta_2 H)} + e^{-tH} \}. \end{aligned} \quad (3.7)$$

To evaluate this, we use the Schwinger perturbative expansion [28]. For  $U(t) = e^{-t(H + H_1)}$ , where  $H_1 = \delta_1 H \ll H$ ,

$$\begin{aligned} \text{Tr} U(t) &= \text{Tr} U_0(t) - t \text{Tr} [H_1 U_0(t)] + \frac{t^2}{2} \int_0^1 du_1 \\ &\quad \times \text{Tr} [H_1 U_0((1-u_1)t) H_1 U_0(u_1 t)] + \dots \end{aligned} \quad (3.8)$$

Using this, we can write the response of the  $\zeta$  function to these perturbations as

$$\delta_2 \delta_1 \zeta_H(s) = \frac{\mu^s}{2\Gamma(s)} \int_0^\infty dt t^{s+1} \int_0^1 du_1 \{ \text{Tr} [(\delta_1 H) U_0((1-u_1)t) (\delta_2 H) U_0(u_1 t)] + \text{Tr} [(\delta_2 H) U_0((1-u_1)t) (\delta_1 H) U_0(u_1 t)] \}. \quad (3.9)$$

As it stands, it is not finite. When the traces are taken involving  $U_0((1-u_1)t)$  and  $U_0(u_1 t)$ , the divergences at  $(1-u_1)t, u_1 t \rightarrow 0$  are present. At this point, we modify the above expression for the second variation of the  $\zeta$  function by introducing the factor  $[u_1(1-u_1)t^2]^\nu$ . At the end of the calculation, once the analytic continuation is found, we set  $\nu=0$ . We view the introduction of this factor as an extension of the  $\zeta$ -function method to the situation where the second variation is needed. The replacement of  $U_0(t) \rightarrow t^\nu U_0(t)$  is the spirit of the usual  $\zeta$ -function method and reproduces the usual results when applied to the traditional problems, such as finding the first variation, which produces the expectation value of the quantum stress-energy tensor. After the change of variables

$$u = (1-u_1)t, \quad v = u_1 t \quad (3.10)$$

the twice-varied  $\zeta$  function transforms to

$$\delta_2 \delta_1 \zeta_H(s) = \frac{\mu^s}{2\Gamma(s)} \int_0^\infty du \int_0^\infty dv (u+v)^s (uv)^\nu \{ \text{Tr} [(\delta_1 H) U_0(u) (\delta_2 H) U_0(v)] + \text{Tr} [(\delta_2 H) U_0(u) (\delta_1 H) U_0(v)] \}. \quad (3.11)$$

Considering the first trace in the above expression, we find

$$\text{Tr} [(\delta_1 H) U_0(u) (\delta_2 H) U_0(v)] = \sum_{n, n'} \langle n' | (\delta_1 H) e^{-uH} | n \rangle \langle n | (\delta_2 H) e^{-vH} | n' \rangle = \sum_{n, n'} e^{-u\lambda_n - v\lambda_{n'}} \langle n' | (\delta_1 H) | n \rangle \langle n | (\delta_2 H) | n' \rangle. \quad (3.12)$$

By defining

$$T_{ab}[\phi_n(x), \phi_n'^*(x)] \equiv -\frac{2}{\sqrt{g(x)}} \left\langle n' \left| \frac{\delta H}{\delta g_{ab}(x)} \right| n \right\rangle = -\frac{2}{\sqrt{g(x)}} \int d\mathbf{z} \phi_n'^*(z) \frac{\delta H}{\delta g_{ab}(x)} \phi_n(z), \quad (3.13)$$

we can now write the stress-energy correlation bitensor as

$$\Delta T_{abcd}^2(x, y) = \frac{1}{2} \frac{d}{ds} \left[ \frac{\mu^s}{\Gamma(s)} \int_0^\infty du \int_0^\infty dv (u+v)^s (uv)^\nu \sum_{n, n'} e^{-u\lambda_n - v\lambda_{n'}} T_{ab}[\phi_n(x), \phi_n'^*(x)] T_{cd}[\phi_n'(y), \phi_n'^*(y)] \right], \quad (3.14)$$

where the  $s, \nu \rightarrow 0$  limit is understood.

For the rest of this paper, we will specialize to the cases of homogeneous spacetimes. This implies  $\Delta T_{abcd}^2(x, y)$  will only depend on  $r = x - y$ . We find it convenient to average over all  $x$ . We can do this while leaving the points separated by  $r$ . Also, the homogeneity will usually lead to a degeneracy of the eigenvalues. Thus the collective quantum number  $n$  can be split into principal and degenerate parts  $n \rightarrow n, m$  and the eigenvalues only depend on  $n$ :  $\lambda_{nm} \rightarrow \lambda_n$ . This allows the sum over the degenerate indices to be done before evaluation of the  $\zeta$  function. Putting all these together the stress-energy two-point function becomes

$$\Delta T_{abcd}^2(r) = \frac{1}{2\Omega} \frac{d}{ds} \left[ \frac{\mu^s}{\Gamma(s)} \int_0^\infty du \int_0^\infty dv (u+v)^s (uv)^\nu \left( \sum_{n, n'} e^{-u\lambda_n - v\lambda_{n'}} \sum_{mm'} \int_{\mathcal{M}} d\mathbf{x} T_{ab}[\phi_{nm}(x), \phi_{n'm'}^*(x)] \right. \right. \\ \left. \left. \times T_{cd}[\phi_{n'm'}(x+r), \phi_{nm}^*(x+r)] \right) \right], \quad (3.15)$$

where  $\Omega = \int_{\mathcal{M}} d\mathbf{x}$ , the volume of the manifold. If the manifold is noncompact, it is understood to be the unit volume.

