Echoing and scaling in Einstein-Yang-Mills critical collapse

Carsten Gundlach*
LAEFF-INTA (Laboratorio de Astrofísica Espacial y Física Fundamental – Instituto Nacional de Tecnología Aerospacial), P.O. Box 50727, 28080 Madrid, Spain
(Received 31 October 1996)

We confirm recent numerical results of echoing and mass scaling in the gravitational collapse of a spherical Yang-Mills field by constructing the critical solution and its perturbations as an eigenvalue problem. Because the field equations are not scale invariant, the Yang-Mills critical solution is asymptotically, rather than exactly, self-similar, but the methods for dealing with discrete self-similarity developed for the real scalar field can be generalized. We find an echoing period $\Delta = 0.73784 \pm 0.00002$ and a critical exponent for the black hole mass $\gamma = 0.1964 \pm 0.0007$. [S0556-2821(97)05010-8]

PACS number(s): 04.25.Dm, 04.20.Dw, 04.40.Nr, 64.60.Ht

I. INTRODUCTION

Recently, Choptuik, Chmaj, and Bizoń [1] (CCB) have studied the gravitational collapse of an SU(2) Yang-Mills (YM) field restricted to spherical symmetry. They were interested in what is now commonly known as ''critical phenomena in gravitational collapse,'' and their method of investigation was the numerical time evolution of a great number of initial data sets. The purpose of the present paper is to confirm some of their results by a different method which does not involve time evolution of initial data, and to calculate two important numbers, the ''echoing period'' Δ and ''critical exponent'' γ, to higher precision. We confirm what is now becoming a standard picture of critical phenomena in gravitational collapse, and generalize that picture to matter models which are not scale invariant.

Critical phenomena occur at the boundary in phase space between initial data which eventually form a black hole and data which do not. Choptuik [2] pioneered the method of numerically evolving initial data taken from one-parameter families of initial data that cross this boundary. Such families are not hard to find. It is enough that they form a black hole for large values of their parameter p (strong data) but not for small values (weak data). In a bisection search, one can then numerically determine the critical value p_* of p for a given family, such that a black hole forms for $p > p_*$, but not for $p < p_*$.

For the spherically symmetric massless scalar field, Choptuik found that the black hole mass could be made arbitrarily small, and scaled like $M \sim (p - p_*)^\gamma$, with $\gamma \simeq 0.37$ the same for all families of data. Furthermore, the time evolution of all data with $|p - p_*|$ sufficiently small, from all families, approaches one universal solution. This solution has the strange property of being periodic in the logarithm of both r and t, or $\phi(r,t) = \phi(e^\Delta r, e^\Delta t)$, with a period of $\Delta \simeq 3.44$. [See Eq. (10) below for the definition of the coordinates (r,t). The metric components a and α show the same scaling behavior as ϕ.] In geometric terms, this symmetry is a discrete self-similarity [3]. The smaller $|p - p_*|$, the more ''echos'' were visible before the black hole formed or before the fields dispersed to infinity. For introductory reviews on critical phenomena in gravitational collapse, see [4–6].

CCB investigated the spherical Einstein-Yang-Mills (EYM) system because it promised a richer structure than the Einstein-scalar system. The matter field equations are nonlinear, and the combined matter and Einstein equations contain a preferred scale formed from the YM coupling e, Newton's constant G, and c. Proceeding as Choptuik did for the scalar field, they found two regions in phase space with qualitatively different behavior. In what they call ''region I,'' they found a ''mass gap'': At $p = p_*$ the black hole mass begins discontinuously at a finite value. This minimum black hole mass is the same for all one-parameter families, and is equal to the mass of the well-known Bartnik-McKinnon (BM) solution [7]. The time evolution in fact approximates the BM solution over some finite time, and this time is the longer, the closer p is to p_*. In ''region II'' they found the mass scaling and echoing familiar from critical collapse of the scalar field and other matter models, here with $\Delta \simeq 0.74$ and $\gamma \simeq 0.20$. The two kinds of behavior are reminiscent of first- and second-order phase transitions, with the black hole mass changing either continuously or discontinuously at the critical point.

In the next section we shall see that each type of behavior can be understood, in dynamical systems terms, through the presence of an intermediate attractor. The intermediate attractor in region I is the BM solution, which is static and asymptotically flat. The region II intermediate attractor is (asymptotically) self-similar and was not known before. It is the technical task of this paper to calculate the type II attractor and its linear perturbations.

Our analytic and numerical methods are a generalization of those developed for the spherical scalar field in [8,3]. There one proceeds as follows. We assume that a solution to the field equations exists which is exactly self-similar. In suitable coordinates, self-similarity is equivalent to periodicity of the conformal metric [see Eq. (21) below]. This provides periodic boundary conditions in one coordinate. Boundary conditions in the other coordinate are obtained by demanding regularity at the center of spherical symmetry, $r = 0$, and at the past light cone of the singularity. We thus

*Present address: Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Schlaatzweg 1, 14473 Potsdam, Germany. Electronic address: gundlach@aei-potsdam.mpg.de

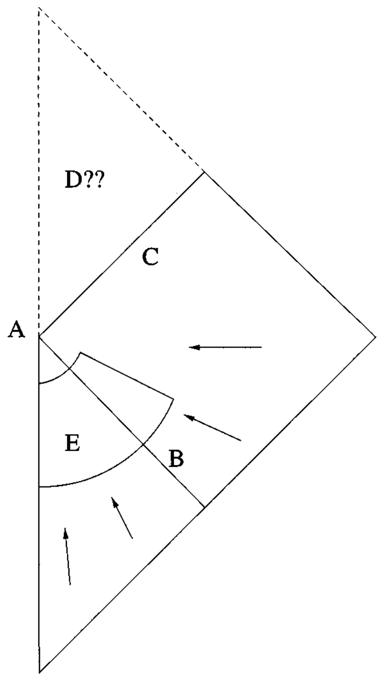

FIG. 1. Global structure of the exactly self-similar solution. The spacetime is spherically symmetric, and the left edge is $r=0$. A: Naked curvature singularity. B: Past Cauchy horizon. C: Future Cauchy horizon. This may be a curvature singularity, or the curvature may be finite except in point A. In that case there is no unique continuation beyond C, but a natural candidate is a self-similar continuation with regular $r=0$ (apart from point A). This is indicated as region D. Note that the spacetime is not asymptotically flat. The arrows give the direction ($\tau \rightarrow -\infty$), in which one linear perturbation mode grows and all others decrease. E is a schematic indication of the region of the critical solution that is visible as an intermediate attractor in collapse simulations (see also Figs. 2 and 3).

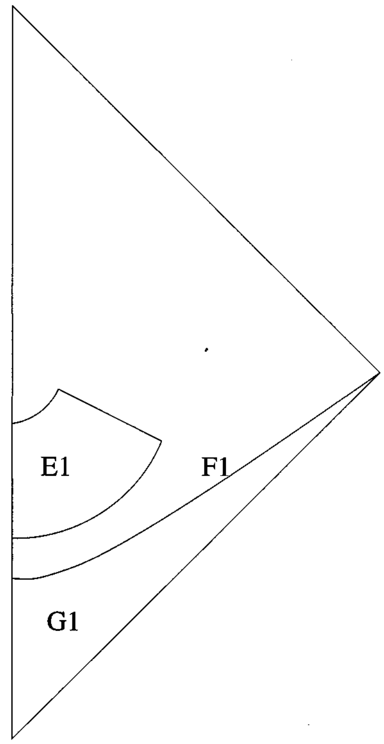

FIG. 2. Global structure of a subcritical collapse solution. F1: Regular, asymptotically flat initial data. G1: Past development (irrelevant for our purposes). E1: In this spacetime region all fields approximate those in region E of the self-similar solution.

obtain a nonlinear boundary value problem for a system of hyperbolic and elliptic equations, with the self-similarity period Δ determined as an eigenvalue. To enforce periodicity in the ansatz, we expand the metric and matter fields in a Fourier series. As the field equations are only $(1+1)$-dimensional because of spherical symmetry, the equations for the Fourier components (with respect to the periodic coordinate) become ordinary differential equations (ODE's) (in the other coordinate). The period Δ now appears as an adjustable parameter in the equations. The Fourier expansion is truncated, so that one deals with a large but finite number of equations, and this ODE eigenvalue problem is solved numerically.

