M theory as a matrix model: A conjecture

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We suggest and motivate a precise equivalence between uncompactified 11-dimensional M theory and the $N=\infty$ limit of the supersymmetric matrix quantum mechanics describing $D0$ branes. The evidence for the conjecture consists of several correspondences between the two theories. As a consequence of supersymmetry the simple matrix model is rich enough to describe the properties of the entire Fock space of massless well separated particles of the supergravity theory. In one particular kinematic situation the leading large distance interaction of these particles is exactly described by supergravity. The model appears to be a nonperturbative realization of the holographic principle. The membrane states required by M theory are contained as excitations of the matrix model. The membrane world volume is a noncommutative geometry embedded in a noncommutative spacetime. [S0556-2821(97)03308-0]

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I. INTRODUCTION

M theory [1] is the strongly coupled limit of type-IIA string theory. In the limit of infinite coupling, it becomes an 11-dimensional theory in a background-infinite flat space. In this paper M theory will always refer to this decompactified limit. We know very little about this theory except for the following two facts. At low energy and large distances, it is described by 11-dimensional supergravity. It is also known to possess membrane degrees of freedom with membrane tension $1/l_P^3$ where l_P is the 11-dimensional Planck length. It seems extremely unlikely that M theory is any kind of conventional quantum field theory. The degrees of freedom describing the short distance behavior are simply unknown. The purpose of this paper is to put forward a conjecture about these degrees of freedom and about the Hamiltonian governing them.

The conjecture grew out of a number of disparate facts about M theory, D branes [2], matrix descriptions of their dynamics [3], supermembranes [4,5,6], the holographic principle [7], and short distance phenomena in string theory [8,9]. Simply stated the conjecture is this. M theory, in the light cone frame, is exactly described by the large N limit of a particular supersymmetric matrix quantum mechanics. The system is the same one that has been used previously used to study the small distance behavior of $D0$ branes [9]. Townsend [10] was the first to point out that the supermatrix formulation of membrane theory suggested that membranes could be viewed as composites of $D0$ branes. Our work is a precise realization of his suggestion.

In what follows we will present our conjecture and some evidence for it. We begin by reviewing the description of string theory in the infinite momentum frame. We then present our conjecture for the full set of degrees of freedom of M theory and the Hamiltonian which governs them. Our strongest evidence for the conjecture is a demonstration that our model contains the excitations which are widely believed to exist in M theory, supergravitons and large metastable classical membranes. These are discussed in Secs. III and V. The way in which these excitations arise is somewhat miraculous, and we consider this to be the core evidence for our conjecture. In Sec. IV we present a calculation of supergraviton scattering in a very special kinematic region and argue that our model reproduces the expected result of low energy supergravity. The calculation depends on a supersymmetric nonrenormalization theorem whose validity we will discuss there. In Sec. VI we argue that our model may satisfy the holographic principle. This raises crucial issues about Lorentz invariance which are discussed there.

We emphasize that there are many unanswered questions about our proposed version of M theory. Nonetheless, these ideas seem of sufficient interest to warrant presenting them here. If our conjecture is correct, this would be the first nonperturbative formulation of a quantum theory which includes gravity.

II. INFINITE MOMENTUM FRAME AND THE HOLOGRAPHIC PRINCIPLE

The infinite momentum frame [11] is the old name for the misnamed light cone frame. Thus far this is the only frame in which it has proved possible to formulate string theory in Hamiltonian form. The description of M theory which we will give in this paper is also in the infinite momentum frame. We will begin by reviewing some of the features of the infinite momentum frame formulation of relativistic quantum mechanics. For a comprehensive review we refer the reader to [11]. We begin by choosing a particular spatial direction x^{11} called the longitudinal direction. The nine-dimensional space transverse to x^{11} is labeled x^i or $x^{\perp}$. Time

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will be indicated by t. Now consider a system of particles with momenta $(p_\perp^a, p_{11}^a)$ where a labels the particle. The system is boosted along the x^{11} axis until all longitudinal momenta are much larger than any scale in the problem. Further longitudinal boosting just rescales all longitudinal momenta in fixed proportion. Quantum field theory in such a limiting reference frame has a number of properties which will be relevant to us.

It is convenient to begin by assuming that the x^{11} direction is compact with a radius R. The compactification serves as an infrared cutoff. Accordingly, the longitudinal momentum of any system or subsystem of quanta is quantized in units of $1/R$. In the infinite momentum frame, all systems are composed of constituent quanta or partons. The partons all carry strictly positive values of longitudinal momentum. It is particularly important to understand what happens to quanta of negative or vanishing p_{11}. The answer is that as the infinite momentum limit is approached, the frequency of these quanta, relative to the Lorentz-time-dilated motion of the boosted system, becomes infinite and the zero and negative momentum quanta may be integrated out. The process of integrating out such fast modes may influence or even determine the Hamiltonian of the remaining modes. In fact, the situation is slightly more complicated in certain cases for the zero momentum degrees of freedom. In certain situations such as spontaneous symmetry breaking, these longitudinally homogeneous modes define backgrounds whose moduli may appear in the Hamiltonian of the other modes. In any case the zero and negative momentum modes do not appear as independent dynamical degrees of freedom.

Thus we may assume all systems have longitudinal momentum given by an integer multiple of $1/R$,

$$p_{11} = N/R, \tag{2.1}$$

with N strictly positive. At the end of a calculation we must let R and N/R tend to infinity to get to the uncompactified infinite momentum limit.

The main reason for the simplifying features of the infinite momentum frame is the existence of a transverse Galilean symmetry which leads to a naive nonrelativistic form for the equations. The role of nonrelativistic mass is played by the longitudinal momentum p_{11}. The Galilean transformations take the form

$$p_i \to p_i + p_{11} v_i . \tag{2.2}$$

As an example of the Galilean structure of the equations, the energy of a free massless particle is

$$E = \frac{p_\perp^2}{2p_{11}}. \tag{2.3}$$

For the 11-dimensional supersymmetric theory we will consider, the Galilean invariance is extended to the super-Galilean group which includes 32 real supergenerators. The supergenerators divide into two groups of 16, each transforming as spinors under the nine-dimensional transverse rotation group. We denote them by Q_α, and q_A, and they obey anticommutation relations

$$[Q_\alpha, Q_\beta]_+ = \delta_{\alpha\beta} H,$$
$$[q_A, q_B]_+ = \delta_{AB} P_{11},$$
$$[Q_\alpha, q_A] = \gamma^i_{A\alpha} P_i . \tag{2.4}$$

The Lorentz generators which do not preserve the infinite momentum frame mix up the two kinds of generators.

Let us now recall some of the features of string theory in the infinite momentum or light cone frame [12]. We will continue to call the longitudinal direction x^{11} even though in this case the theory has only ten space-time directions. The transverse space is of course eight dimensional. To describe a free string of longitudinal momentum p_{11}, a periodic parameter σ which runs from 0 to p_{11} is introduced. To regulate the world sheet theory, a cutoff $\delta\sigma = \epsilon$ is introduced. This divides the parameter space into $N = p_{11}/\epsilon$ segments, each carrying longitudinal momentum ϵ. We may think of each segment as a parton, but unlike the partons of quantum field theory, these objects always carry $p_{11} = \epsilon$. For a multiparticle system of total longitudinal momentum p_{11}(total), we introduce a total parameter space of overall length p_{11}(total), which we allow to be divided into separate pieces, each describing a string. The world sheet regulator is implemented by requiring each string to be composed of an integer number of partons of momentum ϵ. Interactions are described by splitting and joining processes in which the number of partons is strictly conserved. The regulated theory is thus seen to be a special case of Galilean quantum mechanics of N partons with interactions which bind them into long chains and allow particular kinds of rearrangements.

The introduction of a minimum unit of momentum ϵ can be given an interpretation as an infrared cutoff. In particular, we may assume that the x^{11} coordinate is periodic with length $R = \epsilon^{-1}$. Evidently, the physical limit $\epsilon \to 0$, $R \to \infty$ is a limit in which the number of partons N tends to infinity.

It is well known [7] that in this large N limit the partons become infinitely dense in the transverse space and that this leads to extremely strong interactions. This circumstance, together with the Bekenstein bound on entropy, has led to the *holographic* speculation that the transverse density of partons is strictly bounded to about one per transverse Planck area. In other words, the partons form a kind of incompressible fluid. This leads to the unusual consequence that the transverse area occupied by a system of longitudinal momentum p_{11} cannot be smaller than p_{11}/ϵ in Planck units.

The general arguments for the holographic behavior of systems followed from considerations involving the Bekenstein–'t Hooft bound on the entropy of a spatial region [13] and were not specific to string theory. If the arguments are correct, they should also apply to 11-dimensional theories which include gravitation. Thus we should expect that in M theory the radius of a particle such as the graviton will grow with p_{11} according to

$$\rho = \left(\frac{p_{11}}{\epsilon}\right)^{1/9} l_P = (p_{11} R)^{1/9} l_P , \tag{2.5}$$

where l_P is the 11-dimensional Planck length. In what follows we will see quantitative evidence for exactly this behavior.

At first sight the holographic growth of particles appears to contradict the boost invariance of particle interactions. Consider the situation of two low energy particles moving past one another with some large transverse separation, let us say of order a meter. Obviously these particles have negligible interactions. Now boost the system along the longitudinal direction until the size of each particle exceeds their separation. They now overlap as they pass each other. But longitudinal boost invariance requires that the scattering amplitude be still essentially zero. This would seem to require extremely special and unnatural cancellations. We will see below that one key to this behavior is the very special Bogomol'ni-Prasad-Sommerfield (BPS) property of the partons describing M theory. However, we are far from having a complete understanding of the longitudinal boost invariance of our system. Indeed, we view it as the key dynamical puzzle which must be unraveled in understanding the dynamics of M theory.

III. *M* THEORY AND *D*0 BRANES

M theory with a compactified longitudinal coordinate x^{11} is by definition a type-IIA string theory. The correspondences between the two theories include [1] the following.

(1) The compactification radius R is related to the string coupling constant by

$$R = g^{2/3} l_P = g l_s , \tag{3.1}$$

where l_s is the string length scale:

$$l_s = g^{-1/3} l_P . \tag{3.2}$$

(2) The Ramond-Ramond (RR) photon of IIA theory is the Kaluza-Klein (KK) photon which arises upon compactification of 11-dimensional supergravity.