#### IV. FORM FOR THE KLEIN-GORDON FIELD

We now develop the general form for the second variation of the  $\zeta$  function for the Klein-Gordon field. In the Lorentzian sector, we use the Misner-Thorne-Wheeler (MTW) signature convention  $(-1, 1, \dots)$ , and thus in the Euclidean sector, we have the signature  $(1, 1, \dots)$ . We assume the metric can be given the form

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & h_{ij} \end{pmatrix}, \quad (4.1)$$

where  $h_{ij}$  is the metric for the spatial section. We denote the time and spatial variables by  $x = (\tau, \mathbf{x})$ , the invariant spatial volume form by  $d\mathbf{x}$ , and the spatial manifold by  $\Sigma$  (thus  $\mathcal{M} = S^1 \times \Sigma$ ).

The wave operator for the Klein-Gordon field is given by

$$H = -\square + \xi R + m^2 = -\frac{\partial^2}{\partial \tau^2} - \Sigma \Delta + \xi R + m^2. \quad (4.2)$$

Let  $u_n(\mathbf{x})$  be the eigenfunctions of  $\Sigma \Delta$ :  $\Sigma \Delta u_n(\mathbf{x}) = -\kappa_n^2 u_n(\mathbf{x})$ , where  $n$  denotes the (collective) quantum numbers for the spatial part of the spectrum. We assume the  $u_n(\mathbf{x})$  are orthonormal. The Euclidean time is made periodic with a period of  $\beta = 1/T$ , where  $T$  can be interpreted as a temperature. The eigenfunctions are thus given by

$$\phi_{k_0, n}(x) = \frac{e^{-ik_0 \tau}}{\sqrt{\beta}} u_n(\mathbf{x}), \quad k_0 = \frac{2\pi n_0}{\beta}, \quad n_0 = 0, \pm 1, \pm 2, \dots, \quad (4.3a)$$

and the eigenvalues by (see, e.g., [27])

$$\lambda_{k_0, n} = k_0^2 + \kappa_n^2 + \xi R + m^2. \quad (4.3b)$$

From our definition of the stress-energy tensor (3.13) we find

$$T_{ab}[\psi, \phi](x) \equiv \frac{2}{\sqrt{g(x)}} \int \sqrt{g(x')} \psi(x') \left( \frac{\delta H_{x'}}{\delta g^{ab}(x)} \right) dx' \\ = -2 \nabla_a \psi \nabla_b \phi + g_{ab} (\nabla_c \psi \nabla^c \phi + \psi \nabla_c \nabla^c \phi) \\ + 2 \xi \psi \phi R_{ab}. \quad (4.4)$$

This differs from the usual definition of  $T_{ab}[\psi, \phi]$ , but this is to be expected. The eigenvalues used are themselves different, since we have the extra  $\tau$  degree of freedom.

In this paper, we wish to concentrate on computing the autocorrelation of the energy density

$$\Delta \rho^2(x) \equiv \lim_{y \rightarrow x} (\langle \rho(x) \rho(y) \rangle - \langle \rho(x) \rangle \langle \rho(y) \rangle), \quad \rho = T_{00}. \quad (4.5)$$

This will give a measure for the magnitude of the fluctuations of the stress-energy vacuum expectation value. This choice yields

$$\rho[\phi, \psi] = -(\partial_\tau \phi \partial_\tau \psi - \psi \partial_\tau^2 \phi) + (\nabla_i \psi \nabla^i \phi + \psi \Sigma \Delta \phi), \quad (4.6)$$

where  $i$  is summed over the spatial degrees of freedom only. This becomes

$$\rho[\phi_n, \phi_n'^*] = -(k_0^2 + k_0 k_0' + \kappa_n^2) \phi_n \phi_n'^* + (\nabla_i \phi_n) (\nabla^i \phi_n'^*) \quad (4.7)$$

when we specialize to eigenmodes (4.3). When we consider  $|\rho[\phi_n, \phi_{n'}^*]|^2$  the contribution from the  $\phi_n^* \phi_{n'} \nabla_i \phi_n \nabla^i \phi_{n'}^*$ , and its conjugate will vanish when summed over. This will be shown case by case. This assumption provides

$$|\rho[\phi_n, \phi_{n'}^*]|^2 = (k_0^2 + k_0 k_0' + \kappa_n^2)(k_0'^2 + k_0 k_0' + \kappa_{n'}^2) |\phi_n|^2 |\phi_{n'}^*|^2 + |(\nabla_i \phi_n)(\nabla^i \phi_{n'}^*)|^2. \quad (4.8)$$

At this point, we wish to take the zero-temperature limit, i.e.,  $\beta \rightarrow \infty$  whereby  $\sum_{n_0=-\infty}^{\infty} \rightarrow (\beta/2\pi) \int_{-\infty}^{\infty} dk_0$ . We can now do these sums and/or integrals easily, since they amount to calculating the moments of Gaussians. We find

$$\begin{aligned} \sum_{n_0=-\infty}^{\infty} e^{-(2\pi n_0/\beta)^2 u} &\rightarrow \frac{\beta}{2\pi} \int_{-\infty}^{\infty} e^{-uk_0^2} dk_0 = \frac{\beta}{2\sqrt{\pi}} u^{-1/2}, \\ \sum_{n_0=-\infty}^{\infty} \left(\frac{2\pi n_0}{\beta}\right)^2 e^{-(2\pi n_0/\beta)^2 u} &\rightarrow \frac{\beta}{2\pi} \int_{-\infty}^{\infty} k_0^2 e^{-uk_0^2} dk_0 = \frac{\beta}{4\sqrt{\pi}} u^{-3/2}, \end{aligned} \quad (4.9)$$

and there is zero contribution for any odd power of  $k_0$  or  $k_0'$ . The normalization of  $\phi_n$  and  $\phi_{n'}$  provide a factor of  $\beta^{-2}$  which cancels the powers of  $\beta$  introduced in approaching the  $\beta \rightarrow \infty$  limit. Introducing the functions

$$\Xi_{nn'}(u, v) = \frac{1}{8\pi\Omega(uv)^{1/2}} \left\{ \frac{1}{uv} + \frac{\kappa_n^2}{v} + \frac{\kappa_{n'}^2}{u} + 2\kappa_n^2 \kappa_{n'}^2 \right\} \sum_{mm'} \int_{\Sigma} |u_{nm}|^2 |u_{n'm'}|^2 d\mathbf{x}, \quad (4.10a)$$

