Relativistic conservation laws and integral constraints for large cosmological perturbations

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For every mapping of a perturbed spacetime onto a background and with any vector field ξ we construct a strongly, identically conserved covariant vector density $I(\xi)$, which is the divergence of a covariant antisymmetric tensor density, a ''superpotential.'' $I(\xi)$ is linear in the energy-momentum tensor perturbations of matter, which may be large; $I(\xi)$ does not contain the second order derivatives of the perturbed metric. The superpotential is identically zero when perturbations are absent. By integrating strongly conserved vectors over a part Σ of a hypersurface S of the background, which spans a two-surface $\partial\Sigma$, we obtain integral relations between, on the one hand, initial data of the perturbed metric components and the energy-momentum perturbations on Σ and, on the other, the boundary values on $\partial\Sigma$. We show that there are as many such integral relations as there are different mappings, ξ's, Σ's, and $\partial\Sigma$'s. For given boundary values on $\partial\Sigma$, the integral relations may be interpreted as integral constraints on local initial data including the energy-momentum perturbations. Strong conservation laws expressed in terms of Killing fields $\bar{\xi}$ of the background become ''physical'' conservation laws. In cosmology, to each mapping of the time axis of a Robertson-Walker space on a de Sitter space with the same spatial topology there correspond ten conservation laws. The conformal mapping leads to a straightforward generalization of conservation laws in flat spacetimes. Other mappings are also considered. Traschen's ''integral constraints'' for linearized spatially localized perturbations of the energy-momentum tensor are examples of conservation laws with peculiar ξ vectors whose equations are rederived here. In Robertson-Walker spacetimes, the ''integral constraint vectors'' are the Killing vectors of a de Sitter background for a special mapping. [S0556-2821(97)00310-X]

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I. INTRODUCTION

A. Strong conservation laws and cosmology

Background spacetimes are commonly used in perturbation theories in general relativity [1] and play an essential role in cosmology [2]. One ''puzzle'' [3] in the theory of cosmological perturbations is Traschen's ''integral constraints'' for *spatially localized* perturbations [4,5]. These Gauss-type restrictions on the energy-momentum of matter perturbations have significant effects [6]: They point to an important reduction of the Sachs-Wolfe [7] effect on the mean square angular fluctuations at large angles of the cosmic background temperature due to local inhomogeneities in the universe for spatially isolated perturbations.

Traschen's relations remind us of Bergmann's *strong conservation laws* [8] applied to perturbations of isolated systems. Such conservation laws, which were explored in detail by Bergmann and Schiller [9], are, in fact, identities. The identities, which involve an arbitrary vector ξ, have played a basic role in the derivation of weak or Noether conserved currents in general relativity [10] and are still in use [11]. We found it thus interesting to study strong conservation laws on background spacetimes [12] in the context of cosmological perturbations.

Strong conservation laws are obtained from Lagrangians that are scalar densities with not higher than first order derivatives of the fields. There are *no* such *scalar densities* for the metric, and therefore strong conservation laws in general relativity are coordinate dependent. The coordinate dependence can be ''brushed under the rug'' by mapping the spacetime on a flat background [13]. This method offers, for example, the advantage of making the Bondi mass [14] calculable from Einstein's pseudotensor in Bondi coordinates [15] rather than in Minkowski coordinates [16]. But backgrounds are more than a useful tool in relativistic cosmology; they are inevitable in linear and nonlinear perturbation theories.

Here we derive strong conservation laws *with respect to curved backgrounds* along the line indicated by Bergmann. We define a Lagrangian density $\hat{L}_G$ for the gravitational field, quadratic in the first order covariant derivatives of the perturbed metric (the caret means ''density,'' i.e., multiplication by $\sqrt{-g}$). $\hat{L}_G$ is normalized so that $\hat{L}_G=0$ when there are no perturbations. Perturbations do not have to be small. The strong conservation laws, derived from $\hat{L}_G$, are identically conserved vector densities $\hat{I}^\mu(\xi)$, the divergences of covariant superpotential densities $\hat{J}^{\mu\nu}$:

$$\hat{I}^\mu=\partial_\nu \hat{J}^{\mu\nu}, \quad \hat{J}^{\mu\nu}=-\hat{J}^{\nu\mu}. \tag{1.1}$$

The $\hat{I}^\mu$'s are identically conserved independently of whether or not Einstein's equations are satisfied. However, we consider only metrics that satisfy Einstein's equations. The $\hat{I}^\mu$'s are linear in the perturbed energy-momentum tensor, and both $\hat{I}^\mu$ and $\hat{J}^{\mu\nu}$ contain the perturbed metric and its first

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order covariant derivatives (no second order derivatives); both are zero when there are no perturbations. It follows from Eq. (1.1) that, if Σ is any piece of a hypersurface S which spans a two-surface $\partial\Sigma$

$$\int_{\Sigma} \hat{I}^{\mu} d\Sigma_{\mu} = \int_{\partial\Sigma} \hat{J}^{\mu\nu} d\Sigma_{\mu\nu}. \tag{1.2}$$

These exact nonlinear integral identities represent strong global conservation laws if the integration is over the whole hypersurface S. If Σ is only a piece of the total, one may, in the manner of Penrose [17], speak of quasilocal strong conservation laws.

Now suppose that the boundary values, and thus $\hat{J}^{\mu\nu}$, on $\partial\Sigma$ are *given*. Then Eq. (1.2) represents an *integral constraint* on the perturbations of the energy-momentum tensor δT^{μ}_{ν} for given initial perturbations of the metric on Σ. Reciprocally, if δT^{μ}_{ν} is given, Eq. (1.2) represent integral constraints on the initial metric data on Σ. There are many integral constraints: for any mapping, any ξ, and any Σ with the same boundary values and the same ξ and its first derivatives on $\partial\Sigma$. Integral constraints may be used to relate boundary values of the metric to the matter sources on Σ.

Coming back to Traschen's integral constraints for linear perturbations, these represent particular forms of Eq. (1.2) with a class of ''integral constraint vectors'' $\xi^{\mu} = V^{\mu}$ (not necessarily Killing vectors), for which Eq. (1.2) reduces to

$$\int_{\Sigma} \delta T^{\mu}_{\nu} V^{\nu} d\hat{\Sigma}_{\mu} = 0. \tag{1.3}$$

Boundary contributions are by definition vanishing. These equations are the integral constraints on δT^{μ}_{ν} that Traschen [4] and Traschen and Eardley [6] considered for spatially localized perturbations on a Robertson-Walker background. They found that Eq. (1.3) reduces considerably the Sachs-Wolfe [7] effect of δT^{μ}_{ν} on the angular fluctuations of the cosmic background radiation. Different boundary values may have less stringent effects.

B. Noether conservation laws on curved backgrounds

In special relativity [18] as in general relativity [19,20], when the arbitrary vector ξ^{μ} is replaced by a Killing vector of the background, $\bar{\xi}^{\mu}$, the strong conservation laws become physical conservation laws. Noether conserved vectors $\hat{J}^{\mu}$ have a physical content analogous to energy-momentum and angular-momentum conserved currents in electromagnetism. However, contrary to electromagnetism, conserved gravitational currents cannot be made gauge independent, i.e., independent of the mapping.

Noether conservation laws can be applied to asymptotically flat spacetimes. This subject is not dealt with here in detail, but it is noteworthy that our superpotential $\hat{J}^{\mu\nu}$ gives properly the ''standard'' expression for total energy and linear and angular momenta at spatial infinity [19] *and* at null infinity [21] found in the literature [22]. The global conservation laws, *in their superpotential forms*, relate local quantities to boundary values and, if applied globally, give physical interpretations to ''asymptotic parameters'' of solutions. They are also useful in cosmology.

C. Noether conservation laws in cosmology

In cosmology, there are six Noether conservation laws for perturbations in a Robertson-Walker background, corresponding to the six Killing vectors. There are four non-Noether conservation laws for each of the remaining conformal Killing vectors. The ten vectors correspond to the fact that Robertson-Walker spacetimes are conformal to de Sitter spacetimes, which, as is well known, admit ten independent Killing vectors like Minkowski space. In cosmological applications, de Sitter spaces appear more suitable as backgrounds than Minkowski spaces, the more so because in the inflationary scenario, de Sitter spacetimes transform into Robertson-Walker spacetimes. The quasienergy and initial position of the mass center [40] can be associated with the four Killing vectors of de Sitter spaces which do not correspond to the six Killing vectors of Robertson-Walker universes.

Traschen's integral constraints, which we mentioned before, look like conservation laws on a de Sitter space in disguise. Tod [23] has shown that the equations for *integral constraint vectors* (ICV's) V^{μ} are the conditions for Σ to be embeddable in a spacetime with constant curvature of which the V^{μ}'s are the Killing vectors.

In Sec. II we give the general theory of strong conservation laws relative to a curved background for both nonlinear and linearized perturbations. A summary of some of the results of Sec. II appeared already in [12]. Here we give full details and we also include a generalization of the Belinfante-Rosenfeld identities [24]. Section III is devoted to Noether conservation laws; the energy-momentum tensor and the helicity tensor with respect to the background are singled out. Results of applications to asymptotically flat backgrounds are mentioned. Section IV gives details on Noether's conservation laws for Robertson-Walker spaces mapped on de Sitter spaces with the same spatial topology. In Sec. V Traschen's integral constraints are related to strong conservation laws. Integral constraint vectors are shown to be the Killing vectors of a de Sitter background with a particular mapping.

II. STRONGLY CONSERVED CURRENTS

The main result of this section is summarized in Eq. (2.39).

A. Lagrangian density for gravitational fields on a curved background

Let $g_{\mu\nu}(x^{\lambda})$, $\lambda,\mu,\nu,\ldots=0,1,2,3$, be the metric of a spacetime $\mathcal{M}$ with signature -2, and let $\overline{g}_{\mu\nu}(\overline{x^{\lambda}})$ be the metric of the background $\overline{\mathcal{M}}$. Both are tensors with respect to arbitrary coordinate transformations. Once we have chosen a mapping so that points P of $\mathcal{M}$ map into points $\overline{P}$ of $\overline{\mathcal{M}}$, then we can use the convention that $\overline{P}$ and P shall always be given the same coordinates $\overline{x^{\lambda}} = x^{\lambda}$. This convention implies that a coordinate transformation on $\mathcal{M}$ inevitably induces a coordinate transformation with the same functions on $\overline{\mathcal{M}}$. With this convention, such expressions as $g_{\mu\nu}(x^{\lambda}) - \overline{g}_{\mu\nu}(x^{\lambda})$ become true tensors. However, if the particular mapping has been left unspecified, we are still free to change it. The form of the equations for perturbations must inevitably contain a gauge invariance corresponding to this freedom.

