Consistency of dust solutions with div $H=0$

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One of the necessary covariant conditions for gravitational radiation is the vanishing of the divergence of the magnetic Weyl tensor H_{ab}, while H_{ab} itself is nonzero. We complete a recent analysis by showing that in irrotational dust spacetimes the condition div $H=0$ evolves consistently in the exact nonlinear theory. [S0556-2821(97)03108-1]

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Irrotational dust spacetimes, typically considered as models for the late universe or for gravitational collapse, are covariantly characterized by the dust four-velocity u^a, energy density ρ, expansion Θ, and shear σ_{ab}, and by the free gravitational field, described by the electric and magnetic parts of the Weyl tensor C_{abcd}:

$$E_{ab}=C_{acbd}u^c u^d\ ,\quad H_{ab}=\frac{1}{2}\varepsilon_{acd}C^{cd}{}_{be}u^e\ ,$$

where $h_{ab}=g_{ab}+u_a u_b$ is the spatial projector, g_{ab} is the metric tensor, and $\varepsilon_{abc}=\eta_{abcd}u^d$ is the spatial projection of the spacetime permutation tensor η_{abcd} [1]. Gravitational radiation is covariantly described by the nonlocal fields E_{ab}, the tidal part of the curvature which generalizes the Newtonian tidal tensor, and H_{ab}, which has no Newtonian analogue [2]. As such, H_{ab} may be considered as the true gravity wave tensor, since there is no gravitational radiation in Newtonian theory. However, as in electromagnetic theory, gravity waves are characterized by H_{ab} and E_{ab}, where both are divergence-free but neither is curl-free [3,4].

In [1], it was shown that in the generic case, i.e., without imposing any divergence-free conditions, the covariant constraint equations evolve consistently with the covariant propagation equations. These equations are as follows.

Propagation equations:

$$\dot{\rho}+\Theta\rho=0\ , \tag{1}$$

$$\dot{\Theta}+\frac{1}{3}\Theta^2+\frac{1}{2}\rho=-\sigma_{ab}\sigma^{ab}\ , \tag{2}$$

$$\dot{\sigma}_{ab}+\frac{2}{3}\Theta\sigma_{ab}+E_{ab}=-\sigma_{c\langle a}\sigma_{b\rangle}{}^c\ , \tag{3}$$

$$\dot{E}_{ab}+\Theta E_{ab}-\mathrm{curl}H_{ab}+\frac{1}{2}\rho\sigma_{ab}=3\sigma_{c\langle a}E_{b\rangle}{}^c\ , \tag{4}$$

$$\dot{H}_{ab}+\Theta H_{ab}+\mathrm{curl}E_{ab}=3\sigma_{c\langle a}H_{b\rangle}{}^c\ . \tag{5}$$

Constraint equations:

$$D^b\sigma_{ab}-\frac{2}{3}D_a\Theta=0\ , \tag{6}$$

$$\mathrm{curl}\sigma_{ab}-H_{ab}=0\ , \tag{7}$$

$$D^bE_{ab}-\frac{1}{3}D_a\rho=\varepsilon_{abc}\sigma^b{}_dH^{cd}\ , \tag{8}$$

$$D^bH_{ab}=-\varepsilon_{abc}\sigma^b{}_dE^{cd}\ , \tag{9}$$

where $S_{\langle ab\rangle}=h_a{}^ch_b{}^dS_{(cd)}-\frac{1}{3}S_{cd}h^{cd}h_{ab}$ is the projected, symmetric and trace-free part of S_{ab}, the covariant spatial derivative is defined by $D_aS^{b\cdots}{}_{\cdots c}=h_a{}^ph^b{}_q\cdots h_c{}^r\nabla_pS^{q\cdots}{}_{\cdots r}$, the covariant spatial divergence is D^bS_{ab}, and the covariant spatial curl is $\mathrm{curl}\,S_a=\varepsilon_{abc}D^bS^c$ for vectors and $\mathrm{curl}\,S_{ab}=\varepsilon_{cd(a}D^cS_{b)}{}^d$ for tensors. (Further details are given in [1,5].) In the linearized theory of covariant perturbations about a Friedman-Robertson-Walker (FRW) background, the right-hand sides of these equations are all zero.

It was previously claimed that in the exact nonlinear theory, the gravity wave condition

$$D^bH_{ab}=0 \tag{10}$$

implies $H_{ab}=0$ [6]. As shown in [1], this claim arises from a sign error and is incorrect, Bianchi-type V spacetimes providing a counterexample. Here, we complete the analysis of

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[1] by showing that consistency is maintained if Eq. (10) is imposed, without H_{ab} being zero.

The fact that consistency is not automatic is illustrated by the case of silent universes, in which $H_{ab}=0$. For these solutions, consistent evolution of the condition $H_{ab}=0$ imposes a series of nontrivial integrability conditions, which are identically satisfied in the linearized case, but not in the nonlinear case [7,8]. Thus, there is a linearization instability in silent universes. By contrast, when Eq. (10) holds but H_{ab} is not forced to vanish, which includes gravity wave solutions, there is no linearization instability following from the evolution of Eq. (10). An example of consistency conditions arising already at the linearized level is given by purely magnetic spacetimes, $E_{ab}=0$, for which $\Theta\sigma_{ab}=0$ [7].

The proof that Eq. (10) evolves consistently is based on a combination of tetrad methods [6,9] and the covariant methods of [1]. The only direct effect of Eq. (10) on the covariant propagation and constraint equations is an algebraic modification of the constraint (9), which does not change the consistent evolution of the constraints. We have to check only consistent evolution of the new condition (10) itself. It is more convenient to replace Eq. (10) by the equivalent condition that follows from Eq. (9),

$$[\sigma,E]=0\,, \tag{11}$$

where we are using index-free notation for the covariant commutator. In the linearized case, Eq. (11) is identically satisfied since the left-hand side is second order of smallness, and consistency is automatic.