The collapse solutions that CCB study are asymptotically flat (because they choose asymptotically flat initial data), and either have no singularity, or a black hole region. In contrast, the exactly self-similar, critical solution has a naked singularity, and is not asymptotically flat. Nevertheless, it can play the role of an intermediate attractor, as it only has to be a good approximation to collapse solutions in a compact region which includes neither the naked singularity nor infinity. Figures 1–3 clarify this relation between collapse solutions and the critical solution.

In order to check locally that the exactly self-similar solution is an intermediate attractor, we calculate the spectrum of its linear perturbations and show that there is only one growing mode. ''Growing'' here means growing towards the naked singularity, but in the region where collapse solutions agree with the self-similar solution this also means growing with t at constant r (see Fig. 1). As a last step, it can be shown, essentially by dimensional analysis, that the growth rate of the one unstable mode is related to the critical exponent γ, which governs the scaling of the black hole mass in marginally supercritical collapse. This allows an independent, semianalytical calculation of the critical exponent.

An extension of the formalism is required for YM, or indeed any generic, non-scale-invariant matter [9]. In contrast to the scalar field system, the field equations contain a mass and length scale e^{-1} (in units $c=G=1$), where e is the coupling constant in the YM-covariant derivative $D_a = \nabla_a + ieA_a$. The presence of a scale in the field equations excludes the existence of an exactly self-similar solution. Instead, we make a series ansatz for a solution which becomes self-similar asymptotically on spacetime scales much smaller than e^{-1} or, equivalently, for curvatures much greater than e^2. The echoing period Δ is determined by the leading term of the expansion alone. For the linearized equations we also make a series ansatz, but the spectrum $\{\lambda_i\}$ of Lyapunov exponents is once more determined by the first term of that series alone. Moreover, to calculate the first term of the perturbation expansion one only needs to know the first term of the background expansion. Therefore, the higher terms of either expansion are not required in order to calculate both the echoing period Δ and critical exponent γ ex-

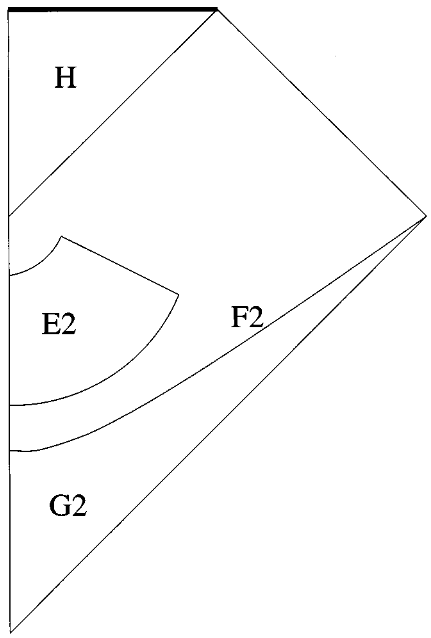

FIG. 3. Global structure of a supercritical solution. The data F2 can be chosen almost equal to F1, in which case G2 will be almost the same as G1, and E2 almost the same as E1 and E, but at late times the solution is qualitatively different, nevertheless. H is the black hole region.

actly, and will not be calculated here.

The remainder of the paper is organized as follows. Section II is a more detailed review, motivating the remaining technical sections. By reviewing type I and type II phenomena in parallel, in similar notation, we hope to make the essential mechanism of critical phenomena stand out more clearly from the technical complications. In Sec. III, we give the field equations for the spherically symmetric EYM system, and go over to coordinates and field variables adapted to self-similarity. In Sec. IV, we construct the type II, self-similar, critical solution as a nonlinear eigenvalue problem. In Sec. V we construct its linear perturbations in another, now linear, eigenvalue problem and verify that only one of them is growing. This allows us to calculate the critical exponent governing the mass scaling semianalytically, without numerical collapse simulations. In Sec. VI we summarize our results, which are in good agreement with collapse simulations, discuss how the EYM system differs from other systems in which critical collapse has previously been studied, and put the present paper into perspective.

II. TYPE I AND TYPE II CRITICAL PHENOMENA

All critical phenomena found in gravitational collapse so far, including the new type I phenomena, can be explained in terms of an intermediate attractor in a dynamical system, as was first suggested by Evans and Coleman [10]. Here we motivate the more technical calculations in the following sections in this language. In order to stress the basic ideas and the mathematical similarities between type I and type II critical behavior, we shall simplify type II in two aspects.

We begin by introducing a compact notation that is used throughout the paper: let Z stand for the vector of variables of some first-order form of the field equations, such that, for example, the complete field equations in spherical symmetry can be compactly written as $F(Z,Z_{,r},Z_{,t})=0$.

The type I critical phenomena are dominated by the BM solution [7]. It is static, spherically symmetric, and asymptotically flat. Let us call it $Z_*^{(\mathrm{I})}(r)$. As it does not depend on t, its general linear perturbation δZ obeys an equation of the form

$$\delta Z_{,t}+A(r)\,\delta Z_{,r}+B(r)\,\delta Z=0. \tag{1}$$

Therefore, δZ must be of the form

$$\delta Z=\sum_{i=1}^{\infty} C_i^{(\mathrm{I})} e^{\lambda_i^{(\mathrm{I})} t}\delta_i Z^{(\mathrm{I})}(r), \tag{2}$$

where the $C_i^{(\mathrm{I})}$ are free constants. Because the background is real, the λ_i and $\delta_i Z$ form complex-conjugate pairs. Of course, we consider only real combinations δZ. We restrict ourselves to the ansatz (12) below for the YM field. The BM solution has exactly one unstable perturbation mode within that ansatz [11], that is, $\mathrm{Re}\lambda_1>0$ and $\mathrm{Re}\lambda_i<0$ for $i=2,3,\ldots$. This makes it an intermediate attractor (of codimension 1) in dynamical systems terms. Furthermore, it is known that the final state arising from initial data $Z_0(r,\epsilon)\equiv Z_*^{(\mathrm{I})}(r)+\epsilon\,\delta_1 Z^{(\mathrm{I})}(r)$ is a black hole for one sign of ϵ, and flat space with outgoing waves for the other. Let p be the parameter of a one-parameter family of initial data such that for $p>p_*$ a black hole forms, and for $p<p_*$ the solution disperses. Then, for $(p-p_*)$ sufficiently small, the time evolution of data from the family enters an intermediate asymptotic regime of the form

$$Z(r,t)\simeq Z_*^{(\mathrm{I})}(r)+\frac{\partial C_1^{(\mathrm{I})}}{\partial p}(p_*)(p-p_*)e^{\lambda_1^{(\mathrm{I})} t}\delta_1^{(\mathrm{I})} Z(r). \tag{3}$$

Here the decaying perturbations $(i\geqslant 2)$ have been neglected because they are already small by assumption, and we have approximated $C_1=C_1(p)$ to leading order, with $C_1(p_*)=0$ by definition. This solution leaves the intermediate asymptotic regime to form a black hole or disperse at a time $t=T$ when the amplitude of the perturbations, $\partial C_1/\partial p(p-p_*)e^{\lambda_1 T}$, has reached some small fiducial value ϵ. This gives a lifetime

$$T=-\lambda\ln(p-p_*)+c \tag{4}$$

of the metastable state, where c depends on the one-parameter family through $\partial C_1/\partial p$, but where $\lambda=1/\lambda_1^{(\mathrm{I})}$ is universal. This was in fact observed by CCB, and used to estimate $\lambda_1^{(\mathrm{I})}$, in good agreement with perturbation theory [11].

The critical phenomena in regime II are dominated by an intermediate attractor that is self-similar instead of static. Here we pretend that it is continuously, rather than discretely, self-similar. This cuts down unessential detail of notation, and clarifies the similarity with the type I critical solution, which also has a continuous symmetry (it is static). We also disregard the fact that the self-similarity holds only asymptotically on small scales, and pretend that it is an exact symmetry. (Both simplifications will be dropped in the following sections.)