Let $R^{\lambda}_{\nu\rho\sigma}$ and $\overline{R}^{\lambda}_{\nu\rho\sigma}$ be the curvature tensors of $\mathcal{M}$ and $\overline{\mathcal{M}}$. These are related as follows [25]:

$$R^{\lambda}_{\nu\rho\sigma}=\overline{D}_{\rho}\Delta^{\lambda}_{\nu\sigma}-\overline{D}_{\sigma}\Delta^{\lambda}_{\nu\rho}+\Delta^{\lambda}_{\rho\eta}\Delta^{\eta}_{\nu\sigma}-\Delta^{\lambda}_{\sigma\eta}\Delta^{\eta}_{\nu\rho}+\overline{R}^{\lambda}_{\nu\rho\sigma}. \tag{2.1}$$

Here $\overline{D}_{\rho}$ are covariant derivatives with respect to $\overline{g}_{\mu\nu}$ and $\Delta^{\lambda}_{\mu\nu}$ is the difference between Christoffel symbols in $\mathcal{M}$ and $\overline{\mathcal{M}}$:

$$\Delta^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}-\overline{\Gamma}^{\lambda}_{\mu\nu}=\tfrac{1}{2}g^{\lambda\rho}(\overline{D}_{\mu}g_{\rho\nu}+\overline{D}_{\nu}g_{\rho\mu}-\overline{D}_{\rho}g_{\mu\nu}). \tag{2.2}$$

Our quadratic Lagrangian density $\hat{\mathcal{L}}_G$ for gravitational perturbations is then defined as

$$\hat{\mathcal{L}}_G=\hat{\mathcal{L}}-\overline{\hat{\mathcal{L}}},\quad \hat{\mathcal{L}}=-\frac{1}{2\kappa}(\hat{R}+\partial_{\mu}\hat{k}^{\mu}),$$

$$\overline{\hat{\mathcal{L}}}=-\frac{1}{2\kappa}\overline{\hat{R}},\quad \kappa=\frac{8\pi G}{c^4}. \tag{2.3}$$

The caret means, as we said before, multiplication by $\sqrt{-g}$, never by $\sqrt{-\overline{g}}$. Thus, if $\hat{R}=\sqrt{-g}R$, $\overline{\hat{R}}$ will unambiguously mean $\sqrt{-g}\overline{R}$. Notice that $\hat{\overline{R}}=\sqrt{-g}\overline{R}\neq\overline{\hat{R}}=\sqrt{-\overline{g}}\hat{\overline{R}}/\sqrt{-g}$. The divergence of the vector density $\hat{k}^{\mu}$,

$$\hat{k}^{\mu}=\frac{1}{\sqrt{-g}}\overline{D}_{\nu}(-gg^{\mu\nu})=\hat{g}^{\mu\rho}\Delta^{\sigma}_{\rho\sigma}-\hat{g}^{\rho\sigma}\Delta^{\mu}_{\rho\sigma}, \tag{2.4}$$

cancels all second order derivatives of $g_{\mu\nu}$ in R. $\hat{\mathcal{L}}$ is the Lagrangian used by Rosen. $\overline{\hat{\mathcal{L}}}$ is $\hat{\mathcal{L}}$ in which $g_{\mu\nu}$ has been replaced by $\overline{g}_{\mu\nu}$. When $g_{\mu\nu}=\overline{g}_{\mu\nu}$, $\hat{\mathcal{L}}_G$ is thus identically zero. The intention here is to obtain conservation laws in the background space so that if $g_{\mu\nu}=\overline{g}_{\mu\nu}$, conserved vectors and superpotentials would be identically zero as in Minkowski space in special relativity. The following formula, deduced from Eqs. (2.3) and (2.1), shows explicitly how $\hat{\mathcal{L}}_G$ is quadratic in the first order derivatives of $g_{\mu\nu}$ or, equivalently, quadratic in $\Delta^{\mu}_{\rho\sigma}$:

$$\hat{\mathcal{L}}_G=\frac{1}{2\kappa}\hat{g}^{\mu\nu}(\Delta^{\rho}_{\mu\nu}\Delta^{\sigma}_{\rho\sigma}-\Delta^{\rho}_{\mu\sigma}\Delta^{\sigma}_{\rho\nu})-\frac{1}{2\kappa}(\hat{g}^{\mu\nu}-\overline{\hat{g}}^{\mu\nu})\overline{R}_{\mu\nu}. \tag{2.5}$$

Notice that if $\overline{R}^{\lambda}_{\nu\rho\sigma}=0$ and coordinates are such that $\overline{\Gamma}^{\lambda}_{\mu\nu}=0$, $\hat{\mathcal{L}}_G$ is nothing else than the familiar ''ΓΓ-ΓΓ'' Lagrangian density [26].

B. Infinitesimal reparametrization in both $\mathcal{M}$ and $\overline{\mathcal{M}}$

Lie differentials are particularly convenient in describing infinitesimal displacements in both $\mathcal{M}$ and $\overline{\mathcal{M}}$; they are thus not associated with a change of mapping. If

$$\Delta x^{\mu}=\xi^{\mu}\Delta\lambda \tag{2.6}$$

represents an infinitesimal one-parameter displacement generated by a sufficiently smooth vector field ξ^{μ}, the corresponding changes in tensors are given in terms of the Lie derivatives with respect to the vector field ξ^{μ}, $\Delta g_{\mu\nu}=\pounds_{\xi}g_{\mu\nu}\Delta\lambda$, etc. The Lie derivatives may be written in terms of ordinary partial derivatives ∂_{μ}, covariant derivative $\overline{D}_{\mu}$ with respect to $\overline{g}_{\mu\nu}$, or covariant derivative D_{μ} with respect to $g_{\mu\nu}$. Thus,

$$\begin{aligned}\pounds_{\xi}g_{\mu\nu}&=g_{\mu\lambda}\partial_{\nu}\xi^{\lambda}+g_{\lambda\nu}\partial_{\mu}\xi^{\lambda}+\xi^{\lambda}\partial_{\lambda}g_{\mu\nu}\\&=g_{\mu\lambda}\overline{D}_{\nu}\xi^{\lambda}+g_{\lambda\nu}\overline{D}_{\mu}\xi^{\lambda}+\xi^{\lambda}\overline{D}_{\lambda}g_{\mu\nu}\\&=g_{\mu\lambda}D_{\nu}\xi^{\lambda}+g_{\lambda\nu}D_{\mu}\xi^{\lambda}.\end{aligned} \tag{2.7}$$

Consider now the Lie differential $\Delta\hat{\mathcal{L}}$ of $\hat{\mathcal{L}}$. With the variational principle in mind, we write $\Delta\hat{\mathcal{L}}=\pounds_{\xi}\hat{\mathcal{L}}\Delta\lambda$ in the form

$$\Delta\hat{\mathcal{L}}=\frac{1}{2\kappa}\hat{G}^{\mu\nu}\Delta g_{\mu\nu}+\partial_{\mu}\hat{A}^{\mu}\Delta\lambda, \tag{2.8}$$

where Einstein's tensor density $\hat{G}^{\mu\nu}=\hat{R}^{\mu\nu}-\frac{1}{2}\hat{g}^{\mu\nu}R$ is the variational derivative of $\hat{\mathcal{L}}$ with respect to $g_{\mu\nu}$ and $\hat{A}^{\mu}$ is a vector density linear in ξ (see below). The Lie derivative of a scalar density like $\hat{\mathcal{L}}$ is just an ordinary divergence $\partial_{\mu}(\hat{L}\xi^{\mu})$. Thus

$$\hat{O}\equiv\pounds_{\xi}\hat{\mathcal{L}}-\partial_{\mu}(\hat{\mathcal{L}}\xi^{\mu})=0. \tag{2.9}$$

Combining Eqs. (2.9) with (2.8), we obtain

$$\hat{O}\equiv\frac{1}{2\kappa}\hat{G}^{\mu\nu}\pounds_{\xi}g_{\mu\nu}+\partial_{\mu}\hat{B}^{\mu}, \tag{2.10}$$

where

$$\hat{B}^{\mu}=\hat{A}^{\mu}-\hat{\mathcal{L}}\xi^{\mu}=\tfrac{1}{2}\hat{\Sigma}^{\mu\rho\sigma}\pounds_{\xi}g_{\rho\sigma}+\hat{\Xi}^{\mu}-\hat{\mathcal{L}}\xi^{\mu}, \tag{2.11}$$

in which

$$\begin{aligned}2\kappa\hat{\Sigma}^{\mu\rho\sigma}&=(g^{\mu\rho}g^{\sigma\nu}+g^{\mu\sigma}g^{\rho\nu}-g^{\mu\nu}g^{\rho\sigma})\hat{\Delta}^{\lambda}_{\nu\lambda}\\&\quad-(g^{\nu\rho}g^{\sigma\lambda}+g^{\nu\sigma}g^{\rho\lambda}-g^{\nu\lambda}g^{\rho\sigma})\hat{\Delta}^{\mu}_{\nu\lambda}\end{aligned} \tag{2.12}$$

and

$$4\kappa\hat{\Xi}^{\mu}=\hat{g}^{\mu\lambda}\partial_{\lambda}Z+\hat{g}^{\rho\sigma}[\overline{D}^{\mu}Z_{\rho\sigma}-(\overline{D}_{\rho}Z^{\mu}_{\sigma}+\overline{D}_{\sigma}Z^{\mu}_{\rho})], \tag{2.13}$$

with

$$Z_{\rho\sigma}\equiv\pounds_{\xi}\overline{g}_{\rho\sigma}=\overline{D}_{\rho}\xi_{\sigma}+\overline{D}_{\sigma}\xi_{\rho},\quad Z=\overline{g}^{\rho\sigma}Z_{\rho\sigma},\quad \xi_{\sigma}=\overline{g}_{\sigma\mu}\xi^{\mu}. \tag{2.14}$$

Indices will be moved up or down with $\overline{g}_{\rho\sigma}$ only, never with $g_{\rho\sigma}$. In Eq. (2.11), $\pounds_{\xi}g_{\rho\sigma}$ may be replaced by its expression (2.7) in terms of D_{ν} derivatives. In this way, $\hat{B}^{\mu}$ contains D_{ν} derivatives only.

