Finite quantum electrodynamics with a gravitationally smeared propagator

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On the basis of qualitative considerations of quantum fluctuations in the space-time geometry, a modification of the Feynman propagator is proposed involving a smearing of the propagator over a space-time region of an extent of about a Planck length, or $\simeq 10^{-33}$ cm, surrounding the light cone. The smeared propagator is a weighted average of modified Bessel functions both in coordinate space and in momentum space. In coordinate space, the smeared propagator has a milder singularity on the light cone than the Feynman propagator, and in momentum space it provides an effective cutoff at the Planck mass. Radiative corrections in QED can be easily calculated with this propagator, and the results are finite. The renormalizations of the electron charge and mass are of the order of a few percent. [S0556-2821(97)03208-6]

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I. INTRODUCTION

The methods for dealing with the ultraviolet divergences that afflict QED have not improved much since the introduction of the Dyson and the Pauli-Villars procedures fifty years ago [1]. It is true that in modern calculations the crude amputations performed by Dyson and Pauli and Villars are often supplanted by the more subtle procedures of proper-time regularization, analytic regularization, dimensional regularization, or ζ-function regularization [2,3,4,5]. But all these procedures are purely formal, that is, they are merely prescriptions for the segregation of infinite terms and finite remainders. The infinite terms are peremptorily discarded (by means of counterterms) and the finite remainders are saved for comparisons with experiments. Why the infinite terms appear in the first place, and why they should be discarded, is a question not addressed by these formal prescriptions [6].

Seeking a physical, rather than formal, resolution to the problem of infinities, Landau [7] and others conjectured early on that such a resolution might be found in quantum gravity. Simple estimates of quantum fluctuations of the geometry suggest that at the scale of the Planck length, $L_* = 1/M_* = \sqrt{\hbar G/c^3} = 1.6\times 10^{-33}$ cm, the fluctuations in the metric $g_{\mu\nu}$ are of the order of magnitude of 1; that is, the geometry becomes highly uncertain and, consequently, the light cone is smeared out. The ultraviolet divergences in quantum field theory arise from the singularities of the propagator functions on the light cone, and a smearing out of the light cone is expected to lead to modifications of the propagator functions at short distances and suppression of the divergences. Deser [8] proposed to calculate the propagator by a functional integral (that is, an average) over all possible geometries and he gave qualitative arguments for how this results in a smearing of the light cone. But our ignorance of the details of the quantum fluctuations in the geometry poses an obstacle to explicit calculation of the modifications of the propagator functions.

Khriplovich [9] attempted to calculate the modifications of the propagator functions by an entirely different approach. He used the naive covariant quantum gravity theory to evaluate gravitational self-energy effects for spin-$\frac{1}{2}$ particles. Following an earlier attempt by DeWitt [10] for spin 0, Khriplovich summed ladder contributions involving $0,1,2,\ldots,\infty$ gravitons. For a spin-$\frac{1}{2}$ particle of zero mass, this led him to a modified propagator

$$\Delta(p) = \not{p}\, g(p) \tag{1}$$

with

$$g(p) = -\frac{2}{L^2p^4} - \frac{4}{L^4p^6} + \frac{4i}{L(p^2)^{3/2}} K_3(-2iL\sqrt{p^2}), \tag{2}$$

where $L=\sqrt{3/4\pi}L_*$ and K_n is a Bessel function of the second kind. The modified propagator approximates the Feynman propagator $1/p^2$ for $|p^2| \ll M_*^2$ but it vanishes as $1/p^4$ in the upper half energy plane for $|p^2| \gg M_*^2$, which is a significant improvement over the Feynman propagator. The difference is even more drastic in coordinate space, where the Fourier transform of Eq. (2) gives

$$g(x^2) = \frac{i}{2\pi^2L^2}\left(1 + \frac{x^2-L^2}{L^2} \ln \frac{x^2-L^2-i\epsilon}{x^2-i\epsilon}\right). \tag{3}$$

This has only a mild singularity at $x^2=0$, in contrast with the highly singular behavior of the zero-mass Feynman propagator,

$$D_F(x^2) = \frac{i}{4\pi^2(x^2-i\epsilon)} = -\left(\frac{1}{4\pi}\right)\delta(x^2) + \frac{i}{4\pi^2x^2}. \tag{4}$$

For a spin-$\frac{1}{2}$ particle of nonzero mass, Khriplovich was not able to find a closed solution for the propagator. However, it is possible to verify from his equations that the singularities for the massive case are no worse than those for the massless case.

