

Perturbation theory with a variational basis: The generalized Gaussian effective potential

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The perturbation theory with a variational basis is constructed and analyzed. The generalized Gaussian effective potential is introduced and evaluated up to second order for self-interacting scalar fields in one and two spatial dimensions. The problem of the renormalization of the mass is discussed in detail. Thermal corrections are incorporated. The comparison between the finite temperature generalized Gaussian effective potential and the finite temperature effective potential is critically analyzed. The phenomenon of the restoration at high temperature of the symmetry broken at zero temperature is discussed. [S0556-2821(97)00306-8]

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I. INTRODUCTION

In recent years the variational Gaussian approximation has played an important role in the nonperturbative study of quantum field theories. In particular, to investigate the spontaneous symmetry-breaking phenomenon in scalar quantum field theories, the Gaussian effective potential has been introduced [1]. The main disadvantage of the variational approach is the absence of the control of the approximation. Moreover, in quantum field theories the presence of ultraviolet divergences often makes the variational calculations useless.

The aim of this paper is to develop a variational scheme in scalar quantum field theories which allows us to evaluate in a systematic manner the corrections to the Gaussian approximation and, at the same time, to keep under control the ultraviolet divergences. To this end, we shall construct a perturbation theory with a variational basis. The method we shall follow is widely used in many-body theory where it is known as the method of correlated basis functions [2]. Using the variational basis we construct a vacuum state $|\Omega\rangle$ which is adiabatically connected to the Gaussian trial vacuum, whereupon we introduce the generalized Gaussian effective potential $V_G(\phi_0)$ defined as the expectation value of the Hamiltonian density on $|\Omega\rangle$ in the presence of the scalar condensate ϕ_0 . We shall give an explicit formula for $V_G(\phi_0)$ which is similar to the usual perturbative expansion of the effective potential by means of Feynman vacuum diagrams. Moreover, we shall show that the variational-perturbation theory developed in this paper offers a solution to the ultraviolet divergence problem in the variational approaches which is analogous to the usual perturbative renormalization theory. For the sake of simplicity, we perform explicit calculations in the case of self-interacting scalar fields in one and two spatial dimensions. Indeed, these theories are superrenormalizable, so that we only need to renormalize the mass. In the second part of the paper we discuss the finite temperature corrections to the generalized Gaussian effective potential. Moreover, we critically compare our approach to the finite temperature effective potential and the Gaussian potential.

The plan of the paper is as follows. In Sec. II we discuss the Gaussian approximation in scalar field theories and introduce the Gaussian effective potential. In Sec. III we set up

the variational basis starting from the trial Gaussian vacuum wave functional. Section IV is devoted to perturbation theory with a variational basis. The generalized Gaussian effective potential is discussed in Sec. V. The calculations of the second order corrections to the Gaussian effective potential are presented in Sec. VI where we discuss in detail the mass renormalization. In Sec. VII we introduce the finite temperature generalized Gaussian effective potential and evaluate the lowest order thermal corrections. The second order thermal corrections are explicitly evaluated in Sec. VIII. Our conclusions are drawn in Sec. IX. Several technical details are relegated in two Appendixes. In Appendix A we perform the high temperature expansions which are relevant for the lowest order thermal corrections. In Appendix B we collect some well-known result on the thermodynamic perturbation theory in the Matsubara's scheme. Moreover, we present some useful results on the thermal propagator.

II. GAUSSIAN APPROXIMATION

In this section we discuss the Gaussian approximation in scalar quantum field theories. In particular we shall focus on the Gaussian effective potential [1] for self-interacting scalar fields in $d = \nu + 1$ space-time dimensions.

The Gaussian approximation in quantum field theories has been widely developed for a long time [3–5]. The Gaussian approximation is a variational method in which one considers trial Gaussian wave functionals as the ground state of the theory.

Let us consider a real scalar field $\phi(x)$ whose Hamiltonian is

$$H = \int d^{\nu}x \left[\frac{\Pi^2(\vec{x})}{2} + \frac{1}{2} [\vec{\nabla} \phi(\vec{x})]^2 + \frac{1}{2} m^2 \phi^2(\vec{x}) + \frac{\lambda}{4!} \phi^4(\vec{x}) \right]. \quad (2.1)$$

In the Schrödinger representation the physical states are wave functionals of ϕ ; the conjugate momentum $\Pi(x)$ acts as a functional derivative:

$$\Pi(x)|\Psi\rangle \rightarrow \frac{1}{i} \frac{\delta}{\delta \phi(\vec{x})} \Psi[\phi]. \quad (2.2)$$

The inner product is defined by

$$\langle \Psi_1 | \Psi_2 \rangle = \int [d\phi] \Psi_1^*[\phi] \Psi_2[\phi]. \quad (2.3)$$

The stationary Schrödinger equation reads

$$\int d^v x \left[-\frac{1}{2} \frac{\delta^2}{\delta \phi(\vec{x}) \delta \phi(\vec{x})} + \frac{1}{2} [\vec{\nabla} \phi(\vec{x})]^2 + \frac{1}{2} m^2 \phi^2(\vec{x}) + \frac{\lambda}{4!} \phi^4(\vec{x}) \right] \Psi[\phi] = E \Psi[\phi]. \quad (2.4)$$

The analogy with ordinary quantum mechanics is evident. In particular, we would like to apply the variational principle which has been successfully developed in quantum mechanics.

In quantum field theories it is the ground state that determines the physical properties of the quantum system. The Gaussian approximation amounts to approximating the vacuum functional with a set of trial Gaussian functionals centered at ϕ_0 :

$$\Psi_0[\phi] = \mathcal{N} \exp \left[-\frac{1}{4} \int d^v x d^v y [\phi(\vec{x}) - \phi_0] G(\vec{x}, \vec{y}) \times [\phi(\vec{y}) - \phi_0] \right], \quad (2.5)$$

where the normalization constant is such that

$$\langle \Psi_0 | \Psi_0 \rangle = 1. \quad (2.6)$$

A well-known method to investigate the structure of a quantum field theory is to use the effective potential $V_{\text{eff}}(\phi_0)$ [6]. In scalar field theories it turns out that the effective potential is the expectation value of the Hamiltonian density in a certain state wherein the expectation value of the scalar field is ϕ_0 [7]. These considerations suggested to introduce the so-called Gaussian effective potential $V_G(\phi_0)$.

The Gaussian effective potential (GEP) is defined by minimizing the Hamiltonian density on the set of wave functionals, Eq. (2.5):

$$V_{\text{GEP}}(\phi_0) = \frac{1}{V} \min_{|\Psi_0\rangle} \langle \Psi_0 | H | \Psi_0 \rangle, \quad (2.7)$$

where V is the spatial volume.

$V_{\text{GEP}}(\phi_0)$, being a variational quantity, not only goes beyond perturbation theory, but often gives a more realistic picture of the qualitative physics than the effective potential. Moreover, the Gaussian effective potential is easily computable. To see this, we note that due to the translation invariance of the vacuum we have

$$G(\vec{x}, \vec{y}) = \int \frac{d^v k}{(2\pi)^v} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} 2g(\vec{k}). \quad (2.8)$$

Let us consider the functional

$$\Psi_0^J[\phi] = \mathcal{N} \exp \left[-\frac{1}{4} \int d^v x d^v y \eta(\vec{x}) G(\vec{x} - \vec{y}) \eta(\vec{y}) + \frac{1}{2} \int d^v x \eta(\vec{x}) J(\vec{x}) \right], \quad (2.9)$$

where

$$\eta(\vec{x}) = \phi(\vec{x}) - \phi_0 \quad (2.10)$$

and \mathcal{N} is fixed by Eq. (2.6). We can easily evaluate the functional

$$I[J] = \langle \Psi_0^J | \Psi_0^J \rangle. \quad (2.11)$$

Indeed, Eq. (2.11) involves a straightforward Gaussian functional integration. We get

$$I[J] = \exp \left[\frac{1}{2} \int d^v x d^v y J(\vec{x}) G^{-1}(\vec{x} - \vec{y}) J(\vec{y}) \right], \quad (2.12)$$

where

$$G^{-1}(\vec{x}, \vec{y}) = \int \frac{d^v k}{(2\pi)^v} \frac{1}{2g(\vec{k})} e^{-i\vec{k}(\vec{x} - \vec{y})}. \quad (2.13)$$

Now, we have

$$\langle \Psi_0 | \eta(\vec{x}_1) \cdots \eta(\vec{x}_n) | \Psi_0 \rangle = \frac{\delta^n I[J]}{\delta J(\vec{x}_1) \cdots \delta J(\vec{x}_n)} \Big|_{J=0}. \quad (2.14)$$