Belinfante [24] and Rosenfeld [24] extracted from Eq. (2.10) various identities and showed how to complete Pauli's canonical energy-momentum tensor to make it symmetrical [27]. Identities (2.10) have been used to construct strong conservation laws in general relativity, without mapping on a background [28] and, more rarely, with a mapping on a flat background [19]. Here we use the identities (2.10) to construct strong conservation laws on curved backgrounds. The Bianchi identities imply $D_\nu G^{\mu\nu}=0$ so that with Eq. (2.7), Eq. (2.10) can be written as the divergence of a vector density

$$\hat{O}=\partial_\mu \hat{j}^\mu=0 \quad \text{where} \quad \hat{j}^\mu=\frac{1}{\kappa}\hat{G}^\mu_\nu \xi^\nu+\hat{B}^\mu. \tag{2.15}$$

Hence $\hat{\mathcal{L}}$ "generates" a vector $\hat{j}^\mu$ that is *identically* conserved. It has been obtained without using Einstein's field equations; Eq. (2.15) is the kind of strong conservation law introduced by Bergmann [8]. We shall, of course, assume that Einstein's equations are satisfied, and replace, $(1/\kappa)G^\mu_\nu$ by the energy-momentum of matter,

$$\frac{1}{\kappa}G^\mu_\nu=T^\mu_\nu, \tag{2.16}$$

so that our strong conservation law (2.15) reads

$$\partial_\mu \hat{j}^\mu=\partial_\mu(\hat{T}^\mu_\nu \xi^\nu+\hat{B}^\mu)=0. \tag{2.17}$$

Equation (2.17) is, strictly speaking, not an identity anymore. Given T^μ_ν, Eq. (2.17) holds only for metrics that satisfy Eq. (2.16). $\hat{j}^\mu$ is linear in ξ and its derivatives up to order 2. If in $\hat{B}^\mu$, the $\overline{D}_\rho\xi_\sigma$ are decomposed into symmetric and antisymmetric parts, using $Z_{\rho\sigma}$ defined in Eq. (2.14),

$$\overline{D}_\rho \xi_\sigma=\partial_{[\rho}\xi_{\sigma]}+\tfrac{1}{2}Z_{\rho\sigma}, \tag{2.18}$$

$\hat{j}^\mu$ takes the form

$$\hat{j}^\mu=\hat{P}^\mu_\nu \xi^\nu+\hat{\sigma}^{\mu[\rho\sigma]}\partial_{[\rho}\xi_{\sigma]}+\hat{Z}^\mu, \tag{2.19}$$

in which

$$\hat{P}^\mu_\nu=\hat{T}^\mu_\nu+\frac{1}{2\kappa}\hat{g}^{\rho\sigma}\overline{R}_{\rho\sigma}\delta^\mu_\nu+\hat{t}^\mu_\nu, \tag{2.20}$$

with

$$\begin{aligned}2\kappa t^\mu_\nu=&\,g^{\rho\sigma}[(\Delta^\lambda_{\rho\lambda}\Delta^\mu_{\sigma\nu}+\Delta^\mu_{\rho\sigma}\Delta^\lambda_{\lambda\nu}-2\Delta^\mu_{\rho\lambda}\Delta^\lambda_{\sigma\nu})\\&-\delta^\mu_\nu(\Delta^\eta_{\rho\sigma}\Delta^\lambda_{\eta\lambda}-\Delta^\eta_{\rho\lambda}\Delta^\lambda_{\eta\sigma})]\\&+g^{\mu\lambda}(\Delta^\sigma_{\rho\sigma}\Delta^\rho_{\lambda\nu}-\Delta^\sigma_{\lambda\sigma}\Delta^\rho_{\rho\nu}),\end{aligned} \tag{2.21}$$

and $\hat{\sigma}^{\mu[\rho\sigma]}$ is the antisymmetric part of $\hat{\sigma}^{\mu\rho\sigma}$ related to $\hat{\Sigma}^{\mu\rho\sigma}$ as

$$\begin{aligned}2\kappa\hat{\sigma}^{\mu\rho\sigma}&=2\kappa\hat{\Sigma}^{\mu\rho\lambda}g_{\lambda\nu}\overline{g}^{\nu\sigma}\\&=(g^{\mu\rho}\overline{g}^{\sigma\nu}+\overline{g}^{\mu\sigma}g^{\rho\nu}-g^{\mu\nu}\overline{g}^{\rho\sigma})\hat{\Delta}^\lambda_{\nu\lambda}\\&\quad-(g^{\nu\rho}\overline{g}^{\sigma\lambda}+\overline{g}^{\nu\sigma}g^{\rho\lambda}-g^{\nu\lambda}\overline{g}^{\rho\sigma})\hat{\Delta}^\mu_{\nu\lambda}\end{aligned} \tag{2.22}$$

(the terms containing $\overline{g}^{\rho\sigma}$ do not contribute to $\hat{\sigma}^{\mu[\rho\sigma]}$) while

$$\begin{aligned}4\kappa\hat{Z}^\mu=&\,(Z^\mu_\rho g^{\rho\sigma}+g^{\mu\rho}Z^\sigma_\rho-g^{\mu\sigma}Z)\Delta^\lambda_{\sigma\lambda}\\&+(g^{\rho\sigma}Z-2g^{\rho\lambda}Z^\sigma_\lambda)\Delta^\mu_{\rho\sigma}+g^{\mu\lambda}\partial_\lambda Z\\&+g^{\rho\sigma}(\overline{D^\mu}Z_{\rho\sigma}-2\overline{D}_\rho Z^\mu_\sigma).\end{aligned} \tag{2.23}$$

C. Superpotentials and strong conservation laws

Since $\hat{j}^\mu$ as given by Eq. (2.15) is identically conserved whatever $g_{\mu\nu}$ is, it must be the divergence of an antisymmetric tensor density that depends on the arbitrary $g_{\mu\nu}$'s too; thus,

$$\hat{j}^\mu=\partial_\nu \hat{j}^{\mu\nu} \quad \text{where} \quad \hat{j}^{\mu\nu}=-\hat{j}^{\nu\mu}. \tag{2.24}$$

$\hat{j}^{\mu\nu}$ is easy to find and has been derived directly from $\hat{\mathcal{L}}$ in [19]. In those papers the background is assumed to be flat, but the derivation of $\hat{j}^{\mu\nu}$ does not depend on that assumption:

$$j^{\mu\nu}=\frac{1}{\kappa}D^{[\mu}\xi^{\nu]}+\frac{1}{\kappa}\xi^{[\mu}k^{\nu]}. \tag{2.25}$$

The term $(1/\kappa)D^{[\mu}\xi^{\nu]}$ will be recognized as $\frac{1}{2}$ Komar's [29] superpotentials. In terms of $\overline{D}$ derivatives,

$$D_\rho\xi^\mu=\overline{D}_\rho\xi^\mu+\Delta^\mu_{\rho\lambda}\xi^\lambda, \tag{2.26}$$

and regarding expression (2.4) for k^μ, $j^{\mu\nu}$ may be written in the form

$$\kappa j^{\mu\nu}=g^{[\mu\rho}\overline{D}_\rho\xi^{\nu]}+g^{[\mu\rho}\Delta^{\nu]}_{\rho\lambda}\xi^\lambda+\xi^{[\mu}g^{\nu]\rho}\Delta^\sigma_{\rho\sigma}-\xi^{[\mu}\Delta^{\nu]}_{\rho\sigma}g^{\rho\sigma}. \tag{2.27}$$

Had we applied the identity (2.9) to $\overline{\hat{\mathcal{L}}}$ instead of $\hat{\mathcal{L}}$, we would have written everywhere $\overline{g}_{\mu\nu}$ instead of $g_{\mu\nu}$, from Eq. (2.9) up to Eq. (2.23). We would have found strong, barred, conserved vector densities $\overline{\hat{j}^\mu}$ and barred superpotentials $\overline{\hat{j}^{\mu\nu}}$:

$$\overline{\hat{j}^\mu}=\left(\overline{\hat{T}^\mu_\nu}+\frac{1}{2\kappa}\overline{\hat{R}}\delta^\mu_\nu\right)\xi^\nu+\overline{\hat{Z}^\mu}=\partial_\nu\overline{\hat{j}^{\mu\nu}}, \tag{2.28}$$

with

$$\overline{\hat{Z}^\mu}=\overline{g^{\mu\lambda}}\partial_\lambda Z+\overline{g^{\rho\sigma}}(\overline{D^\mu}Z_{\rho\sigma}-2\overline{D}_\rho Z^\mu_\sigma) \tag{2.29}$$

and

$$\overline{\hat{j}^{\mu\nu}}=\frac{1}{\kappa}\overline{D^{[\mu}\xi^{\nu]}}. \tag{2.30}$$

Strongly conserved vectors for $\hat{\mathcal{L}}_G=\hat{\mathcal{L}}-\overline{\hat{\mathcal{L}}}$ are thus obtained by subtracting barred vectors and superpotentials from unbarred ones; in this way, we define *relative* vectors and in particular *relative superpotentials* $\hat{j}^{\mu\nu}$—relative to the background space. Setting

$$\hat{I}^\mu=\hat{j}^\mu-\overline{\hat{j}^\mu}, \quad \hat{J}^{\mu\nu}=\hat{j}^{\mu\nu}-\overline{\hat{j}^{\mu\nu}}=-\hat{J}^{\nu\mu}, \tag{2.31}$$

we have

$$\hat{I}^\mu\equiv\hat{J}^\mu+\hat{\zeta}^\mu=\partial_\nu\hat{J}^{\mu\nu}, \quad \partial_\mu\hat{I}^\mu\equiv 0, \tag{2.32}$$

where

$$\hat{J}^\mu=\hat{\theta}^\mu_\nu\xi^\nu+\hat{\sigma}^{\mu[\rho\sigma]}\partial_{[\rho}\xi_{\sigma]}, \tag{2.33}$$

with

$$\hat{\theta}^{\mu}_{\nu} = \delta\hat{T}^{\mu}_{\nu} + \frac{1}{2\kappa}\hat{l}^{\rho\sigma}\overline{R}_{\rho\sigma}\delta^{\mu}_{\nu} + \hat{t}^{\mu}_{\nu}, \tag{2.34}$$

in which

$$\delta\hat{T}^{\mu}_{\nu} \equiv \hat{T}^{\mu}_{\nu} - \overline{\hat{T}^{\mu}_{\nu}}, \quad \hat{l}^{\mu\nu} \equiv \hat{g}^{\mu\nu} - \overline{\hat{g}^{\mu\nu}}, \tag{2.35}$$

and $\hat{\zeta}^{\mu} = \hat{Z}^{\mu} - \overline{\hat{Z}^{\mu}}$ is given by

$$4\kappa\zeta^{\mu} = (Z^{\mu}_{\rho}g^{\rho\sigma} + g^{\mu\rho}Z^{\sigma}_{\rho} - g^{\mu\sigma}Z)\Delta^{\lambda}_{\sigma\lambda} + (g^{\rho\sigma}Z - 2g^{\rho\lambda}Z^{\sigma}_{\lambda})\Delta^{\mu}_{\rho\sigma} + l^{\mu\lambda}\partial_{\lambda}Z + l^{\rho\sigma}(\overline{D^{\mu}}Z_{\rho\sigma} - 2\overline{D}_{\rho}Z^{\mu}_{\sigma}), \tag{2.36}$$

while the superpotential is given by

$$\hat{J}^{\mu\nu} = \frac{1}{\kappa}[D^{[\mu}\hat{\xi}^{\nu]} - \overline{D^{[\mu}\hat{\xi}^{\nu]}} + \hat{\xi}^{[\mu}k^{\nu]}]. \tag{2.37}$$

$J^{\mu\nu}$ can also be written in terms of $g_{\mu\nu}$, $\Delta^{\mu}_{\rho\sigma}$, and ξ^{μ}:

$$\kappa J^{\mu\nu} = l^{[\mu\rho}\overline{D}_{\rho}\xi^{\nu]} + g^{[\mu\rho}\Delta^{\nu]}_{\rho\lambda}\xi^{\lambda} + \xi^{[\mu}g^{\nu]\rho}\Delta^{\sigma}_{\rho\sigma} - \xi^{[\mu}\Delta^{\nu]}_{\rho\sigma}g^{\rho\sigma}, \tag{2.38}$$

in which $l^{\mu\nu} = \hat{l}^{\mu\nu}/\sqrt{-g}$.