$$\Theta_{nn'}(u, v) = \frac{1}{8\pi\Omega(uv)^{1/2}} \left( \frac{1}{u} + 2\kappa_n^2 \right) \sum_{mm'} \int_{\Sigma} u_{nm} (\nabla_i u_{nm}^*) u_{n'm'}^* \nabla^i u_{n'm'} d\mathbf{x}, \quad (4.10b)$$

$$\Pi_{nn'}(u, v) = \frac{1}{4\pi\Omega(uv)^{1/2}} \sum_{mm'} \int_{\Sigma} |(\nabla_i u_{nm})(\nabla^i u_{n'm'}^*)|^2 d\mathbf{x}, \quad (4.10c)$$

and

$$\Psi_{nn'}(u, v) = \Xi_{nn'}(u, v) + \Theta_{nn'}(u, v) + \Theta_{n'n}(v, u) + \Pi_{nn'}(u, v), \quad (4.11)$$

we define the  $\zeta$  function

$$\zeta_{\Psi}(s, \nu) = \sum_{n, n'} \int_0^{\infty} \int_0^{\infty} (u+v)^s (uv)^{\nu} \Psi_{nn'}(u, v) e^{-\lambda_n u - \lambda_{n'} v} dv du. \quad (4.12)$$

From it, we derive an expression for the regularized autocorrelation of the energy density as

$$\Delta\rho^2 = \frac{1}{2} \frac{d}{ds} \left[ \frac{\mu^s}{\Gamma(s)} \zeta_{\Psi}(s, \nu) \right] \Big|_{s=0, \nu=0} = \frac{1}{2} \lim_{\nu \rightarrow 0} \zeta_{\Psi}(0, \nu), \quad (4.13)$$

where we have used  $1/\Gamma(s) \sim s + \gamma s^2 + 0(s^3)$  ( $\gamma$  is Euler's constant) for  $s \sim 0$ . We have assumed  $\nu > d/2 + 1$ , in which case both  $\zeta_{\Psi}(0, \nu)$  and  $d\zeta_{\Psi}(s, \nu)/ds|_{s=0}$  are finite. Thus to find the regularized expression for  $\Delta\rho^2$ , we need to find the analytic continuation of  $\zeta_{\Psi}(0, \nu)$  in  $\nu$  such that it is finite at  $\nu=0$ .

## V. FLUCTUATIONS FOR $\Sigma = R^D \times S^1$

As the first application, we calculate in this section the variance of the energy density for a massless ( $m=0$ ) minimally coupled ( $\xi=0$ ) scalar field on a  $(d+2)$ -dimensional flat ( $R=0$ ) spacetime which is periodic on one spatial dimension with period  $L$ . For this geometry, the spatial eigenfunctions are

$$u_{\mathbf{k}n}(\mathbf{x}, z) = \frac{e^{i\mathbf{k} \cdot \mathbf{x} + ilz}}{\sqrt{(2\pi)^d L}}, \quad \mathbf{k} \in \mathbf{R}^d, \quad n = 0, \pm 1, \pm 2, \dots, \quad (5.1)$$

with  $l = 2\pi/L$ . Denoting by  $\mathbf{x} = (x_1, \dots, x_d)$  the coordinates for the open dimensions and  $z$  for the one compact dimension, we have  $\Sigma \Delta = \sum_{j=1}^d \partial_{x_j}^2 + \partial_z^2$  and hence  $\kappa_{k,n}^2 = k^2 + l^2 n^2$  ( $k^2 = |\mathbf{k}|^2$ ). From this we see we should take  $k$  and  $n$  as the principal indices and the angular degeneracy of  $\mathbf{k}$  as the degenerate indices, i.e.,  $\sum_{m \rightarrow} \int d\Omega_{d-1}$ , integration over the unit  $d-1$  sphere. Where there is no confusion, we will denote the volume of the sphere as  $S^{d-1}$ .

To evaluate Eq. (4.10) we sum over the degenerate indices and perform the volume average

$$\int d\Omega_{d-1} \int d\Omega'_{d-1} \frac{1}{\text{Vol}(\Sigma)} \int d\mathbf{x} dz |u_{kn}|^2 |u_{k'n'}|^2 = \left( \frac{S^{d-1}}{(2\pi)^d L} \right)^2 \quad (5.2)$$

from which Eq. (4.10a) becomes

$$\Xi_{kk'n'}(u, v) = \frac{1}{8\pi} \left( \frac{S^{d-1}}{(2\pi)^d L} \right)^2 \frac{1}{(uv)^{1/2}} \left\{ \frac{1}{uv} + \frac{k^2 + l^2 n^2}{2v} + \frac{k'^2 + l^2 n'^2}{2u} + 2(k^2 + l^2 n^2)(k'^2 + l^2 n'^2) \right\}. \quad (5.3)$$

Also, since  $\nabla_i u_{kn} \nabla^i u_{k'n'}^* = (kk' \cos \gamma + l^2 nn') u_{kn} u_{k'n'}^*$ , where  $\cos \gamma = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'$ , the momentum correlation term (4.10c) is

$$\Pi_{kk'n'}(u, v) = \frac{1}{8\pi} \left( \frac{S^{d-1}}{(2\pi)^d L} \right)^2 \frac{2}{(uv)^{1/2}} \left\{ \frac{k^2 k'^2}{d} + l^4 n^2 n'^2 \right\}. \quad (5.4)$$

Here we have used  $\int d\Omega_{d-1} \int d\Omega'_{d-1} \cos \gamma = 0$  and Eq. (A7) from the Appendix:  $\int d\Omega_{d-1} \int d\Omega'_{d-1} \cos^2 \gamma = (S^{d-1})^2/d$ . Also, when summed over  $n$  and  $n'$ ,  $\Theta_{kk'n'}$  vanishes.