Equations (2.14) and (2.12) allow us to evaluate the expectation values of monomial in $\eta(\vec{x})$ on the Gaussian vacuum functionals. It is now a straightforward exercise to calculate

$$E_0[\phi_0, g(\vec{k})] = \frac{\langle \Psi_0 | H | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}. \quad (2.15)$$

We get

$$E_0[\phi_0, g(\vec{k})] = V \left\{ \frac{1}{4} \int \frac{d^v k}{(2\pi)^v} g(\vec{k}) + \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + \frac{1}{4} \int \frac{\vec{k}^2 + m^2 + \frac{\lambda}{2} \phi_0^2}{g(\vec{k})} + \frac{3}{4} \frac{\lambda}{4!} \left[\int \frac{d^v k}{(2\pi)^v} \frac{1}{g(\vec{k})} \right]^2 \right\}. \quad (2.16)$$

The Gaussian effective potential is obtained by minimizing $E_0[\phi_0, g(\vec{k})]$ with respect to $g(\vec{k})$. By imposing the extremum condition

$$\frac{\delta E_0[\phi_0, g(\vec{k})]}{\delta g(\vec{k})} = 0, \quad (2.17)$$

we obtain

$$g(\vec{k}) = \sqrt{\vec{k}^2 + \mu^2(\phi_0)}, \quad (2.18)$$

where $\mu(\phi)$ satisfies the gap equation

$$\mu^2 = m^2 + \frac{\lambda}{2} \phi_0^2 + \frac{\lambda}{4} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{g(\vec{k})}. \quad (2.19)$$

Inserting Eqs. (2.18) and (2.19) into Eq. (2.16) we get the Gaussian effective potential

$$V_{\text{GEP}}(\phi_0) = \frac{\lambda}{4!} \phi_0^4 + \frac{m^2}{2} \phi_0^2 + \frac{1}{2} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} g(\vec{k}) - \frac{\lambda}{32} \left[\int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{g(\vec{k})} \right]^2. \quad (2.20)$$

Putting [1]

$$I_n(\mu) = \frac{1}{2} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} [g(\vec{k})^2]^{n-1/2}, \quad (2.21)$$

we rewrite Eq. (2.20) in the more compact form

$$V_{\text{GEP}}(\phi_0) = \frac{\lambda}{4!} \phi_0^4 + \frac{m^2}{2} \phi_0^2 + I_1(\mu) - \frac{\lambda}{8} I_0^2(\mu), \quad (2.22)$$

while the gap equation becomes

$$\mu^2 = m^2 + \frac{\lambda}{2} \phi_0^2 + \frac{\lambda}{2} I_0(\mu). \quad (2.23)$$

For later convenience, it is useful to work in units of $\mu_0 = \mu(\phi_0=0)$. Thus we introduce the dimensionless parameters

$$\Phi_0^2 = \frac{\phi_0^2}{\mu_0^{\nu-1}}, \quad (2.24)$$

$$\hat{\lambda} = \frac{\lambda}{4! \mu_0^{3-\nu}}, \quad (2.25)$$

$$x = \frac{\mu^2}{\mu_0^2}. \quad (2.26)$$

Moreover, we redefine the zero of the energy scale by subtracting in $V_{\text{GEP}}(\phi_0)$ the (divergent) quantity $V_{\text{GEP}}(\phi_0=0)$:

$$V_{\text{GEP}}^{\nu+1}(\phi_0) = \frac{V_{\text{GEP}}(\phi_0) - V_{\text{GEP}}(\phi_0=0)}{\mu_0^{\nu+1}}. \quad (2.27)$$

III. THE VARIATIONAL BASIS

In the previous section we introduced the Gaussian effective potential. The most serious problem of the Gaussian effective potential resides in the lack of control on the variational approximation. For these reasons it is desirable to deal with a generalization of the Gaussian effective potential which allows us to compute in a systematic way the corrections to the Gaussian approximation [8]. The problem we are interested in is not an academic one. Indeed, it is well known that the Gaussian approximation does not take into account all the two-loop contributions. As a consequence in non-

Abelian gauge theories the Gaussian approximation breaks gauge invariance [9].

In order to evaluate the corrections to the variational Gaussian approximation we need to set up a variational-perturbation theory. To this end we now construct a variational basis starting from the vacuum wave functional $\Psi_0[\eta]$. To do this we consider $\Psi_0[\eta]$ as the ground state wave functional of a suitable Hamiltonian.

Let us consider the operators

$$a(\vec{p}) = \int \frac{d^{\nu}x}{(2\pi)^{\nu/2}} \frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2g(\vec{p})}} \times \left(\int d^{\nu}y \frac{1}{2} G(x,y) \eta(y) + \frac{\delta}{\delta\eta(\vec{x})} \right), \quad (3.1)$$

$$a^{\dagger}(\vec{p}) = \int \frac{d^{\nu}x}{(2\pi)^{\nu/2}} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2g(\vec{p})}} \times \left(\int d^{\nu}y \frac{1}{2} G(x,y) \eta(y) - \frac{\delta}{\delta\eta(\vec{x})} \right). \quad (3.2)$$

It is easy to see that the only nontrivial commutator is

$$[a(\vec{p}_1), a^{\dagger}(\vec{p}_2)] = \delta(\vec{p}_1 - \vec{p}_2). \quad (3.3)$$

Moreover, we have

$$a(\vec{p}) \Psi_0[\eta] = 0. \quad (3.4)$$

Now we rewrite the annihilation and creation operators, Eqs. (3.1) and (3.2), by means of the Fourier transform of the fluctuation fields $\eta(\vec{x})$:

$$\eta(\vec{p}) = \int \frac{d^{\nu}x}{(2\pi)^{\nu/2}} e^{-i\vec{p}\cdot\vec{x}} \eta(\vec{x}), \quad (3.5)$$

$$\frac{\delta}{\delta\eta(\vec{p})} = \int \frac{d^{\nu}x}{(2\pi)^{\nu/2}} e^{i\vec{p}\cdot\vec{x}} \frac{\delta}{\delta\eta(\vec{x})}. \quad (3.6)$$

We get

$$a(\vec{p}) = \sqrt{\frac{g(\vec{p})}{2}} \left(\eta(-\vec{p}) + \frac{1}{g(\vec{p})} \frac{\delta}{\delta\eta(\vec{p})} \right), \quad (3.7)$$

$$a^{\dagger}(\vec{p}) = \sqrt{\frac{g(\vec{p})}{2}} \left(\eta(\vec{p}) - \frac{1}{g(\vec{p})} \frac{\delta}{\delta\eta(-\vec{p})} \right). \quad (3.8)$$

Consider, now, the Hamiltonian

$$\tilde{H}_0 = \frac{1}{2} \int d^{\nu}p \left[- \frac{\delta}{\delta\eta(\vec{p})} \frac{\delta}{\delta\eta(-\vec{p})} + g^2(\vec{p}) \eta(\vec{p}) \eta(-\vec{p}) \right], \quad (3.9)$$

which can be rewritten as

$$\tilde{H}_0 = \int d^{\nu}p g(\vec{p}) a^{\dagger}(\vec{p}) a(\vec{p}) + E_0, \quad (3.10)$$

where

$$E_0 = \frac{1}{2} \int d^v x \int \frac{d^v p}{(2\pi)^v} g(\vec{p}). \quad (3.11)$$

Let $|0\rangle$ be the vacuum of H_0 in the abstract ket formalism. From Eq. (3.4) it follows

$$\langle \eta|0\rangle = \Psi_0[\eta], \quad (3.12)$$

which is $\Psi_0[\eta]$ is the vacuum of \tilde{H}_0 in the Schrödinger representation. Starting from $|0\rangle$ we can set up the many particle states by acting on the vacuum with the creation operators $a^\dagger(\vec{p})$. In the Schrödinger representation we have, for instance,

$$\begin{aligned} \Psi_1[\eta] &= \langle \eta|\vec{p}\rangle = \langle \eta|a^\dagger(\vec{p})|0\rangle \\ &= \int \frac{d^v x}{(2\pi)^{v/2}} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2g(\vec{p})}} \\ &\quad \times \left(\int d^v y \frac{1}{2} G(x,y) \eta(y) - \frac{\delta}{\delta\eta(\vec{x})} \right) \Psi_0[\eta] \\ &= \int \frac{d^v x d^v y}{(2\pi)^{v/2}} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2g(\vec{p})}} G(\vec{x}-\vec{y}) \eta(\vec{y}) \Psi_0[\eta]. \end{aligned} \quad (3.13)$$

Obviously we also have

$$\tilde{H}_0|\vec{p}\rangle = [E_0 + g(\vec{p})]|\vec{p}\rangle \quad (3.14)$$

and

$$\langle \vec{p}_1|\vec{p}_2\rangle = \delta(\vec{p}_1 - \vec{p}_2) \langle 0|0\rangle. \quad (3.15)$$

In this way we construct the orthonormal set (Fock basis) of wave functionals $\{\Psi_n[\eta]\}$, where $\Psi_n[\eta]$ is obtained by applying n times the creation operator on $\Psi_0[\eta]$.

