

## Dimensional renormalization in $\phi^3$ theory: Ladders and rainbows

R. Delbourgo\* and D. Elliott†

*University of Tasmania, GPO Box 252-21, Hobart, Tasmania 7001, Australia*

D. S. McAnally‡

*University of Queensland, St Lucia, Brisbane, Queensland 4067, Australia*

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The sum of all the ladder and rainbow diagrams in  $\phi^3$  theory near six dimensions leads to self-consistent higher order differential equations in coordinate space which are not particularly simple for arbitrary dimension  $D$ . We have now succeeded in solving these equations, expressing the results in terms of generalized hypergeometric functions; the expansion and representation of these functions can then be used to prove the absence of renormalization factors which are transcendental for this theory and this topology to all orders in perturbation theory. The correct anomalous scaling dimensions of the Green functions are also obtained in the six-dimensional limit. [S0556-2821(97)01208-3]

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### I. INTRODUCTION

In a recent paper [1] we managed to derive closed forms for ladder corrections to self-energy graphs (rainbows) and vertices, in the context of dimensional renormalization. We only succeeded in carrying out this program for Yukawa couplings near four dimensions, although we did obtain the differential equations pertaining to the  $\phi^3$  theory as well; but we were not able to solve the latter in simple terms. We have now managed to obtain closed expressions for  $\phi^3$  theory as well and wish to report the results here. The answers are indeed nontrivial and take the form of  ${}_0F_3$  functions, which perhaps explains why they had eluded us so far. Interestingly, the closed form results for Yukawa-type models lead to Bessel functions with curious indices and arguments; but as these can also be written as  ${}_0F_1$  functions, the analogy with  $\phi^3$  is close after all.

Given the exact form of the results, both for rainbows and ladders, we are able to test out Kreimer's [2] hypothesis about the connection between knot theory and renormalization theory with confidence, fully verifying that the renormalization factors for such topologies are indeed nontranscendental. At the same time we are able to determine the  $Z$  factors to any given order in perturbation theory and show that in the  $D \rightarrow 6$  limit, the correct anomalous dimensions of the Green functions do emerge, which is rather satisfying. The various  $Z$  factors come out as poles in  $1/(D-6)$  when the Green functions are expanded in the normal way as powers of the coupling constant, but the complete result produces the renormalized Green function to all orders in coupling for any dimension  $D$ .

In the next section we treat the vertex diagrams, converting the differential equation for ladders into hypergeometric form. Upon picking the correct solution we are able to do two things: (i) establish that in the  $D \rightarrow 6$  limit one arrives at

the correct anomalous scaling factor for the vertex function, and (ii) obtain the  $Z$  factors through a perturbative expansion of argument of the hypergeometric function. The case (i) is a bit tricky; it requires an asymptotic analysis, because the indices of the hypergeometric function as well as the argument diverge in the six-dimensional limit. The next section contains the analysis of the rainbow graphs; the equations are similar to the vertex case, but different solutions must be selected, resulting in a different anomalous dimension. It is nevertheless true that the self-energy renormalization constant remains nontranscendental. A brief concluding section ends the paper.

### II. LADDER VERTEX DIAGRAMS

We will only treat the massless case, since this is sufficient to specify the  $Z$  factors once an external momentum scale is introduced. To further simplify the problem we shall consider the case where the vertex is at zero-momentum transfer, leaving just one external momentum  $p$ . The equation for the one-particle irreducible vertex  $\Gamma$ , in the ladder approximation, thereby reduces to

$$\Gamma(p) = Z + ig^2 \int \frac{1}{q^2} \Gamma(q) \frac{1}{q^2} \frac{d^D q / (2\pi)^D}{(p-q)^2}. \quad (1)$$