The tensors in Eq. (2.33) have a physical interpretation. On a flat background, in coordinates in which $\overline{\Gamma}^{\lambda}_{\mu\nu} = 0$, t^{μ}_{ν} reduces to Einstein's pseudotensor. $\hat{\theta}^{\mu}_{\nu}$ appears therefore as the energy-momentum tensor of the perturbations with respect to the background. The second tensor in Eq. (2.33), $\hat{\sigma}^{\mu[\rho\sigma]}$, is quadratic in the metric perturbations just like $\hat{t}^{\mu}_{\nu}$. It is also bilinear in the perturbed metric components $(g_{\mu\nu} - \overline{g}_{\mu\nu})$ and their first order derivatives. $\hat{\sigma}^{\mu[\rho\sigma]}$ resembles, in this respect, the helicity tensor density in electromagnetism (see below). The factor of $\partial_{[\rho}\xi_{\sigma]}$ represents thus the helicity tensor density of the perturbations with respect to the background.

It should be noted again that all the components of I^{μ} and of the superpotential $J^{\mu\nu}$ itself are identically zero if $g_{\mu\nu} = \overline{g}_{\mu\nu}$; therefore, strong conservation laws refer to perturbations only and not to the background.

To summarize, the main result obtained so far is the explicit form of strongly conserved vectors $\hat{I}^{\mu}$ and their associated superpotentials $\hat{J}^{\mu\nu}$ on any background:

$$\hat{I}^{\mu} \equiv \hat{\theta}^{\mu}_{\nu}\xi^{\nu} + \hat{\sigma}^{\mu[\rho\sigma]}\partial_{[\rho}\xi_{\sigma]} + \hat{\zeta}^{\mu} = \partial_{\nu}\hat{J}^{\mu\nu}, \tag{2.39}$$

in which $\hat{\theta}^{\mu}_{\nu}$ is given in Eq. (2.34), $\hat{\sigma}^{\mu\rho\sigma}$ in Eq. (2.22), ζ^{μ} in Eq. (2.36), and $J^{\mu\nu}$ in Eq. (2.38). $\hat{I}^{\mu}$ is strongly conserved for any ξ^{μ} and any mapping of $\mathcal{M}$ on $\overline{\mathcal{M}}$.

D. Strong integral conservation laws and integral constraints

We can now integrate Eq. (2.39) on a part Σ of a hypersurface S which spans a two-surface $\partial\Sigma$ and obtain a *strong integral conservation law*:

$$\int_{\Sigma}(\hat{\theta}^{\mu}_{\nu}\xi^{\nu} + \hat{\sigma}^{\mu[\rho\sigma]}\partial_{[\rho}\xi_{\sigma]} + \hat{\zeta}^{\mu})d\Sigma_{\mu} = \int_{\partial\Sigma}\hat{J}^{\mu\nu}d\Sigma_{\mu\nu}. \tag{2.40}$$

On both sides of this equality appear, besides $\delta\hat{T}^{\mu}_{\nu}$, components of the metric perturbations and their first order derivatives. Therefore, Eq. (2.40) is an integral relation between possible metric initial data on Σ, the energy-momentum perturbations δT^{μ}_{ν}, and the boundary values on $\partial\Sigma$. For *fixed* boundary values, and for each ξ^{μ}, Eq. (2.40) gives an *integral constraint* on the metric initial data for given δT^{μ}_{ν}. Reciprocally, for given metric initial data, Eq. (2.40) is an integral constraint on δT^{μ}_{ν}. In particular, if perturbations are ''localized'' in the sense that the boundary integral is zero, then the integral constraints are simply given by

$$\int_{\Sigma}(\hat{\theta}^{\mu}_{\nu}\xi^{\nu} + \hat{\sigma}^{\mu[\rho\sigma]}\partial_{[\rho}\xi_{\sigma]} + \hat{\zeta}^{\mu})d\Sigma_{\mu} = 0 \quad \text{(isolated system)}. \tag{2.41}$$

There exist special vectors ξ^{μ} for which the expression (2.36) for ζ^{μ} takes a somewhat simpler form.

If the background admits conformal Killing vectors, like in Robertson-Walker spacetimes,

$$Z_{\rho\sigma} = \tfrac{1}{4}\overline{g_{\rho\sigma}}Z, \tag{2.42}$$

and Eq. (2.36) becomes

$$8\kappa\zeta^{\mu} = (l^{\mu\rho} + \tfrac{1}{2}\overline{g^{\mu\rho}}l)\partial_{\rho}Z - (g^{\mu\rho}\Delta^{\sigma}_{\rho\sigma} - g^{\rho\sigma}\Delta^{\mu}_{\rho\sigma})Z \quad (\xi^{\mu} \text{ conformal}). \tag{2.43}$$

If ξ^{μ} is a homothetic Killing vector,

$$Z_{\rho\sigma} = \tfrac{1}{4}\overline{g_{\rho\sigma}}C, \quad C = \text{const}, \tag{2.44}$$

Eq. (2.43) reduces to

$$8\kappa\zeta^{\mu} = -(g^{\mu\rho}\Delta^{\sigma}_{\rho\sigma} - g^{\rho\sigma}\Delta^{\mu}_{\rho\sigma})C = -Ck^{\mu} \quad (\xi^{\mu} \text{ homothetic}). \tag{2.45}$$

For Killing vectors of the background, which hereafter will be denoted by $\overline{\xi^{\mu}}$ we get $\zeta^{\mu} = 0$. If, in addition, Killing vectors are tangent to Σ, $\overline{\xi^{\mu}}d\Sigma_{\mu} = 0$, as will be the case in Robertson-Walker spacetimes mapped on de Sitter spaces, the coupling to the background Ricci tensor in Eq. (2.34) disappears, and Eq. (2.40) reduces to

$$\int_{\Sigma}[(\delta\hat{T}^{\mu}_{\nu} + \hat{t}^{\mu}_{\nu})\overline{\xi^{\nu}} + \hat{\sigma}^{\mu[\rho\sigma]}\partial_{[\rho}\overline{\xi_{\sigma]}}]d\Sigma_{\mu} = \int_{\partial\Sigma}\hat{J}^{\mu\nu}d\Sigma_{\mu\nu} \quad (\overline{\xi^{\mu}}d\Sigma_{\mu} = 0). \tag{2.46}$$

E. Belinfante-Rosenfeld identities

Equation (2.32), $\partial_{\mu}\hat{I}^{\mu} = 0$, with $\hat{I}^{\mu}$ depending linearly on ξ^{μ}'s and their first order derivatives, holds for any ξ^{μ}. Therefore, $\partial_{\mu}\hat{I}^{\mu} = 0$ is a linear combination of the ξ^{μ}'s and their derivatives $\overline{D}_{\lambda}\xi^{\mu}$ and $\overline{D}_{(\rho\sigma)}\xi^{\mu}$:

$$\partial_{\mu}\hat{I}^{\mu} = \hat{O}_{\nu}\xi^{\nu} + \hat{O}^{\mu}_{\nu}\overline{D}_{\mu}\xi^{\nu} + \hat{O}^{\rho\sigma}_{\nu}\overline{D}_{(\rho\sigma)}\xi^{\nu} \equiv 0, \tag{2.47}$$

whose coefficients must be identically zero. This gives 60 identities—the Belinfante-Rosenfeld identities generalized to curved backgrounds. Strong integral conservation laws and integral constraints are obtained with linear combinations of these 60 identities with ξ^μ and its derivatives as coefficients. Calculations of the coefficients are somewhat tedious, but straightforward. A useful equation is that which transforms ζ^μ into an expression depending on $\overline{D}_\mu\xi^\nu$ and $\overline{D}_{(\rho\sigma)}\xi^\nu$ rather than $Z_{\rho\sigma}$ and $\overline{D}_\lambda Z_{\rho\sigma}$:

$$\hat{\zeta}^\mu=\left(-\frac{1}{4\kappa}\hat{l}^{\mu\rho}\overline{R}_{\rho\nu}+\frac{1}{2\kappa}\hat{l}^{\rho\sigma}\overline{R}^\mu_{\rho\sigma\nu}\right)\xi^\nu+\hat{\sigma}^{\mu\nu}{}_\lambda\overline{D}_\nu\xi^\lambda+\hat{\beta}^{\mu\rho\sigma}_\lambda\overline{D}_{(\rho\sigma)}\xi^\lambda, \tag{2.48}$$

where $\sigma^{\mu\rho\sigma}$ is defined in Eq. (2.22), while

$$\hat{\beta}^{\mu\rho\sigma}_\lambda=\frac{1}{4\kappa}(\hat{l}^{\mu\rho}\delta^\sigma_\lambda+\hat{l}^{\mu\sigma}\delta^\rho_\lambda-2\hat{l}^{\rho\sigma}\delta^\mu_\lambda)=\hat{\beta}^{\mu\sigma\rho}_\lambda. \tag{2.49}$$

Inserting Eq. (2.48) into Eq. (2.32) leads to the following set of identities, following from Eq. (2.47):

$$O_\nu=\overline{D}_\mu\theta^\mu_\nu+\tfrac{1}{2}\sigma^{\rho\sigma\lambda}\overline{R}_{\lambda\nu\rho\sigma}+\frac{1}{2\kappa}(\overline{D}_\mu l^{\rho\sigma}\overline{R}^\mu_{\rho\sigma\nu}-l^{\rho\sigma}\overline{D}_\nu\overline{R}_{\rho\sigma}-\tfrac{1}{2}\overline{D}_\rho l^{\rho\sigma}\overline{R}_{\sigma\nu})=0, \tag{2.50}$$

$$O^\mu_\nu=\theta^\mu_\nu+\overline{D}_\lambda\sigma^{\lambda\mu}{}_\nu-\frac{1}{\kappa}l^{\mu\rho}\overline{R}_{\rho\nu}=0. \tag{2.51}$$

$$O^{(\rho\sigma)}_\nu=\sigma^{(\rho\sigma)}{}_\nu+\overline{D}_\mu\beta^{\mu\rho\sigma}_\nu=0. \tag{2.52}$$

Equation (2.50) shows that θ^μ_ν, the energy-momentum tensor with respect to a curved background, is in general not ''conserved''; it is not divergenceless. It is, however, divergenceless if the background is flat. Equation (2.51) shows that on a Ricci-flat background, θ^μ_ν is itself the divergence of a tensor; i.e., it derives from a superpotential. The generalized Belinfante-Rosenfeld identities may be useful to check θ^μ_ν and $\sigma^{\mu\rho\sigma}$ calculated independently. Equations (2.50)–(2.52) are a covariant formulation of Goldberg's [28] identities extended to curved backgrounds.