It should be emphasized that the Fock basis is univocally determined by the vacuum functional $\Psi_0[\eta]$. As we will discuss in the next section, the vacuum functional will be fixed with a variational procedure. For this reason the Fock basis $\{\Psi_n(\eta)\}$ will be referred to as a variational basis.

IV. PERTURBATION THEORY WITH THE VARIATIONAL BASIS

In this section we use the variational basis to set up a perturbation theory for the ground state energy. To this end we split our Hamiltonian, Eq. (2.1), into two pieces:

$$H = H_0 + H_I, \quad (4.1)$$

where H_0 will be the *free* Hamiltonian and H_I the *perturbation*. We define H_0 and H_I as

$$(H_0)_{nm} = \langle n|H_0|m\rangle = \delta_{nm} H_{nn}, \quad (4.2)$$

$$(H_I)_{nm} = \langle n|H_I|m\rangle = (1 - \delta_{nm}) H_{nm}, \quad (4.3)$$

where

$$H_{nm} = \langle n|H|m\rangle = \int [d\eta] \Psi_n^*[\eta] H \left(-i \frac{\delta}{\delta\eta}, \eta \right) \Psi_m[\eta]. \quad (4.4)$$

Equations (4.2) and (4.3) show that the perturbation H_I is given by the off-diagonal elements of the full Hamiltonian H with respect to the variational basis. If the wave functional $\Psi_0[\eta]$ is close to the true ground state of H , then we expect that the $(H_I)_{mn}$ are small with respect to $(H_0)_{nm}$, i.e., that H_I is a genuine perturbation.

We recall that in Sec. II we fixed the wave functional $\Psi_0[\eta]$ by minimizing $E[\phi_0, g(\vec{k})] = H_{00}$ on the class of trial Gaussian functionals, Eq. (2.5). In this way we get an optimized perturbation expansion. Moreover, we stress that in our scheme it is unnecessary to start with a small parameter in H . Thus our method goes beyond the usual perturbation theory.

We now address ourselves in the determination of H_0 and H_I . We evaluate, firstly, the diagonal elements of H with respect to the Fock variational basis $\{\Psi_n[\eta]\}$. To this end we rewrite the Hamiltonian (2.1) in terms of the annihilation and creation operators (3.7) and (3.8). A rather lengthy but otherwise straightforward calculation shows that

$$H = H^{(0)} + H^{(1)} + H^{(2)} + H^{(3)} + H^{(4)}, \quad (4.5)$$

where

$$H^{(0)} = E[\phi_0, g(\vec{k})], \quad (4.6)$$

$$H^{(1)} = \left[m^2 \phi_0 + \frac{\lambda}{6} \phi_0^3 + \frac{\lambda}{4} \phi_0 \int \frac{d^v k}{(2\pi)^v} \frac{1}{g(\vec{k})} \right] \int d^v x: \eta(\vec{x}):, \quad (4.7)$$

$$\begin{aligned} H^{(2)} &= \frac{1}{2} \int d^v p \left[g(\vec{p}) + \frac{\vec{p}^2 + m^2 + (\lambda/2)\phi_0^2}{g(\vec{p})} + \frac{\lambda}{4} \int \frac{d^v k}{(2\pi)^v} \frac{1}{g(\vec{p})g(\vec{k})} \right] a^\dagger(\vec{p}) a(\vec{p}) \\ &\quad + \frac{1}{4} \int d^v p \left[-g(\vec{p}) + \frac{\vec{p}^2 + m^2 + (\lambda/2)\phi_0^2 + (\lambda/4) \int [d^v p' / (2\pi)^3] \frac{1}{g(\vec{p}')}}{g(\vec{p})} \right] [a^\dagger(\vec{p}) a^\dagger(-\vec{p}) + a(\vec{p}) a(-\vec{p})], \end{aligned} \quad (4.8)$$

$$H^{(3)} = \frac{\lambda}{3!} \phi_0 \int d^{\nu}x : \eta^3(\vec{x}) :, \quad (4.9)$$

$$H^{(4)} = \frac{\lambda}{4!} \int d^{\nu}x : \eta^4(\vec{x}) :, \quad (4.10)$$

where the colons mean normal ordering with respect to the vacuum $\Psi_0[\eta]$.

In the ket formalism the n -particle wave functionals are given by

$$|n\rangle = |\vec{p}_1 \nu_1 ; \vec{p}_2 \nu_2 \cdots \rangle = \prod_i \frac{(2\pi)^{\nu/2}}{V^{1/2}} \frac{1}{(\nu_i!)^{1/2}} a^{\dagger \nu_i}(\vec{p}_i) |0\rangle, \quad (4.11)$$

with $\sum_i \nu_i = n$. A straightforward calculation gives

$$\langle n | H^{(0)} | n \rangle = E[\phi_0, g(\vec{k})], \quad (4.12)$$

$$\langle n | H^{(1)} | n \rangle = \langle n | H^{(3)} | n \rangle = 0, \quad (4.13)$$

$$\begin{aligned} \langle n | H^{(2)} | n \rangle &= \frac{1}{2} \sum_i \nu_i \left[g(\vec{p}_i) + \frac{\vec{p}_i + m^2 + (\lambda/2)\phi_0^2}{g(\vec{p}_i)} \right] \\ &+ \frac{\lambda}{8} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{g(\vec{k})} \sum_i \nu_i \frac{1}{g(\vec{p}_i)}. \end{aligned} \quad (4.14)$$

As concerns $H^{(4)}$, a rather lengthy but elementary calculation shows that in the thermodynamic limit $V \rightarrow \infty$, we also have

$$\langle n | H^{(4)} | n \rangle = 0. \quad (4.15)$$

Now, if we select the variational basis by minimizing $H_{00} = E_0[\phi_0, g(\vec{k})]$, i.e., if we impose the extremum condition, Eq. (2.17), we find that $H^{(2)}$ reduces to

$$H^{(2)} = \int d^{\nu}p g(\vec{p}) a^{\dagger}(\vec{p}) a(\vec{p}). \quad (4.16)$$

Moreover,

$$\langle n | H | n \rangle = E[\phi_0, g(\vec{k})] + \sum_i \nu_i g(\vec{p}_i), \quad (4.17)$$

where $g(\vec{k}) = \sqrt{\vec{k}^2 + \mu^2(\phi_0)}$ and $\mu(\phi_0)$ satisfies the gap equation (2.19).

Equation (4.16) tells us that $H^{(2)}$ is the normal-ordered Hamiltonian of a free scalar field with mass $\mu(\phi_0)$. Moreover, it is now clear that the off-diagonal elements of the full Hamiltonian are due to $H^{(1)}$, $H^{(3)}$, and $H^{(4)}$. As a consequence we can write (using again the gap equation)

$$H_0 = H^{(0)} + H^{(2)} = E[\phi_0, g(\vec{k})] + H^{(2)}, \quad (4.18)$$

$$\begin{aligned} H_I = \int d^{\nu}x &\left[\left(\mu^2(\phi_0)\phi_0 - \frac{\lambda}{3}\phi_0^3 \right) : \eta(\vec{x}) : + \frac{\lambda}{3!} \phi_0 : \eta^3(\vec{x}) : \right. \\ &\left. + \frac{\lambda}{4!} : \eta^4(\vec{x}) : \right]. \end{aligned} \quad (4.19)$$

We stress once again that our perturbation is given by the off-diagonal elements of the full Hamiltonian H with respect to the variational basis $\{\Psi_n[\eta]\}$. This means that the perturbation expansion that we will discuss in the next section is not a weak coupling expansion. In other words, our variational procedure, which selects the Fock basis $\{\Psi_n[\eta]\}$, minimizes the off-diagonal elements H_{nm} , so that even though the quartic self-coupling is strong, the perturbative expansion gives sensible results. Finally, it is worth mentioning that the simple results, Eq. (4.19), for the perturbation Hamiltonian rely on Eq. (4.15) which is valid only for quantum systems with an infinite number of degrees of freedom.