(3) No perturbative string states carry RR charge. In other words, all perturbative string states carry vanishing momentum along the x^{11} direction. The only objects in the theory which do carry RR photon charge are the $D0$ branes of Polchinski. $D0$ branes are point particles which carry a single unit of RR charge. Equivalently, they carry longitudinal momentum

$$P_{11}^{D0 \ \text{branes}} = 1/R. \tag{3.3}$$

The $D0$ branes carry the quantum numbers of the first massive KK modes of the basic 11-dimensional supergravity multiplet, including 44 gravitons, 84 components of a three-form, and 128 gravitinos. We will refer to these particles as supergravitons. As 11-dimensional objects, these are all massless. As a consequence, they are BPS saturated states in the ten-dimensional (10D) theory. Their 10D mass is $1/R$.

(4) Supergravitons carrying Kaluza-Klein momentum $p_{11} = N/R$ also exist, but are not described as elementary $D0$ branes. As shown in [3], their proper description is as bound composites of N $D0$ branes.

These properties make the $D0$ branes candidate partons for an infinite momentum limit description of M theory. We expect that if, as in quantum field theory, the degrees of freedom with vanishing and negative p_{11} decouple, then M theory in the infinite momentum frame should be a theory whose only degrees of freedom are $D0$ branes. Anti-$D0$ branes carry negative Kaluza-Klein momenta, and strings carry vanishing p_{11}. The decoupling of anti-$D0$ branes is particularly fortunate because brane-antibrane dynamics is something about which we know very little [14]. The BPS property of zero branes ameliorates the conflict between infinitely growing parton wave functions and low energy locality, which we noted at the end of the last section. We will see some partial evidence for this in a nontrivial scattering computation below. We will also discuss below the important point that a model containing only $D0$ branes actually contains large classical supermembrane excitations. Since the conventional story of the M-theoretic origin of strings depicts them as membranes wrapped around the compactified 11th dimension, we have some reason to believe that strings have not really been left out of the system.

All of these circumstances lead us to propose that M theory in the infinite momentum frame is a theory in which the only dynamical degrees of freedom or partons are $D0$ branes. Furthermore, it is clear in this case that all systems are built out of the composites of partons, each of which carries the minimal p_{11}. We note, however, that our system does have a set of degrees of freedom which go beyond the parton coordinates. Indeed, as first advocated in [3], the $D0$ brane coordinates of N partons have to be promoted to matrices. At distance scales larger than the 11-dimensional Planck scale, these degrees of freedom become very massive and largely decouple,[1] but their virtual effects are responsible for all parton interactions. These degrees of freedom are BPS states and are related to the parton coordinates by gauge transformations. Furthermore, when the partons are close together, they become low frequency modes. Thus they cannot be omitted in any discussion of the dynamics of $D0$ branes.

IV. *D*0 BRANE MECHANICS

If the infinite momentum limit of M theory is the theory of $D0$ branes, decoupled from the other string theory degrees of freedom, what is the precise form of the quantum mechanics of the system? Fortunately there is a very good candidate which has been extensively studied in another context in which $D0$ branes decouple from strings [9].

As emphasized at the end of the last section, open strings which connect $D0$ branes do not exactly decouple. In fact, the very short strings which connect the branes when they are practically on top of each other introduce a new kind of coordinate space in which the nine spatial coordinates of a system of N $D0$ branes become nine $N \times N$ matrices $X^i_{a,b}$ [3]. The matrices X are accompanied by 16 fermionic superpartners $\theta_{a,b}$, which transform as spinors under the SO(9) group of transverse rotations. The matrices may be thought of as the spatial components of ten-dimensional super Yang-Mills (SYM) fields after dimensional reduction to zero space directions. These Yang-Mills fields describe the open strings which are attached to the $D0$ branes. The Yang-Mills quantum mechanics has U(N) symmetry and is described (in

[1]Indeed, we will propose that this decoupling is precisely what defines the regime in which the classical notion of distance makes sense.

units with $l_s = 1$) by the Lagrangian

$$L = \frac{1}{2g} \{ \text{tr}\ \dot{X}^i \dot{X}^i + 2\theta^T \dot{\theta} + \tfrac{1}{2}\ \text{tr}[X^i, X^j]^2 - 2\theta^T \gamma_i [\theta, X^i] \}. \tag{4.1}$$

Here we have used conventions in which the fermionic variables are 16-component nine-dimensional spinors.

In [9] this Lagrangian was used to study the short distance properties of $D0$ branes in weakly coupled string theory. The 11D Planck length emerged as a natural dynamical length scale in that work, indicating that the system (4.1) describes some M-theoretic physics. In [9], Eq. (4.1) was studied as a low velocity effective theory appropriate to the heavy $D0$ branes of weakly coupled string theory. Here we propose Eq. (4.1) as the most general infinite momentum frame Lagrangian, with at most two derivatives, which is invariant under the gauge symmetry and the super-Galilean group[2] [15]. It would be consistent with our assumption that matrix $D0$ branes are the only degrees of freedom of M theory to write a Lagrangian with higher powers of first derivatives. We do not know if any such Lagrangians exist which preserve the full symmetry of the infinite momentum frame. What is at issue here is 11-dimensional Lorentz invariance. In typical infinite momentum frame field theories, the naive classical Lagrangian for the positive longitudinal momentum modes is renormalized by the decoupled infinite frequency modes. The criterion which determines the infinite momentum frame Lagrangian is invariance under longitudinal boosts and null plane rotating Lorentz transformations (the infamous angular conditions). Apart from simplicity, our main reason for suggesting the Lagrangian (4.1) is that we have found some partial evidence that the large N limit of the quantum theory it defines is indeed Lorentz invariant. A possible line of argument systematically leading to Eq. (4.1) is discussed in Sec. IX.

Following [9], let us rewrite the action in units in which the 11D Planck length is 1. Using Eqs. (3.1) and (3.2), the change of units is easily made and one finds

$$L = \text{tr}\left\{ \frac{1}{2R} D_t Y^i D_t Y^i - \tfrac{1}{4} R [Y^i, Y^j]^2 - \theta^T D_t \theta - R\theta^T \gamma_i [\theta, Y^i] \right\}, \tag{4.2}$$

where $Y = X/g^{1/3}$. We have also changed the units of time to 11D Planck units. We have restored the gauge field ($\partial_t \to D_t = \partial_t + iA$) to this expression (previously we were in the $A=0$ gauge) in order to emphasize that the supersymmetric (SUSY) transformation laws (here ϵ and ϵ' are two independent 16-component anticommuting SUSY parameters)

$$\delta X^i = -2\epsilon^T \gamma^i \theta, \tag{4.3}$$

$$\delta\theta = \tfrac{1}{2}\{ D_t X^i \gamma_i + \gamma_- + \tfrac{1}{2}[X^i, X^j]\gamma_{ij} \}\epsilon + \epsilon', \tag{4.4}$$

$$\delta A = -2\epsilon^T \theta \tag{4.5}$$

involve a gauge transformation. As a result, the SUSY algebra closes on the gauge generators and only takes on the form (2.4) when applied to gauge-invariant states.

The Hamiltonian has the form

$$H = R\ \text{tr}\left\{ \frac{\Pi_i \Pi_i}{2} - \frac{1}{4}[Y^i, Y^j]^2 + \theta^T \gamma_i [\theta, Y^i] \right\}, \tag{4.6}$$

where Π is the canonical conjugate to Y. Note that in the limit $R \to \infty$, all finite energy states of this Hamiltonian have infinite energy. We will be interested only in states whose energy vanishes like $1/N$ in the large N limit, so that this factor becomes the inverse power of longitudinal momentum which we expect for the eigenstates of a longitudinal-boost-invariant system. Thus, in the correct infinite momentum frame limit, the only relevant asymptotic states of the Hamiltonian should be those whose energy is of order $1/N$. We will exhibit a class of such states below, the supergraviton scattering states. The difficult thing will be to prove that their S-matrix elements depend only on ratios of longitudinal momenta, so that they are longitudinally boost invariant.

To understand how this system represents ordinary particles, we note that when the Y's become large the commutator term in H becomes very costly in energy. Therefore for large distances the finite energy configurations lie on the flat directions along which the commutators vanish. In this system with 16 supercharges,[3] these classical zero energy states are in fact exact supersymmetric states of the system. In contrast to field theory, the continuous parameters which describe these states [the Higgs vacuum expectation values (VEV's) in the language of SYM theory] are not vacuum superselection parameters, but rather collective coordinates. We must compute their quantum wave functions rather than freeze them at classical values. They are, however, the slowest modes in the system, so that we can integrate out the other degrees of freedom to get an effective SUSY quantum mechanics of these modes alone. We will study some aspects of this effective dynamics below.

Along the flat directions the Y^i are simultaneously diagonalizable. The diagonal matrix elements are the coordinates of the $D0$ branes. When the Y are small, the cost in energy for a noncommuting configuration is not large. Thus for small distances there is no interpretation of the configuration space in terms of ordinary positions. Classical geometry and distance are only sensible concepts in this system in regions far out along one of the flat directions. We will refer to this as the long distance regime. In the short distance regime, we have a noncommutative geometry. Nevertheless, the full Hamiltonian (4.6) has the usual Galilean symmetry. To see this we define the center of mass of the system by

[2]The gauge invariance is in fact necessary to supertranslation invariance. The supergenerators close on gauge transformations and only satisfy the supertranslation algebra on the gauge-invariant subspace.

[3]The 16 supercharges which anticommute to the longitudinal momentum act only on the center of mass of the system and play no role in particle interactions.

$$Y(\text{c.m.}) = \frac{1}{N}\,\text{tr}Y. \tag{4.7}$$

A transverse translation is defined by adding a multiple of the identity to Y. This has no effect on the commutator term in L because the identity commutes with all Y. Similarly rotational invariance is manifest.

The center-of-mass momentum is given by

$$P(\text{c.m.}) = \text{tr}\Pi = \frac{N}{R}\,\dot{Y}(\text{c.m.}). \tag{4.8}$$

Using $p_{11} = N/R$ gives the usual connection between transverse velocity and transverse momentum:

$$\frac{1}{p_{11}}\,P(\text{c.m.}) = \dot{Y}(\text{c.m.}). \tag{4.9}$$

A Galilean boost is defined by adding a multiple of the identity to $\dot{Y}$. We leave it to the reader to show that this has no effect on the equations of motion. This establishes the Galilean invariance of H. The super-Galilean invariance is also completely unbroken. The alert reader may be somewhat unimpressed by some of these invariances, since they appear to be properties of the center-of-mass coordinate, which decouples from the rest of the dynamics. Their real significance will appear below when we show that our system possesses multiparticle asymptotic states, on which these generators act in the usual way as a sum of single-particle operators.