We now turn to the principal index sums for  $k$  and  $k'$ . First consider the case when  $n=0$  (or  $n'=0$ ), this leads us to evaluate

$$\int_0^\infty dk k^{d-1} \int_0^\infty du u^{\nu-a} k^b e^{-k^2 u}. \quad (5.5)$$

Here  $(a, b) = (3/2, 0)$  or  $(1/2, 2)$ , and  $2a + b - 3 = 0$ . We assume a boundary on the nonperiodic dimensions at large distance and moved to infinity. This is effected by having  $\int_0^\infty dk \rightarrow \lim_{\epsilon \rightarrow 0} \int_\epsilon^\infty$ . Then Eq. (5.5) becomes

$$\int_\epsilon^\infty dk k^{d-1} \int_0^\infty du u^{\nu-a} k^b e^{-k^2 u} = \Gamma(\nu - a + 1) \int_\epsilon^\infty dk k^{d-2\nu} \xrightarrow{\nu \rightarrow 0} -\frac{\Gamma(1-a)}{d+1} \epsilon^{d+1} \xrightarrow{\epsilon \rightarrow 0} 0, \quad (5.6)$$

where we have used  $\nu > d/2$  in evaluating the  $k$  integration. Thus we find there is no contribution from the  $n=0$  or  $n'=0$  terms in  $\zeta_\Psi$ .

Turning to the case where  $n$  and  $n'$  do not vanish, we consider the function

$$\Psi_{nn'}(u, v) = \int_0^\infty k^{d-1} dk \int_0^\infty k'^{d-1} dk' \Psi_{kk'n'}(u, v) e^{-uk^2 - vk'^2}. \quad (5.7)$$

In the integrand, there are terms with either factors of  $k^{d-1}$  or  $k^{d+1}$ , similarly for  $k'$ . Using

$$\int_0^\infty k^{d-1} dk e^{-uk^2} = \frac{1}{2} \Gamma\left(\frac{d}{2}\right) u^{-d/2} \quad \text{and} \quad \int_0^\infty k^{d+1} dk e^{-uk^2} = \frac{1}{2} \Gamma\left(\frac{d}{2}\right) u^{-d/2} \frac{d}{2u}, \quad (5.8)$$

we have the rule that upon doing the  $k, k'$  integrations, we get an overall factor of  $[\Gamma(d/2)/2]^2 (uv)^{-d/2}$  and each factor of  $k^2$  becomes  $d/2u$ , and  $k'^2$  becomes  $d/2v$ . Applying this rule, Eq. (4.11) has been determined

$$\Psi_{nn'}\left(\frac{u}{l^2}, \frac{v}{l^2}\right) = \frac{1}{8\pi} \left[ \frac{S^{d-1} l^{d+3}}{2(2\pi)^d L} \Gamma\left(\frac{d}{2}\right) \right]^2 \left( \frac{1}{uv} \right)^{(d+3)/2} \left\{ \frac{(d+1)(d+2)}{2} + (1+d)(un^2 + vn'^2) + 4uvn^2 n'^2 \right\} \quad (5.9)$$

and the corresponding  $\zeta$  function is

$$\zeta_\Psi(0, \nu) = B \sum_{n, n'=1}^\infty \int_0^\infty du \int_0^\infty dv (uv)^{\nu - (d+3)/2} \left\{ \frac{(d+1)(d+2)}{2} + (1+d)(un^2 + vn'^2) + 4uvn^2 n'^2 \right\} e^{-n^2 u - n'^2 v}, \quad (5.10)$$

where

$$B = \frac{1}{8\pi} \left[ \frac{S^{d-1} l^{d+1-2\nu}}{(2\pi)^d L} \Gamma\left(\frac{d}{2}\right) \right]^2. \quad (5.11)$$

This form of the  $\zeta$  function now allows us to perform the needed analytic continuation. We make use of the relation

$$\sum_{n=1}^\infty n^a \int_0^\infty t^{s-1} e^{-n^2 t} = \Gamma(s) \zeta_R(2s - a), \quad (5.12)$$

where  $\zeta_R(s)$  is the Riemann zeta function. Recalling Eq. (4.13), the variance of the energy density is

$$\Delta\rho^2 = \frac{B}{2} \zeta_R(-(d+1))^2 \left\{ \frac{(d+1)(d+2)}{2} \Gamma\left(-\frac{d+1}{2}\right)^2 + 2(d+1) \Gamma\left(-\frac{d+1}{2}\right) \Gamma\left(-\frac{d-1}{2}\right) + 4 \Gamma\left(-\frac{d-1}{2}\right)^2 \right\}. \quad (5.13)$$

Since  $\Gamma(-(d-1)/2) = -[(d+1)/2] \Gamma(-(d+1)/2)$ , the second and third terms in the above expression cancel, leaving,

$$\Delta\rho^2 = \frac{(d+1)(d+2)}{2} \left[ \frac{S^{d-1}}{2\pi^{d+1}L^{d+2}} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d+2}{2}\right) \zeta_R(d+2) \right]^2 \quad (5.14)$$

by way of the Riemann zeta function reflection formula,

$$\Gamma\left(\frac{s}{2}\right) \zeta_R(s) = \pi^{s-1/2} \Gamma\left(\frac{1-s}{2}\right) \zeta_R(1-s). \quad (5.15)$$

This is our result for the zero-temperature variance of the vacuum energy density for a massless minimally coupled quantum scalar field on a  $(d+2)$ -dimensional spacetime periodic in one spatial dimension. To get a measure of the fluctuations, we consider the dimensionless quantity

$$\Delta' = \frac{\langle \rho^2 \rangle - \langle \rho \rangle^2}{\langle \rho \rangle^2}. \quad (5.16)$$

For the system at hand, the energy density is [18]

$$\langle \rho \rangle = -\frac{S^{d-1}}{2\pi^{d+1}L^{d+2}} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d+2}{2}\right) \zeta_R(d+2). \quad (5.17)$$

Thus we get one of the main results of this paper:

$$\Delta'(\Sigma = R^d \times S^1) = \frac{(d+1)(d+2)}{2}. \quad (5.18)$$

Kuo and Ford [11] computed the same measure of the fluctuations for the case of  $\Sigma = R^2 \times S^1$  via a different method. Their result of  $\Delta' = 6$  is obtained here for  $d=2$ . It is interesting to note that the relative amount of fluctuation increases quadratically with the dimension of the spacetime.