Khriplovich was aware of several shortcomings of his calculation. He included only ladder diagrams of gravitons and excluded all other graviton diagrams. Furthermore, the form (2) of the propagator hinges in a crucial way on his choice of the gravitational gauge—only for a special, preferred choice

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of gauge does the propagator take the simple form (2). To these criticisms we might add that any attempt at calculating a propagator at the Planck scale via the covariant quantum gravity theory is doomed to failure, since at this scale the covariant theory is expected to fail. For all these reasons, Khriplovich's results cannot be taken very seriously. However, his results do offer some tentative support for the view that quantum gravity suppresses the singularities in the propagator functions.

The earlier calculation by DeWitt [10] attempted to find the gravitational modifications in the propagator for a spin-0 particle. His result did not indicate any suppression of singularities (for $|p^2| \gg m^2$, DeWitt's gravity-modified propagator behaves like the ordinary Feynman propagator). The difference seems to arise from an unfortunate choice of gauge (in fact, DeWitt's result is clearly pathological: for $L_* \to 0$, his gravity-modified propagator fails to reduce to the ordinary Feynman propagator; instead, the mass in the propagator mutates into $1/\sqrt{2}$ times the free-particle mass). If the calculation for the spin-0 case is performed with a choice of gauge similar to that of Khriplovich, then results similar to those of Khriplovich can be obtained (see the Appendix).

We are now at an impasse: we have good physical reasons for supposing that the resolution of the problem of divergences is to be found in the smearing out of the light cone and in the consequent modifications of the propagator functions at distances $\simeq L_*$ (or momenta $\simeq M_*$), but we do not have available any reliable calculation telling us the exact form of these modifications. We can surmount this impasse by recognizing that, for most calculations in QED, *the exact form of the smeared propagator is unimportant: we can adopt any of a wide variety of smeared propagator functions and obtain finite results*. The reason for this insensitivity to the details of the propagator functions is that every acceptable smeared propagator coincides with the Feynman propagator for distances $|x^2| \gg L_*^2$ and it has an effective cutoff in momentum space at $|p^2| \simeq M_*^2$ to suppress the singularities on the light cone. Since the experimentally accessible energies are much less than M_*, the main consequence of replacing the conventional Feynman propagators by such smeared propagators is to cut off all the infinite integrals at $|p^2| \simeq M_*^2$. The details of these cutoffs will vary with the details of the smeared propagator, but such details are subsequently hidden in the renormalization procedure, and they therefore become irrelevant. The central point of this paper is the exploitation of this insensitivity to the details of the smearing.

In Sec. II we will discuss the general properties of smeared propagators and in Sec. III we will use such smeared propagators to obtain finite results for the radiative corrections in QED. This is an attempt at a ''realistic'' approach to QED based on relevant physics, rather than a ''formalistic'' approach based on improper (although unambiguous) mathematical manipulations.

A rough estimate indicates that the charge and mass renormalizations produced by a typical smeared propagator are of the order of 10%. This agrees with the mass renormalization found by Khriplovich; the agreement is not accidental since it turns out that the propagator (2) is a special case of a smeared propagator. Renormalizations of the same order of magnitude were also obtained by Isham, Salam, and Strathdee [11] with their ''superpropagator'' method. But there is no direct relationship between the superpropagator and our smeared propagator (instead of a modification of the propagator *per se*, the superpropagator also involves a gravitational modification of the photon-electron vertex).

II. THE SMEARED PROPAGATOR

The conventional Feynman propagator for a scalar field of mass m is

$$G_F(x^2) = -\frac{m}{4\pi^2}\frac{K_1(im\sqrt{x^2-i\epsilon})}{\sqrt{x^2-i\epsilon}}. \tag{5}$$

We want to smear out this propagator over a region of extent $\Delta x^2 \simeq L_*^2$ surrounding the light cone. For this, we adopt the procedure familiar from the mathematics of smeared fields [12] in quantum field theory: we multiply the propagator function $G_F(x^2-\lambda)$ by a weighting function $f(\lambda)$ and integrate over λ. This yields the smeared propagator

$$\overline{G_F}(x^2) = \int d\lambda\, f(\lambda) G_F(x^2-\lambda)$$
$$= -\frac{m}{4\pi^2}\int d\lambda\, f(\lambda)\,\frac{K_1(im\sqrt{x^2-\lambda-i\epsilon})}{\sqrt{x^2-\lambda-i\epsilon}}. \tag{6}$$

We assume that the weighting function $f(\lambda)$ is non-negative and normalized to 1:

$$\int d\lambda\, f(\lambda) = 1. \tag{7}$$

The weighting function $f(\lambda)$ represents the quantum fluctuations of the geometry in a crude, phenomenological way. The details of the weighting function are unknown; they depend on the unknown details of the quantum fluctuations. As we will see below, causality arguments suggest that $f(\lambda)$ should vanish for negative values of λ; consequently, $f(\lambda)$ is expected to be nonzero over a range from $\lambda = 0$ to $\lambda \simeq L_*^2 > 0$. We will assume that all kinds of particles have the same weighting function; this seems plausible, since the weighting function represents an underlying, universal geometric effect.