V. A CONJECTURE

Our conjecture is thus that M theory formulated in the infinite momentum frame is exactly equivalent to the $N \to \infty$ limit of the supersymmetric quantum mechanics described by the Hamiltonian (4.6). The calculation of any physical quantity in M theory can be reduced to a calculation in matrix quantum mechanics followed by an extrapolation to large N. In what follows we will offer evidence for this surprising conjecture.

Let us begin by examining the single-particle spectrum of the theory. For $N=1$ the states with $p_\perp = 0$ are just those of a single $D0$ brane at rest. The states form a representation of the algebra of the 16 θ's with 2^8 components. These states have exactly the quantum numbers of the 256 states of the supergraviton. For nonzero $p_\perp$ the energy of the object is

$$E = \frac{R}{2}\,p_\perp^2 = \frac{p_\perp^2}{2p_{11}}. \tag{5.1}$$

For states with $N > 1$, we must study the U(N)-invariant Schrödinger equation arising from Eq. (4.6). H can easily be separated in terms of center-of-mass and relative motions. Define

$$Y = \frac{Y(\text{c.m.})}{N}\,I + Y_{\text{rel}}, \tag{5.2}$$

where I is the unit matrix and Y_{rel} is a traceless matrix in the adjoint of SU(N) representing relative motion. The Hamiltonian then has the form

$$H = H_{\text{c.m.}} + H_{\text{rel}}, \tag{5.3}$$

with

$$H_{\text{c.m.}} = \frac{P(\text{c.m.})^2}{2p_{11}}. \tag{5.4}$$

The Hamiltonian for the relative motion is the dimensional reduction of the supersymmetric 10D Yang-Mills Hamiltonian. Although a direct proof based on the Schrödinger equation has not yet been given, duality between IIA strings and M theory requires the relative Schrödinger equation to have normalizable threshold bound states with zero energy [3]. The bound system must have exactly the quantum numbers of the 256 states of the supergraviton. For these states the complete energy is given by Eq. (5.4). Furthermore, these states are BPS saturated. No other normalizable bound states can occur. Thus we find that the spectrum of stable single-particle excitations of Eq. (4.6) is exactly the supergraviton spectrum with the correct dispersion relation to describe massless 11-dimensional particles in the infinite momentum frame.

Next let us turn to the spectrum of widely separated particles. That a simple quantum mechanical Hamiltonian like Eq. (4.6) should be able to describe arbitrarily many well-separated particles is not at all evident and would certainly be impossible without the special properties implied by supersymmetry. Begin by considering commuting block diagonal matrices of the form

$$\begin{pmatrix} Y_1^i & 0 & 0 & \cdots \\ 0 & Y_2^i & 0 & \\ 0 & 0 & Y_e^i & \end{pmatrix}, \tag{5.5}$$

where Y_a^i are $N_a \times N_a$ matrices and $N_1 + N_2 + \cdots + N_n = N$. For the moment suppose all other elements of the Y's are constrained to vanish identically. In this case the Schrödinger equation obviously separates into n uncoupled Schrödinger equations for the individual block degrees of freedom. Each equation is identical to the original Schrödinger equation for the system of N_a $D0$ branes. Thus the spectrum of this truncated system includes collections of noninteracting supergravitons.

Now let us suppose the supergravitons are very distant from one another. In other words, for each pair the relative distance, defined by

$$R_{a,b} = \left|\frac{\text{tr}Y_a}{N_a} - \frac{\text{tr}Y_b}{N_b}\right|, \tag{5.6}$$

is asymptotically large. In this case the commutator terms in Eq. (4.6) cause the off-diagonal blocks in the Y's to have very large potential energy proportional to $R_{a,b}^2$. This effect can also be thought of as the Higgs effect giving mass to the broken generators (''W bosons'') of U(N) when the symmetry is broken to U(N_1)×U(N_2)×U(N_3)$\cdots$. Thus one might naively expect the off-diagonal modes to leave the spectrum of very widely separated supergravitons unmodified. However, this is not correct in a generic situation. The off-diagonal modes behave like harmonic oscillators with frequency of order $R_{a,b}$, and their zero point energy will generally give rise to a potential energy of similar magnitude. This effect would certainly preclude an interpretation of the matrix model in terms of well-separated independent particles.

Supersymmetry is the ingredient which rescues us. In a well-known way, the fermionic partners of the off-diagonal bosonic modes exactly cancel the potential due to the bosons, leaving exactly flat directions. We know this from the nonrenormalization theorems for supersymmetric quantum mechanics with 16 supergenerators [15]. The effective Lagrangian for the collective coordinates along the flat directions must be supersymmetric, and the result of [15] guarantees that up to terms involving at most two derivatives the Lagrangian for these coordinates must be the dimensional reduction of $U(1)^n$ SYM theory, where n is the number of blocks (i.e., the number of supergravitons). This is just the Lagrangian for free motion of these particles. Furthermore, since we are doing quantum mechanics and the analogue of the Yang-Mills coupling is the dimensional quantity l_p^3, the coefficient of the quadratic term is uncorrected from its value in the original Lagrangian.

There are residual virtual effects at order p^4 from these heavy states which are the source of parton interactions. Note that the off-diagonal modes are manifestly nonlocal. The apparent locality of low energy physics in this model must emerge from a complex interplay between SUSY and the fact that the frequencies of the nonlocal degrees of freedom become large when particles are separated. We have only a limited understanding of this crucial issue, but in the next section we will provide some evidence that local physics is reproduced in the low energy, long distance limit.

The center of mass of a block of size $N^{(a)}$ is defined by Eq. (5.2). It is easy to see that the Hamiltonian for an asymptotic multiparticle state, when written in terms of center-of-mass transverse momenta, is just

$$H_{\text{asymp}}=\sum_a \frac{R\mathbf{P}^{(a)^2}}{N^{(a)}}=\sum_a \frac{\mathbf{P}^{(a)^2}}{p_{11}^{(a)}}. \tag{5.7}$$

Note that the dispersion relation for the asymptotic particle states has the fully 11-dimensional Lorentz-invariant form. This is essentially due to the BPS nature of the asymptotic states. For large relative separations, the supersymmetric quantum state, corresponding to the supersymmetric classical flat direction in which the gauge symmetry is ‘‘broken’’ into n blocks, will be precisely the product of the threshold bound-state wave functions of each block subsystem. Each individual block is a BPS state. Its dispersion relation follows from the SUSY algebra and is relativistically invariant even when (e.g., for finite N) the full system is not.

We also note that the statistics of multisupergraviton states comes out correctly because of the residual block permutation gauge symmetry of the matrix model. When some subset of the blocks are in identical states, the original gauge symmetry instructs us to mod out by the permutation group, picking up minus signs depending on whether the states are constructed from an odd or even number of Grassmann variables. The spin statistics connection is the conventional one.

Thus the large N matrix model contains the Fock space of asymptotic states of 11-dimensional supergravity, and the free propagation of particles is described in a manner consistent with 11-dimensional Lorentz invariance. The field theory Fock space is, however, embedded in a system which, as we shall see, has no ultraviolet divergences. Particle statistics is embedded in a continuous gauge symmetry. We find the emergence of field theory as an approximation to an elegant finite structure one of the most attractive features of the matrix model approach to M theory.

VI. LONG RANGE SUPERGRAVITON INTERACTIONS

The first uncanceled interactions in the matrix model occur in the effective action at order $\dot{y}^4$ where $\dot{y}$ is the velocity of the supergravitons [9]. These interactions are calculated by thinking of the matrix model as SYM theory and computing Feynman diagrams. At one loop one finds an induced quartic term in the velocities which corresponds to an induced $F_{\mu\nu}^4$ term. The precise term for two $D0$ branes is given by

$$\frac{A[\dot{y}(1)-\dot{y}(2)]^4}{R^3 r^7}, \tag{6.1}$$

where r is the distance between the $D0$ branes and A is a coefficient of order 1, which can be extracted from the results of [9]. This is the longest range term which governs the interaction between the $D0$ branes as r tends to infinity. Thus the effective Lagrangian governing the low energy, long distance behavior of the pair is

$$L=\frac{\dot{y}(1)^2}{2R}+\frac{\dot{y}(2)^2}{2R}-A\,\frac{[\dot{y}(1)-\dot{y}(2)]^4}{R^3 r^7}. \tag{6.2}$$

The calculation is easily generalized to the case of two well-separated groups of N_1 and N_2 branes forming bound states. Keeping only the leading terms for large N (planar graphs), we find

$$L=\frac{N_1\dot{y}(1)^2}{2R}=\frac{N_2\dot{y}(2)^2}{2R}-AN_1N_2\,\frac{[\dot{y}(1)-\dot{y}(2)]^4}{R^3 r^7}. \tag{6.3}$$

To understand the significance of Eq. (6.3), it is first useful to translate it into an effective Hamiltonian. To leading order in inverse powers of r, we find

$$H_{\text{eff}}=\frac{p_\perp(1)^2}{2p_{11}(1)}+\frac{p_\perp(2)^2}{2p_{11}(2)}+A\left[\frac{p_\perp(1)}{p_{11}(1)}-\frac{p_\perp(2)}{p_{11}(2)}\right]^4 \times\frac{p_{11}(1)p_{11}(2)}{r^7 R}. \tag{6.4}$$

From Eq. (6.4) we can compute a scattering amplitude in the Born approximation. Strictly speaking, the scattering amplitude is defined as a 10D amplitude in the compactified theory. However, it contains information about the 11D amplitude in the special kinematic situation where no longitudinal momentum is exchanged. The relation between the

10D amplitude and 11D amplitude at vanishing Δp_{11} is simple. They are essentially the same thing except for a factor of R, which is needed to relate the 10- and 11-dimensional phase space volumes. The relation between amplitudes is $A_{11}=RA_{10}$. Thus from Eq. (6.4) we find the 11D amplitude

$$A\left[\frac{p_{\perp}(1)}{p_{11}(1)}-\frac{p_{\perp}(2)}{p_{11}(2)}\right]^4 \frac{p_{11}(1)p_{11}(2)}{r^7}. \tag{6.5}$$

The expression in Eq. (6.5) is noteworthy for several reasons. First of all, the factor r^{-7} is the 11D Green function (in space) after integration over x^{11}. In other words, it is the scalar Feynman propagator for vanishing longitudinal momentum transfer. Somehow the simple matrix Hamiltonian ''knows'' about massless propagation in 11D spacetime. The remaining momentum-dependent factors are exactly what is needed to make Eq. (6.5) identical to the single (super) graviton exchange diagram[4] in 11D. Even the coefficient is correct. This is closely related to a result reported in [9] where it was shown that the annulus diagram governing the scattering of two $D0$ branes has exactly the same form at very small and very large distances, which can be understood by noting that only BPS states contribute to this process on the annulus. This plus the usual relations between couplings and scales in type-IIA string theory and M theory guarantees that we obtain the correct normalization of 11-dimensional graviton scattering in supergravity (SUGRA). In the weak coupling limit, very long distance behavior is governed by single supergraviton exchange, while the ultrashort distances are governed by the matrix model. In [9] the exact equivalence between the leading interactions computed in these very different manners was recognized, but its meaning was not clear. Now we see that it is an important consistency criterion in order for the matrix model to describe the infinite momentum limit of M theory.