F. Linearized strong conservation laws on a curved background

In the linear approximation, we write $g_{\mu\nu}=\overline{g}_{\mu\nu}+h_{\mu\nu}$ and thereafter we shall omit the overbar on $\overline{g}_{\mu\nu}$; $\overline{D}_\mu$ becomes D_μ, and terms quadratic in $h_{\mu\nu}$ and $D_\lambda h_{\mu\nu}$ or $D_\lambda h_{\mu\nu}$ are neglected. The right-hand side of Eq. (2.35) becomes

$$\hat{l}^{\mu\nu}=\sqrt{-g}(-h^{\mu\nu}+\tfrac{1}{2}g^{\mu\nu}h),\quad h=g^{\mu\nu}h_{\mu\nu}; \tag{2.53}$$

indices are now displaced with $g_{\mu\nu}$, for instance, $h^{\mu\nu}=g^{\mu\rho}g^{\nu\sigma}h_{\rho\sigma}$.

The right-hand side of Eq. (2.40), with the superpotential $J^{\mu\nu}$ given by Eq. (2.38), can now be written entirely in terms of $l^{\mu\nu}$ because, in the linear approximation, $\Delta^\mu_{\rho\sigma}$, defined in Eq. (2.2), becomes

$$\Delta^\mu_{\rho\sigma}=\tfrac{1}{2}(D_\rho h^\mu_\sigma+D_\sigma h^\mu_\rho-D^\mu h_{\rho\sigma}). \tag{2.54}$$

If we substitute this expression for $\Delta^\mu_{\rho\sigma}$ into Eq. (2.38), we obtain, after a few rearrangements, the perturbed superpotential density $\hat{J}^{\mu\nu}\cong\sqrt{-g}J^{\mu\nu}$ which is linear in $l^{\mu\nu}$ and its first derivatives:

$$\kappa J^{\mu\nu}=l^{[\mu\rho}D_\rho\xi^{\nu]}+\xi^{[\mu}D_\rho l^{\nu]\rho}-D^{[\mu}l^{\nu]}_\rho\xi^\rho. \tag{2.55}$$

The left-hand side of Eq. (2.40) contains two terms quadratic in the perturbations: t^μ_ν [cf. Eq. (2.21)] and $\sigma^{\mu[\rho\sigma]}$ [cf. Eq. (2.22)]. These two terms are now neglected. With Eq. (2.54), the linearized expression for ζ^μ [cf. Eq. (2.36)] reduces to

$$4\kappa\zeta^\mu=Z^{\rho\sigma}(2D_\rho l^\mu_\sigma-D^\mu l_{\rho\sigma})-l^{\rho\sigma}(2D_\rho Z^\mu_\sigma-D^\mu Z_{\rho\sigma})+(l^{\mu\rho}D_\rho Z-D_\rho l^{\mu\rho}Z). \tag{2.56}$$

Like $J^{\mu\nu}$, ζ^μ is linear in $l^{\mu\nu}$ and its first derivatives. The linearized form of the strong conservation law (2.40) is thus as follows:

$$\int_\Sigma\left(\delta\hat{T}^\mu_\nu\xi^\nu+\frac{1}{2\kappa}\hat{l}^{\rho\sigma}R_{\rho\sigma}\xi^\mu+\hat{\zeta}^\mu\right)d\Sigma_\mu=\int_{\partial\Sigma}\hat{J}^{\mu\nu}d\Sigma_{\mu\nu}, \tag{2.57}$$

with $\hat{J}^{\mu\nu}$ given by Eq. (2.55) and ζ^μ by Eq. (2.56). The linearized integral identities can also be written in terms of δT^μ_ν rather than $\delta\hat{T}^\mu_\nu$. Since from Eq. (2.53) we deduce that

$$\delta\sqrt{-g}=\tfrac{1}{2}\sqrt{-g}l=\tfrac{1}{2}\sqrt{-g}h\quad\text{with}\quad l=g_{\mu\nu}l^{\mu\nu},\quad h=g_{\mu\nu}h^{\mu\nu}, \tag{2.58}$$

we can replace $\delta\hat{T}^\mu_\nu$ in Eq. (2.57) by

$$\delta\hat{T}^\mu_\nu=\sqrt{-g}\,\delta T^\mu_\nu+\tfrac{1}{2}\sqrt{-g}T^\mu_\nu h,\quad \delta T^\mu_\nu=T^\mu_\nu-\overline{T}^\mu_\nu, \tag{2.59}$$

and obtain, using Einstein's equations for the background,

$$\int_\Sigma\left[\delta T^\mu_\nu\xi^\nu+\frac{1}{2\kappa}(R^\mu_\nu\delta^\sigma_\rho-R^\sigma_\rho\delta^\mu_\nu)h^\rho_\sigma\xi^\nu+\zeta^\mu\right]d\hat{\Sigma}_\mu=\int_{\partial\Sigma}J^{\mu\nu}d\hat{\Sigma}_{\mu\nu}. \tag{2.60}$$

Equations (2.57) and (2.60) are useful forms of the linearized strong integral conservation laws.

Simplifications occur when ζ^μ simplifies; in particular, if the background admits conformal Killing vectors, like in Robertson-Walker spacetimes [see Eq. (2.42)], in which case Eq. (2.56) becomes

$$8\kappa\zeta^\mu=(l^{\mu\rho}+\tfrac{1}{2}g^{\mu\rho}l)\partial_\rho Z-ZD_\rho(l^{\mu\rho}+\tfrac{1}{2}g^{\mu\rho}l)\quad(\xi_\mu\ \text{conformal}). \tag{2.61}$$

If ξ^μ is a homothetic Killing vector [see Eq. (2.44)], then Eq. (2.61) reduces to

$$8\kappa\zeta^{\mu}=-CD_{\rho}(l^{\mu\rho}+\tfrac{1}{2}g^{\mu\rho}l),\quad C=\text{const}$$
$$(\xi^{\mu}\ \text{homothetic}). \tag{2.62}$$

For Killing vectors of the background, $\zeta^{\mu}=0$. If, in addition, Killing vectors are tangent to Σ, $\overline{\xi^{\mu}}d\Sigma_{\mu}=0$, as may be the case in Robertson-Walker spacetimes, Eq. (2.60) reduces then to

$$\int_{\Sigma}\left(\delta T^{\mu}_{\nu}+\frac{1}{2\kappa}R^{\mu}_{\nu}h\right)\overline{\xi^{\nu}}d\hat{\Sigma}_{\mu}=\int_{\partial\Sigma}J^{\mu\nu}d\hat{\Sigma}_{\mu\nu}, \tag{2.63}$$

with $J^{\mu\nu}$ given by Eq. (2.55).

III. NOETHER CONSERVATION LAWS

We now return to Eq. (2.32) and consider what happens when arbitrary ξ^{μ}'s are replaced by Killing vectors $\overline{\xi^{\mu}}$ of the background.

A. Conserved current J^{μ}

$\hat{J}^{\mu}$, which contains the physics of the conservation laws, is not, in general, a conserved vector density since

$$\partial_{\mu}\hat{J}^{\mu}=-\partial_{\mu}\hat{\zeta}^{\mu}. \tag{3.1}$$

However, when ξ^{μ} is a Killing vector $\overline{\xi^{\mu}}$ of the background, then $Z_{\rho\sigma}=0$ [cf. Eq. (2.14)], $\hat{\zeta}^{\mu}=0$, and $\hat{J}^{\mu}(\overline{\xi})$ *is* conserved. Hence we can speak about ''physical conservation laws.'' We should bear in mind, however, that in general the conserved quantities will depend on the choice of the background.

$\hat{J}^{\mu}$ has been derived in the same way as ''Noether's theorem'' in classical field theory [18]. Thus, by replacing ξ^{μ} in strongly conserved currents by Killing vectors $\overline{\xi^{\mu}}$ of the background, we obtain Noether conserved vector densities. These are *exact* with mappings on curved backgrounds:

$$J^{\mu}(\overline{\xi})=\theta^{\mu}_{\nu}\overline{\xi^{\nu}}+\sigma^{\mu[\rho\sigma]}\partial_{[\rho}\overline{\xi_{\sigma]}},\quad \partial_{\mu}\hat{J}^{\mu}(\overline{\xi})=0. \tag{3.2}$$

The interpretation of θ^{μ}_{ν} and $\sigma^{\mu\rho\sigma}$ is suggested by electromagnetic conserved currents in special relativity. For an electromagnetic field, with

$$\hat{\mathcal{L}}_{\rm EM}=-\frac{1}{16\pi}\sqrt{-g}F^{\mu\nu}F_{\mu\nu},\quad F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}, \tag{3.3}$$

one finds

$$\hat{J}^{\mu}_{\rm EM}=\hat{\theta}^{\mu}_{\nu\rm EM}\overline{\xi^{\nu}}-\frac{1}{4\pi}\hat{F}^{\mu\rho}A^{\sigma}\partial_{[\rho}\overline{\xi_{\sigma]}}, \tag{3.4}$$

where $\overline{\xi^{\nu}}$ are Killing vectors of Minkowski space which is here described in arbitrary coordinates. In Eq. (3.4), the expression

$$\hat{\theta}^{\mu}_{\nu\rm EM}=\frac{\partial\hat{\mathcal{L}}_{\rm EM}}{\partial(\partial_{\mu}A_{\rho})}\partial_{\nu}A_{\rho}-\hat{\mathcal{L}}_{\rm EM}\delta^{\mu}_{\nu} \tag{3.5}$$

represents Pauli's canonical energy-momentum tensor density. It is not the standard symmetric electromagnetic energy-momentum tensor density

$$\hat{T}^{\mu}_{\nu\rm EM}=\frac{1}{4\pi}\sqrt{-g}[F^{\mu\rho}F_{\rho\nu}+\tfrac{1}{4}\delta^{\mu}_{\nu}F^{\rho\sigma}F_{\rho\sigma}]. \tag{3.6}$$

Indeed,

$$\hat{J}^{\mu}_{\rm EM}=\hat{T}^{\mu}_{\nu\rm EM}\overline{\xi^{\nu}}-\partial_{\rho}\left(\frac{1}{4\pi}\hat{F}^{\mu\rho}A_{\nu}\overline{\xi^{\nu}}\right). \tag{3.7}$$

The second term in Eq. (3.7) is gauge dependent and its divergence is zero. It is generally assumed that the appropriate boundary values ensure that this second term does not contribute to the global conserved quantity. However, if, with each displacement vector ξ^{ν}, we associate a gauge transformation $A_{\mu}\to A_{\mu}+\partial_{\mu}\zeta$ such that $(A_{\mu}+\partial_{\mu}\zeta)\xi^{\mu}=0$, the gauge-dependent term in Eq. (3.7) will disappear.