## VI. FLUCTUATIONS FOR $\Sigma = S^3$

As a second example, we calculate the fluctuations of energy density for a massless ( $m=0$ ) conformally coupled ( $\xi=1/6$ ) scalar field on a three-dimensional space of constant curvature  $S^3$  with radius  $a$ . The spacetime  $\mathcal{M}$  is then the Einstein universe.

We start by writing the spatial metric as [29]

$$ds^2 = \gamma_{ab} \sigma^a \sigma^b = \sum_{a=1}^3 l_a^2 (\sigma^a)^2, \quad (6.1)$$

where the  $\sigma^a$ 's are the basis one-forms on the three sphere satisfying the structure relations

$$d\sigma^a = \frac{1}{2} \epsilon_{bc}^a \sigma^b \sigma^c. \quad (6.2)$$

Here  $\epsilon_{bc}^a$ , components of the totally antisymmetric tensor, are the structure constants for the rotation group  $SO(3)$ . The  $\gamma_{ab} = l_a^2 \delta_{ab}$  are constants of the space (the principal curvature radii in a static mixmaster universe [30]) and for the Einstein universe  $\Sigma = S^3$  we have  $l_1 = l_2 = l_3 = a/2$ . The curvature scalar is  $R = 6/a$  and the volume is  $\Omega = 2\pi^2 a^3$ .

Using the Euler angle parametrization, the basis forms are

$$\begin{aligned} \sigma^1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\ \sigma^2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi, \\ \sigma^3 &= d\psi + \cos\theta d\phi, \end{aligned} \quad (6.3)$$

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 4\pi.$$

Taking  $e_a$  as the invariant vectors dual to  $\sigma^a$  and defining the angular momentum operators  $L_a = i e_a$ , the spatial Laplacian becomes

$$\Delta = \sum_{a=1}^3 l_a^{-2} (e_a)^2 = -\frac{4}{a^2} (L_1^2 + L_2^2 + L_3^2) = -\frac{4L^2}{a^2}. \quad (6.4)$$

For the harmonic functions on  $S^3$  we can use the  $SO(3)$  representation (Wigner) functions  $D_{KM}^J(\theta, \psi, \phi)$ , where  $J=0, \frac{1}{2}, 1, \frac{3}{2}, \dots$  are the principal quantum numbers and  $K, M = -J, -J+1, \dots, J-1, J$  are the degenerate quantum numbers. We will also find it convenient to use the  $SO(4)$  representation functions, the hyperspherical harmonics with principal quantum number  $n=2J$ . From  $L^2 D_{KM}^J = J(J+1) D_{KM}^J$  we find

$$\kappa_n^2 = \frac{4J(J+1)}{a^2} = \frac{n(n+2)}{a^2}, \quad \lambda_n = \kappa_n^2 + \xi R = \frac{(n+1)^2}{a^2} \quad (6.5)$$

along with the spatial eigenmodes

$$u_{JKM} = \sqrt{\frac{n+1}{\Omega}} D_{KM}^J(\theta, \psi, \phi) \quad (6.6)$$

for  $S^3$ .

To compute  $\Xi_{nn'}$ , we first use the sum rule  $\sum_{M''} D_{MM''}^J (D_{M''M'}^J)^* = \delta_{M,M'}$  to get  $\sum_{MK} D_{MK}^J (D_{MK}^J)^* = 2J+1$ . Since

$$\sum_{\text{angular}} \int d\mathbf{x} |u_{nKM} u_{n'K'M'}|^2 = \frac{(n+1)(n'+1)}{\Omega} \int d\mathbf{x} \sum_{KM} |D_{KM}^J|^2 \sum_{K'M'} |D_{K'M'}^{J'}|^2 = \frac{(n+1)^2(n'+1)^2}{\Omega}, \tag{6.7}$$

Eq. (4.10a) is given by

$$\Xi_{nn'} = \frac{(n+1)^2(n'+1)^2}{8\pi\Omega^2(uv)^{1/2}} \left[ \frac{1}{uv} + \frac{n(n+2)}{a^2v} + \frac{n'(n'+2)}{a^2u} + \frac{2n(n+2)n'(n'+2)}{a^4} \right]. \tag{6.8}$$

We now turn to the momentum correlation term  $\Pi_{nn'}$ . To facilitate the evaluation of the spatial derivative terms, we use properties of the generators of the Lie algebra, which for SO(3), is just the quantum theory of angular momentum (see, e.g. [31]). We have

$$\nabla_i u_{nm} \nabla^i u_{n'm'}^* = \gamma^{ab} e_a(u_{nm}) e_b(u_{n'm'}^*) = \left(\frac{2}{a}\right)^2 \frac{1}{\Omega} \sqrt{(n+1)(n'+1)} \sum_{a=1}^3 L_a(D_{KM}^J) L_a(D_{K'M'}^{J'}). \tag{6.9}$$

We also recast the spatial volume measure:

$$\int_{\mathcal{M}} d\mathbf{x} = \frac{a^3}{4} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\psi = \frac{a^3}{4} \int d\Omega, \tag{6.10}$$

where we use the notation of [31] for  $d\Omega$ . We assume the integrand is invariant under  $\psi \rightarrow \psi + 2\pi$ . This condition is satisfied by  $D_{KM}^J$ .