If $Z_*^{(\mathrm{II})}$ is continuously self-similar, in suitable coordinates it depends only on r/t. Its general linear perturbation $\delta Z^{(\mathrm{II})}$ then obeys an equation of the form

$$\delta Z_{,t}+A\left(\frac{r}{t}\right)\delta Z_{,r}+B\left(\frac{r}{t}\right)\delta Z=0. \tag{5}$$

Therefore, δZ must be of the form

$$\delta Z=\sum_{i=1}^{\infty} C_i^{(\mathrm{II})}t^{\lambda_i^{(\mathrm{II})}}\delta_i Z^{(\mathrm{II})}\left(\frac{r}{t}\right). \tag{6}$$

Once more, there is exactly one growing mode. (We will demonstrate that explicitly in Sec. IV.) The intermediate asymptotic regime for type II behavior is

$$Z(r,t)\simeq Z_*^{(\mathrm{II})}(r/t)+\frac{\partial C_1^{(\mathrm{II})}}{\partial p}(p_*)(p-p_*)t^{\lambda_1^{(\mathrm{II})}}\delta_1 Z^{(\mathrm{II})}(r/t). \tag{7}$$

Once more, let T be the value of t where the amplitude of the perturbation, $\partial C_1^{(\mathrm{II})}/\partial p(p_*)(p-p_*)t^{\lambda_1^{(\mathrm{II})}}$, has reached a small fiducial value ϵ. One now argues from scale invariance via dimensional analysis [12–14,3] that the black hole mass M is proportional to T, and obtains for the black hole mass

$$\ln M=\gamma\ln(p-p_*)+c, \tag{8}$$

where $\gamma=-1/\lambda_1^{(\mathrm{II})}$ is universal, and c is a family-dependent constant.

If the scale invariance is only asymptotic, as it is for scalar electrodynamics or EYM, the scaling argument to calculate the black hole mass goes through unchanged [9]. If the critical solution is discretely self-similar, as for the scalar field or the model considered here, with an echoing period of Δ in the logarithm of the length and time scales, the analysis is also unchanged in its basic idea, but a periodic ''wiggle'' [3] or ''fine structure'' [15] is found to be superimposed on the mass scaling law, which becomes

$$\ln M=\gamma\ln(p-p_*)+c+\Psi[\ln(p-p_*)+c/\gamma], \tag{9}$$

where Ψ is a universal periodic function with period $\Delta/(2\gamma)$. (Note that the one family-dependent constant c appears twice in the formula.) The form of the critical solution and its perturbations is also more complicated, and will be discussed in Secs. IV A and V A, respectively.

III. FIELD EQUATIONS AND SCALING VARIABLES

In this section we write down the field equations for the spherically symmetric EYM system, and then introduce coordinates and field variables that are adapted to type II behavior, where scale invariance plays a crucial role. In the following we consider only type II behavior, and no longer write the index (II). We adopt the conventions and notation of CCB, which include making both the YM field and the coordinates r and t dimensionless by absorbing suitable factors of G, c, and e into them.

The spherically symmetric spacetime metric is written as

$$ds^2\equiv-\alpha^2dt^2+a^2dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{10}$$

where a and α depend only on r and t. The most general spherically symmetric ansatz for the SU(2) YM connection is [7,16]

$$\begin{aligned}A=&A_0\tau_3dt+A_1\tau_3dr+(\phi_1\tau_1+\phi_2\tau_2)d\theta\\&+(-\phi_2\sin\theta\tau_1+\phi_1\sin\theta\tau_2+\cos\theta\tau_3)d\varphi,\end{aligned} \tag{11}$$

where the τ_i are the generators of SU(2), that is, the Pauli matrices. The fields A_0, A_1, ϕ_1, and ϕ_2 depend only on r and t. One can always set $A_0=0$ by a gauge transformation. Moreover, in the BM solution only ϕ_1 does not vanish. The perturbations of the BM solution decouple into two classes: in one of them, the ''gravitational sector,'' ϕ_2 and A_1 remain strictly zero, and only ϕ_1 and the metric are perturbed. In the other, the ''sphaleron sector,'' ϕ_1 and the metric remain unperturbed. The BM solution has one unstable mode in each sector. (In fact, ''the'' BM solution is only the first of a discrete family of static, asymptotically flat EYM solutions, the nth of which has $2n$ unstable modes, n each in the sphaleron and gravitational sectors [16].) It is a consistent truncation to set $A_0=A_1=\phi_2=0$ and retain only ϕ_1, as CCB have done. This means that one includes the BM solution, and one of its two unstable modes. In order to reproduce CCB's results for type II critical phenomena, we make the same restriction. After renaming ϕ_1 to $W(r,t)$, we have for the YM connection

$$A=W(\tau_1d\theta+\tau_2\sin\theta d\varphi)+\tau_3\cos\theta d\varphi \tag{12}$$

and the YM field strength is

$$\begin{aligned}F&=dA+A\wedge A\\&=dW\wedge(\tau_1d\theta+\tau_2\sin\theta d\varphi)-(1-W^2)\tau_3d\theta\wedge\sin\theta d\varphi.\end{aligned} \tag{13}$$

In order to write the field equations in first-order form, we define

$$\Phi\equiv W_{,r}, \quad \Pi\equiv\frac{a}{\alpha}W_{,t}. \tag{14}$$

The complete field equations, reduced to spherical symmetry, are

$$r\Phi_{,t}=r\left(\frac{\alpha}{a}\Pi\right)_{,r}, \tag{15}$$

$$r\Pi_{,t}=r\left(\frac{\alpha}{a}\Phi\right)_{,r}+a\alpha r^{-1}W(1-W^2), \tag{16}$$

$$r\frac{a_{,r}}{a}=\tfrac{1}{2}(1-a^2)+\Phi^2+\Pi^2+\tfrac{1}{2}a^2r^{-2}(1-W^2)^2, \tag{17}$$

$$r\frac{\alpha_{,r}}{\alpha}=\tfrac{1}{2}(a^2-1)+\Phi^2+\Pi^2-\tfrac{1}{2}a^2r^{-2}(1-W^2)^2, \tag{18}$$

$$r\frac{a_{,t}}{\alpha}=2\Pi\Phi. \tag{19}$$

These equations are the YM equation, and three of the four algebraically independent components of the Einstein equations. The fourth component is obtained by combining derivatives of the other three and is, therefore, redundant.

In order to construct a discretely self-similar solution, we follow [8,3] in defining new coordinates

$$\tau\equiv\ln(-t),\quad \zeta\equiv\ln\left(-\frac{r}{t}\right)-\xi_0(\tau), \tag{20}$$

where ξ_0 is a periodic function to be determined, with period Δ. (This definition differs slightly from [3] in that t and r are dimensionless, and that t is negative.) The resulting spacetime metric is

$$ds^2=e^{2\tau}\{-\alpha^2 d\tau^2+e^{2(\zeta+\xi_0)}[a^2(d\zeta+(1+\xi_0')d\tau)^2+d\theta^2+\sin^2\theta d\varphi^2]\}, \tag{21}$$

where a and α are now functions of ζ and τ, and where $\xi_0\equiv\xi_0(\tau)$ and $\xi_0'\equiv d\xi_0/d\tau$. As discussed in [3], discrete self-similarity is equivalent to a and α being periodic in τ. In the field equations we make the replacements

$$r\frac{\partial}{\partial r}=\frac{\partial}{\partial\zeta},\quad r\frac{\partial}{\partial t}=e^{\zeta+\xi_0(\tau)}\left[-\frac{\partial}{\partial\tau}+(1+\xi_0'(\tau))\frac{\partial}{\partial\zeta}\right],$$
$$r=e^{\tau+\zeta+\xi_0(\tau)} \tag{22}$$

to transform to the new coordinates.

We shall be looking for a solution in which a and α are periodic. What does this mean for the matter variables Φ, Π, and W? The Einstein equations suggest that Φ and Π should be periodic too, but W cannot be periodic because of the explicit presence of the factors e^{τ} in the equations. Nor can we simply absorb such a factor into the definition of W to make it periodic. This means that the equations have no nontrivial self-similar (periodic) solutions. The physical reason is the presence of the length scale e^{-1} in the problem, which is only hidden by the dimensionless variables. Following a suggestion by Choptuik [17], we define a new field S by

$$W\equiv 1-rS. \tag{23}$$

With this definition, the two potential terms arising in the field equations

$$r^{-1}W(1-W^2)=(1-rS)(2-rS)S,$$
$$r^{-2}(1-W^2)^2=(2-rS)^2S^2 \tag{24}$$

split into the sum of a term which no longer contains r explicitly, plus terms containing only positive powers of r, which become negligible on small spacetime scales (as $r\to0$ or as $\tau\to-\infty$). What we have done here is to expand around the vacuum solution $W=1$, because we expect our self-similar solution to oscillate around W on smaller and smaller scales. The explicit factor of r expresses our expectation that the amplitude of the oscillations in W will decrease as they occur on smaller scales, while S oscillates with constant amplitude.