V. GENERALIZED GAUSSIAN EFFECTIVE POTENTIAL

In the previous section we were able to split the Hamiltonian H into two pieces: the free Hamiltonian H_0 and the interaction H_I . If we neglect H_I , we see that the ground state of H_0 is the wave functional $\Psi_0[\eta]$ and the Gaussian effective potential, Eq. (2.22), is the ground state energy density. In other words, $V_{\text{GEP}}(\phi_0)$ is the lowest order term of the vacuum energy density in the perturbation expansion generated by H_I . Thus the corrections to the Gaussian effective potential can be readily obtained by means of the standard perturbation expansion for the ground state energy. For the ground state energy we may use the well-known Brueckner-Goldstone formula [10]

$$\begin{aligned} E_{\text{GS}}(\phi_0) &= E_0(\phi_0, g(\vec{k})) \\ &+ \sum_{n=0}^{\infty} \left[\langle 0 | H_I \left(\frac{1}{E_0 - H_0} H_I \right)^n | 0 \rangle \right]_{\text{conn}}. \end{aligned} \quad (5.1)$$

For instance, up to second order in H_I and using Eqs. (4.2) and (4.3), we have

$$E_{\text{GS}}(\phi_0) = E_0 + \sum_{n>0} (H_{00} - H_{nn})^{-1} |\langle n | H | 0 \rangle|^2. \quad (5.2)$$

Higher order terms can be analyzed by means of the so-called Goldstone diagrams [10,11].

However, in order to show that E_{GS} in Eq. (5.1) gives correctly the correction to the Gaussian effective potential, we must ascertain that it exists in a state $|\Omega\rangle$ such that

$$E_{\text{GS}}(\phi_0) = \frac{\langle \Omega | H | \Omega \rangle}{\langle \Omega | \Omega \rangle}, \tag{5.3}$$

with the constraint

$$\frac{\langle \Omega | \eta(\vec{x}) | \Omega \rangle}{\langle \Omega | \Omega \rangle} = 0. \tag{5.4}$$

To do this, we use the Gell-Mann–Low theorem on the ground state [12]. Let us consider the Hamiltonian

$$H_\epsilon = H_0 + H_I e^{-\epsilon|t|}, \quad \epsilon \rightarrow 0^+. \tag{5.5}$$

Next, we introduce the temporal evolution operator

$$U_\epsilon(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \cdots \int_{t_0}^t dt_n e^{-\epsilon(|t_1| + \cdots + |t_n|)} \times T[H_I(t_1) \cdots H_I(t_n)], \tag{5.6}$$

where $H_I(t)$ is the perturbation Hamiltonian in the interaction representation. The Gell-Mann–Low theorem says that if the following quantity exists to all order in perturbation theory,

$$\lim_{\epsilon \rightarrow 0^+} \frac{U_\epsilon(0, -\infty)|0\rangle}{\langle 0|U_\epsilon(0, -\infty)|0\rangle} \equiv \frac{|\Omega\rangle}{\langle 0|\Omega\rangle}, \tag{5.7}$$

then

$$H \frac{|\Omega\rangle}{\langle 0|\Omega\rangle} = E \frac{|\Omega\rangle}{\langle 0|\Omega\rangle}. \tag{5.8}$$

Note that the denominator in Eqs. (5.7) and (5.8) is crucial, for the numerator and the denominator do not separately exist as $\epsilon \rightarrow 0^+$.

From Eq. (5.8) it follows that

$$E = \frac{\langle \Omega | H | \Omega \rangle}{\langle \Omega | \Omega \rangle}. \tag{5.9}$$

Now we show that $E_{\text{GS}} = E$. Indeed, from Eqs. (5.8) and (5.5) we get

$$E - E_0 = \frac{\langle 0 | H_I | \Omega \rangle}{\langle 0 | \Omega \rangle}, \tag{5.10}$$

where we have taken into account that $H_0|0\rangle = E_0|0\rangle$.

Now a standard manipulation [11] shows that

$$\begin{aligned} \langle 0 | H_I | \Omega \rangle &= \langle 0 | \Omega \rangle \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \\ &\times \int_{-\infty}^0 dt_1 \cdots \int_{-\infty}^0 dt_n e^{[-\epsilon(t_1 + \cdots + t_n)]} \\ &\times \langle 0 | T[H_I(0)H_I(t_1) \cdots H_I(t_n)] | 0 \rangle_{\text{conn}}, \end{aligned} \tag{5.11}$$

where the subscript means that we need to take into account only the connected terms. In order to carry out the time integrations, we consider the n th order contribution in Eq. (5.11). Observing that

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} \tag{5.12}$$

and that the $n!$ possible time orderings give identical contributions, we get

$$\begin{aligned} (E - E_0)^{(n)} &= (-i)^n \int_{-\infty}^0 dt_1 \cdots \int_{-\infty}^0 dt_n e^{-\epsilon(|t_1| + \cdots + |t_n|)} \\ &\times \langle 0 | H_I e^{iH_0 t_1} H_I e^{-iH_0(t_1 - t_2)} \cdots H_I \\ &\times e^{-iH_0(t_{n-1} - t_n)} H_I e^{-iH_0 t_n} | 0 \rangle_{\text{conn}}. \end{aligned} \tag{5.13}$$

By changing variables to relative times,

$$x_1 = t_1, \quad x_2 = t_2 - t_1, \dots, x_n = t_n - t_{n-1}, \tag{5.14}$$

one finally obtains

$$\begin{aligned} (E - E_0)^{(n)} &= \langle 0 | H_I \frac{1}{E_0 - H_0 + i n \epsilon} H_I \cdots H_I \\ &\times \frac{1}{E_0 - H_0 + i \epsilon} H_I | 0 \rangle_{\text{conn}}. \end{aligned} \tag{5.15}$$

Because of the limitation to connected contributions, the limit $\epsilon \rightarrow 0^+$ is harmless. Hence we get

$$E - E_0 = \sum_{n=0}^{\infty} \langle 0 | H_I \left(\frac{1}{E_0 - H_0} H_I \right)^n | 0 \rangle_{\text{conn}}, \tag{5.16}$$

which shows that indeed $E = E_{\text{GS}}$.

We can finally write down the generalization of the Gaussian effective potential we are looking for [8]:

$$V_G(\phi_0) = \frac{1}{V} \frac{\langle \Omega | H | \Omega \rangle}{\langle \Omega | \Omega \rangle}, \tag{5.17}$$

with the constraint

$$\frac{\langle \Omega | \eta(\vec{x}) | \Omega \rangle}{\langle \Omega | \Omega \rangle} = 0. \tag{5.18}$$

Note that Eq. (5.18) assures that the expectation value of the scalar field $\phi(\vec{x})$ on the state $|\Omega\rangle$ is ϕ_0 .

Several remarks are in order. Equation (5.16) shows that E reduces to E_0 in the zeroth order due to the normal ordering of the interaction Hamiltonian. Thus, in that approximation $V_G(\phi_0)$ coincides with the Gaussian effective potential.

Higher order contributions to the generalized Gaussian effective potential $V_G(\phi_0)$ can be evaluated by the Brueckner-Goldstone formula, Eq. (5.16). In this case one deals with an expansion in terms of the Goldstone diagrams [11]. However, one can do better if one uses Eqs. (5.10) and (5.11):

$$\begin{aligned} V_G(\phi_0) &= V_{\text{GEP}}(\phi_0) + \frac{1}{V} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^0 dt_1 \cdots \int_{-\infty}^0 dt_n \\ &\times \langle 0 | T[H_I(0)H_I(t_1) \cdots H_I(t_n)] | 0 \rangle_{\text{conn}}, \end{aligned} \tag{5.19}$$

where we have performed the harmless limit $\epsilon \rightarrow 0^+$. Indeed a given term in Eq. (5.19) can be easily evaluated by means

of the standard Feynman diagrams. It is evident that the free Feynman propagator coincides with the propagator of a scalar field with mass $\mu(\phi_0)$.

It should be stressed that it is convenient to analyze the expansion (5.19) by means of the Feynman diagrams. Indeed, it is well known that a Feynman diagram with k vertices contains $k!$ Goldstone diagrams, corresponding to the number of permutations of the times t_1, \dots, t_k .

The expansion, Eq. (5.19), gives rise to a set of vacuum Feynman diagrams. Let us consider the term of order n in Eq. (5.19). In the time-ordered product there are $n+1$ interaction Hamiltonians. This means that, unlike the usual vacuum diagrams the factorial factor $n!$ is canceled by the number of permutations of the independent integration variables. As a consequence, Eq. (5.19) allows a diagrammatic expansion of the higher order corrections which is amenable to a diagrammatic resummation.

Finally, we point out that in terms of Feynman diagrams, the constraint, Eq. (5.18), sets to zero the tadpolelike diagrams that are due to the linear term in the interaction Hamiltonian.