Letting  $\Gamma(p) \equiv p^4 G(p)$ , the equation can be Fourier transformed into the coordinate space equation for  $G$ :

$$[\delta^4 - ig^2 \Delta_c(x)] G(x) = Z \delta^D(x), \quad (2)$$

where  $\Delta_c$  is the causal Feynman propagator for arbitrary dimension  $D$ . Since the coupling  $g$  is dimensionful when  $D \neq 6$ , it is convenient to introduce a mass scale  $\mu$  and define a dimensionless coupling parameter  $a$  via

$$\frac{g^2}{4\pi^{D/2}} \frac{\Gamma(D/2-1)}{(-x^2)^{1-D/2}} \equiv \frac{4a(\mu r)^{6-D}}{r^4}.$$

Then, rotating to Euclidean space ( $r^2 = -x^2$ ), the ladder vertex equation simplifies to

\*Electronic address: bob.delbourgo@phys.utas.edu.au

†Electronic address: david.elliott@math.utas.edu.au

‡Electronic address: dsm@maths.uq.oz.au

$$\left[ \left( \frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} \right)^2 - \frac{4a(\mu r)^{6-D}}{r^4} \right] G(r) = Z \delta^D(r). \quad (3)$$

This is trivial to solve when  $D=6$ , since it becomes homogeneous for  $r>0$  and the appropriate solution is

$$G(r) \propto r^b, \quad b = -1 - \sqrt{5 - 2\sqrt{4+a}},$$

reducing to  $G(r) \propto r^{-2}$  or  $\Gamma(p)=1$  in the free field case ( $a=0$ ); it represents a useful limit when analyzing the full equation (3), to which we now turn.

Let us define the scaling operator  $\Theta_r = r(d/dr)$ . This allows us to rewrite the square of the d'Alembertian operator as

$$\begin{aligned} \partial^4 &= \left[ \frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} \right]^2 \\ &= r^{-4} (\Theta_r - 2) \Theta_r (\Theta_r + D - 4) (\Theta_r + D - 2). \end{aligned} \quad (4)$$

Hence for  $r>0$  the original equation (3) reduces to the simpler form

$$[\Theta_\rho (\Theta_\rho - 2) (\Theta_\rho + D - 4) (\Theta_\rho + D - 2) - 4a\rho^{6-D}] G = 0, \quad (5)$$

where  $\rho = \mu r$  and  $\Theta_\rho$  is the corresponding scaling operator. Next, rescaling the argument to  $t = 4a\nu^4 \rho^{-1/\nu}$ , with  $\nu \equiv 1/(D-6)$ , we obtain the hypergeometric equation

$$[\Theta_t (\Theta_t + 2\nu) (\Theta_t - 1 - 2\nu) (\Theta_t - 1 - 4\nu) - t] G = 0. \quad (6)$$

Being of fourth order, there are four linearly independent solutions:

$$\begin{aligned} &{}_0F_3(b_1, b_2, b_3; t), \\ &t^{1-b_1} {}_0F_3(2-b_1, b_2-b_1+1, b_3-b_1+1; t), \\ &t^{1-b_2} {}_0F_3(2-b_2, b_3-b_2+1, b_1-b_2+1; t), \\ &t^{1-b_3} {}_0F_3(2-b_3, b_1-b_3+1, b_2-b_3+1; t), \end{aligned}$$

where  $b_1 \equiv 1 + 2\nu$ ,  $b_2 = -4\nu$ , and  $b_3 = -2\nu$ . The appropriate solution, which near  $t=0$  behaves as  $r^{4-D}$  when  $a=0$ , is the last choice: namely,

$$G \propto t^{1+2\nu} {}_0F_3(2+2\nu, 2+4\nu, 1-2\nu; t).$$

Near  $r=0$  this behaves like  $r^{-2-1/\nu}$ . Finally, renormalizing the Green function  $G$  to equal  $\mu^{D-4}$  when  $r=1/\mu$ , the scale we introduced previously for the coupling constant, and restoring the original variables, we end up with the exact result

$$G(r) = r^{4-D} \frac{{}_0F_3\left(2 - \frac{2}{6-D}, 2 - \frac{4}{6-D}, 1 + \frac{2}{6-D}; \frac{4a(\mu r)^{6-D}}{(6-D)^4}\right)}{{}_0F_3\left(2 - \frac{2}{6-D}, 2 - \frac{4}{6-D}, 1 + \frac{2}{6-D}; \frac{4a}{(6-D)^4}\right)}. \quad (7)$$