The smearing in Eq. (6) is designed to preserve the Poincaré invariance of the propagator. Actually, the exact propagator that describes the propagation from a space-time point x_1 to a space-time point x_2 involves the detailed geometry of the intermediate space-time regions, and since this geometry is not Poincaré invariant on the Planck scale, we expect that the exact propagator is not Poincaré invariant. The smeared propagator is presumably related to the (unknown) exact propagator by some kind of averaging over all possible geometries, and this averaging restores Poincaré invariance.

The smeared propagators are not Green's functions for the standard free-particle wave equation. But all smeared propagators approach the Feynman propagator asymptotically for $|x^2| \gg L_*^2$. In any case, we will assume that the smeared propagators can be identified with the time-ordered vacuum expectation values of field operators, to be used in the evaluation of S-matrix elements according to the usual Feynman rules. The smeared propagators have no spectral Lehmann-Symanzik-Zimmermann (LSZ) representation, and none is to be expected since this representation fails in the presence of gravitational interactions [9].

The propagator of Khriplovich is an example of a smeared propagator. It is easy to verify that Eq. (6) with $m=0$ leads to Eq. (3) if the weighting function is taken as

$$f(\lambda)=\frac{2}{L^2}\left(1-\frac{\lambda}{L^2}\right)\quad \text{for}\ \ 0\leqslant\lambda\leqslant L^2. \tag{8}$$

In contrast, the ''regularized'' propagator of Feynman, Pauli and Villars [1], and Bogoliubov [13],

$$\operatorname{reg}G_F(x^2)=-\frac{m}{4\pi^2}\frac{K_1(im\sqrt{x^2-i\epsilon})}{\sqrt{x^2-i\epsilon}}+\frac{M}{4\pi^2}\frac{K_1(iM\sqrt{x^2-i\epsilon})}{\sqrt{x^2-i\epsilon}}, \tag{9}$$

is not a smeared propagator, since it does not approach the Feynman propagator for $|x^2|\gg 1/M^2$.

For spacelike x, the global commutativity theorem [14] demands $[\phi(x),\phi(0)]=0$, which requires $\overline{G_F}=0$, and it follows that λ in Eq. (6) must be positive. This means that the light cone $x^2=0$ is smeared toward the interior of the future and the past space-time regions; the smeared light cone forms a ''light shell,'' extending from $x^2=0$ to $x^2\simeq L_*^2$.

For timelike x (and $t>0$), the commutator implied by the smeared propagator is

$$\begin{aligned}[\phi(x),\phi(0)]&=2i\ \operatorname{Re}\overline{G_F}(x^2)\\ &=-\frac{i}{2\pi}f(x^2)+\frac{im}{4\pi}\int d\lambda\, f(\lambda)\,\theta(x^2-\lambda)\\ &\quad\times\frac{J_1(m\sqrt{x^2-\lambda})}{\sqrt{x^2-\lambda}}.\end{aligned} \tag{10}$$

In contrast, the commutator for the conventional Feynman propagator is

$$[\phi(x),\phi(0)]=-\frac{i}{2\pi}\,\delta(x^2)+\frac{im}{4\pi}\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}}. \tag{11}$$

From Eq. (10) we see that, if $f'(0)$ is finite, the equal-time commutator $[\partial\phi(\mathbf{x},0)/\partial t,\phi(0)]$ implied by the smeared propagator equals zero, in contrast to the conventional canonical commutator $[\partial\phi(\mathbf{x},0)/\partial t,\phi(0)]=-i\,\delta^3(\mathbf{x})$. This looks like a fatal conflict with a basic postulate of quantum mechanics. However, if we perform the Fourier decomposition of ϕ, we find that for modes of energy $E_p\ll M_*$ the commutation relations of the operators $\mathbf{a}_p$ and $\mathbf{a}\dagger_p$ retain their standard form. The Fourier decomposition of $\phi(\mathbf{x},t)$ and of $\phi(0)$ implies

$$\begin{aligned}[\mathbf{a}_p,\mathbf{a}\dagger_{p'}]&=\delta^3(\mathbf{p}-\mathbf{p}')\int d^3x\,\frac{e^{i(E_pt-\mathbf{p}\cdot\mathbf{x})}}{2E_p}\\ &\quad\times\{E_p{}^2[\phi(\mathbf{x},t),\phi(0)]\\ &\quad+2iE_p[\partial\phi(\mathbf{x},t)/\partial t,\phi(0)]\\ &\quad-[\partial^2\phi(\mathbf{x},t)/\partial t^2,\phi(0)]\}.\end{aligned} \tag{12}$$