Let us next consider possible corrections to the effective action coming from higher loops. In particular, higher loops can potentially correct the quartic term in velocities. Since our interest lies in the large N limit, we may consider the leading (planar) corrections. Doing ordinary large N counting, one finds that the $\dot{y}^4$ term may be corrected by a factor which is a function of the ratio N/r^3. Such a renormalization by $f(N/r^3)$ could be dangerous. We can consider several cases which differ in the behavior of f as N/r^3 tends to infinity. In the first two, the function tends to zero or infinity. The meaning of this would be that the coupling to gravity is driven either to zero or infinity in the infinite momentum limit. Either behavior is intolerable. Another possibility is that the function f tends to a constant not equal to 1. In this case the gravitational coupling constant is renormalized by a constant factor. This is not supposed to occur in M theory. Indeed, supersymmetry is believed to protect the gravitational coupling from any corrections. The only other possibility is that $f\rightarrow 1$. The simplest way in which this can happen is if there are no corrections at all other than the one-loop term, which we have discussed.

We believe that there is a nonrenormalization theorem for this term which can be proven in the context of SUSY quantum mechanics with 16 generators. The closest thing we have been able to find in the literature is a nonrenormalization theorem for the $F_{\mu\nu}{}^4$ term in the action of ten-dimensional string theory[5] which has been proved by Tseytlin [16]. In the quantum-mechanical context, we believe that it is true and that the scattering of two supergravitons at large transverse distance and zero longitudinal momentum is exactly given in the matrix model by low energy 11D supergravity perturbation theory. Dine [17] has constructed the outlines of an argument which demonstrates the validity of the nonrenormalization theorem.

We have considered amplitudes in which vanishing longitudinal momentum is exchanged. Amplitudes with nonvanishing exchange of p_{11} are more complicated. They correspond to processes in 10D in which RR charge is exchanged. Such collisions involve rearrangements of the $D0$ branes in which the collision transfers $D0$ branes from one group to the other. We are studying such processes, but we have no definitive results as yet.

We have thus presented some evidence that the dynamics of the matrix model respects 11-dimensional Lorentz invariance. If this is correct, then the model reduces exactly to supergravity at low energies. It is clear, however, that it is much better behaved in the ultraviolet than a field theory. At short distances, as shown extensively in [9], restoration of the full matrix character of the variables cuts off all ultraviolet divergences. The correspondence limit by which M theory reduces to supergravity indicates that we are on the right track.

VII. SIZE OF A SUPERGRAVITON

As we have pointed out in Sec. II, the holographic principle requires the transverse size of a system to grow with the number of constituent partons. It is therefore of interest to estimate the size of the threshold bound state describing a supergraviton of longitudinal momentum N/R. According to the holographic principle, the radius should grow like $N^{1/9}$ in 11D Planck units. We will use a mean field approximation in which we study the wave function of one parton in the field of N others. We therefore consider the effective Lagrangian (6.3) for the case $N_1=1$, $N_2=N$. The action simplifies for $N\gg 1$ since in this case the N-particle system is much heavier than the single-particle system. Therefore we may set its velocity to zero. The Lagrangian becomes

$$L_1=\frac{\dot{y}^2}{2R}-N\,\frac{\dot{y}^4}{R^3y^7}, \tag{7.1}$$

[4]In [9] the amplitude was computed for $D0$ branes which have momenta orthogonal to their polarizations (this was not stated explicitly there, but was implicit in the choice of boundary state). The spin dependence of the amplitude is determined by the supersymmetric completion of the v^4/r^7 amplitude, which we have not computed. In principle, this gives another check of 11-dimensional Lorentz invariance. We suspect that the full answer follows by applying the explicit supersymmetries of the light cone gauge to the amplitude we have computed.

[5]We thank C. Bachas for pointing out this theorem to us.

where y refers to the relative coordinate between the two systems. We can remove all N and R dependence from the action $S=\int L_1 dt$ by scaling:

$$y \rightarrow N^{1/9} y, \qquad t \rightarrow \frac{N^{2/9}}{R} t. \tag{7.2}$$

The characteristic length, time, and velocity ($v=\dot{y}$) scales are

$$y \sim N^{1/9}, \qquad t \sim \frac{N^{2/9}}{R}, \qquad v \sim \frac{y}{t} \sim N^{-1/9} R. \tag{7.3}$$

That the size of the bound state wave function scales like $N^{1/9}$ is an indication of the incompressibility of the system when it achieves a density of order one degree of freedom per Planck area. This is in accordance with the holographic principle.

This mean field picture of the bound state, or any other description of it as a simple cluster, makes the problem of longitudinal boost invariance mentioned earlier very concrete. Suppose we consider the scattering of two bound states with N_1 and N_2 constituents, respectively, $N_1 \sim N_2 \sim N$. The mean field picture strongly suggests that scattering will show a characteristic feature at an impact parameter corresponding to the bound-state size $\sim N^{1/9}$. But this is not consistent with longitudinal boosts which take $N_1 \rightarrow \alpha N_1$, $N_2 \rightarrow \alpha N_2$. Boost invariance requires physics to depend only on the ratio N_1/N_2 or, said another way, only on the ratio of the bound-state sizes. This strongly suggests that a kind of scale invariance must be present in the dynamics that is clearly absent in the simple picture discussed above. In the string case the scale-invariant world sheet dynamics is crucial for longitudinal boost invariance.

The possibility that partons might form subclusters within the bound state was ignored in mean field discussion. A preliminary discussion of a hierarchical clustering model with many length scales is presented in Appendix A. Note also that wave functions of threshold bound states are power law behaved.

Understanding the dynamics of these bound states well enough to check longitudinal boost invariance reliably is an important subject for future research.

VIII. MEMBRANES

In order to be the strong coupling theory of IIA string theory, M theory must have membranes in its spectrum. Although in the decompactified limit there are no truly stable finite energy membranes, very-long-lived large classical membranes must exist. In this section we will show how these membranes are described in the matrix model, a result first found[6] in [5]. Townsend [10] first pointed out the connection between the matrix description of $D0$ brane dynamics and the matrix description of membranes, and speculated that a membrane might be regarded as a collective excitation of $D0$ branes. Our conjecture supplies a precise realization of Townsend's idea.

The formulation we will use to describe this connection is a version of the methods introduced in [5,18,19].

Begin with a pair of unitary operators U,V satisfying the relations

$$UV = e^{2\pi i/N} VU, \qquad U^N = 1, \qquad V^N = 1. \tag{8.1}$$

These operators can be represented on an N-dimensional Hilbert space as clock and shift operators. They form a basis for all operators in the space. Any matrix Z can be written in the form

$$Z = \sum_{n,m=1}^{N} Z_{nm} U^m V^m. \tag{8.2}$$

U and V may be thought of as exponentials of canonical variables p and q:

$$U = e^{ip}, \qquad V = e^{iq}, \tag{8.3}$$

where p,q satisfy the commutation relations

$$[q,p] = \frac{2\pi i}{N}. \tag{8.4}$$

From Eq. (8.2) we see that only periodic functions of p and q are allowed. Thus the space defined by these variables is a torus. In fact, there is an illuminating interpretation of these coordinates in terms of the quantum mechanics of particles on a torus in a strong background magnetic field. The coordinates of the particle are p,q. If the field is strong enough, the existence of a large gap makes it useful to truncate the space of states to the finite dimensional subspace of lowest Landau levels. On this subspace the commutation relation (8.4) is satisfied. The lowest Landau wave packets form minimum uncertainty packets which occupy an area $\sim 1/N$ on the torus. These wave packets are analogous to the ''Planckian cells'' which make up quantum phase space. The p,q space is sometimes called the noncommuting torus, the quantum torus, or the fuzzy torus. In fact, for large enough N we can choose other bases of N-dimensional Hilbert space which correspond to the lowest Landau levels of a charged particle propagating on an arbitrary Riemann surface wrapped by a constant magnetic field. For example, in [5], de Wit *et al.* construct the finite dimensional Hilbert space of

[6]We are grateful to M. Green for pointing out this paper to us when a preliminary version of this work was presented at the Santa Barbara Strings '96 conference.

lowest Landau levels on a sphere. This connection between finite matrix models and two-dimensional surfaces is the basis for the fact that the large N matrix model contains membranes. For finite N, the model consists of maps of quantum Riemann surfaces into a noncommuting transverse superspace; i.e., it is a model of a noncommuting membrane embedded in a noncommutative space.[7]

In the limit of large N, the quantum torus behaves more and more like classical phase space. The following correspondences connect the two.

(1) The quantum operators Z defined in Eq. (8.2) are replaced by their classical counterparts. Equation (8.2) becomes the classical Fourier decomposition of a function on phase space.

(2) The operation of taking the trace of an operator goes over to N times the integral over the torus:

$$\mathrm{tr}Z \to N\int Z(p,q)dp\ dq. \tag{8.5}$$

(3) The operation of commuting two operators is replaced by $1/N$ times the classical Poisson brackets:

$$[Z,W] \to \frac{1}{N}[\partial_q Z\ \partial_p W - \partial_q W\ \partial_p Z]. \tag{8.6}$$

We may now use the above correspondence to formally rewrite the matrix model Lagrangian. We begin by representing the matrices Y^i and θ as operator functions $Y^i(p,q)$ and $\theta(p,q)$. Now apply the correspondences to the two terms in Eq. (4.2). This gives

$$L = \frac{p_{11}}{2}\int dp\ dq\ Y^i(p,q)^2 - \frac{1}{p_{11}}\int dp\ dq \times[\partial_q Y^i \partial_p Y^j - \partial_q Y^j \partial_p Y^i]^2 + \text{fermionic terms} \tag{8.7}$$

and a Hamiltonian

$$H = \frac{1}{2p_{11}}\int dp\ dq\ \Pi_{ii}(p,q)^2 + \frac{1}{p_{11}}\int dp\ dq \times[\partial_q Y^i \partial_p Y^j - \partial_q Y^j \partial_p Y^i]^2 + \text{fermionic terms}. \tag{8.8}$$

Equation (8.8) is exactly the standard Hamiltonian for the 11D supermembrane in the light cone frame. The construction shows us how to build configurations in the matrix model which represent large classical membranes. To do so we start with a classical embedding of a toroidal membrane described by periodic functions $Y^i(p,q)$. The Fourier expansion of these functions provides us with a set of coefficients Y^i_{mn}. Using Eq. (8.2), we then replace the classical Y's by operator functions of U,V. The resulting matrices represent the large classical membranes.