To check that the poles in  $(D-6)$  cancel out at any given order in perturbation theory, one simply expands the numerator and denominator in Eq. (7) to any particular power in the dimensionless coupling  $a$  and take the limit as  $D \rightarrow 6$ . For instance, to order  $a^3$ , with a little work one arrives at

$$\begin{aligned} r^2 G(r) &\rightarrow 1 + \frac{a}{4} \ln(\mu r) + \frac{a^2}{64} \ln(\mu r) [1 + 2 \ln(\mu r)] \\ &+ \frac{a^3}{1536} \ln(\mu r) [9 + 6 \ln(\mu r) + 4 \ln^2(\mu r)] \\ &+ O(a^4). \end{aligned} \quad (8)$$

It is most gratifying that this agrees perfectly with the expansion of the scaling index  $b$  obtained previously at  $D=6$ . The most significant point is that there is no sign of a transcendental constant in the singularities of the perturbation expansion for  $\phi^3$  theory near six dimensions, signifying that the renormalization constant  $Z$  is free of them, in agreement with the Kreimer hypothesis based on knot theory.

One last (rather difficult) check on our work is to see what happens directly to Eq. (7) as  $D$  approaches six, without having to invoke perturbation theory. For that an asymptotic analysis [3] based on the method of steepest descent (see for

example, de Bruijn [4]) is needed. We start by making use of the Barnes integral representation [5,6] of the hypergeometric function:

$$\begin{aligned} &{}_0F_3(b_1, b_2, b_3; t) \\ &= \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(b_1+z)\Gamma(b_2+z)\Gamma(b_3+z)} \Gamma(-z) t^z dz. \end{aligned}$$

In our case the  $b$  arguments lead us to evaluate the integral

$$\begin{aligned} I_\nu(r) &\equiv \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Gamma(2+4\nu)\Gamma(2+2\nu)\Gamma(1-2\nu)\Gamma(-z)}{\Gamma(2+4\nu+z)\Gamma(2+2\nu+z)\Gamma(1-2\nu+z)} \\ &\times [4a\nu^4 \rho^{-1/\nu}]^z dz \end{aligned} \quad (9)$$

in the limit as  $\nu \rightarrow \infty$ . We shall show that as a function of  $\rho$ ,  $I_\nu$  behaves like  $\rho^{1-\sqrt{5-2\sqrt{4+a}}}$ . Remember that  $G(r) \propto r^{-2-1/\nu} I_\nu(r)$ .

For the method of steepest descents, suppose we write  $I_\nu$  as

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} g_\nu(z) \exp[f_\nu(z)] dz,$$

where  $g_\nu$  is a ‘‘slowly varying’’ function. We see from Eq. (9) that all the poles of the integrand lie on the positive real axis. If  $\zeta$  is such that  $\text{Re}(\zeta) < 0$  and  $f'_\nu(\zeta) = 0$ , then an approximate evaluation of  $I_\nu$  is given by

$$\alpha g_\nu(\zeta) \exp[f_\nu(\zeta)] / \sqrt{2\pi |f''_\nu(\zeta)|},$$

where

$$\alpha \equiv \exp[-i \arg(f''_\nu(\zeta))/2].$$

On applying the reflection formula for the gamma function [5] to both  $\Gamma(1-2\nu)$  and  $\Gamma(1-2\nu+z)$  appearing in Eq. (9), we find that we can write

$$g_\nu(z) = \rho^{-z/\nu} \frac{\sin\pi(2\nu-z)}{\sin(2\pi\nu)},$$

provided  $\nu$  is not an integer, and

$$\exp[f_\nu(z)] = \frac{\Gamma(2+4\nu)\Gamma(2+2\nu)\Gamma(2\nu-z)\Gamma(-z)}{\Gamma(2+4\nu+z)\Gamma(2+2\nu+z)\Gamma(2\nu)} (4a\nu^4)^z.$$