This assumes that $\mathbf{a}_p$ and $\mathbf{a}\dagger_p$ are independent of time, an assumption that is valid for $E_p\ll M_*$, since the wave functions for such modes approximately obey the standard wave equation. The integrals on the right side involve the function $f(t^2-\mathbf{x}^2)$, or $f(t^2-r^2)$ in polar coordinates. For the evaluation, it is convenient to select a time such that $L_*\ll t\ll 1/E_p$, say, $t\simeq\sqrt{L_*/E_p}$. The function $f(t^2-r^2)$ is then sharply peaked between $r^2=t^2$ and $r^2\simeq t^2-L_*^2$, and the integrals can be evaluated by saddle-point integration over the peak. For instance,

$$\int_0^\infty dr\,r^n f(t^2-r^2)\simeq\tfrac{1}{2}t^{n-1}. \tag{13}$$

With these approximations, Eq. (12) yields $[\mathbf{a}_p,\mathbf{a}\dagger_{p'}]\simeq\delta^3(\mathbf{p}-\mathbf{p}')$. Likewise, $[\mathbf{a}_p,\mathbf{a}_{p'}]\simeq 0$. The errors in these commutation relations are of order $E_p{}^2t^2$, or E_pL_*. Thus, for modes with $E_p\ll M_*$, the unequal-time commutator (10) is sufficient to recover the usual interpretation of $\mathbf{a}\dagger_p$ and $\mathbf{a}_p$ as creation and destruction operators. An incidental advantage of relying on the unequal-time commutator is that we avoid the highly singular limit $t\to 0$ and $\partial t\to 0$ implicit in the conventional canonical equal-time commutator, which has long rendered this commutator suspect in the eyes of mathematical field theorists. In contrast, the unequal-time commutator (10) and its time derivative are quite nonsingular.

In momentum space, the smeared propagator is

$$\begin{aligned}\overline{G_F}(p)&=\int d^4x\,e^{ip\cdot\mathbf{x}}\overline{G_F}(x^2)\\ &=-i\int d\lambda\, f(\lambda)\,\frac{\sqrt{\lambda}K_1[-i\sqrt{\lambda(p^2-m^2+i\epsilon)}]}{\sqrt{p^2-m^2+i\epsilon}}.\end{aligned} \tag{14}$$

The smeared propagators in coordinate space and in momentum space exhibit a remarkable duality: both involve weighted averages of modified Bessel functions of order 1 and, apart from an overall numerical factor, they are related by the simple exchanges $x\leftrightarrow p$, $\sqrt{\lambda}\leftrightarrow -m$.

The analytic properties of the smeared propagator are similar to those described by Khriplovich for his propagator. In the complex p^2 plane, the smeared propagator has a branch cut along the real axis from m^2 to ∞. In the complex p_0 plane, instead of the usual two isolated poles at $p_0=\pm\sqrt{\mathbf{p}^2+m^2}$, the smeared propagator has two branch cuts extending from $p_0=\pm\sqrt{\mathbf{p}^2+m^2}$ to $\pm\infty$. Asymptotically, the smeared propagator vanishes exponentially in all directions in the complex plane except along the edges of the branch cuts, where it is an oscillating function with a gradually decreasing amplitude.

If $|p^2| \ll M_*^2$, the dominant term in K_1 is $i/\sqrt{\lambda(p^2-m^2+i\epsilon)}$ and the smeared propagator approaches the Feynman propagator,

$$\overline{G_F}(p) \to \frac{1}{p^2-m^2+i\epsilon}. \tag{15}$$

This makes it obvious that probability amplitudes and wave packets that contain only such ''low'' momentum components propagate according to the conventional Feynman propagator, in the expected way. However, for momenta $|p^2| \simeq M_*^2$, $\overline{G_F}$ differs drastically from G_F and the propagator does not obey the standard wave equation. At the Planck scale, it is impossible to regard the propagator as a Green's function of any kind of wave equation, since in the absence of a well-defined background geometry, differential operators, such as $\Box^2$, are meaningless or, at the least, they are meaningless as c numbers. Thus, at the Planck scale, the conventional initial-value problem is meaningless.

III. RADIATIVE CORRECTIONS IN QED

As an illustration of how the smeared modified propagator leads to finite results we present the calculations of the second-order radiative corrections in QED, that is, the vacuum polarization, the electron self-energy, and the photon vertex.