[7]Note that it is clear in this context that membrane topology is not conserved by the dynamics. Indeed, for fixed N a given matrix can be thought of as a configuration of many different membranes of different topology. It is only in the large N limit that stable topological structure may emerge in some situations.

If the matrix model membranes described above are to correspond to M-theory membranes, their tensions must agree. Testing this involves keeping track of the numerical factors of order 1 in the above discussion. We present this calculation in Appendix B where we show that the matrix model membrane tension exactly agrees with the M-theory membrane tension. This has also been verified by Berkooz and Douglas [20] using a different technique.

We do not expect static finite energy membranes to exist in the uncompactified limit. Nevertheless, let us consider the conditions for such a static solution. The matrix model equations of motion for static configurations is

$$[Y^i,[Y^j,Y^i]] = 0. \tag{8.9}$$

It is interesting to consider a particular limiting case of an infinite membrane stretched out in the 8,9 plane for which a formal solution of Eq. (8.9) can be found. We first rescale p and q:

$$P = \sqrt{N}p,$$

$$Q = \sqrt{N}q,$$

$$[Q,P] = 2\pi i. \tag{8.10}$$

In the $N\to\infty$ limit, the P,Q space becomes an infinite plane. Now consider the configuration

$$Y^8 = R_8 P,$$

$$Y^9 = R_9 Q, \tag{8.11}$$

with all other $Y^i = 0$. Here R_i is the length of the corresponding direction, which should of course be taken to infinity. Since $[Y^8,Y^9]$ is a c number, Eq. (8.9) is satisfied. Thus we find the necessary macroscopic membranes require by M theory. This stretched membrane has the requisite ''wrapping number'' on the infinite plane. On a general manifold one might expect the matrix model version of the wrapping number of a membrane on a two-cycle to be

$$W = \frac{1}{N}\mathrm{Tr}\omega_{ij}(X^i(p,q))[X^i,X^j], \tag{8.12}$$

where ω is the two-form associated with the cycle. This expression approaches the classical winding number as we take the limit in which Poisson brackets replace commutators.

Another indication that we have found the right representation of the membrane comes from studying the supersymmetry transformation properties of our configuration.[8] The supermembrane should preserve half of the supersymmetries of the model. The SUSY transformation of the fermionic coordinates is

[8]This result was derived in collaboration with Seiberg, along with a number of other observations about supersymmetry in the matrix model, which will appear in a future publication.

$$\delta\theta = (P^i\gamma_i + [X^i, X^j]\gamma_{ij})\epsilon + \epsilon'. \tag{8.13}$$

For our static membrane configuration, $P^i=0$, and the commutator is proportional to the unit matrix; so we can choose ϵ' to make this variation vanish. The unbroken supergenerators are linear combinations of the IMF q_A and Q_α.

It is interesting to contemplate a kind of duality and complementarity between membranes and $D0$ branes. According to the standard light cone quantization of membranes, the longitudinal momentum p_{11} is uniformly distributed over the area of the p,q parameter space. This is analogous to the uniform distribution of p_{11} along the σ axis in string theory. As we have seen, the p,q space is a noncommuting space with a basic indivisible quantum of area. The longitudinal momentum of such a unit cell is $1/N$ of the total. In other words, the unit phase space cells that result from the noncommutative structure of p,q space are the $D0$ branes with which we began. The $D0$ branes and membranes are dual to one another. Each can be found in the theory of the other.

The two kinds of branes also have a kind of complementarity. As we have seen, the configurations of the matrix model which have classical interpretations in terms of $D0$ branes are those for which the Y's commute. On the other hand, the configurations of a membrane which have a classical interpretation are the extended membranes of large classical area. The area element is the Poisson bracket which in the matrix model is the commutator. Thus the very classical membranes are highly nonclassical configurations of $D0$ branes.

In the paper of de Wit, Luscher, and Nicolai [6], a pathology of membrane theory was reported. It was found that the spectrum of the membrane Hamiltonian is continuous. The reason for this is the existence of the unlifted flat directions along which the commutators vanish. Previously, it had been hoped that membranes would behave like strings and have discrete level structure and perhaps be the basis for a perturbation theory which would generalize string perturbation theory. In the present context this apparent pathology is exactly what we want. M theory has no small coupling analogous to the string splitting amplitude. The bifurcation of membranes when the geometry degenerates is expected to be an order-1 process. The matrix model, if it is to describe all of M theory, must inextricably contain this process. In fact, we have seen how important it is that supersymmetry maintains the flat directions. A model of a single noncommutative membrane actually contains an entire Fock space of particles in flat 11-dimensional space-time.

Another pathology of conventional membrane theories which we expect to be avoided in M theory is the nonrenormalizability of the membrane world volume field theory. For finite N, it is clear that ultraviolet divergences on the world volume are absent because the noncommutative nature of the space defines a smallest volume cell, just like a Planck cell in quantum-mechanical phase space (but we should emphasize here that this is a classical rather than quantum-mechanical effect in the matrix model). The formal continuum limit which gives the membrane Hamiltonian is clearly valid for describing the classical motion of a certain set of metastable semiclassical states of the matrix model. It should not be expected to capture the quantum mechanics of the full large N limit. In particular, it is clear that the asymptotic supergraviton states would look extremely singular and have no real meaning in a continuum membrane formalism. We are not claiming here to have a proof that the large N limit of the matrix quantum mechanics exists, but only that the issues involved in the existence of this limit are not connected to the renormalizability of the world volume field theory of the membrane.

There is one last point worth making about membranes. It involves evidence for 11D Lorentz invariance of the matrix model. We have considered in some detail the Galilean invariance of the infinite momentum frame and found that it is satisfied. But there is more to the Lorentz group. In particular, there are generators J^i which in the light cone formalism rotate the lightlike surface of initial conditions. The conditions for invariance under these transformations are the notorious angular conditions. We must also impose longitudinal boost invariance. The angular conditions are what makes Lorentz invariance so subtle in light cone string theory. It is clearly important to determine if the matrix model satisfies the angular conditions in the large N limit. In the full quantum theory the answer is not yet clear, but at the level of the classical equations of motion the answer is yes. The relevant calculations were done by de Wit, Marquard, and Nicolai [21]. The analysis is too complicated to repeat here, but we can describe the main points.

The equations for classical membranes can be given in covariant form in terms of a Nambu-Goto-type action. In the covariant form the generators of the full Lorentz group are straightforward to write down. In passing to the light cone frame, the expressions for the nontrivial generators become more complicated, but they are quite definite. In fact, they can be expressed in terms of the $Y(p,q)$ and their canonical conjugates $\Pi(p,q)$. Finally, using the correspondence between functions of p,q and matrices, we are led to matrix expressions for the generators. The expressions, of course, have factor-ordering ambiguities, but these, at least formally, vanish as $N\rightarrow\infty$. In fact, according to [21], the violation of the angular conditions goes to zero as $1/N^2$. Needless to say, a quantum version of this result would be very strong evidence for our conjecture.

We cannot refrain from pointing out that the quantum version of the arguments of [21] is apt to be highly nontrivial. In particular, the classical argument works for every dimension in which the classical supermembrane exists, while, by analogy with perturbative string theory, we only expect the quantum Lorentz group to be recovered in 11 dimensions. Further, the longitudinal boost operator of [21] is rather trivial and operates only on a set of zero mode coordinates, which we have not included in our matrix model. Instead, we expect the longitudinal boost generator to involve rescaling N in the large N limit and, thus, to relate the Hilbert spaces of different SUSY quantum-mechanical models. We have already remarked in the previous section that, as anticipated in [7], longitudinal boost invariance is the key problem in our model. We expect it to be related to a generalization of the conformal invariance of perturbative string theory.

IX. TOWARDS DERIVATION AND COMPACTIFICATION

In this section we would like to present a line of argument which may lead to a proof of the conjectured equivalence between the matrix model and M theory. It relies on a stringy extension of the conjectured nonrenormalization theorem combined with the possibility that all velocities in the large N cluster go to zero as $N\to\infty$.

Imagine that R stays fixed as $N\to\infty$. Optimistically, one might imagine that finite R errors are as small as in perturbative 11D supergravity, meaning that they are suppressed by powers of $k_{11}R$ or even $\exp(-k_{11}R)$ where k_{11} is the center-of-mass longitudinal momentum transfer. So for $k_{11}\sim 1/l_P$ we could imagine R fixed at a macroscopic scale and have very tiny errors. The mean field estimates discussed in Sec. VII give the velocity $v\sim N^{-1/9}R$, which, with R fixed, can be made arbitrarily small at large N. Although it is likely that the structure of the large N cluster is more complicated than the mean field description, it is possible that this general property of vanishing velocities at large N continues to hold. In particular, in Appendix A we present arguments that the velocities of the coordinates along some of the classical flat directions of the potential are small. We suspect that this can be generalized to all of the flat directions. If that is the case, then the only high velocities would be those associated with the ''core'' wave function of Appendix A. The current argument assumes that the amplitude for the core piece of the wave function vanishes in the large N limit.

Non-Abelian field strength is the correct generalization of velocity for membrane-type field configurations like those discussed in Sec. VIII. For classical configurations at least these field strengths are order $1/N$ and so are also small.

We have previously conjectured that the v^4 terms in the quantum-mechanical effective action are not renormalized beyond one loop. For computing the 11-dimensional supergravity amplitude, we needed this result in the matrix quantum mechanics, but is possible that this result holds in the full string perturbation theory. For example, the excited open string states can be represented as additional non-BPS fields in the quantum mechanics. These do not contribute to the one loop v^4 term because they are not BPS. Perhaps they do not contribute to higher loops for related reasons.

If these two properties hold, then the conjecture follows. The scattering of large N clusters of $D0$ branes can clearly be computed at small g (small R) using quantum mechanics. But these processes, by assumption, only involve low velocities independent of g and so only depend on the v^4 terms in the effective action which, by the stringy extension of the nonrenormalization theorem, would not receive g corrections. So the same quantum-mechanical answers would be valid at large g (large R).

This would prove the conjecture.

From this point of view, we have identified a subset of string theory processes (large-N $D0$ brane scattering) which are unchanged by stringy loop corrections and so are computable at strong coupling.