Since

$$f'_\nu(z) = \ln(4a\nu^4) - [\psi(2\nu-z) + \psi(-z) + \psi(2+2\nu+z) + \psi(2+4\nu+z)],$$

where  $\psi$  denotes the psi (or digamma) function, we look for a zero at  $z = -\xi\nu$  say, where  $0 < \xi < 2$ . Since we assume  $\nu \gg 1$  and since for  $x \gg 1$ ,  $\psi(x) = \ln x + O(1/x)$ , we find that  $\xi$  must satisfy the quartic

$$\xi(\xi+2)(\xi-2)(\xi-4) = 4a.$$

The four solutions of this equation are

$$\xi = 1 \pm \sqrt{5 \pm 2\sqrt{4+a}}$$

and are all real if  $0 \leq a < 9/4$ . In particular we shall choose the zero  $\beta$  say in (0,2) which is closest to the origin; that is

$$\beta = 1 - \sqrt{5 - 2\sqrt{4+a}}.$$

With this value of  $\beta$  we find

$$f''_\nu(-\beta\nu) \approx (1-\beta)(4+2\beta-\beta^2)/(a\nu).$$

Since in fact  $0 < \beta < 1$ , we have that  $\arg f''_\nu(-\beta\nu) = 0$  so that  $\alpha = 1$ . Again,

$$g_\nu(-\beta\nu) = \frac{\sin[(2+\beta)\pi\nu]}{\sin(2\pi\nu)} \rho^\beta$$

and, after some algebra,

$$\begin{aligned} \exp[f_\nu(-\beta\nu)] &= \frac{4\sqrt{2\pi}\beta(\beta+2)}{a^{3/2}\nu^{1/2}} \left[ \frac{16(2+\beta)}{(4-\beta)^2(2-\beta)} \right]^{2\nu} \\ &\times \exp(-4\beta\nu), \end{aligned}$$

approximately. Consequently, for  $\nu \gg 1$  but not an integer, we find

$$\begin{aligned} &{}_0F_3(2+2\nu, 2+4\nu, 1-2\nu; 4a\nu^4 \rho^{-1/\nu}) \\ &\sim \frac{\rho^\beta \sin[(2+\beta)\pi\nu]}{a \sin(2\pi\nu)} \frac{4\beta(\beta+2) \exp(-4\beta\nu)}{(1-\beta)^{1/2}(4+2\beta-\beta^2)^{1/2}} \\ &\times \left[ \frac{16(2+\beta)}{(4-\beta)^2(2-\beta)} \right]^{2\nu}. \end{aligned}$$

Using this asymptotic expansion, we obtain simply from Eq. (7) that

$$G(r) = \mu^\beta r^{5-D-\sqrt{5-2\sqrt{4+a}}}, \quad (10)$$

which is just the scaling behavior at six dimensions which we were seeking. We have therefore fully verified the correctness of Eq. (7) in all the limits. The last step is to convert the answer to Minkowski space by making the familiar substitution  $r^2 \rightarrow -x^2 + i\epsilon$ .

### III. RAINBOW DIAGRAMS

Let  $\Delta_R(p)$  denote the renormalized  $\phi$  propagator in rainbow approximation, so that  $p^2 \Delta_R(p) = 1 - \Sigma_R(p)/p^2$ , where  $\Sigma_R$  is the rainbow self-energy. The propagator obeys the integral equation in momentum space

$$p^4 \Delta_R(p) = Z p^2 + i g^2 \int \frac{d^D k}{(2\pi)^D} \frac{\Delta_R(p-k)}{k^2}, \quad (11)$$

where  $Z$  now refers to the wave-function renormalization constant. As always we convert this into an  $x$ -space differential equation:

$$[\partial^4 - i g^2 \Delta_c(x)] \Delta_R(x) = -Z \partial^2 \delta^D(x). \quad (12)$$

Interestingly, this is exactly the same equation as Eq. (2), apart from the right-hand side, and it can therefore be converted into hypergeometric form by following the same steps as before. The only difference is that we should look for a different solution, because as  $g \rightarrow 0$ ,  $\Delta_R(p) \rightarrow 1/p^2$ , or  $\Delta_R(x) \sim (x^2)^{1-D/2}$ .