VI. CORRECTION TO THE GAUSSIAN APPROXIMATION

In the previous section we introduced the generalized Gaussian effective potential which allows us to compute in a systematic way the corrections to the Gaussian approximation. Presently we focus on the second order corrections. Let us consider the first nontrivial term in the perturbative expansion, Eq. (5.19). We have

$$V_G(\phi_0) = V_{\text{GEP}}(\phi_0) + \Delta V_G(\phi_0), \quad (6.1)$$

with

$$\Delta V_G(\phi_0) = \frac{-i}{V} \int_{-\infty}^0 dt \langle 0 | T H_I(0) H_I(t) | 0 \rangle_{\text{conn}}. \quad (6.2)$$

Taking into account Eqs. (5.18) and (4.19) we get

$$\Delta V_G(\Phi_0) = \frac{-i}{V} \int_{-\infty}^0 dt \int d^{\nu} x d^{\nu} y \left[\left(\frac{\lambda \phi_0}{3!} \right)^2 \langle 0 | T(\cdot \eta^3(x) \cdot \eta^3(y) \cdot) | 0 \rangle + \left(\frac{\lambda}{4!} \right)^2 \langle 0 | T(\cdot \eta^4(x) \cdot \eta^4(y) \cdot) | 0 \rangle \right], \quad (6.3)$$

where $x = (0, \vec{x})$ and $y = (t, \vec{y})$. Therefore we have (see Fig. 1)

$$\Delta V_G(\phi_0) = \frac{-i}{V} \int_{-\infty}^0 dt \int d^{\nu} x d^{\nu} y \left[\frac{\lambda^2 \phi_0^2}{3!} [iG_F(x, y)]^3 + \frac{\lambda^2}{4!} [iG_F(x, y)]^4 \right], \quad (6.4)$$

where the Feynman propagator is

$$G_F(x, y) = \int \frac{d^{\nu+1} k}{(2\pi)^{\nu+1}} \frac{e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon}. \quad (6.5)$$

Inserting Eq. (6.5) into Eq. (6.4) and performing the time and spatial integrations we recast Eq. (6.4) into

$$\Delta V_G(\phi_0) = -\frac{\lambda^2 \phi_0^2}{3!} \int \prod_{i=1}^3 \frac{d^{\nu} k_i}{(2\pi)^{\nu} 2g(\vec{k}_i)} \frac{(2\pi)^{\nu} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)}{g(\vec{k}_1) + g(\vec{k}_2) + g(\vec{k}_3)} - \frac{\lambda^2}{4!} \int \prod_{i=1}^4 \frac{d^{\nu} k_i}{(2\pi)^{\nu} 2g(\vec{k}_i)} \frac{(2\pi)^{\nu} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)}{g(\vec{k}_1) + g(\vec{k}_2) + g(\vec{k}_3) + g(\vec{k}_4)}. \quad (6.6)$$

Note that the lowest order contribution in the loop expansion is the two-loop diagram [diagram (a) in Fig. 1] which was lost in the Gaussian approximation. In non-Abelian gauge theories this diagram is crucial in order to maintain the gauge invariance in the variational Gaussian approximation [9].

Now we discuss the second order corrections in the case of scalar fields in $\nu = 1, 2$ spatial dimensions [13].

A. Scalar fields in 1+1 dimensions

To start with, we consider the Gaussian effective potential in one spatial dimension $\nu = 1$:

$$V_{\text{GEP}}(\phi_0) = \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + I_1(\mu) - \frac{\lambda}{8} I_0^2(\mu), \quad (6.7)$$

where

$$I_0(\mu) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{g(k)}}, \quad (6.8)$$

$$I_1(\mu) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sqrt{g(k)}. \quad (6.9)$$

Introducing an ultraviolet cutoff Λ , it is not difficult to see that the integrals (6.8) and (6.9) display quadratic and logarithmic divergences. Subtracting the energy density of the $\phi_0 = 0$ vacuum $V_{\text{GEP}}(\phi_0 = 0)$, one is left with a logarithmic divergence which can be corrected by renormalizing the

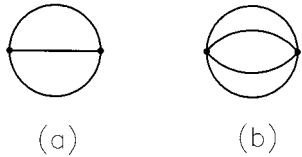


FIG. 1. Second order corrections to the Gaussian effective potential.

mass. In the Gaussian approximation the renormalized mass is defined as [1]

$$m_R^2 \equiv \left. \frac{\partial^2 V_{\text{GEP}}(\phi_0)}{\partial \phi_0^2} \right|_{\phi_0=0} = m^2 + \frac{\lambda}{2} I_0(\mu_0) = \mu_0^2, \quad (6.10)$$

where we recall that $\mu_0 = \mu(\phi_0=0)$. However, we shall see that once we consider the corrections to the Gaussian approximation the prescription Eq. (6.10) needs modification.

A serious problem of the variational approximation in quantum field theories is due to the presence of ultraviolet divergences [14]. The variational-perturbation theory discussed in this paper offers a natural solution to the ultraviolet divergence problem which parallels the perturbative renormalization theory. As a matter of fact, we showed that the generalized Gaussian effective potential $V_G(\phi_0)$ is the energy density of the vacuum $|\Omega\rangle$ with scalar condensate $\phi_0 = \langle \Omega | \phi | \Omega \rangle / \langle \Omega | \Omega \rangle$. Observing that only the energy differences are of importance, our renormalization prescription will be to reabsorb the ultraviolet divergences in the field theory without scalar condensate. Moreover, for $\phi_0=0$ the Hamiltonian, Eqs. (4.18) and (4.19), reduces to the one of a scalar field with mass μ_0 and normal-ordered quartic self-interaction. In the case of one and two spatial dimensions that theory is superrenormalizable and we only need to renormalize the mass. We define the renormalized mass by means of the zero-momentum two-point proper vertex $\Gamma^{(2)}(p; \phi_0=0)$:

$$m_R^2 = -\Gamma^{(2)}(0; \phi_0=0). \quad (6.11)$$

In the Gaussian approximation the $\phi_0=0$ Hamiltonian coincides with the free Hamiltonian of a scalar field with mass μ_0 , so that Eq. (6.11) gives

$$m_R^2 = \mu_0^2, \quad (6.12)$$

which agrees with Eq. (6.10). In one spatial dimension Eq. (6.12) eliminates completely the ultraviolet divergences, for the higher order corrections are finite.

From Eq. (6.12) and the gap equation we get

$$m^2 = m_R^2 - \frac{\lambda_0}{2} I_0(m_R). \quad (6.13)$$

Inserting Eq. (6.13) into Eq. (6.7) and using Eqs. (2.24)–(2.27) and (2.22) one obtains [1]

$$V_{\text{GEP}}^{1+1}(\phi_0) = -2\hat{\lambda}\Phi_0^4 + \frac{x-1}{24\hat{\lambda}} \left[1 + \frac{3\hat{\lambda}}{\pi} + \frac{x-1}{2} \right], \quad (6.14)$$

with the gap equation

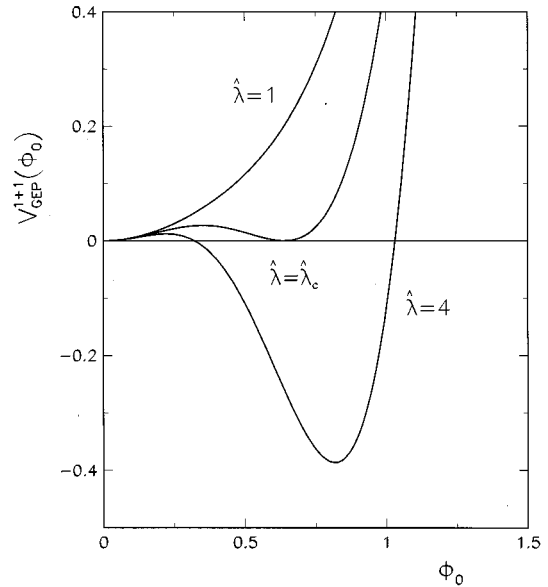


FIG. 2. The Gaussian effective potential in one spatial dimension for three different values of $\hat{\lambda}$.

$$x - 1 + \frac{3}{\pi} \ln x = 12\hat{\lambda}\Phi_0^2. \quad (6.15)$$

These results have been obtained for the first time by Chang [15]. In Fig. 2 we display $V_{\text{GEP}}^{1+1}(\Phi_0)$ as a function of Φ_0 for various values of the dimensionless coupling $\hat{\lambda}$. Several features are worth mentioning. First, $\Phi_0=0$ is always a local minimum of $V_{\text{GEP}}(\Phi_0)$. For $\hat{\lambda} < \hat{\lambda}_c$, with $\hat{\lambda}_c \approx 2.5527$, the $\Phi_0=0$ vacuum is the true ground state. On the other hand, for $\hat{\lambda} > \hat{\lambda}_c$ the ground state is for $\Phi_0 \neq 0$. As a consequence, a first order phase transition occurs at $\hat{\lambda}_c$. However, Chang [16] pointed out that the Simon-Griffiths theorem [17] rules out the possibility of a first order phase transition in one-dimensional $\lambda\phi^4$ field theory. Moreover, Chang [16] showed that there is no contradiction between the existence of a second order transition and the Simon-Griffiths theorem.