Two further definitions, namely, $\Pi_\pm\equiv\Pi\pm\Phi$ and $g\equiv a/\alpha$, will be useful because g alone determines the ingoing and outgoing null geodesics, and Π_+ and Π_- are the components of the matter field propagating along them.

In the following, we use the coordinates ζ and τ, and the fields $Z\equiv\{a,g,\Pi_+,\Pi_-,S\}$. In these variables, the complete field equations, including the definitions of Π_+ and Π_- in terms of S, are

$$\Pi_{\pm,\zeta}=\frac{\mp e^{\zeta+\xi_0}g\Pi_{\pm,\tau}+C\Pi_\pm\mp a^2(1-e^{\tau+\zeta+\xi_0}S)(2-e^{\tau+\zeta+\xi_0}S)S}{1\mp(1+\xi_0')e^{\zeta+\xi_0}g}, \tag{25}$$

$$a_{,\zeta}=\frac{a}{2}(C+\Pi_+^2+\Pi_-^2), \tag{26}$$

$$g_{,\zeta}=Cg, \tag{27}$$

$$S_{,\zeta}=-S-\tfrac{1}{2}(\Pi_+-\Pi_-), \tag{28}$$

$$0=a_{,\tau}+e^{-(\zeta+\xi_0)}g^{-1}\frac{a}{2}(\Pi_+^2-\Pi_-^2)-(1+\xi_0')\frac{a}{2}(C+\Pi_+^2+\Pi_-^2), \tag{29}$$

$$0=S_{,\tau}-e^{-(\zeta+\xi_0)}g^{-1}\underline{1}2(\Pi_++\Pi_-)+(1+\xi_0')[S+\tfrac{1}{2}(\Pi_+-\Pi_-)], \tag{30}$$

where

$$C\equiv 1-a^2+a^2(2-e^{\tau+\zeta+\xi_0}S)^2S^2. \tag{31}$$

As suggested by the way we have written the equations, Eqs. (25)–(28) can be treated as evolution equations in ζ, with periodic boundary conditions in τ, and Eqs. (29) and (30) as constraints which are propagated by the evolution equations. Note that now only positive powers of e^{τ} appear explicitly, so that in the limit $\tau\to-\infty$ we are left with a set of nontrivial, scale-invariant equations for Z. The terms multiplied by e^{τ} are ''irrelevant'' in the language of renormalization group theory [18].

The equations are invariant under $W\to-W$, and both $W=1$ and $W=-1$ are vacuum, or pure gauge, solutions, with $F=0$. In Eq. (23) we have implicitly assumed that

$W \to 1$ asymptotically. A solution tending to $W = -1$ can be trivially obtained from one tending to $W=1$ by changing the sign of W, Φ, and Π, while leaving S, a, and α unchanged. The field equations are left unchanged.

IV. BACKGROUND SOLUTION

A. The eigenvalue problem

In this section we construct the solution $Z_*(r,t)$ which dominates type II behavior. In the last section we noted that the field equations do not admit an exactly self-similar solution. Physically, this follows from the presence of a preferred scale in the equations. In the dimensionless variables we have introduced, this scale survives, hidden in the factors e^τ that appear at several places in the field equations. The physical interpretation of e^τ is the scale on which the fields echo, divided by the underlying scale of the equations. In the limit $e^\tau \to 0$, on very small spacetime scales, the equations become scale invariant. We want to find the solution that becomes self-similar in that limit. This leads us to the ansatz

$$Z_*(\zeta,\tau) = \sum_{n=0}^{\infty} e^{n\tau} Z_{*n}(\zeta,\tau), \tag{32}$$

where each Z_{*n} is periodic in τ with period Δ. In the limit $e^\tau \to 0$, Z_{*0} dominates the solution, which then becomes self-similar. Z_{*0} is the solution of a nonlinear eigenvalue problem, with eigenvalue Δ, and boundary conditions arising from certain regularity requirements. Z_{*1} is the solution of an inhomogeneous nonlinear boundary value problem, with source terms depending on Z_{*0}. Similar boundary value problems completely determine all higher Z_{*n} recursively, so that the ansatz is consistent.

In the following we are interested only in the equations for Z_{*0}, and from now on we suppress the suffix $_{*0}$ on the components of Z_{*0}, denoting Π_{+*0} simply by Π_+, etc. (In the compact formal notation Z_{*0} we keep the suffix.) The equations for Z_{*0} are derived from those for Z above by setting the factor e^τ equal to zero at each explicit occurrence. We choose to evolve only Π_+, Π_-, and g in ζ, with Eqs. (25) and (27), and to determine a and S at each new value of ζ from the constraints, Eqs. (29) and (30). The set of equations we solve numerically is then, in simplified notation,

$$\Pi_{\pm,\zeta} = \frac{\mp e^{\zeta+\xi_0} g \Pi_{\pm,\tau} + (1-a^2+4a^2S^2)\Pi_\pm \mp 2a^2 S}{1 \mp (1+\xi_0') e^{\zeta+\xi_0} g}, \tag{33}$$

$$g_{,\zeta} = (1-a^2+4a^2S^2) g, \tag{34}$$

$$0 = a_{,\tau} + e^{-(\zeta+\xi_0)} g^{-1} \frac{a}{2}(\Pi_+^2 - \Pi_-^2) - (1+\xi_0')\frac{a}{2}(1-a^2+4a^2S^2+\Pi_+^2+\Pi_-^2), \tag{35}$$

$$0 = S_{,\tau} - e^{-(\zeta+\xi_0)} g^{-1} \tfrac{1}{2}(\Pi_+ + \Pi_-) + (1+\xi_0')[S + \tfrac{1}{2}(\Pi_+ - \Pi_-)]. \tag{36}$$

All fields are periodic in τ with a period Δ that is to be determined as an eigenvalue. Here as in the example of the scalar field [3], the field equations are complemented by regularity conditions at the center $r=0$ (for $t<0$), and at the past self-similarity horizon [the past light cone of the point $(r=0, t=0)$, or $r \simeq -t$]. One can solve these boundary conditions in terms of free parameters.

To make $r=0 \Leftrightarrow \zeta = -\infty$ a regular center, we impose $a=1$ and $g=1$ there. We expand in powers of e^ζ, and notice that a, g, and Π are even in that expansion (because they are even in r at $r=0$), while S and Φ are odd. We label the orders of this expansion by a suffix in parentheses to distinguish them from the orders in the expansion (32). The expansion coefficients can be given recursively in terms of one free periodic function $S_{(1)}(\tau)$. To order $e^{3\zeta}$ they are, giving Π and Φ instead of Π_+ and Π_-,

$$a_{(0)}(\tau) = 1, \tag{37}$$

$$g_{(0)}(\tau) = 1, \tag{38}$$

$$\Pi_{(0)}(\tau) = 0, \tag{39}$$

$$S_{(1)}(\tau) = \text{free}, \tag{40}$$

$$\Phi_{(1)}(\tau) = -2S_{(1)}, \tag{41}$$

$$a_{(2)}(\tau) = 2S_{(1)}^2, \tag{42}$$

$$g_{(2)}(\tau) = 0, \tag{43}$$

$$\Pi_{(2)}(\tau) = e^{\xi_0}[S_{(1)}' - (1+\xi_0')S_{(1)}], \tag{44}$$

$$S_{(3)}(\tau) = \frac{1}{10}\{e^{\xi_0}[\Pi_{(2)}' - 2(1+\xi_0')\Pi_{(2)}] + 8S_{(1)}^3\}, \tag{45}$$

$$\Phi_{(3)}(\tau) = -4S_{(3)}. \tag{46}$$

These expressions are used to impose the asymptotic boundary condition at $\zeta \to -\infty$ at some small value of ζ, say $\zeta = \zeta_{\text{left}}$.

We use the remaining coordinate freedom, the choice of $\xi_0(\tau)$, to move the self-similarity horizon to the coordinate surface $\zeta=0$ by means of the coordinate condition

$$[1-(1+\xi_0')e^{\xi_0} g]_{\zeta=0} = 0, \tag{47}$$

which means that $\zeta=0$ is null, and impose analyticity there by the condition

$$[-e^{\xi_0} g \Pi_{+,\tau} + (1-a^2+4a^2S^2)\Pi_+ - 2a^2 S]_{\zeta=0} = 0. \tag{48}$$

(This is a regular and sufficient condition, by the same argument already used in [3].)