A simple analysis shows the correct solution is

$$t^{1+4\nu} {}_0F_3(2+4\nu, 2+6\nu, 1+2\nu; t);$$

$$t = 4a\nu^4 (\mu r)^{-1/\nu}, \quad \nu = 1/(D-6),$$

because this reduces to  $r^{2-D}$  when  $a=0$ . Actually we can solve Eq. (12) directly at  $D=6$  when  $a \neq 0$  because it is a simple homogeneous equation leading to

$$\Delta_R(x) \propto r^{-1-\sqrt{5+2\sqrt{4+a}}}$$

and thereby determine the anomalous dimension from the exponent of  $r$ . Anyhow, the exact solution of the rainbow sum for any  $D$  and renormalized at  $r=1/\mu$  is here obtained to be

$$\Delta_R(r) = r^{2-D} \frac{{}_0F_3\left(2 - \frac{4}{6-D}, 2 - \frac{6}{6-D}, 1 - \frac{2}{6-D}; \frac{4a(\mu r)^{6-D}}{(6-D)^4}\right)}{{}_0F_3\left(2 - \frac{4}{6-D}, 2 - \frac{6}{6-D}, 1 - \frac{2}{6-D}; \frac{4a}{(6-D)^4}\right)}. \quad (13)$$

The numerator and denominator of Eq. (13), when expanded in powers of  $a$  will reproduce the (renormalized) perturbation series; to third order we find, in the limit as  $D \rightarrow 6$ , that all poles disappear and

$$r^4 \Delta_R(r) = 1 - \frac{a}{12} \ln(\mu r) + \frac{a^2}{1728} \ln(\mu r) [11 + 6 \ln(\mu r)] - \frac{a^3}{124416} \ln(\mu r) [103 + 66 \ln(\mu r) + 12 \ln^2(\mu r)] + O(a^4).$$

This coincides perfectly with the expansion of the scaling exponent at  $D=6$ .

Lastly we need to show that the  $D \rightarrow 6$  limit of Eq. (13) collapses to the scaling behavior found above, via an asymptotic analysis of the Barnes representation. We have indicated how this can be proven in the previous section and thus we skip the formal details to avoid boring the reader. The long and the short of the analysis is that no transcendentals enter into the above expressions for the self-energy (including their singularities, which are tied to the wavefunction renormalization constant). These results confirm nicely the Kreimer [2] hypothesis that the  $Z$  factors will be simple rationals for such topologies.

#### IV. CONCLUSIONS

We have succeeded in evaluating an all-orders solution of Green functions for ladder and rainbow diagrams for any dimension  $D$  in  $\phi^3$  theory; the results are nontrivial, involving  ${}_0F_3$  hypergeometric functions. We have demonstrated that, in the limit as  $D \rightarrow 6$ , the correct six-dimensional scaling behavior (which can be separately worked out) is reproduced. One can likewise determine the exact solutions for

massless bubble ladder exchange in  $\phi^4$  theory, because the equations are very similar: they are also of fourth order and can be converted into hypergeometric form too [7].

More intriguing is the question of what happens when self-energy and ladder insertions are considered, so far as renormalization constants are concerned. A recent paper by Kreimer [8] has shown that such topologies with their disjoint divergences can produce transcendental  $Z$  in accordance with link diagrams that are of the  $(2, q)$  torus knot variety, where the highest  $q$  is determined by the loop number. It would be interesting to show this result without resorting to perturbation theory by summing all those graphs exactly, as we have done in this paper. (Kreimer cautions that multiplicative renormalization may screen his new findings.) The generalization to massive propagators [9] does not seem beyond the realms of possibility either, although it has a marginal bearing on  $Z$ -factors.

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