Remarkably, it turns out that the two-loop correction, diagram (a) in Fig. 1, gives rise to a second order phase transition. Indeed that correction is finite:

$$\Delta V_G(\Phi_0) = -a \frac{\hat{\lambda}^2}{x} \Phi_0^2 \mu_0^2, \quad (6.16)$$

with

$$a = \frac{3}{\pi^2} \int_{-\infty}^{+\infty} dx dy \frac{1}{\sqrt{(x^2+1)(y^2+1)}[(x+y)^2+1]} \times \frac{1}{[\sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{(x+y)^2+1}]} \approx 0.7136, \quad (6.17)$$

so that in this approximation we get

$$\frac{V_G(\Phi_0)}{\mu_0^2} = V_{\text{GEP}}^{1+1}(\Phi_0) - a \frac{\hat{\lambda}^2}{x} \Phi_0^2. \quad (6.18)$$

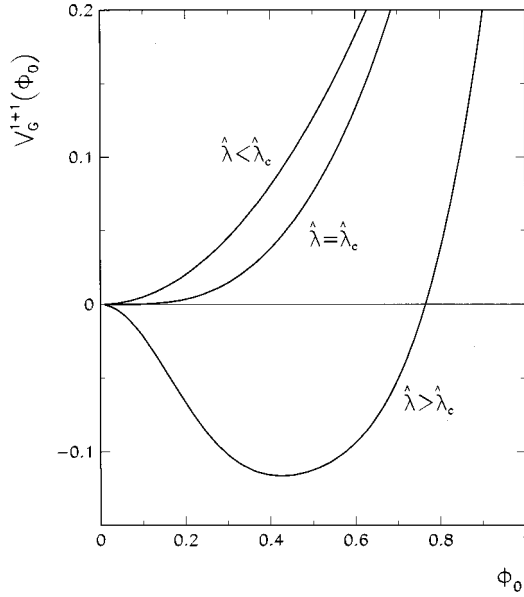


FIG. 3. The two-loop Gaussian effective potential for $\nu=1$ and three different values of $\hat{\lambda}$.

In Fig. 3 we display Eq. (6.18). We see that now there is a second order phase transition at $\hat{\lambda}_c = 1/\sqrt{2a} \approx 0.8371$. This is confirmed by considering the mass gap of the $\Phi_0=0$ vacuum:

$$m_{\text{phys}}^2 = \mu_0^2 + \Sigma(0), \quad (6.19)$$

where $\Sigma(p)$ is the proper self-mass of the $\phi_0=0$ theory. In the second order approximation $\Sigma(p)$ is given by the so-called setting sun diagram [18]. It is easy to show that

$$\Sigma(0) = -\frac{\lambda^2}{3!} \int d^2x [G_E(x)]^3, \quad (6.20)$$

where $G_E(x)$ is the Euclidean Feynman propagator:

$$G_E(x) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx}}{k^2 + \mu_0^2}. \quad (6.21)$$

$$b = \frac{3}{16\pi^2} \int_{-\infty}^{+\infty} dx dy dz \frac{1}{\sqrt{(x^2+1)(y^2+1)(z^2+1)[(x+y+z)^2+1]}} \frac{1}{\sqrt{x^2+1}\sqrt{y^2+1}\sqrt{z^2+1}\sqrt{(x+y+z)^2+1}} \approx 0.0509. \quad (6.28)$$

We would like to stress that now Eq. (6.24) is no longer valid. In the present case this does not matter due to the fact that the second order corrections are ultraviolet finite. However, in the case of two spatial dimensions these corrections are divergent. Thus, adopting the renormalization prescription, Eq. (6.24), instead of Eq. (6.11), it may lead to an incongruous result.

In Fig. 4 we contrast the generalized Gaussian effective potential in the two-loop approximation (solid lines) and in the full second order approximation (dashed lines).

A straightforward calculation gives

$$\Sigma(0) = -2a\hat{\lambda}^2\mu_0^2. \quad (6.22)$$

Thus, from Eqs. (6.19) and (6.22) we get

$$\frac{m_{\text{phys}}^2}{\mu_0^2} = \left(1 - \frac{\hat{\lambda}^2}{\hat{\lambda}_c^2}\right). \quad (6.23)$$

One can easily check that in this case

$$m_{\text{phys}}^2 = \left. \frac{\partial^2 V_G(\phi_0)}{\partial \phi_0^2} \right|_{\phi_0=0}. \quad (6.24)$$

A remarkable consequence of Eq. (6.24) is that the mass renormalization of the Gaussian effective potential extends to $V_G(\phi_0)$ in the two-loop approximation. Equation (6.23) tells us that the $\phi_0=0$ vacuum is stable for $\hat{\lambda} < \hat{\lambda}_c$. Moreover, near the critical coupling we have

$$\frac{m_{\text{phys}}}{\mu_0} \sim (\hat{\lambda}_c - \hat{\lambda})^{1/2}, \quad (6.25)$$

so that the correlation length $\xi = 1/m_{\text{phys}}$ diverges as

$$\xi \sim (\hat{\lambda}_c - \hat{\lambda})^{-\nu}, \quad \nu = \frac{1}{2}. \quad (6.26)$$

Our results are in agreement with previous studies [16,19,20]. However, our generalized Gaussian effective potential relies on a firm field-theoretical basis which allows us to take care of the higher order corrections. Moreover, in our scheme there are no ambiguities in the renormalization of the ultraviolet divergences.

Let us consider now the contribution due to diagram (b) in Fig. 1. We have

$$\Delta V_G(\phi_0) = -a \frac{\hat{\lambda}^2}{x} \phi_0^2 \mu_0^2 - b \frac{\hat{\lambda}^2}{x}, \quad (6.27)$$

where

A few comments are in order. The order of the transition is not modified by the three-loop second order correction. Moreover, the critical coupling $\hat{\lambda}_c \approx 1.1486$ is quite close to our previous value. As a matter of fact, in the critical region $\hat{\lambda} \approx 1$ the effects of the three-loop correction do not substantially change the shape of the potential. Therefore we can safely conclude that the most important contributions in the critical region are given by the two loop term. This suggests that higher order corrections do not modify the order of the transition.

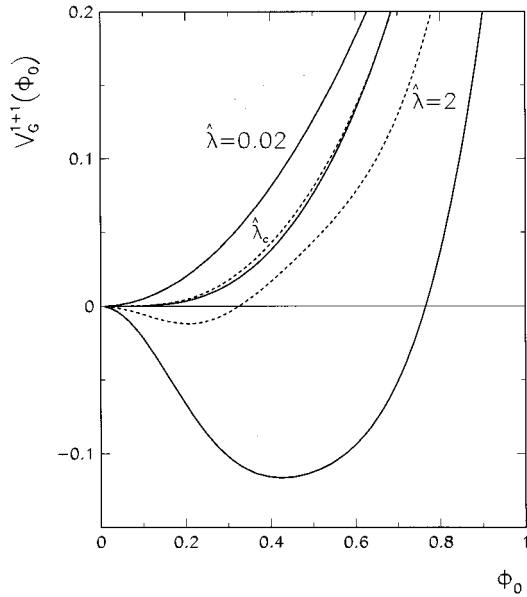


FIG. 4. The two-loop (solid lines) and second order (dashed lines) generalized Gaussian effective potentials for $\nu=1$ and three different values of $\hat{\lambda}$.

B. Scalar fields in 2+1 dimensions

In the case of two spatial dimensions the Gaussian effective potential is

$$V_{\text{GEP}}(\phi_0) = \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + I_1(\mu) - \frac{\lambda}{8} I_0^2(\mu), \quad (6.29)$$

where, now,

$$I_0(\mu) = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{g(\vec{k})}, \quad (6.30)$$

$$I_1(\mu) = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} g(\vec{k}). \quad (6.31)$$

Subtracting the energy density of the $\phi_0=0$ vacuum we get

$$V_{\text{GEP}}(\phi_0) - V_{\text{GEP}}(0) = \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + I_1(\mu) - I_1(\mu_0) - \frac{\lambda}{8} [I_0^2(\mu) - I_0^2(\mu_0)], \quad (6.32)$$

which is still affected by ultraviolet divergences.