A. Vacuum polarization

For a photon of momentum q, the vacuum polarization tensor is

$$\Pi_{\mu\nu}(q) = -\frac{e^2}{(2\pi)^4}\int d^4p \ \mathrm{Tr}[\gamma_\mu(\not p+m)\gamma_\nu(\not p-\not q+m)] \times \overline{G_F}(p)\overline{G_F}(p-q), \tag{16}$$

where the electron propagator $(\not p+m)\overline{G_F}(p)$ is obtained from the scalar propagator in the usual way, by multiplication with $(\not p+m)$. The integral (16) can be conveniently evaluated by taking advantage of the integral representation for the Bessel functions [15]:

$$\frac{K_n(\xi)}{\xi^n} = \int_0^\infty \frac{\exp(-z-\xi^2/4z)}{(2z)^{n+1}}\, dz. \tag{17}$$

With a 90° rotation of the contour of integration this yields

$$\overline{G_F}(p) = -i\int d\lambda\, f(\lambda)\int_0^\infty dz \ \exp[i\lambda/4z + i(p^2-m^2+i\epsilon)z]. \tag{18}$$

This integral representation is analogous to the familiar exponential representation

$$\frac{1}{p^2-m^2+i\epsilon} = -i\int_0^\infty dz \ \exp[i(p^2-m^2+i\epsilon)z] \tag{19}$$

often exploited in the calculations with the conventional Feynman propagator. The close correspondence between Eqs. (18) and (19) permits us to perform the evaluation of $\Pi_{\mu\nu}$ with the smeared propagator by following exactly the same steps as in conventional QED (see, e.g., Ref. [16]), but with a finite result. As in conventional QED, the result for the polarization tensor separates into a gauge invariant part and noninvariant part. The gauge invariant part is

$$\Pi^{(1)}_{\mu\nu}(q) = -\frac{2i\alpha}{\pi}(q^2 g_{\mu\nu}-q_\mu q_\nu)\int d\lambda_1\int d\lambda_2 f(\lambda_1)f(\lambda_2) \times \int_0^1 dz\, 2z(1-z)K_0(\xi), \tag{20}$$

where

$$\xi = \sqrt{\frac{\lambda_1(1-z)+\lambda_2 z}{z(1-z)}[m^2-q^2z(1-z)]}. \tag{21}$$

For any momentum in the range of experimental interest, $|q^2| \ll 1/L_*^2$ and the Bessel function K_0 is then approximately $K_0(\xi) \simeq -\gamma - \ln \xi/2$, so Eq. (20) reduces to the familiar result of conventional QED, except for an alteration of the charge renormalization factor:

$$\Pi^{(1)}_{\mu\nu}(q) = i(q^2 g_{\mu\nu}-q_\mu q_\nu) \times \left\{ (Z_3-1) + \frac{2\alpha}{\pi}\int_0^1 dz\, z(1-z) \times \ln\left[1-\frac{q^2}{m^2}z(1-z)\right]\right\} \tag{22}$$

with

$$Z_3 - 1 = -\frac{\alpha}{\pi}\left\{\frac{2}{3}\ln 2 - \frac{2}{3}\gamma - \frac{5}{9} + 2\int_0^1 dz\int d\lambda_1\int d\lambda_2 f(\lambda_1)f(\lambda_2) \times z(1-z)\ln\frac{1}{m^2[\lambda_1(1-z)+\lambda_2 z]}\right\}. \tag{23}$$

If the weighting function $f(\lambda)$ is more or less concentrated at $\lambda \simeq L_*^2$, we can estimate

$$Z_3 - 1 \simeq -\frac{\alpha}{\pi}\ln\frac{1}{mL_*} \simeq -0.1. \tag{24}$$

Thus, the charge renormalization is about 10%. As we will see below, the other renormalizations in QED are also of this order of magnitude.

The gauge noninvariant part of $\Pi_{\mu\nu}$ is

$$\Pi^{(2)}_{\mu\nu}(q) = -\frac{i\alpha}{\pi}g_{\mu\nu}\int_0^1 dz\int d\lambda_1\int d\lambda_2 f(\lambda_1)f(\lambda_2)2K_2(\xi) \times [m^2-q^2z(1-z)]. \tag{25}$$

This term is $\simeq -i\alpha g_{\mu\nu}/L_*^2$ and indicates a large photon mass and a large violation of the conservation of the electron current induced in the vacuum. The finite value of this term represents an improvement as compared with the infinite result found in conventional QED with the conventional Feynman propagator (before regularization). However, even a finite gauge-noninvariant term is unacceptable, and we need to find some way of eliminating or canceling this term. Several possibilities come to mind.

(i) Our smeared propagator is merely a first approximation that takes into account the quantum fluctuations of the geometry only in a rough, average way. The exact propagator is likely to exhibit a more complicated behavior near $x^2=0$, which remains to be discovered by functional integration over all possible geometries as proposed by Deser [8] or by some other means. Any such alteration of the behavior near $x^2=0$ will affect the value of the gauge noninvariant term and possibly eliminate it (the gauge noninvariant term is more sensitive to the behavior at $x^2=0$ than is the gauge invariant term).