The first term $\hat{T}^{\mu}_{\nu\rm EM}\xi^{\nu}$, has a proper local meaning. On a space like hypersurface extending to infinity,

$$\begin{aligned}\int_{\Sigma}\hat{J}^{\mu}_{\rm EM}d\Sigma_{\mu}&=\int_{\Sigma}\hat{\theta}^{\mu}_{\nu\rm EM}\overline{\xi^{\nu}}d\Sigma_{\mu}\\&\quad+\int_{\partial\Sigma\to\infty}\left(-\frac{1}{4\pi}\hat{F}^{\mu\rho}A_{\nu}\overline{\xi^{\nu}}\right)d\Sigma_{\mu\rho}\\&=\int_{\Sigma}\hat{T}^{\mu}_{\nu\rm EM}\overline{\xi^{\nu}}d\Sigma_{\mu}.\end{aligned} \tag{3.8}$$

Here Eq. (3.8) represents the *total* energy-momentum for Killing vectors ξ^{μ}'s of translations. It gives the total angular momentum if ξ^{μ}'s describe spatial rotations; the integral of the second term on the right-hand side of Eq. (3.2) represents, in this case, the spin of the electromagnetic field. This means that $\hat{T}^{\mu}_{\nu\rm EM}\xi^{\nu}$ contains also the spin density.

By analogy with electromagnetism, we shall give similar interpretations to the two terms on the right-hand side of Eq. (3.2). θ^{μ}_{ν} is the (relative) energy-momentum tensor with respect to a given background for a given mapping and, similarly, $\sigma^{\mu[\rho\sigma]}$ can be interpreted as the (relative) spin tensor. As in electromagnetism, the conserved vector density $\hat{J}^{\mu}$ may not have a well-defined local meaning even for a given mapping. However, $\hat{J}^{\mu}$ generates global conservation laws which are advantageously associated with a superpotential. Global quantities with appropriate mappings near the *boundary* of the domain of integration may, and indeed have, interesting physical meaning in certain cases as we shall see below.

B. Conservation laws in asymptotically flat spacetimes

Locally conserved quantities are related to boundary values through the superpotential to which we now turn our attention. Global conservation laws derived from $J^{\mu\nu}$ have been discussed in [19] and in [21]; they will not be analyzed here. The results of those applications are, however, illuminating and worth summarizing. They strengthen the interpretation of J^{μ} as a Noether conserved vector of energy, linear, and angular momentum.

Spacetimes that are asymptotically flat admit asymptotic Killing vectors. Each space may be mapped on a flat background that is identified with the spaces itself at infinity.

To calculate globally conserved quantities, the mapping can be defined only asymptotically. To each Killing vector $\overline{\xi^\mu}$ of the background, the total amount of the corresponding conserved quantity "in the whole space at a given time" is the integral of $\hat{J}^\mu$ over a spacelike hypersurface Σ extending to infinity:

$$P(\overline{\xi})=\int_\Sigma \hat{J}^\mu d\Sigma_\mu=\int_{\partial\Sigma\to\infty}\hat{J}^{\mu\nu}d\Sigma_{\mu\nu}. \tag{3.9}$$

1. Results at spatial infinity

We may use the asymptotic solution representing an isolated system, as given in [30] to calculate energy and linear and angular momenta at $t=$const. The corresponding quantities $P(\overline{\xi})$ show which parameters in the asymptotic solutions are commonly interpreted as energy and linear and angular momenta of such a system. It is worth noting that $J^{\mu\nu}$ provides both linear and angular momenta as does the pseudotensor of Landau and Lifshitz [26]. Our $J^{\mu\nu}$ is, however, derived from a real Noether conserved vector. In contrast to the Landau-Lifshitz pseudotensor, it can be calculated in arbitrary coordinates whereas the Landau-Lifshitz pseudotensor (or the Einstein pseudotensor for energy and linear momentum) gives meaningful results only in coordinates which become Lorentzian at infinity in such a manner that $g_{\mu\nu}\to\eta_{\mu\nu}+O(r^{-1})$ (see, however, [15]).

2. Results at null infinity

Here, for axisymmetric [14] or general [31] outgoing radiation asymptotic solutions, it is advantageous to use Newman-Unti [32] coordinates $x^\lambda\equiv(x^0=t-r,r,x^2,x^3)$ conformally flat in x^2,x^3. The solutions have asymptotic symmetries represented by the Bondi-Metzner-Sachs group [33]. The BMS group contains supertranslations $u\to u+\alpha(x^2,x^3)$. For the Killing vectors of translations in the background, we identify $P(\overline{\xi})$, respectively, with the Bondi [34] mass $P_0(\overline{\xi})$ and with Sachs [33] linear momentum [35] $P_k(\overline{\xi})$. $P_\alpha(\overline{\xi})$ $(\alpha,\beta,\ldots=0,1,2,3)$ behaves like a vector under Lorentz transformations of coordinates in the flat background, and the fluxes dP_α/du are invariant under supertranslations. Similarly, for Killing vectors of spatial rotations in the background, $P(\overline{\xi})$ is the same [21] as the standard definition of the angular momentum $L_k(\overline{\xi})$ [22], but without an "anomalous factor of 2." The angular momentum transforms as a vector for rotations in the background, but dL_k/du depends on the mapping and, in particular, on supertranslations.

The conserved quantities $P(\overline{\xi})$ have one outstanding property worth noting. They are given by a superpotential, not an "asymptotic superpotential." That is, $P(\overline{\xi})$ is obtained from a *differential conservation law* and is directly related through Einstein's equations to the energy-momentum tensor of the matter. No other differential conservation law has been given so far (with or without a background) that gives the standard expressions of the total energy and linear and angular momenta at null infinity.

C. Linearized conservation laws

In the linear approximation, the formulas of Sec. II E are valid. Noether's conserved currents follow from Eq. (2.57) or (2.60) by replacing ξ^μ with Killing vectors $\overline{\xi^\mu}$ for which $\zeta^\mu=0$. Thus the linearized form of the global Noether conservation laws (3.9) becomes

$$\delta P(\overline{\xi})=\int_\Sigma\left[\delta T^\mu_\nu+\frac{1}{2\kappa}(R^\mu_\nu\delta^\sigma_\rho-R^\sigma_\rho\delta^\mu_\nu)h^\rho_\sigma\right]\overline{\xi^\nu}d\hat{\Sigma}_\mu$$
$$=\int_{\delta\Sigma}J^{\mu\nu}(\overline{\xi})d\hat{\Sigma}_{\mu\nu}, \tag{3.10}$$

with

$$\kappa J^{\mu\nu}=l^{[\mu\rho}D_\rho\overline{\xi^{\nu]}}+\overline{\xi^{[\mu}}D_\rho l^{\nu]\rho}-D^{[\mu}l^{\nu]}_\rho\overline{\xi^\rho}. \tag{3.11}$$

IV. CONSERVATION LAWS IN COSMOLOGY WITH RESPECT TO de SITTER BACKGROUNDS

A. Spatially conformal mappings on de Sitter space

Strongly or weakly perturbed Robertson-Walker spacetimes are related, by definition, to a Robertson-Walker background. Robertson-Walker spacetimes admit six Killing vectors, each of these vectors generate a conserved Noether current [36].

We may, however, map Robertson-Walker spacetimes (perturbed or not) on a de Sitter space which has ten Killing vectors and thus ten conserved currents. It is interesting to ask what are the four Noether currents for Killing vectors of the de Sitter space that are not Killing vectors of a Robertson-Walker spacetime. They replace energy and the initial center-of-mass position. To elucidate this, we map closed $(k=1)$, flat $(k=0)$, or open $(k=-1)$ hypersurfaces (at constant cosmic time) of Robertson-Walker spacetimes on the corresponding hypersurfaces, with the same topology, in de Sitter spaces.

Let Robertson-Walker spacetimes be described in coordinates (t,x^k) in which the metric reads

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=\phi^2dt^2+g_{kl}dx^kdx^l$$
$$=\phi^2dt^2-a^2f_{kl}dx^kdx^l, \tag{4.1}$$

where $f_{kl}(x^m)$ have particular forms for closed, flat or open $t=$const hypersurfaces; x^k may be any of suitable coordinates, and ϕ and a are functions of t. The metric of the de Sitter background in these coordinates has a similar form

$$d\overline{s^2}=\overline{g_{\mu\nu}}dx^\mu dx^\nu=\psi^2dt^2+\overline{g_{kl}}dx^kdx^l$$
$$=\psi^2dt^2-\overline{a^2}f_{kl}dx^kdx^l; \tag{4.2}$$

here, ψ and $\overline{a}$ are also functions of t. The "cosmic (proper) times" T are thus given by $dT=\phi(t)dt$ in the Robertson-Walker spacetimes and by $dT=\psi(t)dt$ in de Sitter space. Hypersurfaces with the same t are mapped on one another. Choosing both functions ϕ and ψ fixes the mapping of the cosmic times up to a constant. For the moment we shall fix neither of them.