Introducing the renormalized mass

$$m_R^2 = \left. \frac{\partial^2 V_{\text{GEP}}(\phi_0)}{\partial \phi_0^2} \right|_{\phi_0=0} = m^2 + \frac{\lambda}{2} I_0(\mu_0) = \mu_0^2 \quad (6.33)$$

and using the gap equation

$$\mu^2(\phi_0) = m^2 + \frac{\lambda}{2} \phi_0^2 + \frac{\lambda}{2} I_0(\mu), \quad (6.34)$$

we get the finite result [1]

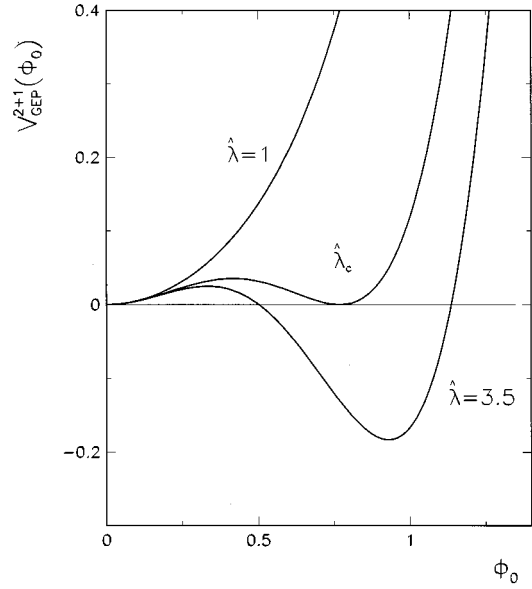


FIG. 5. The Gaussian effective potential in two spatial dimensions for three different values of $\hat{\lambda}$.

$$V_{\text{GEP}}(\Phi_0) = \frac{\Phi_0^2}{2} + \hat{\lambda} \Phi_0^4 - \frac{(\sqrt{x}-1)^2}{24\pi} \left[1 + \frac{9}{2\pi} \hat{\lambda} + 2\sqrt{x} \right]. \quad (6.35)$$

Similarly we can rewrite the gap equation Eq. (6.34) as

$$x = 1 + 12\hat{\lambda} \Phi_0^2 - \frac{3}{\pi} \hat{\lambda} (\sqrt{x}-1), \quad (6.36)$$

whose explicit positive solution is

$$\sqrt{x} = -\frac{3\hat{\lambda}}{2\pi} + \sqrt{\left(1 + \frac{3\hat{\lambda}}{2\pi}\right)^2 + 12\hat{\lambda} \Phi_0^2}. \quad (6.37)$$

In Fig. 5 we show $V_{\text{GEP}}^{2+1}(\Phi_0)$ versus Φ_0 for three different values of $\hat{\lambda}$. Again we find a first order phase transition at the critical coupling $\hat{\lambda} \approx 3.0784$ [1]. Note that in two spatial dimensions there are no rigorous results which could exclude a first order transition. Nevertheless, it is important to investigate the effects of the second order corrections to the Gaussian effective potential. From Eq. (6.4) we have

$$\Delta V_G(\Phi_0) = -i \int_{-\infty}^0 dx \int d^2z \left\{ \frac{\lambda^2 \phi_0^2}{3!} [iG_F(z)]^2 + \frac{\lambda^2}{4!} [iG_F(z)]^4 \right\}, \quad (6.38)$$

where $z = (t, \vec{x})$. Now, observing that the Feynman propagator is an even function and performing the Wick rotation, we obtain

$$\Delta V_G(\Phi_0) = -\frac{1}{2} \int d^3z_E \left\{ \frac{\lambda^2 \phi_0^2}{3!} G_E^3(z_E) + \frac{\lambda^2}{4!} G_E^4(z_E) \right\}, \quad (6.39)$$

where [21]

$$G_E(z_E) = \int \frac{d^3 k_E}{(2\pi)^3} \frac{e^{-ik_E \cdot z_E}}{k_E^2 + \mu^2} = \frac{\mu}{(2\pi)^{3/2} (\mu z)^{1/2}} K_{1/2}(\mu z), \quad (6.40)$$

with $z = |z_E|$ and $K_{1/2}$ is the modified Bessel function of order $\frac{1}{2}$:

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}. \quad (6.41)$$

Thus we get, for the Euclidean Feynman propagator,

$$G_E(z_E) = \frac{e^{-\mu z}}{4\pi z}. \quad (6.42)$$

Introducing

$$J_3(\mu) = \int d^3 z_E G_E^3(z_E) = \int \frac{d^3 z_E}{(4\pi)^3} \frac{e^{-3\mu z}}{z^3}, \quad (6.43)$$

$$J_4(\mu) = \int d^3 z_E G_E^4(z_E) = \int \frac{d^3 z_E}{(4\pi)^4} \frac{e^{-4\mu z}}{z^4}, \quad (6.44)$$

we rewrite Eq. (6.39) as

$$\Delta V_G(\phi_0) = -\frac{\lambda^2 \phi_0^2}{12} J_3(\mu) - \frac{\lambda^2}{48} J_4(\mu). \quad (6.45)$$

From Eqs. (6.43) and (6.44) we see that J_3 and J_4 are divergent. We regularize the integrals by means of an ultraviolet cutoff $\epsilon \sim 1/\Lambda$ [22]:

$$\begin{aligned} J_3(\mu, \epsilon) &= \frac{1}{16\pi^2} \int_\epsilon^\infty dz \frac{e^{-3\mu z}}{z} \\ &= -\frac{1}{16\pi^2} [\ln(\mu\epsilon) + \ln 3 + \gamma] + O(\epsilon), \end{aligned} \quad (6.46)$$

$$\begin{aligned} J_4(\mu, \epsilon) &= \frac{1}{(4\pi)^3} \int_\epsilon^\infty dz \frac{e^{-4\mu z}}{z^2} = -\frac{1}{(4\pi)^3} \left\{ \frac{1}{\epsilon} + 4\mu [\ln(\mu\epsilon) \right. \\ &\quad \left. + \ln 4 + \gamma - 1] \right\} + O(\epsilon), \end{aligned} \quad (6.47)$$

where γ is the Euler-Mascheroni constant. Putting it all together we obtain

$$\begin{aligned} V_G(\phi_0) - V_G(0) &= \frac{1}{2} m^2 \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 + I_1(\mu) - I_1(\mu_0) - \frac{\lambda}{8} [I_0^2(\mu) - I_0^2(\mu_0)] + \frac{\lambda^2 \phi_0^2}{192\pi^2} [\ln(\mu\epsilon) + \ln 3 + \gamma] \\ &\quad - \frac{\lambda^2}{768\pi^3} (\mu - \mu_0) [\ln 4 + \gamma - 1] - \frac{\lambda^2}{768\pi^3} [\mu \ln(\mu\epsilon)] - \mu_0 \ln(\mu_0\epsilon). \end{aligned} \quad (6.48)$$

Now we show that the logarithmic divergences are cured by renormalizing the mass. To this end, we observe that [1]

$$I_1(\mu) - I_1(\mu_0) = \frac{1}{2} (\mu^2 - \mu_0^2) I_0(\mu_0) - \frac{\mu_0^3}{8\pi} \left[\frac{1}{3} \left(\sqrt{\frac{\mu^2}{\mu_0^2} - 1} \right)^2 \left(2 \sqrt{\frac{\mu^2}{\mu_0^2} - 1} \right) \right], \quad (6.49)$$

$$I_0(\mu) - I_0(\mu_0) = -\frac{\mu_0}{4\pi} \left(\sqrt{\frac{\mu^2}{\mu_0^2}} \right). \quad (6.50)$$

Inserting Eqs. (6.49) and (6.50) into Eq. (6.48), and using the gap equation (6.34), we rewrite Eq. (6.48) as

$$\begin{aligned} V_G(\phi_0) - V_G(0) &= \frac{1}{2} \mu_0^2 \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 - \frac{\mu_0^3}{24\pi} \left(\sqrt{\frac{\mu^2}{\mu_0^2} - 1} \right)^2 \left(2 \sqrt{\frac{\mu^2}{\mu_0^2} - 1} \right) - \frac{\lambda}{128\pi^2} \mu_0^2 \left(\sqrt{\frac{\mu^2}{\mu_0^2} - 1} \right)^2 \\ &\quad + \frac{\lambda^2 \phi_0^2}{192\pi^2} [\ln(\mu\epsilon) + \ln 3 + \gamma] - \frac{\lambda^2}{768\pi^3} [(\mu - \mu_0)(\ln 4 + \gamma - 1) + \mu \ln(\mu\epsilon) - \mu_0 \ln(\mu_0\epsilon)]. \end{aligned} \quad (6.51)$$

As we have already discussed, the renormalized mass is

$$m_R^2 = -\Gamma(0, \phi_0 = 0) = \mu_0^2 + \Sigma(0). \quad (6.52)$$

In the lowest order Gaussian approximation we have

$$\Sigma(0) = \delta m^2 - \frac{\lambda^2}{3!} \int d^3 x_E G_E^3(x_E), \quad (6.53)$$

where the second order term is due to the ‘‘setting-sun’’ diagram. Explicitly, by using the previous regularization, we find

$$\Sigma(0) = \delta m^2 + \frac{\lambda^2}{96\pi^2} [\ln(\mu_0\epsilon) + \ln 3 + \gamma] + O(\epsilon). \quad (6.54)$$