(ii) In Eq. (16) for the polarization, we assumed that the two propagators in the integrand are smeared individually, rather than jointly. Since these propagators describe two particles that are created (and later destroyed) at the same spacetime point, the fluctuating geometry will affect both in the same way, at least initially and finally. Accordingly, some kind of joint smearing of the two propagators may be required (this is reminiscent of the joint, overall regularization that Pauli and Villars adopted for their treatment of the polarization).

(iii) The gauge noninvariant term (25) and the consequent violation of electron charge conservation might represent a real physical effect which is compensated by another gauge noninvariant term arising from charged entities such as miniature charged black holes, generated by the quantum fluctuations in the geometry. A miniature black hole (BH) can absorb or emit an electron, $e+(\text{BH of charge } Q)\leftrightarrow(\text{BH of charge } Q-e)$. The distinctive attributes of the electron are lost when it is absorbed by the black hole, that is, in this process the electron ceases to exist and its electric charge is transferred to the black hole. Correspondingly, there is a violation of the electron current conservation, $\partial_\mu j^\mu\neq 0$, and a corresponding violation of gauge invariance for the polarization (25) calculated from the electron current, $q^\mu \Pi^{(2)}_{\mu\nu}\times(q)\neq 0$. However, the total current (the sum of the current of electrons and the current associated with charged black holes) is still conserved and we therefore expect that the polarization calculated from the total current is gauge invariant. This means that besides the electron contribution to the polarization there is an extra contribution from charged black holes, which cancels Eq. (25). A simple estimate indicates that the electron and black-hole contributions are of the same order of magnitude, as required for cancellation. To see this, we evaluate the Fourier transform of Eq. (25) and find the polarization $\Pi_{\mu\nu}(x)$ in coordinate space. The resulting function is $\simeq e^2 g_{\mu\nu}/L_*{}^6$. Since $\Pi_{\mu\nu}(x)\propto\langle T[j_\mu(x)j_\nu(0)]\rangle$, this indicates that the rms vacuum fluctuations of the electron charge density are $\simeq e/L_*{}^3$. Now consider quantum fluctuations in the geometry on a scale L_*. These fluctuations have a mass M_* and since the corresponding Schwarzschild radius is $GM_*=L_*$, most of these fluctuations are in the form of black holes. The density of such black holes is expected to be $\simeq 1/L_*^3$. Emission of electrons and positrons (of energy $\simeq M_*$) by these black holes by the Hawking process leads to charge fluctuations $\pm e$ in volumes $1/L_*^3$, which is indeed of the same order of magnitude as the electron charge fluctuations calculated from the polarization.

(iv) We could try to impose gauge invariance on the polarization by adopting some modification of the photon-electron vertex. However, it seems futile to pursue this ''brute-force'' approach, as long as we do not know how much the quantum fluctuations in the geometry contribute to the solution of the gauge problem via alternatives (i)–(iii). For now, the best we can do is shelve the gauge problem, and hope that it will resolve itself if and when we learn more about the quantum fluctuations in the geometry.

B. Electron self-energy

The electron self-energy can be calculated by similar methods, with the result

$$-i\Sigma=-\frac{\alpha i}{4\pi}\int_0^1 dz\int d\lambda_1\int d\lambda_2 f(\lambda_1)f(\lambda_2)\times[-2\not{p}(1-z)+4m]\times\left\{2\ln 2-2\gamma-\ln\frac{zm^2+z(1-z)p^2}{(1-z)z}-\ln[\lambda_1 z+\lambda_2(1-z)]\right\}. \tag{26}$$

From this we identify the electron wave function renormalization factor,

$$Z_2^{-1}-1=\frac{\alpha}{2\pi}\left\{\frac{1}{2}+\ln 2-\gamma+\frac{1}{2}\int_0^1 dz\int d\lambda_1\int d\lambda_2\times f(\lambda_1)f(\lambda_2)(1-z)\ln\frac{1}{m^2[\lambda_1 z+\lambda_2(1-z)]}+\int_0^1\frac{2(1-z^2)}{z}\,dz\right\} \tag{27}$$

and the mass renormalization

$$\frac{\delta m}{m}=\frac{\alpha}{2\pi}\left\{-\frac{1}{2}+3\ln 2-3\gamma+\int_0^1 dz\int d\lambda_1\int d\lambda_2\times f(\lambda_1)f(\lambda_2)2(1+z)\ln\frac{1}{m^2[\lambda_1 z+\lambda_2(1-z)]}\right\}. \tag{28}$$

If we ignore the usual infrared divergence in Eq. (27), $Z_2^{-1}-1\simeq(\alpha/\pi)\ln 1/mL_*$ and $\delta m/m\simeq(\alpha/\pi)\ln 1/mL_*\simeq 0.1$.