If this line of argument is correct, it gives us an unambiguous prescription for compactification. We take the quantum mechanics which describes 0-brane motion at weak string coupling in the compactified space and then follow it to strong coupling. This approach to compactification requires us to add extra degrees of freedom in the compactified theory. We will discuss an alternative approach in the next section.

For toroidal compactifications, it is clear at weak coupling that one needs to keep the strings which wrap any number of times around each circle. These unexcited wrapped string states are BPS states, and so they do contribute to the v^4 term and hence must be kept. In fact, these are the states which, in the annulus diagram, correct the power law in graviton scattering to its lower dimensional value and are crucial in implementing the various T dualities.

To be specific let us discuss the case of one coordinate X^9 compactified on a circle on radius R_9. Here we should keep the extra string winding states around X^9. An efficient way to keep track of them is to T dualize the X^9 circle. This converts $D0$ branes to $D1$ branes and winding modes to momentum modes. The collection of N $D1$ branes is described by a $(1+1)$-dimensional SU(N) super Yang-Mills quantum field theory with coupling $g^2_{\mathrm{SYM}}=R^2/(R_9 l_P^3)$ on a space of T-dual radius $R_{\mathrm{SYM}}=l_P^3/(RR_9)$. The dimensionless effective coupling of the super Yang-Mills theory is then $g^2_{\mathrm{SYM}}R^2_{\mathrm{SYM}}=(l_P/R_9)^3$, which is independent of R. For p-dimensional tori we get systems of Dp branes described by $(p+1)$-dimensional SU(N) super Yang-Mills theory. Related issues have also recently been discussed in [22].

For more general compactifications the rule would be to keep every BPS state which contributes to the v^4 term at large N. We are currently investigating such compactifications, including ones with less supersymmetry.

The line of argument presented in this section raises a number of questions. Is it permissible to hold R finite or to let it grow very slowly with N? Are there nonperturbative corrections to the v^4 term? Large N probably prohibits instanton corrections in the quantum mechanics, but perhaps not in the full string theory. This might be related to the effect of various wrapped branes in compactified theories.

Does the velocity stay low? A key problem here is that in the mean field theory cloud the 0 branes are moving very slowly. If two 0 branes encounter each other, their relative velocity is much less than the typical velocity in a bound pair $(v\sim R)$. It seems that the capture cross section to go into the pair bound state should be very large. Why is there no clumping into pairs? One factor which might come into play is the following. If the velocity is very low, the de Broglie wavelength of the particles might be comparable to the whole cluster (this is true in the mean field), and so there could be delicate phase correlations across the whole cluster—some kind of macroscopic quantum coherence. Whenever a pair is trying to form, another 0 brane might get between them and disrupt them. This extra coherent complexity might help explain the Lorentz invariance puzzle.

X. ANOTHER APPROACH TO COMPACTIFICATION

The conjecture which we have presented refers to an exact formulation of M theory in uncompactified 11-dimensional space-time. It is tempting to imagine that we can regain the compactified versions of the theory as particular collections of states in the large N limit of the matrix model. There is ample ground for suspicion that this may not be the case and that degrees of freedom that we have thrown away in the uncompactified theory may be required for compactification. Indeed, in IMF field theory the only general method for discussing theories with moduli spaces of vacua is implementable only when the vacua are visible in the classical approximation. Then we can shift the fields and do IMF quantization of the shifted theory. Different vacua correspond to different IMF Hamiltonians for the same degrees of freedom. The proposal of the previous section is somewhat in the spirit of IMF field theory. Different Hamiltonians and, indeed, different sets of degrees of freedom are required to describe each compactified vacuum.

We have begun a preliminary investigation of the alternative hypothesis, that different compactifications are already present in the model we have defined. This means that there must be collections of states which, in the large N limit, have S matrices which completely decouple from each other. Note that the large N limit is crucial to the possible existence of such superselection sectors. The finite N quantum mechanics cannot possibly have superselection rules. Thus the only way in which we could describe compactifications for finite N would be to add degrees of freedom or change the Hamiltonian. We caution the reader that the approach we will describe below is very preliminary and highly conjectural.

This approach to compactification is based on the idea that there is a sense in which our system defines a single "noncommuting membrane." Consider compactification of a membrane on a circle. Then there are membrane configurations in which the embedding coordinates do not transform as scalars under large diffeomorphisms of the membrane volume, but rather are shifted by large diffeomorphisms of the target space. These are winding states. A possible approach to identifying the subset of states appropriate to a particular compactification is to first find the winding states and then find all states which have nontrivial scattering from them in the large N limit. In fact, our limited study below seems to indicate that all relevant states, including compactified supergravitons, can be thought of as matrix model analogues of membrane winding states.

Let us consider compactification of the ninth transverse direction on a circle of radius $2\pi R_9$. A winding membrane is a configuration which satisfies

$$X^9(q,p+2\pi)=X^9(q,p)+2\pi R_9, \tag{10.1}$$

and the winding sector is defined by a path integral over configurations satisfying this boundary condition. A matrix analogue of this is

$$e^{-iNq}X^9e^{iNq}=X^9+2\pi R_9. \tag{10.2}$$

It is easy to see (by taking the trace) that this condition cannot be satisfied for finite N. However, if we take the large N limit in such a way that $q\to\sigma/N\otimes\mathbf{1}_{M\times M}$, with σ an angle variable, then this equation can be satisfied, with

$$X^9=\frac{2\pi R_9}{i}\frac{\partial}{\partial\sigma}\otimes\mathbf{1}_{M\times M}+x^9(\sigma), \tag{10.3}$$

where x^9 is an $M\times M$-matrix-valued function of the angle variable. The other transverse bosonic coordinates and all of the θ's are $M\times M$-matrix-valued functions of σ. These equations should be thought of as limits of finite matrices. Thus $2\pi R_9\mathcal{P}\equiv(2\pi R_9/i)\partial/\partial\sigma$ can be thought of as the limit of the finite matrices diag$(-2\pi PR_9\ldots2\pi PR_9)$, with σ the obvious tridiagonal matrix in this representation. The total longitudinal momentum of such a configuration is $(2P+1)M/R$, and the ratio M/P is an effectively continuous parameter characterizing the states in the large N limit. We are not sure of the meaning of this parameter.

To get a feeling for the physical meaning of this proposal, we examine the extreme limits of large and small R_9. For large R_9 it is convenient to work in the basis where $\mathcal{P}$ is diagonal. If we take all of the coordinates X^i independent of σ, then our winding membrane approaches a periodic array of $(2P+1)$ collections of $D0$ branes, each with longitudinal momentum M/R. We can find a solution of the BPS condition by putting each collection into the M zero-brane threshold bound-state wave function. For large R_9 configurations of the X^i which depend on σ have very high frequency and can be integrated out. Thus, in this limit, the BPS state in this winding sector is approximately a periodic array of supergravitons. We identify this with the compactified supergraviton state. This state will have the right long range gravitational interactions (at scales larger than R_9) in the eight uncompactified dimensions. To obtain the correct decompactified limit, it would appear that we must rescale R, the radius of the longitudinal direction by $R\to(2P+1)R$, as we take P and M to infinity. With this rescaling, all trace of the parameter M/P seems to disappear in the decompactification limit.

For small R_9, our analysis is much less complete. However, string duality suggests an approximation to the system in which we keep only configurations with $M=1$ and $P\to\infty$. In this case the σ dependence of X^9 is pure gauge and the X^i all commute with each other. The matrix model Hamiltonian becomes the Hamiltonian of the Green-Schwarz type-IIA string:

$$H\to\int d\sigma(\dot{\mathbf{X}})^2+\left(\frac{\partial\mathbf{X}}{\partial\sigma}\right)^2+\theta^T\gamma_9\frac{\partial\theta}{\partial\sigma}. \tag{10.4}$$

As in previous sections, we will construct multiwound membrane states by making large block diagonal matrices, each block of which is the previous single-particle construction. Lest such structures appear overly baroque, we remind the reader that we are trying to make explicit constructions of the wave functions of a strongly interacting system with an infinite number of degrees of freedom. For large R_9 it is fairly clear that the correct asymptotic properties of multiparticle states will be guaranteed by the BPS condition (assuming that everything works as conjectured in the uncompactified theory).

If our ansatz is correct for small R_9, it should be possible to justify the neglect of fluctuations of the matrix variables away from the special forms we have taken into account, as well as to show that the correct string interactions (for multistring configurations defined by the sort of block diagonal construction we have used above) are obtained from the matrix model interactions. In this connection it is useful to note that in taking the limit from finite matrices, there is no meaning to the separation of configurations into winding sectors which we have defined in the formal large N limit. In particular, X^9 should be allowed to fluctuate. But we have seen that shifts of X^9 by functions of σ are pure gauge, so that all fluctuations around the configurations which we have kept give rise to higher derivative world sheet interactions. Since the $P\to\infty$ limit is the world sheet continuum limit, we should be able to argue that these terms are irrelevant operators in that limit. We have less understanding about how the sum over world sheet topologies comes out of our formalism, but it is tempting to think that it is in some way connected with the usual topological expansion of large N matrix models. In the Appendix we show that in 11 dimensions, dimensional analysis guarantees the dominance of planar graphs in certain calculations. Perhaps, in ten dimensions, the small dimensionless parameter, R_9/l_P must be scaled with a power of N in order to obtain the limit of the matrix model which gives IIA string theory.

These ideas can be extended to compactification on multidimensional tori. A wrapping configuration of a toroidal membrane can be characterized by describing the cycles on the target torus on which the a and b cycles of the membrane are mapped. This parametrization is redundant because of the SL(2,Z) modular invariance which exchanges the two membrane cycles. We propose that the analogue of these wrapping states, for a d-torus defined by modding out R^d by the shifts $X^i \rightarrow X^i + 2\pi R^i_a$, is defined by the conditions

$$e^{-i\sigma}X^i_{(\mathbf{m},\mathbf{n})}e^{i\sigma} = X^i_{(\mathbf{m},\mathbf{n})} + 2\pi R^i_a n^a, \qquad (10.5)$$

$$e^{2\pi i\mathcal{P}}X^i_{(\mathbf{m},\mathbf{n})}e^{-2\pi i\mathcal{P}} = X^i_{(\mathbf{m},\mathbf{n})} + 2\pi R^i_a m^a, \qquad (10.6)$$

where $\mathbf{n}$ and $\mathbf{m}$ are d-vectors of integers. The solutions to these conditions are

$$X^i_{(\mathbf{m},\mathbf{n})} = [2\rho R^i_a n^a \mathcal{P} + R^i_a m^a \sigma] \otimes \mathbf{1}_{M\times M} + x^i_{(\mathbf{m},\mathbf{n})}(\sigma), \qquad (10.7)$$

where the x^i are periodic $M\times M$-matrix-valued functions of σ. The fermionic and noncompact coordinates are also matrix-valued functions of σ.