Defining

$$B_{JJ'} = \sum_{K,M=-J}^J \sum_{K',M'=-J'}^{J'} \int d\Omega \left| \sum_{a=1}^3 L_a(D_{KM}^J) L_a(D_{K'M'}^{J'}) \right|^2, \tag{6.11}$$

the momentum term (4.10c) becomes

$$\Pi_{nn'} = \frac{(n+1)(n'+1)}{8\pi\Omega^2(uv)^{1/2}} \frac{4}{a^4\pi^2} B_{JJ'}. \tag{6.12}$$

Returning to Eq. (6.11), we can express it in terms of the angular momentum operators as

$$B_{JJ'} = \sum_{KMK'M'} \int d\Omega \left| \frac{1}{2} L_+ D_{KM}^J L_- D_{K'M'}^{J'} + \frac{1}{2} L_- D_{KM}^J L_+ D_{K'M'}^{J'} + L_3 D_{KM}^J L_3 D_{K'M'}^{J'} \right|^2, \tag{6.13}$$

where  $L_\pm = L_1 \pm iL_2$  are the raising and lowering operators. Introducing the convenient notation  $L_\alpha$ ,  $\alpha \in (+, -, 0)$  where  $L_0$  replaces  $L_3$  and the symbols

$$C_K^\pm = \sqrt{J(J+1) - K(K \pm 1)}, \quad C_K^0 = K, \tag{6.14}$$

the action of the angular momentum operators on the harmonic functions is neatly given by

$$L_\alpha D_{KM}^J = (-i)^{-\alpha} C_K^{-\alpha} D_{K-\alpha,M}^J, \quad L_\alpha D_{KM}^{J*} = -i^{-\alpha} C_K^\alpha D_{K+\alpha,M}^J. \tag{6.15}$$

Introducing

$$\langle \alpha\beta; \delta\epsilon \rangle = \sum_{K,M=-J}^J \sum_{K',M'=-J'}^{J'} \int d\Omega L_\alpha D_{KM}^J L_\beta D_{K'M'}^{J*} L_\delta D_{K'M'}^{J'} L_\epsilon D_{K'M'}^{J'*}, \tag{6.16}$$

we can express Eq. (6.11) as

$$B_{JJ'} = \langle 00; 00 \rangle + \frac{1}{4} (\langle ++; -- \rangle + \langle --; ++ \rangle + \langle +-; +- \rangle + \langle -+; -+ \rangle) + \frac{1}{2} (\langle 0-; +0 \rangle + \langle 0+; -0 \rangle + \langle +0; 0- \rangle + \langle -0; 0+ \rangle). \tag{6.17}$$

Using Eq. (6.15), we can write Eq. (6.16) as

$$\langle \alpha\beta; \delta\epsilon \rangle = (-1)^{\beta+\epsilon} i^{\alpha+\beta+\delta+\epsilon} \sum_{KMK'M'} C_K^{-\alpha} C_K^{\beta} C_{K'}^{-\delta} C_{K'}^{\epsilon} \int d\Omega D_{K+\beta, M}^{J*} D_{K'+\epsilon, M'}^{J'*} D_{K-\alpha, M}^J D_{K'-\delta, M'}^{J'} \cdot \quad (6.18)$$

The integral of the fourfold products of the Wigner function is given by

$$\int d\Omega D_{K_1 M_1}^{J_1} D_{K_2 M_2}^{J_2} D_{K_3 M_3}^{J_3*} D_{K_4 M_4}^{J_4*} = 8\pi^2 \sum_{JKM} (2J+1) \begin{pmatrix} J_1 & J_2 & J \\ K_1 & K_2 & K \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J \\ M_1 & M_2 & M \end{pmatrix} \begin{pmatrix} J_3 & J_4 & J \\ K_3 & K_4 & K \end{pmatrix} \begin{pmatrix} J_3 & J_4 & J \\ M_3 & M_4 & M \end{pmatrix}. \quad (6.19)$$

This can readily be seen by two applications of the sum rule

$$D_{K_1 M_1}^{J_1} D_{K_2 M_2}^{J_2} = \sum_{JKM} (2J+1) \begin{pmatrix} J_1 & J_2 & J \\ K_1 & K_2 & K \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J \\ M_1 & M_2 & M \end{pmatrix} D_{KM}^{J*} \quad (6.20)$$

and the orthogonality property

$$\int d\Omega D_{KM}^{J*} D_{K'M'}^{J'} = \frac{8\pi^2}{2J+1} \delta_{JJ'} \delta_{KK'} \delta_{MM'}. \quad (6.21)$$

Utilizing this result, Eq. (6.16) has the form

$$\begin{aligned} \langle \alpha\beta; \delta\epsilon \rangle &= 8\pi^2 (-1)^{\beta+\epsilon} i^{\alpha+\beta+\delta+\epsilon} \sum_{K, M=-J}^J \sum_{K', M'=-J'}^{J'} C_K^{-\alpha} C_K^{\beta} C_{K'}^{-\delta} C_{K'}^{\epsilon} \sum_{J'', K'', M''} (2J''+1) \begin{pmatrix} J & J' & J'' \\ K+\beta & K'+\epsilon & K'' \end{pmatrix} \\ &\times \begin{pmatrix} J & J' & J'' \\ K-\alpha & K'-\delta & K'' \end{pmatrix} \begin{pmatrix} J & J' & J'' \\ M & M' & M'' \end{pmatrix} \begin{pmatrix} J & J' & J'' \\ M & M' & M'' \end{pmatrix}. \end{aligned} \quad (6.22)$$

Using the orthogonality property of the 3- $j$  symbols,

$$\sum_{M, M'} \begin{pmatrix} J & J' & J_1 \\ M & M' & m_1 \end{pmatrix} \begin{pmatrix} J & J' & J_2 \\ M & M' & m_2 \end{pmatrix} = (2J_1+1)^{-1} \delta_{J_1, J_2} \delta_{m_1, m_2}, \quad (6.23)$$

we can reduce the above fourfold product of 3- $j$  symbols to a twofold product by doing the  $M, M', M''$  sums to find the final form that is most useful to us:

$$\langle \alpha\beta; \delta\epsilon \rangle = 8\pi^2 (-1)^{\beta+\epsilon} \sum_{J''} (2J''+1) \sum_{KK'K''} C_K^{-\alpha} C_K^{\beta} C_{K'}^{-\delta} C_{K'}^{\epsilon} \begin{pmatrix} J & J' & J'' \\ K+\beta & K'+\epsilon & K'' \end{pmatrix} \begin{pmatrix} J & J' & J'' \\ K-\alpha & K'-\delta & K'' \end{pmatrix}. \quad (6.24)$$

The triangularity of the 3- $j$  symbols implies the condition  $\alpha+\beta+\delta+\epsilon=0$  for  $\langle \alpha\beta; \delta\epsilon \rangle$  not to vanish.