These two constraints can be solved recursively after expanding, this time in powers of ζ. We denote the components of this expansion also by subscripts in parentheses. The two free parameters here are the periodic functions $g_{(0)}(\tau)$ and $\Pi_{-(0)}(\tau)$. From Eq. (47), one obtains the algebraic identity

$$g_{(0)} = [e^{\xi_0}(1+\xi_0')]^{-1}. \tag{49}$$

To obtain the leading-order coefficients of the other fields, we substitute Eq. (49) into Eqs. (36), (35), and (33) (upper sign),

$$S'_{(0)}+(1+\xi_0')S_{(0)}-(1+\xi_0')\Pi_{-(0)}=0, \tag{50}$$

$$a'_{(0)}-(1+\xi_0')[\Pi^2_{-(0)}+\tfrac{1}{2}]a_{(0)}-(1+\xi_0')[2S^2_{(0)}-\tfrac{1}{2}]a^3_{(0)}=0, \tag{51}$$

$$\Pi'_{+(0)}-(1+\xi_0')[1-a^2_{(0)}+4a^2_{(0)}S^2_{(0)}]\Pi_{+(0)} + (1+\xi_0')2a^2_{(0)}S_{(0)}=0, \tag{52}$$

and consider these as linear ODE's for $S_{(0)}$, $[a_{(0)}]^{-2}$, and $\Pi_{+(0)}$, respectively.

As in the scalar field case, we make the assumption that the metric variables a and g contain only even frequencies in τ, and the matter variables Π_+, Π_-, and S only odd frequencies. This is compatible with the equations for Z_{*0}, although it is not with the equations for the general Z. If this symmetry did not hold, the right-hand side of Eq. (30) would contain even terms in τ, and among them generically a term constant in τ. Then S would not be periodic in τ, but would have a term linear in τ, and through the Einstein equations this would be in contradiction to the periodicity of α and g, and hence the self-similarity of Z_{*0}.

The equivalent of the field S here is the scalar field ϕ in the scalar field model, and for a massive or self-interacting ϕ a similar argument holds. The equations for a massless ϕ, however, do not contain ϕ itself but only its derivatives $\Pi_\pm$. Therefore, a linear dependence of ϕ on τ would not clash with spacetime self-similarity. Such solutions exist, and have been investigated by Brady [19], but surprisingly the critical solution for the massless field is not of this kind, and the massless and massive (or self-interacting) scalar field are, therefore, in the same universality class.

B. Numerical construction

Our numerical method has been described in detail elsewhere [3], and can be applied to the EYM system without modification: By decomposing all fields in Fourier components with respect to τ, the partial differential equations (PDE's) in τ and ζ go over into a (large) system of ODE's in the variable ζ for the Fourier components. ODE's in τ alone, in the boundary conditions and the constraints, go over into algebraic equations which can be solved in closed form. Δ now appears as a parameter in the Fourier transformation of the τ derivatives. The boundary value problem is solved numerically by relaxation.

A solution of the field equations and boundary conditions exists only for isolated values of Δ, and we have found precisely one. The convergence radius of our relaxation algorithm is smaller than for the scalar field, probably because of the shorter period Δ, and instead of an *ad hoc* initial guess we had to use collapse data kindly provided by Choptuik [17] to obtain a good enough starting value for the relaxation algorithm.

We find good agreement of Z_{*0} with the Z of a critical collapse simulation for $-3.00\lesssim\tau\lesssim-2.22$ [17], which is not very surprising as we started our numerical search with these data but, nevertheless, confirms that the ansatz (32) for Z_* is consistent and converges for small enough τ, with Z_{*0} the dominant term.

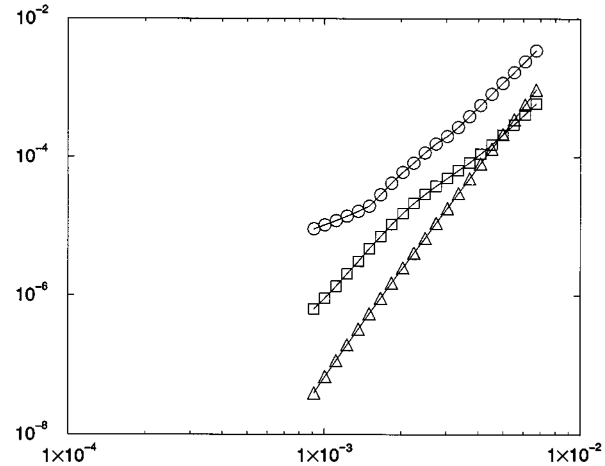

FIG. 4. Quartic convergence of Z_{*0} with $\exp\zeta_{\rm left}$. Assuming that the error is due to a finite value of $\zeta_{\rm left}$ is $E\simeq A\exp 4\,\zeta_{\rm left}$, the difference between two numerical solutions obtained with $\zeta_{\rm left}$ and $\zeta_{\rm left}+\Delta\zeta_{\rm left}$ is $\Delta E\simeq 4A\Delta\zeta_{\rm left}\exp 4\zeta_{\rm left}$. Therefore, we plot here $\Delta E/(4\Delta\zeta_{\rm left})\simeq E$, against $\exp\zeta_{\rm left}$. Circles denote the maximal error, over all grid points and Fourier components, squares the root-mean-square error, and triangles the error in Δ. $N=64$ and $\Delta\zeta=0.1$. The production value $\zeta_{\rm left}=-6.4$ corresponds to the fifth point from the left.

To obtain error bars on the solution, we have checked convergence with the numerical parameters $\zeta_{\rm left}$, the number N of Fourier components, and the grid spacing $\Delta\zeta$, by varying one of them at a time.

Figure 4 demonstrates quartic convergence with $\exp\zeta_{\rm left}$, as expected from our expansion to order $O(\exp 3\zeta_{\rm left})$. This convergence breaks down at very small values of $\exp\zeta_{\rm left}$, because all fields become very small.

Figure 5 demonstrates quadratic convergence with grid spacing in ζ, as expected from centered differencing of the ζ derivatives. This convergence breaks down at very small values of $\Delta\zeta$, probably because grid points get very close to 76the regular singular point $\zeta=0$.

Convergence with N is rapid: The difference between results for $N=64$ and $N=128$ is already of order 10^{-6}. $N=64$ is surprisingly small, given that it means only 16 odd Fourier components each to represent $\Pi_+(\tau)$ and $\Pi_-(\tau)$ and 16 even components for $g(\tau)$ and 15 for $\xi_0(\tau)$. [The component $\cos(4\pi/\Delta)$ of $\xi_0(\tau)$ is taken to be zero to fix the translation invariance in τ of the equations for Z_{*0}.]

For the production run we have chosen $\zeta_{\rm left}=-6.4$, $\Delta\zeta=(1/80)$ (that is, 513 grid points), and $N=128$. The solution $Z_{*0}(\zeta,\tau)$ has an estimated maximal error of $\pm 2.3\times10^{-4}$ and root-mean-square error of $\pm 3.6\times10^{-5}$, in the region $-6.4\leqslant\zeta\leqslant 0$. We obtain $\Delta=0.73784\pm 0.00002$. All three error estimates are dominated by the error from finite differencing in ζ, with the estimated error from expanding around $\zeta=-\infty$ somewhat smaller, and the error from using a finite number of Fourier components in τ much smaller.