C. Vertex correction

As in conventional QED, the expression for the vertex correction is somewhat complicated, and we give only the zero-order term in q^2:

$$\Lambda_\mu(0)=(Z_1^{-1}-1)\gamma_\mu \tag{29}$$

with a vertex renormalization factor

$$Z_1^{-1}-1=\frac{\alpha}{2\pi}\left\{\ln 2-\gamma+\int_0^1 dz\int_0^{1-z}dz'\int d\lambda_1\int d\lambda_2 \right.$$
$$\times\int d\lambda_3 f(\lambda_1)f(\lambda_2)f(\lambda_3)$$
$$\left.\times\ln\frac{1}{m^2[\lambda_1/z+\lambda_2/z'+\lambda_3/(1-z-z')](z+z')}\right\}$$
$$\simeq\frac{\alpha}{\pi}\ln\frac{1}{mL_*}. \tag{30}$$

In conventional electrodynamics, the exact equality of Z_1 and Z_2 is implied by the Ward identity $\partial\Sigma/\partial p^\mu=-\Lambda_\mu$. According to Eqs. (30) and (27), Z_1 and Z_2 are approximately equal but they are probably not exactly equal [the difference depends on the details of the weighting function $f(\lambda)$]. A failure of the Ward identity would not come as a surprise, since this identity relies on the canonical commutation relation and current conservation, both of which fail for the smeared propagator. The Ward identity is often used to demonstrate that QED can be renormalized to all orders, but it is not indispensable for this demonstration.

IV. OTHER APPLICATIONS AND CONCLUSIONS

If the external momenta are small compared with M_*, the S-matrix elements calculated with smeared propagators agree with conventional QED. However, if the external momenta are of the order of M_*, then the results differ significantly. For instance, the electron-electron scattering cross section differs from the usual Mott formula by factors arising from the smearing of the photon propagator. Thus, at high energies QED retains an explicit dependence on the weighting function, and the effects of the smearing cannot any more be hidden in renormalizations.

The smeared propagator can also be applied to the calculation of the conventionally divergent Feynman diagrams that occur in other field theories, including nonrenormalizable field theories. Both nonrenormalizable and renormalizable theories are rendered finite, but the difference is that the nonrenormalizable theories retain an explicit dependence on the weighting function $f(\lambda)$, whereas renormalizable theories permit us to hide this dependence on $f(\lambda)$ in the renormalizations [except when the external energies are comparable with M_*, when even the renormalizable theories retain an explicit dependence on $f(\lambda)$]. Although the higher-order terms in nonrenormalizable theories will be finite, they will usually be large. The coupling constant of a nonrenormalizable theory has dimensions of some negative power of mass, and the higher-order terms contain a factor of some power of the coupling constant multiplied by some positive power of M_*. Such large terms will have to be compensated by counterterms or by some other fine tuning. Presumably the smeared propagator can also be applied to the naive covariant quantum gravity theory. The fact that the smeared propagator generates an effective cutoff at M_* might serve as a bridge between ‘‘covariant’’ quantization and the more fundamental quantization of the geometry itself. The cutoff makes the covariant theory fade away at the Planck scale, when we enter the domain of quantum geometry.

The smeared propagator can yield finite results for vacuum expectation values of products of fields and products of derivatives of fields, such as occur in calculations of the mass of the Higgs boson and the energy-momentum tensor of the vacuum. For example, for a scalar field,

$$\langle 0|\phi(x)\phi(x')+\phi(x')\phi(x)|0\rangle=-2\,\mathrm{Im}\overline{G_F}(x-x'), \tag{31}$$

$$\partial_\mu\partial'_\nu\langle 0|\phi(x)\phi(x')+\phi(x')\phi(x)|0\rangle$$
$$=-2\partial_\mu\partial'_\nu\,\mathrm{Im}\,\overline{G_F}(x-x'), \tag{32}$$

so the limit $x\to x'$ gives

$$\langle 0|\phi(x)\phi(x)|0\rangle$$
$$=\int d\lambda\, f(\lambda)\left[\frac{1}{4\pi^2\lambda}+\frac{m^2}{8\pi^2}\left(\gamma+\frac{1}{2}+\ln\frac{m\sqrt{\lambda}}{2}\right)\right.$$
$$\left.+\ \ldots\right], \tag{33}$$

$$\langle 0|\partial_\mu\phi(x)\partial_\nu\phi(x)|0\rangle$$
$$=g_{\mu\nu}\int d\lambda\, f(\lambda)\left(-\frac{1}{2\pi^2\lambda^2}+\frac{m^2}{8\pi^2\lambda}+\cdots\right). \tag{34}$$

If the integrals of $f(\lambda)/\lambda$ and $f(\lambda)/\lambda^2$ are finite [17], these vacuum expectation values are finite and of the order of m^2/L_*^2 and $g_{\mu\nu}/L_*^4$, respectively. By contrast, in conventional field theory, these vacuum expectation values diverge quadratically and quartically. Although finite, the vacuum expectation values (33) and (34) are large, and to reduce them to a tolerable level we need some kind of fine tuning or some subtraction procedure. For instance, to bring the value of the vacuum energy momentum tensor into agreement with observational cosmology, we need to cancel the large values (33) and (34) by adopting an (almost) equally large value of the cosmological constant [18]. Thus, smeared propagators do not solve the problem of large vacuum expectation values or, more generally, the problem of large radiative corrections in quantum field theories. But by providing us with finite expressions for these quantities, smeared propagators at least permit us to state more sharply what it is we need to cancel.