With this relation, we can evaluate the terms we need for Eq. (6.17):

$$\langle 00; 00 \rangle = 8\pi^2 \sum_{J''} (2J''+1) \sum_{KK'K''} \left[ KK' \begin{pmatrix} J & J' & J'' \\ K & K' & K'' \end{pmatrix} \right]^2 \quad (6.25a)$$

$$\langle ++; -- \rangle = 8\pi^2 \sum_{J''} (2J''+1) \sum_{KK'K''} C_K^- C_K^+ C_{K'}^- C_{K'}^+ \begin{pmatrix} J & J' & J'' \\ K+1 & K'-1 & K'' \end{pmatrix} \begin{pmatrix} J & J' & J'' \\ K-1 & K'+1 & K'' \end{pmatrix} \quad (6.25b)$$

$$= \langle ++; -- \rangle, \quad (6.25c)$$

$$\langle +-; +- \rangle = 8\pi^2 \sum_{J''} (2J''+1) \sum_{KK'K''} \left[ C_K^- C_{K'}^- \begin{pmatrix} J & J' & J'' \\ K-1 & K'-1 & K'' \end{pmatrix} \right]^2, \quad (6.25d)$$

$$\langle -+; -+ \rangle = 8\pi^2 \sum_{J''} (2J''+1) \sum_{KK'K''} \left[ C_K^+ C_{K'}^+ \begin{pmatrix} J & J' & J'' \\ K+1 & K'+1 & K'' \end{pmatrix} \right]^2, \quad (6.25e)$$

$$\langle 0-; +0 \rangle = 8\pi^2 \sum_{J''} (2J''+1) \sum_{KK'K''} KK' C_K^- C_{K'}^- \begin{pmatrix} J & J' & J'' \\ K-1 & K' & K'' \end{pmatrix} \begin{pmatrix} J & J' & J'' \\ K & K'-1 & K'' \end{pmatrix} \quad (6.25f)$$

$$= \langle +0; 0- \rangle, \quad (6.25g)$$

$$\langle 0+; -0 \rangle = 8\pi^2 \sum_{J''} (2J''+1) \sum_{KK'K''} KK' C_K^+ C^{+K'} \begin{pmatrix} J & J' & J'' \\ K+1 & K' & K'' \end{pmatrix} \begin{pmatrix} J & J' & J'' \\ K & K'+1 & K'' \end{pmatrix} \quad (6.25h)$$

$$= \langle +0; 0- \rangle. \quad (6.25i)$$

We expect  $B_{JJ'}$  to assume the form

$$B_{JJ'} = 8\pi^2 (aJ^3 + bJ^2 + cJ + d)(aJ'^3 + bJ'^2 + cJ' + d). \quad (6.26)$$

Determining the coefficients by evaluating  $B_{JJ'}$  for four different pairs of  $J, J'$ , we find

$$B_{JJ'} = 8\pi^2 J(J+1)(J+2)J'(J'+1)(J'+2) = \frac{\pi^2}{2} n(n+1)(n+2)n'(n'+1)(n'+2) \quad (6.27)$$

and

$$\Pi_{nn'} = \frac{(n+1)^2(n'+1)^2}{8\pi\Omega^2(uv)^{1/2}} \frac{2}{a^4} n(n+2)n'(n'+2), \quad (6.28)$$

along with

$$\Psi_{nn'} = \frac{(n+1)^2(n'+1)^2}{8\pi\Omega^2(uv)^{1/2}} \left[ \frac{1}{uv} + \frac{n(n+2)}{a^2v} + \frac{n'(n'+2)}{a^2u} + \frac{4n(n+2)n'(n'+2)}{a^4} \right]. \quad (6.29)$$

Combining these results, the needed  $\zeta$  function is

$$\zeta_{\Psi}(0, \nu) = \frac{1}{8\pi\Omega^2 a^{2-4\nu}} \sum_{n, n'=1}^{\infty} n^2 n'^2 \int_0^{\infty} du \int_0^{\infty} dv (uv)^{\nu-3/2} (1 + (n^2-1)u + (n'^2-1)v + 4(n^2-1)(n'^2-1)uv) e^{-n^2u - n'^2v}. \quad (6.30)$$

Using relation (5.12), the analytic continuation takes the form

$$\begin{aligned} \zeta_{\Psi}(0, \nu) = & \frac{1}{8\pi\Omega^2 a^{2-4\nu}} \left\{ \Gamma\left(\nu - \frac{1}{2}\right)^2 \zeta_R(2\nu-3)^2 + 2\Gamma\left(\nu - \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \zeta_R(2\nu-3)(\zeta_R(2\nu-3) - \zeta_R(2\nu-1)) \right. \\ & \left. + 4\Gamma\left(\nu + \frac{1}{2}\right)^2 (\zeta_R(2\nu-3) - \zeta_R(2\nu-1))^2 \right\}. \end{aligned} \quad (6.31)$$

Setting  $\nu=0$ , we get finally the variance of the energy density of a scalar field on the Einstein universe:

$$\Delta\rho^2 = \frac{37}{76800\pi^4 a^8}. \quad (6.32)$$

Using the result [32] for the energy density,

$$\rho = \frac{1}{480\pi^2 a^4}, \quad (6.33)$$

we find the dimensionless measure of the fluctuations in the energy density (5.16) is given by

$$\Delta'(\Sigma = S^3) = 111. \quad (6.34)$$

Thus for the Einstein universe, the fluctuations in the energy density are indeed quite large. Effects due to the fluctuations of the metric will become important *before* the size of the Universe approaches that of the Planck scale.

## VII. DISCUSSION

In this paper we have shown how the correlation function for the quantum stress-energy tensor is related to the second metric variation of the effective action. This parallels the definition of the expectation value of the quantum stress-energy as the first metric variation of the effective action. A physically meaningful expectation value is derived from a regularized or renormalized effective action. Likewise, the correlation function is defined here in terms of the second variation of the regularized effective action. The correlation of the stress-energy tensor computed for two distinct points is finite regardless of whether the effective action is regularized or not (excluding lightlike separated points for a massless quantum field). It is only when the autocorrelation is computed that the issue of regularization arises. Nonetheless, we choose to use the correlation function defined in terms of the regularized effective action. This is a consistent approach since then it is defined as the second variation of the same object for which the expectation value is the first variation.