C. Global structure

We have explicitly constructed the critical solution only in the past light cone of the point $(t=0, r=0)$. In this paper

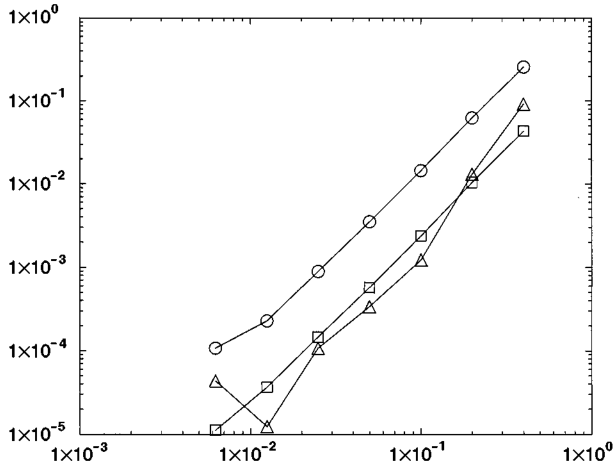

FIG. 5. Quadratic convergence of Z_{*0} with the numerical grid spacing $\Delta\zeta$. As a measure of the error at $\Delta\zeta$, we compare it with $\Delta\zeta/2$. Circles denote the maximal error, over all grid points and Fourier components, squares the root-mean-square error, and triangles the error in Δ. $N=64$ and $\zeta_{\rm left}=-6.4$. The production value $\Delta\zeta=(1/80)$ corresponds to the second point from the left.

we are not concerned with its global structure, only its existence, echoing period, and perturbation spectrum. From the ansatz (21) for the metric and periodicity in τ it follows that the point $(t=0, r=0)$ is a curvature singularity, with curvature blowing up as $\exp(-2\tau)$. It is also clear that the spacetime cannot be asymptotically flat. Neither fact is relevant for its role as an intermediate attractor. We can guess that the global structure is the same as that of the scalar field critical solution [3]. Assuming analyticity at the past light cone, there is a unique self-similar continuation up to $t=0$. We expect that continuation to agree with the critical solution found by CCB. The line $t=0$ is expected to reveal itself as coordinate singularity, with a unique analytic continuation up to the future light cone. The future light cone could *a priori* be a null curvature singularity. In the scalar field case it turns out to be a very mild singularity, with all components of the Riemann tensor finite, and only limited differentiability of the metric components and matter fields. We are not aware of any reason why the same behavior should apply to all kinds of matter, however.

V. LINEAR PERTURBATIONS AND CRITICAL EXPONENT

A. The eigenvalue problem

In this section we construct the one linear perturbation of the critical solution that grows with decreasing spacetime scale, as $\tau\to-\infty$, with the purpose of calculating the critical exponent for the black hole mass in critical collapse.

The linearized evolution equations for a linear perturbation δZ of any background solution are of the general form

$$\delta Z_{,\zeta}=A\,\delta Z_{,\tau}+(B+e^{\tau}C+e^{2\tau}D)\,\delta Z. \qquad (53)$$

Here the explicit powers of e^{τ} are the same that appear in the full field equations. The linearized constraints are of the same general form, but with the left-hand side equal to zero, and the following considerations apply equally to them as well. The perturbation equations differ from those for the scalar field model through the explicit appearance of e^{τ} in the equations, and the fact that the coefficients A, B, C, and D are not periodic in τ even if the background solution we perturb around is the critical solution Z_*, because Z_* is not periodic. However, the coefficients A, B, C, and D admit an expansion of the form

$$A=\sum_{n=0}^{\infty} e^{n\tau}A_n(\zeta,\tau), \qquad (54)$$

where the A_n are periodic. In this expansion, the leading terms A_0, B_0, etc. depend only on the leading term Z_{*0} of the background expansion.

As for the scalar field model, we make the ansatz [3]

$$\delta Z(\zeta,\tau)=\sum_{i=1}^{\infty} C_i e^{\lambda_i\tau}\delta_i Z(\zeta,\tau), \qquad (55)$$

where the C_i are free coefficients, and the λ_i are a discrete set of complex numbers, which are determined as eigenvalues of a new, linear boundary value problem. Clearly, the $\delta_i Z$ obey the equation

$$\delta_i Z_{,\zeta}=A\,\delta_i Z_{,\tau}+(B+\lambda_i A+e^{\tau}C+e^{2\tau}D)\,\delta_i Z. \qquad (56)$$

In the massless scalar field model, the $\delta_i Z$ could be assumed to be periodic in τ. In the presence of a scale, this is no longer possible, and we have to expand each $\delta_i Z$ once more as [9]

$$\delta_i Z(\zeta,\tau)=\sum_{n=0}^{\infty} e^{n\tau}\delta_{in}Z(\zeta,\tau), \qquad (57)$$

where only the individual coefficients $\delta_{in}Z$ are periodic. This expansion is exactly analogous to Eq. (32). The $\delta_{in}Z$ obey a coupled set of equations which can be derived from Eq. (56) in a straightforward bookkeeping exercise, after inserting the expansion (54). These equations are complemented by regularity conditions at $\zeta=-\infty$ and $\zeta=0$. The equations for the $\delta_{i0}Z$ are simply

$$\delta_{i0}Z_{,\zeta}=A_0\,\delta_{i0}Z_{,\tau}+(B_0+\lambda_i A_0)\,\delta_{i0}Z. \qquad (58)$$

This equation, together with the boundary conditions, already determines the spectrum $\{\lambda_i\}$. The other $\delta_{in}Z$ obey inhomogeneous equations and can be determined recursively, but here we are interested only in the spectrum. This also means that we only need the leading term Z_{*0} of the background expansion.

Writing down the field equation (58) for the $\delta_{i0}Z$ is straightforward. As we have seen, one simply linearizes Eqs. (33)–(36) for Z_{*0}, and then makes the replacements $\delta Z_{*0}\to\delta Z_{i0}$, but $\delta Z_{*0,\tau}\to\delta_{i0}Z_{,\tau}+\lambda_i\delta_{i0}Z_{,\tau}$, which follow from the definition (55). Writing a for a_{*0}, etc., and δa for $\delta_{i0}a$, etc., to keep the notation simple, we obtain

$$\delta\Pi_{\pm,\zeta}=[1\mp(1+\xi_0')e^{\zeta+\xi_0}]^{-1}\times\{\mp e^{\zeta+\xi_0}[\Pi_{\pm,\tau}\delta g+g(\delta\Pi_{\pm,\tau}+\lambda_i\delta\Pi_\pm)]+(1-a^2+4a^2S^2)\delta\Pi_\pm+2a(4S^2-1)\Pi_\pm\delta a+8a^2S\Pi_\pm\delta S\mp 2a^2\delta S\mp 4aS\,\delta a\pm(1+\xi_0')e^{\zeta+\xi_0}\Pi_{\pm,\zeta}\delta g\},\tag{59}$$

$$\delta g_{,\zeta}=(1-a^2+4a^2S^2)\delta g+2a(4S^2-1)g\,\delta a+8a^2Sg\,\delta S,\tag{60}$$

$$0=\delta a_{,\tau}+[\lambda_i+\tfrac{1}{2}e^{-(\zeta+\xi_0)}g^{-1}(\Pi_+^2-\Pi_-^2)-\tfrac{1}{2}(1+\xi_0')\times(1-3a^2+12a^2S^2+\Pi_+^2+\Pi_-^2)]\delta a+\{e^{-(\zeta+\xi_0)}[-\tfrac{1}{2}(\Pi_+^2-\Pi_-^2)g^{-2}a\,\delta g+g^{-1}a(\Pi_+\delta\Pi_+-\Pi_-\delta\Pi_-)]-(1+\xi_0')[4a^3S\,\delta S+a(\Pi_+\delta\Pi_++\Pi_-\delta\Pi_-)]\},\tag{61}$$

$$0=\delta S_{,\tau}+(\lambda_i+1+\xi_0')\delta S+\frac{1}{2}\{e^{-(\zeta+\xi_0)}[g^{-2}(\Pi_++\Pi_-)\delta g-g^{-1}(\delta\Pi_++\delta\Pi_-)]+(1+\xi_0')(\delta\Pi_+-\delta\Pi_-)\}.\tag{62}$$