B. Killing vectors of the de Sitter background

The ten Killing vectors of the de Sitter background, $\overline{\xi^\mu}=(\overline{\xi^0},\overline{\xi^k})$, satisfy the Killing equation

$$\overline{Z_{\mu\nu}}=\overline{D_\mu\xi_\nu}+\overline{D_\nu\xi_\mu}=0, \tag{4.3}$$

which in the $1+3$ decomposition given by Eq. (4.2) implies

$$\overline{Z_{00}}=0\Rightarrow\overline{\xi^0}=\frac{1}{\psi}\,\tilde{\xi}^0(x^k), \tag{4.4}$$

$$\overline{Z_{0k}}=0\Rightarrow\dot{\overline{\xi^k}}=-\psi^2\overline{g^{kl}}\nabla_l\overline{\xi^0}, \tag{4.5}$$

where $\tilde{\xi}^0$ is a function whose equation is given below [see Eq. (4.9)], ∇_l is a $\overline{g_{kl}}$ (or g_{kl} or f_{kl}) covariant derivative, and a overdot denotes a derivative with respect to t. It may be useful to remind the reader that indices are displaced by $\overline{g_{\mu\nu}}$. Finally, the spatial part of the Killing equations gives

$$-\frac{\overline{Z_{kl}}}{\overline{a^2}}=f_{mk}\nabla_l\overline{\xi^m}+f_{ml}\nabla_k\overline{\xi^m}+2\psi\overline{H}f_{kl}\overline{\xi^0}=0, \tag{4.6}$$

where

$$\overline{H}=\frac{\dot{\overline{a}}}{\psi\overline{a}} \tag{4.7}$$

is the Hubble ''constant'' of de Sitter space; $\overline{H}$ satisfies the relation

$$\frac{1}{\psi}\,\dot{\overline{H}}=\frac{k}{\overline{a^2}}, \tag{4.8}$$

which follows from Einstein's equations or, as the integrability condition of Eq. (4.3). If we take a partial t derivative of Eq. (4.6) and make use of Eq. (4.5), we obtain

$$\nabla_{kl}\overline{\xi^0}+kf_{kl}\overline{\xi^0}=0 \quad \text{or} \quad \nabla_{kl}\tilde{\xi}^0+kf_{kl}\tilde{\xi}^0=0. \tag{4.9}$$

This equation can be solved. Having $\tilde{\xi}^0$, we can obtain $\overline{\xi^k}$ from Eqs. (4.5) and (4.6). Explicit expressions for $\overline{\xi^\mu}$ and finite group transformations are given in Weinberg [37]. Using Weinberg's coordinates *adapted* to $t=$const slices, f_{kl} becomes

$$f_{kl}=\delta_{kl}+\frac{kx^kx^l}{1-kr^2}, \qquad f^{kl}=\delta^{kl}-kx^kx^l,$$

$$r^2=(x^1)^2+(x^2)^2+(x^3)^2. \tag{4.10}$$

Any $\overline{\xi^\mu}$ is a linear combination with constant coefficients of the following ten vectors.

(a) Quasitranslations in $t=$const:

$$\underset{(r)}{\overline{\xi^0}}=0, \quad \underset{(r)}{\overline{\xi^k}}=\delta^k_r\sqrt{1-kr^2}, \quad r=1,2,3. \tag{4.11}$$

(b) Quasirotations in $t=$const:

$$\underset{[rs]}{\overline{\xi^0}}=0, \quad \underset{[rs]}{\overline{\xi^k}}=\delta^{kr}x^s-\delta^{ks}x^r, \quad r,s=1,2,3. \tag{4.12}$$

(c) Time quasitranslations:

$$\underset{(0)}{\overline{\xi^0}}=\frac{1}{\psi}\sqrt{1-kr^2}, \quad \underset{(0)}{\overline{\xi^k}}=-\overline{H}x^k\sqrt{l-kr^2}. \tag{4.13}$$

(d) Lorentz quasirotations:

$$\underset{[r]}{\overline{\xi^0}}=\frac{1}{\psi}\,x^r, \quad k=0\Rightarrow\underset{[r]}{\overline{\xi^k}}=\overline{H}[\tfrac{1}{2}\delta^{kr}(r^2-\tau^2)-x^kx^l], \tag{4.14}$$

$$k=\pm1\Rightarrow\underset{[r]}{\overline{\xi^k}}=\overline{H}[k\delta^{kr}-x^kx^r], \tag{4.15}$$

where, in Eq. (4.14),

$$\tau=\frac{\psi}{\dot{\overline{a}}} \quad (k=0). \tag{4.16}$$

The Killing vectors (4.11) and (4.12) are also the Killing vectors of Robertson-Walker spacetimes. The vectors (4.13), (4.14), and (4.15) are conformal Killing vectors of Robertson-Walker spacetimes.

C. Superpotentials and conserved vectors

To obtain the superpotentials we follow the calculations outlined in Sec. II C. With the metric components $g_{\mu\nu}$ of Eq. (4.1) and $\overline{g_{\mu\nu}}$ of Eq. (4.2), we calculate the quantities $l^{\rho\sigma}=\hat{l}^{\rho\sigma}/\sqrt{-g}$,

$$l^{00}=\frac{1}{\phi^2}\left(1-\frac{\phi\overline{a}^3}{\psi a^3}\right), \quad l^{kl}=g^{kl}\left(l-\frac{\psi\overline{a}}{\phi a}\right), \tag{4.17}$$

and the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ and $\overline{\Gamma^\lambda_{\mu\nu}}$ and their differences $\Delta^\lambda_{\mu\nu}$,

$$\Gamma^0_{00}=\frac{\dot\phi}{\phi}, \quad \overline{\Gamma^0_{00}}=\frac{\dot\psi}{\psi}\Rightarrow\Delta^0_{00}=\frac{\dot\phi}{\phi}-\frac{\dot\psi}{\psi}\equiv\phi\mathcal{T}. \tag{4.18}$$

The function $\mathcal{T}$ just defined describes a relative shift in times measured in Robertson-Walker cosmic time units. Next,

$$\Gamma^k_{0l}=\phi H\delta^k_l,$$

$$\overline{\Gamma^k_{0l}}=\psi\overline{H}\delta^k_l\Rightarrow\Delta^k_{0l}=\phi\left(H-\frac{\psi}{\phi}\overline{H}\right)\delta^k_l=\phi\mathcal{H}\delta^k_l, \tag{4.19}$$

where

$$H=\frac{\dot a}{\phi a}; \tag{4.20}$$

$\mathcal{H}$ is the relative Hubble function measured in Robertson-Walker cosmic time. Finally,

$$\Gamma^0_{kl}=-\frac{H}{\phi}\,g_{kl},$$

$$\overline{\Gamma^0_{kl}}=-\frac{\overline{H}}{\psi}\,\overline{g_{kl}}\Rightarrow\Delta^0_{kl}=-\frac{1}{\phi}\left(H-\frac{\overline{a^2}\phi}{a^2\psi}\,\overline{H}\right)g_{kl}. \tag{4.21}$$

With $\overline{\xi^\mu}$ given by Eqs. (4.11)–(4.16), $l^{\mu\nu}$ by Eq. (4.17), $\Delta^\lambda_{\mu\nu}$ by Eqs. (4.18)–(4.21), the superpotential, defined in Eq. (2.38), has the form:

$$2\kappa J^{0k} = \mathcal{A}\overline{g^{kl}}\nabla_l\overline{\xi^0} + \mathcal{B}\overline{\xi^k}, \tag{4.22}$$

$$2\kappa J^{kl} = \mathcal{C}\overline{g^{[km}}\nabla_m\overline{\xi^{l]}}, \tag{4.23}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are functions of t:

$$\mathcal{A}(t) = -\frac{\overline{a}^2}{a^2} + 2\frac{\psi\overline{a}^3}{\phi a^3} - \frac{\psi^2}{\phi^2},$$

$$\mathcal{B}(t) = \left(3\frac{\overline{a}^2}{a^2} - 2\frac{\psi\overline{a}^3}{\phi a^3} - \frac{\psi^2}{\phi^2}\right)\frac{\overline{H}}{\psi} - \frac{4}{\phi}\mathcal{H},$$

$$\mathcal{C}(t) = 2\frac{\overline{a}^2}{a^2} - 2\frac{\psi\overline{a}^3}{\phi a^3}. \tag{4.24}$$

The components of the conserved vector density $\hat{J}^\mu$ can be calculated either from the superpotential since $\hat{J}^\mu = \partial_\nu \hat{J}^{\mu\nu}$ or directly from Eq. (2.33). With the usual notations for the T^μ_ν components

$$T^0_0 = \rho, \quad T^k_l = -\delta^k_l P, \quad \overline{T^\mu_\nu} = \Lambda\delta^\mu_\nu, \tag{4.25}$$

the zero component of the current then reads

$$J^0 = \left[\left(\rho - \frac{\psi\overline{a}^3}{\phi a^3}\frac{\Lambda}{\kappa}\right) - \frac{1}{2\kappa}l\Lambda - \frac{3}{\kappa}\mathcal{H}^2\right]\overline{\xi^0} \equiv \mathcal{U}(t)\overline{\xi^0}, \tag{4.26}$$

where

$$l = l^{\rho\sigma}\overline{g_{\rho\sigma}} = 3\frac{\overline{a}^2}{a^2} - 4\frac{\psi\overline{a}^3}{\phi a^3} + \frac{\psi^2}{\phi^2} = \mathcal{C} - \mathcal{A}, \tag{4.27}$$

and the spatial part is given by

$$J^k = \left[\left(-P - \frac{\psi\overline{a}^3}{\phi a^3}\frac{\Lambda}{\kappa}\right) - \frac{1}{2\kappa}l\Lambda + \frac{3}{\kappa}\mathcal{H}^2 - \frac{3\phi}{2\kappa\psi}\left(\frac{\overline{a}^2}{a^2} - \frac{\psi^2}{\phi^2}\right)\right.$$
$$\left.\times(\mathcal{T}+\mathcal{H})\overline{H}\right]\overline{\xi^k} + \frac{1}{2\kappa\psi}\frac{\phi}{}\left(\frac{\overline{a}^2}{a^2} - \frac{\psi^2}{\phi^2}\right)(\mathcal{T}+\mathcal{H})\overline{g^{kl}}\nabla_l(\psi\overline{\xi^0}). \tag{4.28}$$

The first set of parentheses in the brackets of Eqs. (4.26) and (4.28) represents the "relative mass-energy density" and "relative pressure" respectively. The second term is the coupling to the background. The other terms are associated with field energy and helicity and they depend on the mapping of the time axes.

D. Mappings

As a consequence of Eq. (4.26) and $\overline{\xi^0}$'s as given in Eqs. (4.11)–(4.19), the conserved quantities in a volume V enclosed by a sphere of radius r are all equal to zero except the "energy" P_0, associated with time quasitranslations $\underset{(0)}{\overline{\xi^0}}$ given by Eq. (4.13). The "energy" reads

$$P_0 = \frac{4\pi}{3}\overline{a}^3 r^3\mathcal{U}, \tag{4.29}$$

where $\mathcal{U}$ is given by Eq. (4.26). The most appealing mapping is one that gives $\mathcal{U}=0$ so that $P_0=0$. With such a mapping there are ten conserved quantities for *perturbations* of Robertson-Walker spacetimes only; it adds "energy-momentum" to the *perturbed* Robertson-Walker spacetimes that have no quasispacetime translation invariance.