In addition to offering a ‘‘realistic’’ and mathematically consistent treatment of the problem of infinities associated with the singularities on the light cone, the properties of smeared propagator suggest some of the modifications to be expected in quantum field theories at the Planck scale. Field equations, Green’s functions, and canonical commutation relations will probably disappear, but vacuum expectation values of field operators and probability amplitudes will perhaps remain.

APPENDIX

We here apply the method of Khriplovich to the case of a massless scalar field. For such a field, the gravitationally modified propagator $g(k^2)$ produced by the sum of ladder diagrams with 0,1,2, . . . , gravitons obeys the recursive equation

$$k^2 g(k^2)k^2 = k^2 - \frac{i\kappa^2}{(2\pi)^4}\int d^4q\left(q^\mu k^\nu - \frac{\eta^{\mu\nu}}{2}\,q\cdot k\right)\left[\frac{1}{p^2}\,(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}) + \frac{a}{p^4}\,(\eta_{\mu\alpha}p_\nu p_\beta+\eta_{\mu\beta}p_\nu p_\alpha+\eta_{\nu\beta}p_\mu p_\alpha+\eta_{\nu\alpha}p_\mu p_\beta)+\frac{b}{p^4}\,(\eta_{\mu\nu}p_\alpha p_\beta+\eta_{\alpha\beta}p_\mu p_\nu) + \frac{c}{p^6}\,(p_\mu p_\nu p_\alpha p_\beta)\right]\left(q^\alpha k^\beta - \frac{\eta^{\alpha\beta}}{2}\,q\cdot k\right)g(q^2). \tag{A1}$$

Here $\kappa^2=16\pi G$, $p=k-q$ is the momentum of the graviton, $(q^\mu k^\nu-\frac{1}{2}\eta^{\mu\nu}q\cdot k)$ is the vertex corresponding to graviton emission according to the linearized theory, and the middle bracket is the graviton propagator in a gauge characterized by arbitrary constants a,b,c. After multiplying out all the terms, we change the contour of integration for q_0 so it lies along $-i\infty$ to $+i\infty$, and we then perform all the integrations over angles in four-dimensional Euclidean space. With $q_0=iq_4$ and $R^2=-q^2=\mathbf{q}^2+q_4^2$, a lengthy calculation yields

$$k^2g(k^2)k^2=k^2+\frac{\pi^2\kappa^2}{(2\pi)^4}\int_0^\infty dR\left[\frac{R^6}{-k^2}\,\theta(-k^2-R^2)+k^4\theta(R^2+k^2)\right]\left[\left(-2-2a+\frac{b}{2}-\frac{c}{8}\right)\frac{R^3}{-k^2}+\left(-\frac{3}{2}\,b-\frac{3}{8}\,c\right)R\right]g(-R^2). \tag{A2}$$

The essential step in the method of Kriplovich is to choose the gauge so that the most highly divergent terms in Eq. (A2) disappear from the equation. This demands $-2-2a+b/2-c/8=0$. If in addition we choose $3b/2+3c/8=1/8$, we obtain

$$x^2g(-x)=-x-\frac{\kappa^2}{(16\pi)^2}\int_0^\infty dy\left[\frac{y^3}{x}\,\theta(x-y)+x^2\theta(y-x)\right]g(-y), \tag{A3}$$

where $x=-k^2$ and $y=R^2$. This integral equation is the same as that obtained by Khriplovich in the spin-$\frac{1}{2}$ case [see his Eq. (14)].

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[17] Whether the integrals are finite or not depends on the behavior of $f(\lambda)$ near zero. Fluctuations on a scale shorter than L_* would have energies larger than M_* and, therefore, Schwarzschild radii *larger* than L_*. This suggests an inconsistency and perhaps indicates that such fluctuations do not occur. Correspondingly, the weighting function perhaps drops off sharply when λ decreases below L_*^2, so $f(\lambda)/\lambda^2$ is integrable. The weighting function (8) for the Khriplovich propagator does *not* conform to this desideratum.
[18] Supersymmetry may be of some help here; it brings about the cancellation of boson and fermion terms of order $1/\lambda^2$ and of order m^2/λ in the vacuum energy. But this still leaves fairly large terms of order m^4.