Similarly, we obtain the expansion around $\zeta=-\infty$ of the $\delta_{i0}Z$ by linearizing Eqs. (37)–(46) and then making the same replacement, at each order in e^{ζ}. The nonvanishing expansion coefficients to $O(e^{3\zeta})$ are

$$\delta S_{(1)}(\tau)=\text{free},\tag{63}$$

$$\delta\Phi_{(1)}(\tau)=-2\,\delta S_{(1)},\tag{64}$$

$$\delta a_{(2)}(\tau)=4S_{(1)}\delta S_{(1)},\tag{65}$$

$$\delta\Pi_{(2)}(\tau)=e^{\xi_0}[\delta S'_{(1)}+(\lambda-1-\xi_0')\delta S_{(1)}],\tag{66}$$

$$\delta S_{(3)}(\tau)=\frac{1}{10}\{e^{\xi_0}[\delta\Pi'_{(2)}+(\lambda-2-2\xi_0')\delta\Pi_{(2)}\delta]+24S_{(1)}^2\delta S_{(1)}\},\tag{67}$$

$$\delta\Phi_{(3)}(\tau)=-4\,\delta S_{(3)}.\tag{68}$$

As the linearized regularity condition at $\zeta=0$ we impose the vanishing of the numerator of Eq. (59), (upper sign). There is no linearized equivalent of the coordinate condition (47), as we have already fixed the coordinate system when the background was calculated. (This means that $\zeta=0$ is exactly null in the background spacetime, but not in the perturbed spacetime.) The one boundary condition at $\zeta=0$ can be solved recursively in terms of two free periodic functions $\delta g_{(0)}(\tau)$ and $\delta\Pi_{-(0)}(\tau)$, from

$$\delta S'_{(0)}+(\lambda_i+1+\xi_0')\delta S_{(0)}+(1+\xi_0')\times[-\delta\Pi_{-(0)}+\tfrac{1}{2}g^{-1}(\Pi_{+(0)}+\Pi_{-(0)})\delta g_{(0)}]=0,\tag{69}$$

$$\delta a'_{(0)}+[\lambda_i+(1+\xi_0')(-\Pi_{-(0)}^2-\tfrac{1}{2}+\tfrac{3}{2}a_{(0)}^2-6a_{(0)}^2S_{(0)}^2)]\delta a_{(0)}-(1+\xi_0')[\tfrac{1}{2}g_{(0)}^{-1}a_{(0)}(\Pi_{+(0)}^2-\Pi_{-(0)}^2)\delta g_{(0)}+4a_{(0)}^3S_{(0)}\times\delta S_{(0)}+2a_{(0)}\Pi_{-(0)}\delta\Pi_{-(0)}]=0,\tag{70}$$

$$\delta\Pi'_{+(0)}+[\lambda_i-(1+\xi_0')(1-a_{(0)}^2+4a_{(0)}^2S_{(0)}^2)]\delta\Pi_{+(0)}+g_{(0)}^{-1}[\Pi'_{+(0)}-(1+\xi_0')\Pi_{+(1)}]\delta g_{(0)}+(1+\xi_0')\{2a_{(0)}[(1-4S_{(0)}^2)\Pi_{+(0)}+2S_{(0)}]\delta a_{(0)}+2a_{(0)}^2[1-4S_{(0)}\Pi_{+(0)}]\delta S_{(0)}\}=0.\tag{71}$$

The suffix (0) denotes the leading term in an expression in powers of ζ around $\zeta=0$. We still need to calculate the background term $(\Pi_{+,\zeta})_{(0)}=\Pi_{+(1)}$ in Eq. (72). To do this, we expand Eqs. (33), (lower sign), (28), and (30) to $O(\zeta)$, and evaluate the resulting algebraic expressions

$$g_{(1)}=C_{(0)}g_{(0)},\quad \text{where } C_{(0)}\equiv 1-a_{(0)}^2+4a_{(0)}^2S_{(0)}^2,\tag{72}$$

$$\Pi_{-(1)}=\frac{1}{2}[(1+\xi_0')^{-1}\Pi'_{-(0)}+C_0\Pi_{-0}+2a_{(0)}^2S_{(0)}],\tag{73}$$

$$S_{(1)}=-S_{(0)}-\tfrac{1}{2}(\Pi_{+(0)}-\Pi_{-(0)}),\tag{74}$$

$$a_{(1)}=\frac{a_{(0)}}{2}(C_{(0)}+\Pi_{+(0)}^2+\Pi_{-(0)}^2),\tag{75}$$

[alternatively, we could have obtained $S_{(1)}$ and $a_{(1)}$ from expanding the constraints (61) and (62)], and finally solve the linear ODE

$$\Pi'_{+(1)}-(1+\xi_0')(1+2C_{(0)})\Pi_{+(1)}+(1+C_{(0)})\Pi'_{+(0)}\tag{76}$$

$$+(1+\xi_0')\{2a_{(0)}[(1-4S_{(0)}^2)\Pi_{+(0)}+2S_{(0)}]a_{(1)}+2a_{(0)}^2[1-4S_{(0)}\Pi_{+(0)}]S_{(1)}\}=0\tag{77}$$

for $\Pi_{+(1)}$.

Linear perturbations which have the same τ symmetry as that of the background Z_{*0} (S and $\Pi_\pm$ odd frequencies, a and g even frequencies), decouple from those with the opposite symmetry. We call them even and odd perturbations, respectively, and can treat them separately in the numerical calculation of the spectrum $\{\lambda_i\}$.

B. Numerical construction

Our numerical method is the same as in [3]. We evolve a basis of all linear perturbations compatible with the constraints at either one of the boundaries to a matching point, and look for zeros of the determinant of the combined bases as a function of λ. A zero indicates the existence of a perturbation consistent with both sets of boundary conditions for that value of λ. We have implemented this algorithm for both real and complex λ. We have checked our results, for real λ and even perturbations, with a relaxation algorithm that is partially independent numerically, and in which λ figures as an additional variable, which is balanced by fixing the perturbations as an additional boundary condition. The determinant in question is in fact a holomorphic function of λ (because the field equations are real), and this can be used to find its zeros and poles efficiently.

We expect certain zeros and poles in the λ plane from the following considerations. Z_{*0} is scale invariant and, therefore, invariant under the infinitesimal transformation

$$Z_{*0}(r,t)\rightarrow Z_{*0}[(1+\epsilon)r,(1+\epsilon)t]\simeq Z_{*0}(\zeta,\tau+\epsilon)$$
$$\simeq Z_{*0}(\zeta,\tau)+\epsilon Z_{*0,\tau}. \quad (78)$$

This corresponds to a gauge linear perturbation mode with $\lambda_i=0$ and $\delta_i Z=Z_{*0,\tau}$. Z_{*0} is also invariant under time translation,

$$Z_{*0}(r,t)\rightarrow Z_{*0}(r,t+\epsilon)$$
$$\simeq Z_{*0}(\zeta,\tau)+\epsilon e^{-\tau}[(1+\xi_0')Z_{*0,\zeta}-Z_{*0,\tau}], \quad (79)$$

corresponding to a gauge mode with $\lambda_i=-1$. Both gauge modes are even according to our classification.

The ODE's, Eqs. (69)–(71), are all of the form $f'+gf+h=0$, where f stands for $\delta S_{(0)}$, $\delta a_{(0)}$, and $\delta\Pi_{+(0)}$, respectively. In all three equations g depends only on the background solution and is even, while h is linear in the perturbations, and has the same τ symmetry as that of f. It can be shown [3] that this type of equation has no solution when the average value (in τ) of the coefficient g vanishes. As g in each case is of the form $\lambda+(\text{background fields})$, this corresponds to a simple pole in the λ plane. These poles are there not just because of the breakdown of a particular numerical method but indicate that for these values of λ no perturbations exist which obey the boundary condition at $\zeta=0$. The poles arise only when the inhomogeneous term h, and in consequence the unknown f, have a nonvanishing average, that is when they are even.

Calculating the average value of g for each of the three equations, we find that they vanish for $\lambda=-1$, $\lambda=-1-A$, and $\lambda=-A$, respectively, where A is the average value of $2(1+\xi_0')\Pi^2_{-(0)}$, with numerical value $A\simeq0.1726$. (We have used the fact that

$$(\ln a_{(0)})'=(1+\xi_0')[\Pi^2_{-(0)}+\tfrac{1}{2}+(2S^2_{(0)}-\tfrac{1}{2})a^2_{(0)}] \quad (80)$$

has a vanishing average value, as it is the derivative of a periodic function, to simplify the averages.) In summary, for even perturbations we expect zeros at $\lambda=0$ and $\lambda=-1$ (gauge modes), one more zero on the negative real line (the unstable mode), and a pole at $\lambda\simeq-1.17$. For odd perturbations we expect poles at $\lambda=-1$ and $\lambda\simeq-0.17$.

The numerical calculation of the perturbation determinant as a function of λ largely confirms the predictions: For even perturbations, on the negative real line we find a zero at $\lambda_1\simeq-5.0$, corresponding to the expected physical unstable mode, with a critical exponent of $-1/\lambda_1\simeq0.2$, as found in collapse simulations. We also find the expected zero at $\lambda_2=0$. We have verified that the corresponding $\delta_i Z\propto Z_{*0,\tau}$ to high precision. We find the expected pole at $\lambda\simeq-1.17$, but accompanied by a zero very close by. For odd perturbations, on the negative real line we find the expected pole at $\lambda\simeq-0.17$.