In the exact nonlinear case, using only the shear propagation equation (3) and its covariant time derivative, we find that

$$[\sigma,\dot{E}]=-[\sigma,\ddot{\sigma}]+\frac{2}{3}\Theta[\sigma,E]-\sigma[\sigma,E]$$

and

$$[\dot{\sigma},E]=-\frac{2}{3}\Theta[\sigma,E]+\sigma[\sigma,E]\,.$$

Adding these equations gives

$$[\sigma,E]^{\cdot}=-[\sigma,\ddot{\sigma}]\,. \tag{12}$$

Now, the right-hand side may be shown to vanish identically without differentiating Eq. (11), i.e., using only the algebraic content of Eq. (11), as follows.

From the shear propagation equation (3), Eq. (11) is equivalent to

$$[\sigma,\dot{\sigma}]=0\,. \tag{13}$$

We choose an orthonormal tetrad [10] $\{\mathbf{e}_0=\mathbf{u}\,,\mathbf{e}_\mu\}$, with $\{\mathbf{e}_\mu\}$ a shear eigenframe, so that

$$\sigma_{0a}=0=\partial_0\sigma_{0a}\,,\quad \sigma_{\mu\nu}=0=\partial_0\sigma_{\mu\nu}\quad \text{if}\ \mu\neq\nu\,, \tag{14}$$

where ∂_0 denotes the directional derivative along $\mathbf{e}_0=\mathbf{u}$. Then, we have

$$[\sigma,\dot{\sigma}]_{ab}=(\sigma_{aa}-\sigma_{bb})\dot{\sigma}_{ab}\quad (\text{no sum})\,. \tag{15}$$

At all points where the shear is nondegenerate, i.e., where $\sigma_{aa}\neq\sigma_{bb}$ when $a\neq b$, Eqs. (15) and (13) show that $\dot{\sigma}_{ab}$ is diagonal, and thus E_{ab} is also diagonal, by Eq. (3). In fact, diagonality still holds at points of degeneracy, as follows from the tetrad form of the covariant derivative:

$$\dot{\sigma}_{ab}=\partial_0\sigma_{ab}-\Gamma^c{}_{0b}\sigma_{ac}-\Gamma^c{}_{0a}\sigma_{cb}\,,$$

where the Ricci rotation coefficients are $\Gamma_{abc}=\mathbf{e}_a\cdot\nabla_b\mathbf{e}_c=-\Gamma_{cba}$. Using Eqs. (14) and (3), we get

$$a\neq b\Rightarrow\dot{\sigma}_{ab}=(\sigma_{aa}-\sigma_{bb})\Gamma_{b0a}=-E_{ab}\quad(\text{no sum})\,, \tag{16}$$

so that $\dot{\sigma}_{ab}$ is diagonal also where $\sigma_{aa}=\sigma_{bb}$ $(a\neq b)$. Thus, *the shear eigenframe simultaneously diagonalizes* σ_{ab}, $\dot{\sigma}_{ab}$, *and* E_{ab}. This regains a result given in [11].

It also follows from Eqs. (13), (15), and (16) that

$$\Gamma_{a0b}=0 \tag{17}$$

holds at all points where the shear is nondegenerate. [Note that Eq. (17) is an identity for $a=b$.] At points of degeneracy, i.e., where $\sigma_{11}=\sigma_{22}$, we can use the remaining tetrad freedom of a rotation in the $\{\mathbf{e}_1,\mathbf{e}_2\}$ plane to set $\Gamma_{102}=0$, so that Eq. (17) still holds. Specifically, such a rotation through an angle α preserves Eq. (14) and the degeneracy, while

$$\Gamma_{102}\rightarrow\Gamma_{102}-\partial_0\alpha\,.$$

Thus, we can ensure that Eq. (17) holds throughout spacetime, by specializing the eigenframe where necessary. Then, Eq. (17) shows that $\ddot{\sigma}_{ab}$ is also diagonal in this frame, since

$$a\neq b\Rightarrow\ddot{\sigma}_{ab}=(\dot{\sigma}_{aa}-\dot{\sigma}_{bb})\Gamma_{b0a}=0\quad(\text{no sum})\,,$$

where we have used the fact that $\partial_0\dot{\sigma}_{ab}$ is diagonal. The covariant (frame-independent) consequence of the simultaneous diagonalizability of σ_{ab} and $\ddot{\sigma}_{ab}$ is

$$[\sigma,\ddot{\sigma}]=0\,,$$

which shows that the right-hand side of Eq. (12) does indeed vanish identically, consistent with and independent of the derivative of Eq. (11). Thus, the first covariant time derivative of the condition (11) imposes no consistency conditions. It is clear from the above argument that all the subsequent covariant time derivatives of σ_{ab} are also diagonal in the eigenframe, so that these higher derivatives all commute with the shear and among themselves. It follows that the second and higher covariant time derivatives of the condition (11) also vanish without further conditions.

This establishes that *the covariant condition* $\operatorname{div} H=0$ *evolves consistently in the exact nonlinear theory*. The question whether such consistency extends to the further covariant gravity wave condition $\operatorname{div} E=0$ is more difficult, and under investigation.

Finally, we note that, by virtue of Eq. (17) and the propagation equation (4), $\operatorname{curl} H_{ab}$ is also diagonal in the eigenframe that diagonalizes σ_{ab} and E_{ab}, i.e., *there is a shear eigenframe such that* σ_{ab}, E_{ab}, $\operatorname{curl} H_{ab}$, *and all their covariant time derivatives are diagonal and, therefore commute.*

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