In order to discuss more complicated compactifications, we would have to introduce coordinates and find a group of large diffeomorphisms associated with one and two cycles around which membranes can wrap. Then we would search for embeddings of this group into the large N gauge group. Presumably, different coordinate systems would correspond to unitarily equivalent embeddings. We can even begin to get a glimpse of how ordinary Riemannian geometry would emerge from the matrix system. If we take a large manifold which breaks sufficient supersymmetries, the effective action for supergravitons propagating on such a manifold would be obtained, as before, by integrating over the off-diagonal matrices. Now, however, the nonrenormalization theorem would fail and the kinetic term for the gravitons would contain a metric. The obvious conjecture is that this is the usual Riemannian metric on the manifold in question. If this is the case, our prescription for compactification in the noncommutative geometry of the matrix model would reduce to ordinary geometry in the large radius limit.

A question which arises is whether the information about one and two cycles is sufficient to characterize different compactifications. We suspect that the answer to this is no. The moduli of the spaces that arise in string-theoretic compactifications are all associated with the homology of the space, but in general higher dimensional cycles (e.g., three cycles in Calabi-Yau threefolds) are necessary to a complete description of the moduli space. Perhaps in order to capture this information we will have to find the correct descriptions of five branes in the matrix model. If the theory really contains low energy SUGRA, then it will contain solitonic five branes, but it seems to us that the correct prescription is to define five branes as the D branes of membrane theory. We do not yet understand how to introduce this concept in the matrix model.

Finally, we would like to comment on the relation between the compactification schemes of this and the previous sections. For a single circle, if we take P to infinity and substitute the formula (10.3) into the matrix model Hamiltonian (as well as the prescription that all other coordinates and supercoordinates are functions of σ), then we find the Hamiltonian of $(1+1)$-dimensionally reduced 10 D SYM theory in $A_0 = 0$ gauge, with x^9 playing the role of the spatial component of the vector potential. Thus the prescription of the previous section appears to be a particular rule for how the large N limit should be taken in the winding configurations we have studied here. P is taken to infinity first, and then M is taken to infinity. The relation between the two approaches is reminiscent of the Eguchi-Kawai [23] reduction of large N gauge theory. It is clear once again that much of the physics of the matrix model is buried in the subtleties of the large N limit. For multidimensional tori, the relationship between the formalisms of this and the previous section is more obscure.

XI. CONCLUSIONS

Although the evidence we have given for the conjectured exact equivalence between the large N limit of supersymmetric matrix quantum mechanics and uncompactified 11-dimensional M theory is not definitive, it is quite substantial. The evidence includes the following.

(1) The matrix model has exact invariance under the super-Galilean group of the infinite momentum frame description of 11D Lorentz-invariant theories.

(2) Assuming the conventional duality between M theory and IIA string theory, the matrix model has normalizable marginally bound states for any value of N. These states have exactly the quantum numbers of the 11D supergraviton multiplet. Thus the spectrum of single-particle states is exactly that of M theory.

(3) As a consequence of supersymmetric nonrenormalization theorems, asymptotic states of any number of noninteracting supergravitons exist. These well-separated particles propagate in a Lorentz-invariant manner in 11 dimensions. They have the statistics properties of the supergravity Fock space.

(4) The matrix model exactly reproduces the correct long range interactions between supergravitons implied by 11D supergravity, for zero longitudinal momentum exchange. This one-loop result could easily be ruined by higher loop effects proportional to four powers of velocity. We believe that a highly nontrivial supersymmetry theorem protects us against all higher loop corrections of this kind.

(5) By examining the pieces of the bound-state wave function in which two clusters of particles are well separated from each other, a kind of mean field approximation, we find that the longest range part of the wave function grows with N exactly as required by the holographic principle. In par-

ticular, the transverse density never exceeds one parton per Planck area.

(6) The matrix model describes large classical membranes as required by M theory. The membrane world volume is a noncommutative space with a fundamental unit of area analogous to the Planck area in phase space. These basic quanta of area are the original $D0$ branes from which the matrix model was derived. The tension of this matrix model membrane is precisely the same as that of the M-theory membrane.

(7) At the classical level the matrix model realizes the full 11D Lorentz invariance in the large N limit.

Of course, many unanswered questions remain. Locality is extremely puzzling in this system. Longitudinal boost invariance, as we have stressed earlier, is very mysterious. Resolving this issue, perhaps by understanding the intricate dynamics it seems to require, will be crucial in deciding whether or not this conjecture is correct.

One way of understanding Lorentz invariance would be to search for a covariant version of the matrix model in which the idea of noncommutative geometry is extended to all of the membrane coordinates. An obvious idea is to consider functions of angular momentum operators and try to exploit the connection between spin networks and three-dimensional diffeomorphisms. Alternatively, one could systematically study quantum corrections to the angular conditions.

It is likely that more tests of the conjecture can be performed. In particular, it should be possible to examine the large distance behavior of amplitudes with nonvanishing longitudinal momentum transfer and to compare them with supergravity perturbation theory.

It will be important to to try to make precise the line of argument outlined in Sec. IX that may lead to a proof of the conjecture. The approaches to compactification discussed in Secs. IX and X should be explored further.

If the conjecture is correct, it would provide us with the first well-defined nonperturbative formulation of a quantum theory which includes gravitation. In principle, with a sufficiently large and fast computer any scattering amplitude could be computed in the finite N matrix model with arbitrary precision. Numerical extrapolation to infinite N is in principle, if not in practice, possible. The situation is much like that in QCD where the only known definition of the theory is in terms of a conjectured limit of lattice gauge theory. Although the practical utility of the lattice theory may be questioned, it is almost certain that an extrapolation to the continuum limit exists. The existence of the lattice gauge Hamiltonian formulation ensures that the theory is unitary and gauge invariant.

One can envision the matrix model formulation of M theory playing a similar role. It would, among other things, ensure that the rules of quantum mechanics are consistent with gravitation. Given that the classical long distance equations of 11D supergravity have black hole solutions, a Hamiltonian formulation of M theory would, at last, lay to rest the claim that black holes lead to a violation of quantum coherence

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APPENDIX A

In this appendix we will report on a preliminary investigation of the threshold bound-state wave function of N zero branes in the large N limit. In general, we may expect a finite probability for the N brane bound state to consist of p clusters of $N_1,\dots,N_p$ branes separated by large distances along one of the flat directions of the potential. We will try to take such configurations into account by writing a recursion relation relating the N cluster to a k and $N-k$ cluster. This relation automatically incorporates multiple clusters since the pairs into which the original cluster is broken up will themselves contain configurations in which they are split up into further clusters. There may, however, be multiple cluster configurations which cannot be so easily identified as two such superclusters. We will ignore these for now, in order to get a first handle on the structure of the wave function.

The configuration of a pair of widely separated clusters has a single collective coordinate whose Lagrangian we have already written in our investigation of supergraviton scattering. The Lagrangian is

$$L=\frac{1}{2}\,\frac{k(N-k)}{N}\,v^2+\frac{k(N-k)}{r^7}\,v^4, \tag{A1}$$

where r is the distance between the clusters and v is their relative velocity. By scaling, we can write the solution of this quantum-mechanical problem as $\phi\{r(k[N-k])^{2/9}/N^{1/3}\}$, where ϕ is the threshold bound-state wave function of the Lagrangian

$$\frac{1}{2}\,v^2+\frac{v^4}{r^7}. \tag{A2}$$

This solution is valid when $r \gg l_P$.

We are now motivated to write the recursion relation

$$\Psi_N=\Psi_N^{(c)}+\frac{1}{2}P\sum_{k=1}^{N-1}A_{N,k}\Psi_k\Psi_{N-k}\times\phi\left(\frac{r(k[N-k])^{2/9}}{N^{1/3}}\right)e^{-r\,\mathrm{Tr}W_k^\dagger W_k}. \quad \text{(A3)}$$

Here we have chosen a gauge in order to make a block-diagonal splitting of our matrices. Ψ_j is the exact normalized threshold bound-state wave function for j zero branes. W_k are the off-diagonal $k\times N-k$ matrices which generate interactions between the two clusters. P is the gauge-invariant projection operator which rotates our gauge choice among all gauge-equivalent configurations. The $A_{N,k}$ are normalization factors, which in principle we would attempt to find by solving the Schrödinger equation. $\Psi_N^{(c)}$ is the ''core'' wave function, which describes configurations in which all of the zero branes are at a distance less than or equal to l_P from each other. We will describe some of its properties below. In this regime, the entire concept of distance breaks down, since the noncommuting parts of the coordinates are as large as the commuting ones.

The interesting thing which is made clear by this ansatz is that the threshold bound state contains a host of internal distance scales, which becomes a continuum as $N\rightarrow\infty$. This suggests a mechanism for obtaining scale-invariant behavior for large N, as we must if we are to recover longitudinal boost invariance. Note that the typical distance of cluster separation is largest as N goes to infinity when one of the clusters has only a finite number of partons. These are the configurations which give the $N^{1/9}$ behavior discussed in the text, which saturates the Bekenstein bound. By the uncertainty principle, these configurations have internal frequencies of the bound state $\sim N^{-2/9}$. Although these go to zero as N increases, they are still infinitely higher than the energies of supergraviton motions and interactions, which are of order $1/N$. As in perturbative string theory, we expect that this association of the large distance part of the wave function with modes of very high frequency will be crucial to a complete understanding of the apparent locality of low energy physics.

As we penetrate further in to the bound state, we encounter clusters of larger and larger numbers of branes. If we look for separated clusters carrying finite fractions of the total longitudinal momentum, the typical separation falls as N increases. Finally, we encounter the core $\Psi_N^{(c)}$, which we expect to dominate the ultimate short distance and high energy behavior of the theory in noncompact 11-dimensional spacetime.

It is this core configuration to which the conventional methods of large N matrix models, which have so far made no appearance in our discussion, apply. Consider first gauge-invariant Green's functions of operators like $\mathrm{Tr}X_i^{2k}$, where i is one of the coordinate directions. We can construct a perturbation expansion of these Green's functions by conventional functional integral methods. When the time separations of operators are all short compared to the 11-dimensional Planck time, the terms in this expansion are well behaved. We can try to resum them into a large N series. The perturbative expansion parameter (the analogue of g_{YM}^2 if we think of the theory as dimensionally reduced Yang-Mills theory) is $R^3/l_P^6E^3$. Thus the planar Green's functions are functions of $R^3N/l_P^6E^3$.