Since we have only considered geometries for which an

Euclidean section exists, we can regularize the effective action via the  $\zeta$ -function method. For a quantum system its generalized  $\zeta$  function is given as the Mellin transform of the trace of its heat kernel. The effective action is then given by the derivative of the  $\zeta$  function. This constituted our starting point for computing the second variation. The key to the  $\zeta$ -function method is to control the divergence of the heat kernel present when the Schwinger proper time vanishes. This is done by introducing positive powers of the proper time and then performing an analytic continuation in powers of the proper time. At the end of the calculation, the variable of analytic continuation is set to zero. This is consistent in that the introduced power can be relaxed to zero at any point of the calculation to recover the initial formal expression.

The second variation of the generalized  $\zeta$  function is facilitated by the Schwinger perturbative expansion, which shows how the trace of the heat kernel responds. Once we have this response, we only need the trace of a pair of heat kernels. To make this resultant expression meaningful, the key idea of the  $\zeta$  function is for each of these traces we introduce a power of the proper time variable for that trace. The stress-energy correlation function is expressed in terms of traces of the system's heat kernel, regularized via the generalized  $\zeta$ -function method. This is one of the main results of this paper. The  $\zeta$ -function method allows us to relax the introduced powers at any time and recover the formal expression for the correlation function.

For geometries admitting a mode decomposition of the invariant operator, we display the correlation function explicitly in terms of these modes. The specialization to homogeneous geometries is considered and the simplification this entails is explored.

Recent results [11,19] have suggested that quantum fluctuations of the energy density may be significant for systems with nonzero vacuum energy density. Our result confirms this assertion and goes beyond. In particular, the variance of the energy density for a massless scalar field is found. (The variance is the coincidence limit of the energy-energy correlation function.) This measure of the quantum fluctuations is calculable since we have developed the correlation function in terms of the zeta function regularized effective action. Our results are in excellent agreement with the results of [11] for Minkowski spacetime with one compact dimension. We have extended this work to flat spacetimes of arbitrary dimension with one periodic dimension and found that the variance grows quadratically with the dimension of spacetime. This may have unexpected implications for Kaluza-Klein theory. We also found the fluctuations for the Einstein universe, which turns out to be more than ten times larger than the energy density. This shows that quantum fluctuations will become important at energy scales below the Planck scale and supports the suggestion [26] that critical dynamics at such scales could reveal interesting new phenomena. Knowledge of the higher order correlation functions of the quantum stress energy may be necessary to account for the full back reaction effect of these large fluctuations of the quantum fields on the dynamics of the geometry (see, e.g., [22]) and for investigating the issue of the viability of semiclassical theories [12,13] at the Planck scale [33].

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## APPENDIX: CALCULATION OF $\int d\Omega_d \int d\Omega'_d \cos^2 \gamma$

In this appendix, we calculate  $\beta$  defined by

$$B_d = \int_{S^d} d\Omega_d \int_{S^d} d\Omega'_d \cos^2 \gamma = \beta \text{Vol}^2(S^d), \quad (\text{A1})$$

where  $d\Omega_d$  is the volume measure on the  $d$  sphere  $S^d$  and  $\cos \gamma = |\vec{x} \cdot \vec{x}'|$ . Here  $\vec{x}$  and  $\vec{x}'$  are unit vectors in  $R^{d+1}$  (i.e., they are "points" on  $S^d$ ). If we parametrize  $S^d$  with the Euler angles  $\theta_i$ ,  $i=1, \dots, d$  with  $0 \leq \theta_1 \leq 2\pi$  and  $0 \leq \theta_j \leq \pi$ ,  $j \neq 1$ , we have

$$d\Omega_d = \sin^{d-1} \theta_d \cdots \sin \theta_2 d\theta_d \cdots d\theta_1. \quad (\text{A2})$$

Using

$$\int_0^\pi \sin^{k-1} \theta d\theta = \frac{\sqrt{\pi} \Gamma(k/2)}{\Gamma[(k+1)/2]} \quad (\text{A3})$$

we get

$$\text{Vol}(S^d) = \int_{S^d} d\Omega_d = \frac{2\pi^{(d+1)/2}}{\Gamma[(d+1)/2]}. \quad (\text{A4})$$

Let  $\vec{x}'$  be a point on  $S^d$  and define  $\alpha(\vec{x}') = \int d\Omega_d |\vec{x} \cdot \vec{x}'|^2$ . We have  $\alpha(\vec{x}')$  is independent of  $\vec{x}'$  and hence

$$B_d = \int_{S^d} d\Omega'_d \alpha(\vec{x}') = \text{Vol}(S^d) \alpha(\vec{x}_0) \quad \text{any } \vec{x}_0 \in S^d. \quad (\text{A5})$$

We take  $\vec{x}_0 = (0, \dots, x_{0,d+1} = 1)$ . Then  $|\vec{x} \cdot \vec{x}'|^2 = \cos^2 \theta_d = 1 - \sin^2 \theta_d$  and

$$\begin{aligned} \alpha(\vec{x}_0) &= \int_{S^d} d\Omega_d - \int_{S^d} d\Omega_d \sin^2 \theta_d \\ &= \text{Vol}(S^d) \left[ 1 - \frac{\int_0^\pi \sin^{d+1} \theta_d d\theta_d}{\int_0^\pi \sin^{d-1} \theta_d d\theta_d} \right] \\ &= \text{Vol}(S^d) \left[ 1 - \frac{\Gamma[(d+2)/2] \Gamma[(d+1)/2]}{\Gamma[(d)/2] \Gamma[(d+3)/2]} \right] \\ &= \text{Vol}(S^d) \frac{1}{d+1}. \end{aligned} \quad (\text{A6})$$

Thus we get

$$\beta = \frac{1}{d+1}. \quad (\text{A7})$$

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