One may also consider a conformal mapping, for which we take

$$\psi = 1, \quad \phi = \frac{a}{\overline{a}}, \tag{4.30}$$

so that

$$ds^2 = \frac{a^2}{\overline{a}^2}d\overline{s^2}. \tag{4.31}$$

Then,

$$\mathcal{U}(t) = \left(\rho - \phi^{-4}\frac{\Lambda}{\kappa}\right) + 2\phi^{-4}(1-\phi^2)\frac{\Lambda}{\kappa} - \frac{3}{\kappa}\mathcal{H}^2. \tag{4.32}$$

In this case the total energy of a closed space ($k=1$) is also zero, but for open or flat sections it is infinite. The mean "energy density" $P_0[(4\pi/3)\overline{a}^3 r^3]$ of a $k=0$ Robertson-Walker spacetime mapped on de Sitter is given by $\mathcal{U}(t)\sqrt{-g}/\sqrt{-\overline{g}}$. The mean "energy density" in a $k=-1$ space is, however, infinite because P_0 grows faster than the proper volume as $r\to\infty$.

V. TRASCHEN'S INTEGRAL CONSTRAINTS

A. Equations for integral constraint vectors

Let us now go back to Traschen's integral constraints that were written in Eq. (1.3) for spatially localized linear perturbations. For general linear perturbations, Traschen [5] showed that for certain vectors V^μ, called "integral constraint vectors" (ICV's), that satisfy 12 equations on a particular hypersurface S, there exist Gauss-like integrals of the form

$$\int_\Sigma \delta T^\mu_\nu V^\nu d\hat{\Sigma}_\mu = \int_{\delta\Sigma} B^{\mu\nu}d\hat{\Sigma}_{\mu\nu}, \tag{5.1}$$

where $B^{\mu\nu}$ is given in terms of $h_{\mu\nu}$, V^μ, and their first order derivatives. If the perturbed metric gives no contribution to the right-hand side of Eq. (5.1), the expression reduces to Traschen's integral constraints (1.3). At first sight, Eq. (5.1) appears to be a strong conservation law for linear perturbations similar to Eq. (2.60), some terms of the left-hand side of Eq. (2.60) having been transformed into boundary integrals.

The 12 Traschen equations for ICV's were deduced from Einstein's constraint equations [38]. We shall here show that Traschen's ICV equations can be derived from the strong conservation laws (2.60).

The problem is to find the conditions on ξ^μ's for which Eq. (2.60) takes the form (5.1). In doing so, we shall not only obtain the Traschen equations for V^μ, but also find under what conditions Eq. (5.1) holds on a *family* of hypersurfaces S rather than on a particular hypersurface.

Let us write Eq. (2.60) in synchronous coordinates around S. In these coordinates, S is defined by t=const and the metric takes the form

$$ds^2=dt^2+g_{kl}(t,x^m)dx^k dx^l, \quad k,l,m, \ldots=1,2,3. \tag{5.2}$$

It is always possible to keep the gauge synchronous for the perturbations, namely, to take

$$h_{00}=h_{0k}=0, \tag{5.3}$$

because h_{00} and h_{0k} depend on the mapping above and below S. Here we are interested in conditions on one particular hypersurface S (to begin with). On t=const, Eq. (2.60) can be written as

$$\int_\Sigma \left[\delta T^0_\nu \xi^\nu + \frac{1}{2\kappa}(R^0_\nu \xi^\nu h - R^\sigma_\rho h^\rho_\sigma \xi^0) + \zeta^0 \right] dV = \int_{\partial\Sigma} J^{0k} dS_k, \tag{5.4}$$

where

$$dV = d\hat{\Sigma}_0 = \sqrt{-g}\, dx^1 dx^2 dx^3,$$
$$dS_k = d\hat{\Sigma}_{0k} = \sqrt{-g}\, \varepsilon_{klm} dx^{[l} dx^{m]}. \tag{5.5}$$

The component ζ^0 [cf. Eq. (2.56)] is linear in $Z_{\mu\nu}$ and $D_\rho Z_{\mu\nu}$. It is also linear in h_{mn}, $\dot{h}_{mn}=\partial_t h_{nm}$, and $\nabla_k h_{mn}$ (the covariant derivatives of h_{mn} with respect to the three-metric g_{kl}). Thus ζ^0 is of the form

$$\zeta^0 = A^{kl} h_{kl} + B^{mkl} \nabla_m h_{kl} + C^{kl} \dot{h}_{kl}$$
$$= \nabla_m (B^{mkl} h_{kl}) + E^{kl} h_{kl} + C^{kl} \dot{h}_{kl}. \tag{5.6}$$

Inserting Eq. (5.6) into Eq. (5.4) and taking account of Eq. (5.3), we obtain an expression of the form

$$\int_\Sigma \left[\delta T^0_\nu \xi^\nu + \frac{1}{2\kappa} Y^{kl} \tilde{h}_{kl} + \frac{1}{4\kappa} Z^{kl} \partial_t \tilde{h}_{kl} \right] dV$$
$$= \int_{\partial\Sigma} (J^{0m} - B^{mkl} h_{kl}) dS_m, \tag{5.7}$$

in which

$$\tilde{h}_{kl} = h_{kl} - g_{kl} h^m_m, \tag{5.8}$$

and the Z^{kl}'s are the spatial components of the $Z_{\mu\nu}$ tensor defined in Eq. (2.14). The left-hand side of Eq. (5.7) takes the form (5.1) when the factors of $\tilde{h}_{kl}$ and of its time derivative $\partial_t \tilde{h}_{kl}$ vanish:

$$Z_{kl}=0, \quad Y_{kl}=0. \tag{5.9}$$

The first of these equations can be written in a 1+3 decomposition as

$$Z_{kl} = \nabla_k \xi_l + \nabla_l \xi_k + \dot{g}_{kl} \xi^0 = 0 \tag{5.10}$$

(remember that in general $D_k \neq \nabla_k$). With $Z_{kl}=0$, the equation $Y_{kl}=0$ reduces to

$$Y_{kl} = \tfrac{1}{2}(\nabla_k Z^0_l + \nabla_l Z^0_k) + \tfrac{1}{4}\dot{g}_{kl} Z^0_0 - (R_{kl} + g_{kl} G^0_0)\xi^0$$
$$- \tfrac{1}{2} R^0_m \xi^m g_{kl} = 0, \tag{5.11}$$

where G^0_0 is a component of Einstein's tensor. Accordingly, if ICV's satisfy Eqs. (5.10) and (5.11), then Eq. (5.4) or (5.7) has the form (5.1). It is now easy to see that Eqs. (5.10) and (5.11) are equivalent to Traschen's equations (3a) and (3b) [39]. Inserting the explicit expressions of B^{mkl} into Eq. (5.7), we obtain

$$\int_\Sigma \delta T^0_\nu \xi^\nu dV = \int_{\partial\Sigma} \left(J^{0k} - \frac{1}{4\kappa} h^m_m Z^k_0 \right) dS_k, \tag{5.12}$$

where J^{0k} is given in terms of ξ^μ, $h_{\mu\nu}$, and their first order derivatives by Eq. (2.55).

How is Eq. (5.12) modified if we consider perturbations in a nonsynchronous gauge? The answer is that, instead of Eq. (5.12), Eq. (5.4) becomes

$$\int_\Sigma \left[\delta T^0_\nu \xi^\nu + \frac{1}{2\kappa} R^0_k (h^0_0 \xi^k - 2 h^k_0 \xi^0) \right]$$
$$= \int_{\partial\Sigma} \left[J^{0k} - \frac{1}{4\kappa} h^m_m Z^k_0 + \frac{1}{4\kappa}(Z^k_0 h^0_0 - Z^0_0 h^k_0) \right] dS_k. \tag{5.13}$$

Equation (5.13) shows that if Eqs. (5.10) and (5.11) hold *and* if $R^0_k=0$ in synchronous coordinates, Eq. (2.60) has the desired form (5.1) independently of any gauge condition, as pointed out by Traschen. Robertson-Walker spacetimes have that property, but, in general, R^0_k does not vanish. In a synchronous gauge, Eq. (2.60) has the form of Eq. (5.1), not only on a particular S, but on all nearby hypersurfaces.

B. ICV's in Robertson-Walker spacetimes

With a metric of the form (4.1) and with $\phi=1$, Eq. (5.10) can be written as

$$-\frac{Z_{kl}}{a^2} = f_{mk} \nabla_l \xi^m + f_{ml} \nabla_k \xi^m + 2 f_{kl} H \xi^0 = 0 \quad \text{where} \quad H = \frac{\dot{a}}{a} \tag{5.14}$$

and $Y_{kl}=0$ or, equivalently,

$$Y_{kl} - \tfrac{1}{2} a^2 \partial_l \left(\frac{Z_{kl}}{a^2} \right) \equiv \nabla_{kl} \xi^0 + k f_{kl} \xi^0 = 0. \tag{5.15}$$

Equations (5.14) and (5.15) are Traschen's equations (15a) and (15b) in [5].

We notice that Eq. (5.15) for ξ^0 is the same as Eq. (4.9) for ξ^0 and that Eq. (5.14) for ξ^0 and ξ^k is the same as Eq. (4.6) for ξ^0 and ξ^k in the de Sitter space provided that

$$\psi = \frac{H}{\dot{H}}. \tag{5.16}$$

Therefore, a set of solutions for Traschen's equations is given by the ten Killing vectors $\underset{a}{\xi}^{\mu}$ ($a=1,2,\ldots,10$) of the de Sitter space (4.11)–(4.16) with ψ replaced by $H/\overline{H}$. Linear combinations of the ten Killing vectors $\underset{a}{\xi}^{\mu}$'s with coefficients that are functions of t, are also solutions of Traschen's equations. In effect, the ten ICV's, say $\underset{a}{V}^{a}$, given by Traschen, are the following combinations of the Killing vectors:

$$\underset{a}{V}^{\mu}=\psi\,\overline{\underset{a}{\xi}^{\mu}}=\frac{H}{\overline{H}}\overline{\underset{a}{\xi}^{\mu}}\,, \tag{5.17}$$

with the exception of quasi-Lorentz rotations in the flat Robertson-Walker spacetime ($k=0$) for which Traschen's ICV's are equal to

$$\underset{[r]}{V}^{\mu}=\psi\Big(\overline{\underset{[r]}{\xi}^{\mu}}+\tfrac{1}{2}\tau^{2}\,\overline{\underset{(r)}{\xi}^{\mu}}\Big), \tag{5.18}$$

where $\overline{\underset{[r]}{\xi}^{\mu}}$ is given by Eq. (4.14), $\overline{\underset{(r)}{\xi}^{\mu}}$ by Eq. (4.11), and τ by Eq. (4.16).

Equations (5.17) and (5.18) suggest, and it has been shown explicitly in [41], that in fact Traschen's integral constraint (1.3) is an expression of conservation laws for a perturbed Robertson-Walker spacetime *mapped on a de Sitter spacetime* and with the mapping given by the conditions

$$\phi=1, \quad \psi=\frac{H}{\overline{H}}. \tag{5.19}$$

This is at variance with Traschen and Eardley's [6] interpretation of Eq. (1.3) as conditions of energy-momentum conservation defined with respect to the Robertson-Walker background.

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