The perturbative expansion, of course, diverges term by term as $E\rightarrow 0$. If we imagine that, as suggested by our discussion above, these Green's functions should be thought of as measuring properties of the core wave function of the system, there is no physical origin for such an infrared divergence. If, as in higher dimensions, the infrared cutoff is found already in the leading order of the $1/N$ expansion, then it must be of order $\omega_c\sim Rl_P^{-2}N^{1/3}$. Note that this is much larger than any frequency encountered in our exploration of the parts of the wave function with clusters separated along a flat direction.

Now let us apply this result to the computation of the infrared-divergent expectation values of single gauge-invariant operators in the core of the bound-state wave function. The idea is to evaluate the graphical expansion of such an expression with an infrared cutoff and then insert the above estimate for the cutoff to obtain the correct large N scaling of the object. The combination of conventional large N scaling and dimensional analysis then implies that planar graphs dominate even though we are not taking the ''gauge coupling'' $R^3l_P^{-6}$, to zero as we approach the large N limit. Dimensional analysis controls the otherwise unknown behavior of the higher order corrections in this limit. The results are

$$\left\langle\frac{1}{N}\,\mathrm{tr}X^{2k}\right\rangle\sim N^{2k/3}, \quad \text{(A4)}$$

$$\left\langle\frac{1}{N}\,\mathrm{tr}[X^i,X^j]^{2k}\right\rangle\sim N^{4k/3}, \quad \text{(A5)}$$

$$\left\langle\frac{1}{N}\,\mathrm{tr}(\overline{\theta[}\gamma_iX^i,\theta])^{2k}\right\rangle\sim N^{4k/3}, \quad \text{(A6)}$$

$$\left\langle\frac{1}{N}\,\mathrm{tr}\dot{X}^{2k}\right\rangle\sim N^{4k/3}. \quad \text{(A7)}$$

In the first of these expressions, X refers to any component of the transverse coordinates. In the second the commutator refers to any pair of the components. The final expression, whose lowest order perturbative formula has an ultraviolet divergence, is best derived by combining Eqs. (A5) and (A6) and the Schrödinger equation which says that the threshold bound state has zero binding energy. Note that these expressions are independent of R, the compactification radius of the 11th dimension. This follows from a cancellation between the R dependence of the infrared cutoff, that of the effective coupling and that of the scaling factor, which relates the variables X to conventionally normalized Yang-Mills fields.

The first of these equations says that the typical eigenvalue of any one coordinate matrix is of order $N^{1/3}$, much larger than the $N^{1/9}$ extension along the flat directions. The second tells us that this spectral weight lies mostly along the nonflat directions. In conjunction, the two equations can be read as a kind of ''uncertainty principle of noncommutative geometry.'' The typical size of matrices is controlled by the size of their commutator. The final equation fits nicely with our estimate of the cutoff frequency. The typical velocity is such that the transit time of a typical distance[9] is the inverse of the cutoff frequency. It is clear that the high velocities encountered in the core of the wave function could invalidate our attempt to derive the matrix model by extrapolating from weakly coupled string theory. We must hope that the overall amplitude for this part of the wave function vanishes in the large N limit, relative to the parts in which zero branes are separated along flat directions.

It is important to realize that these estimates do not apply along the flat directions, but in the bulk of the N^2 dimensional configuration space. In these directions, it does not make sense to multiply together the ''sizes'' along different coordinate directions to make an area since the different coordinates do not commute. Thus there is no contradiction between the growth of the wave function in nonflat directions and our argument that the size of the bound state in conventional geometric terms saturates the Bekenstein bound.

APPENDIX B

In this appendix we compute the matrix model membrane tension and show that it exactly agrees with the M-theory membrane tension. The useful summary in [24] gives the tension of a Dp brane T_p in IIA string theory or M theory as

$$T_p=\frac{(2\pi)^{1/2}}{g_s}(2\pi)^{-p/2}\left(\frac{1}{2\pi\alpha'}\right)^{(1+p)/2}, \tag{B1}$$

where g_s is the fundamental string coupling and $1/2\pi\alpha'$ is the fundamental string tension.

The membrane tension T_2 is defined so that the mass $\mathcal{M}$ of a stretched membrane of area A is given by $\mathcal{M}=T_2A$. The mass squared for a light cone membrane with no transverse momentum described by the map $\mathcal{X}^i(\sigma_1,\sigma_2)$, $i=1,\ldots,9$, can be written

$$\mathcal{M}^2=(2\pi)^4T_2^2\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\int_0^{2\pi}\frac{d\sigma_2}{2\pi}\sum_{i<j}\{\mathcal{X}^i,\mathcal{X}^j\}^2, \tag{B2}$$

where the Poisson brackets of two functions $A(\sigma_1,\sigma_2),B(\sigma_1,\sigma_2)$ are defined by

$$\{A,B\}\equiv\frac{\partial A}{\partial\sigma_1}\frac{\partial B}{\partial\sigma_2}-\frac{\partial B}{\partial\sigma_1}\frac{\partial A}{\partial\sigma_2}. \tag{B3}$$

The coefficients in Eq. (B2) are set by demanding that $\mathcal{M}^2$ for the map $\mathcal{X}^8=(\sigma_1/2\pi)L$, $\mathcal{X}^9=(\sigma_2/2\pi)L$ is given by $\mathcal{M}^2=(T_2L^2)^2$.

To understand the relation to the matrix model, we write, as in Sec. VIII, the map $\mathcal{X}^i$ as a Fourier series:

$$\mathcal{X}^i(\sigma_1,\sigma_2)=\sum_{n_1,n_2}x^i_{n_1n_2}e^{i(n_1\sigma_1+n_2\sigma_2)}. \tag{B4}$$

[9]In the space of eigenvalues, which in this noncommutative region is not to be confused with the classical geometrical distance between $D0$ branes.

Then the corresponding matrices X^i are given by

$$X^i=\sum_{n_1,n_2}x^i_{n_1n_2}U^{n_1}V^{n_2}, \tag{B5}$$

where the matrices U,V are elements of SU(N), have spectrum $\mathrm{spec}(U)=\mathrm{spec}(V)=\{1,\omega,\omega^2,\ldots,\omega^{N-1}\}$, and obey $UV=\omega VU$, where $\omega=\exp(2\pi i/N)$. In a specific basis, $U=\mathrm{diag}(1,\omega,\omega^2,\ldots,\omega^{N-1})$ and V is a cyclic forward shift.

The scale of Eq. (B5) is fixed since spec(U) and spec(V) go over as $N\to\infty$ to the unit circle $\exp(i\sigma)$. Note that $\mathcal{X}^i$ is real and so X^i is Hermitian.

The dynamics of the matrix model is governed by the Lagrangian

$$\frac{T_0}{2}\mathrm{Tr}\left(\sum_i\dot{X}^i\dot{X}^i+C\sum_{i<j}[X^i,X^j]^2\right). \tag{B6}$$

The normalizations here are fixed by the requirement that Eq. (B6) describe $D0$-brane dynamics. The first term, for diagonal matrices describing $D0$-brane motion, is just the nonrelativistic kinetic energy $(m_0/2)\Sigma_{a=1}^N v_a^2$ since the 0-brane mass $m_0=T_0$. The rest energy of the system is just Nm_0, which in the M-theory interpretation is just p_{11}, and so

$$p_{11}=NT_0. \tag{B7}$$

The coefficient C is fixed by requiring that the small fluctuations around diagonal matrices describe harmonic oscillators whose frequencies are precisely the masses of the stretched strings connecting the 0 branes. This ensures that Eq. (B6) reproduces long range graviton interactions correctly. Expanding Eq. (B6) to quadratic order we find that $C=(1/2\pi\alpha')^2$.

The energy of a matrix membrane configuration with zero transverse momentum is given by the commutator term in Eq. (B6). We can evaluate this commutator in a semiclassical manner at large N as in Sec. VIII by introducing angular operators q with spectrum the interval $(0,2\pi)$ and $p=(2\pi/Ni)\partial/\partial q$ with the spectrum the discretized interval $(0,2\pi)$ so that $[p,q]=2\pi i/N$. The matrices U,V become $U=e^{ip}$, $V=e^{iq}$. By the Baker-Campbell-Hausdorff theorem, we see $UV=\omega VU$. The formal $\hbar$ in this algebra is given by $\hbar=2\pi/N$. Semiclassically, we have

$$[X,Y]\to i\hbar\{\mathcal{X},\mathcal{Y}\},$$

$$\mathrm{Tr}\to\int_0^{2\pi}\int_0^{2\pi}\frac{dp\,dq}{2\pi\hbar}. \tag{B8}$$

So we get

$$\mathrm{Tr}[X^i,X^j]^2\to-\frac{(2\pi)^2}{N}\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\int_0^{2\pi}\frac{d\sigma_2}{2\pi}\{\mathcal{X}^i,\mathcal{X}^j\}^2. \tag{B9}$$

This commutator can also be evaluated for a given finite N matrix configuration explicitly with results that agree with Eq. (B9) as $N\to\infty$.

Now we can perform the check. The value of the matrix model Hamiltonian on a configuration with no transverse momentum is

$$H=\frac{T_0}{2}\left(\frac{1}{2\pi\alpha'}\right)^2\frac{(2\pi)^2}{N}\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\int_0^{2\pi}\frac{d\sigma_2}{2\pi}\sum_{I<j}\{\mathcal{X}^i,\mathcal{X}^j\}^2. \tag{B10}$$

The conjecture interprets the matrix model Hamiltonian H as the infinite momentum frame energy $\sqrt{p_{11}^2+\mathcal{M}^2}-p_{11}\simeq\mathcal{M}^2/2p_{11}$. So the matrix membrane mass squared is $\mathcal{M}_{\rm mat}^2=2p_{11}H$. Using Eq. (B7), we find

$$\mathcal{M}_{\rm mat}^2=T_0^2\left(\frac{1}{2\pi\alpha'}\right)^2(2\pi)^2\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\int_0^{2\pi}\frac{d\sigma_2}{2\pi}\sum_{i<j}\{\mathcal{X}^i,\mathcal{X}^j\}^2. \tag{B11}$$

From Eq. (B2) we can now read off the matrix model membrane tension as

$$(T_2^{\rm mat})^2=T_0^2\left(\frac{1}{2\pi\alpha'}\right)^2\frac{1}{(2\pi)^2}. \tag{B12}$$

So we can write

$$\frac{T_2^2}{(T_2^{\rm mat})^2}=(2\pi)^2(2\pi\alpha')^2\left(\frac{T_2}{T_0}\right)^2=(2\pi)^2\left(\frac{1}{2\pi}\right)^2=1. \tag{B13}$$

So the M-theory and matrix model membrane tensions exactly agree.

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