TABLE I. Convergence of λ with step size in ζ. λ_1 is the Lyapunov exponent of the one growing mode. Its negative inverse is the critical exponent γ. λ_2 is the exponent of the scale change (τ-translation) gauge mode. It must be zero and serves as a check on the numerical error. Note that the numerical grid (and number of steps) is the same for the background as that for the perturbations in each case. The range of ζ is $-6.4\leqslant\zeta\leqslant0$ in each case.

Number of steps	λ_1	λ_2
32	-5.1318584162589	0.13720816860828
64	-5.0828194924873	$-1.4932102080912\times10^{-2}$
128	-5.0891816495598	$-1.1519697001693\times10^{-2}$
256	-5.0910562847918	$4.7061190684163\times10^{-3}$
512	-5.0913625725286	$1.6758611037677\times10^{-2}$

At $\lambda=-1$, for both even and odd perturbations, we do not find the expected zero and pole, respectively, because of a numerical problem which is discussed in the appendix. It does not affect our calculation of the perturbation determinant for values of λ not close to -1. The unstable mode at $\lambda\simeq-5.0$ and gauge mode at $\lambda=0$ are clear enough, and we can use their convergence properties to obtain an estimate of the numerical error.

Table I gives the values of λ for the unstable mode λ_1 and the scale change gauge mode λ_2 as a function of the step size $\Delta\zeta$. The deviation of the numerical value of λ_2 from zero serves as one estimate of numerical error. It is larger than the other estimate, from the convergence of λ_1 and we, therefore, adopt it as our definitive error estimate for λ_1. We obtain $\lambda_1=-5.091\pm0.017$, from which we obtain for the critical exponent $\gamma=-1/\lambda_1=0.1964\pm0.0007$.

VI. CONCLUSIONS

We have obtained the asymptotic form of the type II critical solution of EYM collapse, its echoing period, and the critical exponent for the black hole mass, in a calculation similar to the one we made for the massless scalar field [8,3]. The major new feature is the presence of the length scale in the EYM field equations. In consequence, the critical solution and its linear perturbations are no longer self-similar, but become so only asymptotically on spacetime scales much smaller than the length scale of the field equations (on the order of mass of the BM solution). Here we have only calculated the leading term in the asymptotic expansions for the critical solution and its perturbations, but this is sufficient to calculate both the echoing period Δ and critical exponent γ exactly. We find $\Delta=0.73784\pm0.00002$ and $\gamma=0.1964\pm0.0007$, while Choptuik, Chmaj, and Bizoń [1] find $\Delta\simeq0.74$ and $\gamma\simeq0.20$ in collapse simulations.[1]

[1] Data files of the background and unstable mode from the production run are available through the WWW address http://www.aei-potsdam.mpg.de/~gundlach.

As explained in Sec. II, following [1] we have not made the most general ansatz for the YM field in spherical symmetry, but a consistent restriction. For the type I critical solution, the BM solution, one misses one unstable perturbation mode in doing this. In the full phase space it is an attractor of codimension two, and one would have to fine-tune two parameters in the initial data to make the time evolution approximate it for a long time. Similarly, the type II critical solution we have found here may be an attractor of higher codimension than one in the full phase space.

In the formalism we have developed to deal with the presence of a length scale in the equations, both the background solution, and its linear perturbations, are expanded in powers of that length scale. The leading term of the latter series obeys field equations which are simply the linearization of the field equations for the leading term of the former series. Both sets of equations, therefore, consistently describe one new physical system which is scale invariant, and which is obtained from the original, scale-dependent model in the limit where all fields vary on spacetime scales much smaller than the intrinsic scale of the field equations. In the language of renormalization group theory, these equations are the short-scale fixed point of a renormalization group transformation acting on the original field equations.

What matter system is described by the renormalized equations? In the case of a massless or self-interacting scalar field the fixed point is the massless scalar field [13]. For scalar electrodynamics, it is the massless complex scalar without electromagnetism [9]. For YM matter, the fixed point field equations do not seem to describe a previously known model. They could best be described as an Abelian YM model which acquires a mass through symmetry breaking. In the small-scale limit, the effective YM equation is linear in the rescaled gauge potential S, and the effective stress tensor is quadratic in S. The effective theory is no longer full YM theory, with up to quartic self-interaction terms, but it retains a quadratic self-interaction term in S, which one must think of as an expansion of the original cubic and quartic self-interactions of YM matter around the vacuum background solution $W=1$, with higher than second powers of S suppressed in the small-scale limit.

The example of EYM collapse shows that one does not need scale invariance of the field equations to have critical phenomena with the famous relation $M\sim(p-p_*)^\gamma$. Rather, they can be found in some region of phase space for any system, when the typical scales of the initial data are much smaller than the typical scales of the Einstein-matter field equations. For astrophysical matter, these initial data are simply not realized in astrophysical collapse.

In the present paper we have developed a general formalism for dealing with critical collapse restricted to spherical symmetry, allowing for discrete self-similarity and the presence of a length scale. The generalization to more than one scale is trivial: the various scales can be written as a single scale times dimensionless numbers. The general formalism has already given rise to new physics: the calculation of critical exponents not only for the black hole mass but also its charge in critical collapse of scalar electrodynamics [9].

Most remaining questions in critical phenomena go beyond the restriction to spherical symmetry. Do the spherical critical solutions found so far act also as critical solutions for generic, nonspherical, initial data? What is the angular momentum of the black hole formed from data with angular momentum in the limit where the black hole mass is fine-tuned to zero? Are there qualitatively new phenomena away from spherical symmetry?

ACKNOWLEDGMENTS

I would like to thank Piotr Bizoń and Matt Choptuik for helpful discussions, and Matt Choptuik for making his data from collapse simulations available. This work was supported by the Ministry of Education and Science (Spain).

APPENDIX A: NUMERICAL PROBLEMS AT $\lambda\simeq-1$

Calculating the determinant of even perturbations as a function of λ, we do not find the expected zero but a pole at $\lambda=-1$, with an alternating series of poles and zeros accumulating towards -1 from below. There are no further poles or zeros immediately above. The positions of the poles are the same in the real and complex algorithms, but depend on the values of the numerical parameters N and $\delta\zeta$. A qualitatively similar picture arises for odd perturbations. These features can be explained as a numerical artifact as follows.

We have checked explicitly that the gauge mode (79) obeys the constraint (61). When we try to reconstruct δa of this mode from the constraint, however, the numerical result blows up at small ζ. To understand this, consider the equation $\delta a'+g\,\delta a+h=0$, with g and h defined by Eq. (61). This equation has no solution a that is periodic in τ if the average in τ of g vanishes. The Fourier algorithm that we use to solve this for δa at each ζ needs to divide the average of h by the average of g. As $\zeta\to-\infty$, the average of g over τ as a function of λ and ζ is $\lambda+1+O(e^{2\zeta})$, where the last term is positive. As $\lambda\to-1$ from below, this goes through zero at some small value of ζ. In the exact perturbation mode (79), the average value of h vanishes at the same rate with ζ as that of g, but with small numerical errors this cancellation fails, and small numerical errors are magnified. In the calculation of the perturbation determinant this results in the observed, essentially random behavior for $\lambda\lesssim-1$. For $\lambda>-1$ the problem does not arise, as then the average value of g does not vanish for any ζ.

We have not found a simple way of fixing this problem, as our algorithm relies in an essential way on reconstructing S and a and δS and δa from the constraints at each ζ. It does not affect numerical results, however, unless where the average values of both the coefficients g and h are very small, that is, for $\lambda\lesssim-1$. (If only the average value of g is small, the resulting blowup in the perturbations is physical, as in the other poles we have discussed.) Calculating the perturbation determinant is not a goal in itself, but only a means of finding the spectrum of linear perturbations. With the present method we can say with confidence that there is a zero at $\lambda\simeq-5.0$, and no other zeros for negative real λ, apart from the two gauge modes. We could in principle be missing a zero (physical growing mode) at $\lambda\lesssim-1$, where the code is unreliable and, therefore, we have to rely on evidence from collapse simulations that there is only one unstable mode.

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