We fix the mass counterterm by imposing that

$$m_R^2 = \mu_0^2. \quad (6.55)$$

This results in

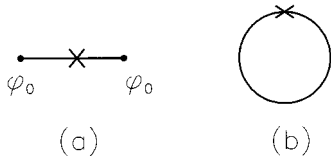


FIG. 6. Mass counterterm contributions to the generalized Gaussian effective potential in the second order approximation.

$$\delta m^2 = -\frac{\lambda^2}{96\pi^2} [\ln(\mu_0\epsilon) + \ln 3 + \gamma]. \tag{6.56}$$

As a consequence, we must introduce the following counter-term Hamiltonian in the $\phi_0=0$ theory:

$$\delta H = \frac{1}{2} \delta m^2 \int d^2x \phi^2(\vec{x}). \tag{6.57}$$

After writing $\phi(\vec{x}) = \phi_0 + \eta(\vec{x})$ and using the constraint, Eq. (5.18), it turns out that δH adds to the second order generalized Gaussian effective potential the further contributions depicted in Fig. 6:

$$\delta V_G^{(a)} = \frac{1}{2} \phi_0^2 \delta m^2, \tag{6.58}$$

$$\delta V_G^{(b)} = \frac{1}{2} \delta m^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{2g(\vec{k})}. \tag{6.59}$$

Now, observing that

$$\frac{1}{2g(\vec{k})} = \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{1}{k_0^2 + g^2(\vec{k})}, \tag{6.60}$$

we have ($\epsilon \rightarrow 0$)

$$\delta V_G^{(b)} = \frac{1}{2} \delta m^2 G_E(\epsilon) = \frac{1}{8\pi} \delta m^2 \left[\frac{1}{4\pi\epsilon} - \mu \right] + O(\epsilon). \tag{6.61}$$

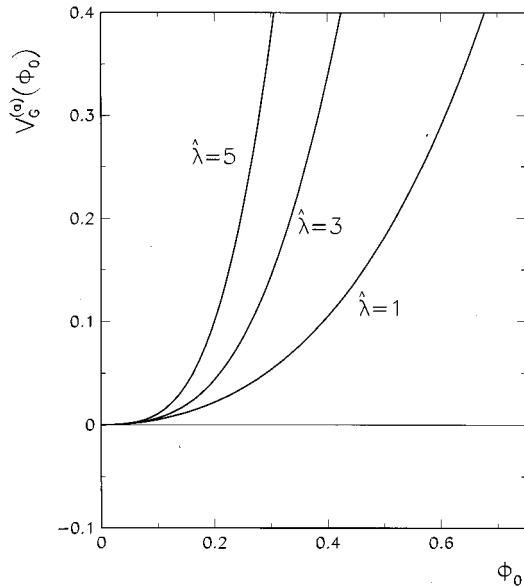


FIG. 7. The generalized Gaussian effective potential for $\nu=2$ and three different values of $\hat{\lambda}$ with the two-loops correction.

Finally, using Eq. (6.56) we get

$$\delta V_G^{(a)} = -\frac{\lambda^2 \phi_0^2}{192\pi^2} [\ln(\mu_0\epsilon) + \ln 3 + \gamma], \tag{6.62}$$

$$\delta V_G^{(b)} = -\frac{\lambda^2}{768\pi^3} [\ln(\mu_0\epsilon) + \ln 3 + \gamma] \left[\frac{1}{4\pi\epsilon} - \mu \right]. \tag{6.63}$$

Now, it is easy to see that $\delta V_G^{(a)}$ eliminates the ultraviolet divergences of the two-loop second order correction, whereas $\delta V_G^{(b)}$ makes finite the three-loop second order correction. Thus we are left with the finite result

$$\begin{aligned} V_G(\phi_0) - V_G(0) &= \frac{1}{2} \mu_0^2 \phi_0^2 + \frac{\lambda}{4!} \phi_0^4 - \frac{\mu_0^3}{24\pi} \left(\sqrt{\frac{\mu^2}{\mu_0^2}} - 1 \right)^2 \\ &\times \left(2\sqrt{\frac{\mu^2}{\mu_0^2}} - 1 \right) - \frac{\lambda}{128\pi^2} \mu_0^2 \left(\sqrt{\frac{\mu^2}{\mu_0^2}} - 1 \right)^2 \\ &+ \frac{\lambda^2 \phi_0^2}{192\pi^2} \ln\left(\frac{\mu}{\mu_0}\right) + \frac{\lambda^2}{768\pi^3} \mu \left[1 - \ln\left(\frac{\mu}{\mu_0}\right) \right. \\ &\left. - \ln\frac{4}{3} \right] - \frac{\lambda^2}{768\pi^3} \mu_0 \left[1 - \ln\frac{4}{3} \right]. \end{aligned} \tag{6.64}$$

In terms of the scaled variables, Eqs. (2.23) and (2.24), we rewrite Eq. (6.64) as

$$\begin{aligned} V_G(\Phi_0) &= V_{\text{GEP}}^{2+1}(\Phi_0) + \frac{3\hat{\lambda}^2}{\pi^2} \Phi_0^2 \ln\sqrt{x} + \frac{3}{4\pi^3} (\sqrt{x} - 1) \\ &\times \left(1 - \ln\frac{4}{3} \right) - \frac{3}{4\pi^3} \hat{\lambda}^2 \sqrt{x} \ln\sqrt{x}. \end{aligned} \tag{6.65}$$

It is worthwhile to study separately the effects of the two second order corrections. If we take into account the two-loop correction we get

$$V_G^{(a)}(\Phi_0) = V_{\text{GEP}}^{2+1}(\Phi_0) + \frac{3\hat{\lambda}^2}{\pi^2} \Phi_0^2 \ln\sqrt{x}. \tag{6.66}$$

In Fig. 7 we display Eq. (6.66) for $\hat{\lambda}=1, 3,$ and 5 . As is evident there is no spontaneous symmetry breaking [13]. Comparing Fig. 7 with Fig. 5, we see that the two-loop correction adds to the Gaussian effective potential a positive contribution which is important in the region $\Phi_0 \sim 1$ and overcomes the negative minimum displayed by $V_{\text{GEP}}^{2+1}(\Phi_0)$ in that region. On the other hand, considering the three-loop second order correction, we have

$$\begin{aligned} V_G^{(b)}(\Phi_0) &= V_{\text{GEP}}^{2+1}(\Phi_0) + \frac{3\hat{\lambda}^2}{4\pi^3} (\sqrt{x} - 1) \left(1 - \ln\frac{4}{3} \right) \\ &- \frac{3}{4\pi^3} \hat{\lambda}^2 \sqrt{x} \ln\sqrt{x}. \end{aligned} \tag{6.67}$$

From Fig. 8, where display Eq. (6.67) for three different

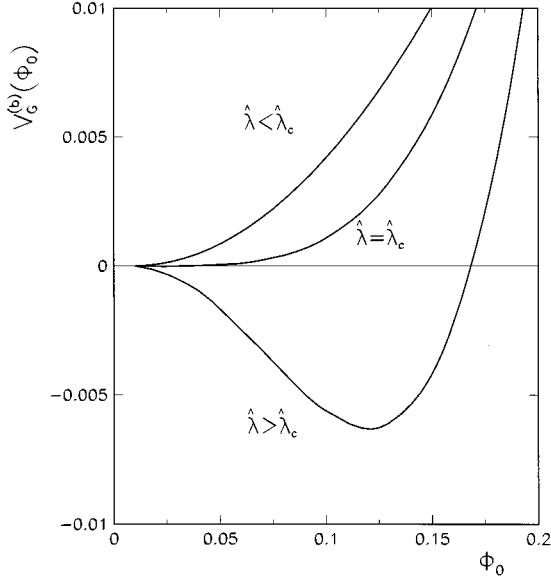


FIG. 8. The generalized Gaussian effective potential for $\nu=2$ and three different values of $\hat{\lambda}$ with the three-loops correction.

values of $\hat{\lambda}$, we deduce that the most important effects of the three-loop second order correction are near the origin where one gets

$$\Delta V_G^{(b)}(\Phi_0) \sim -\frac{18}{4\pi^3} \ln \frac{4}{3} \frac{\hat{\lambda}^3}{1+3\hat{\lambda}/2\pi} \Phi_0^2. \quad (6.68)$$

Indeed, Fig. 8 shows that $V_G^{(b)}(\Phi_0)$ undergoes a continuous phase transition at $\hat{\lambda}_c = 3.0959$. This feature persists even for the full second order generalized Gaussian effective potential Eq. (6.65) (see Fig. 9). Our result is in qualitative agreement with Refs. [23] and [24]. However, from Fig. 9 we see that the condensation energy is very small. Moreover, the critical

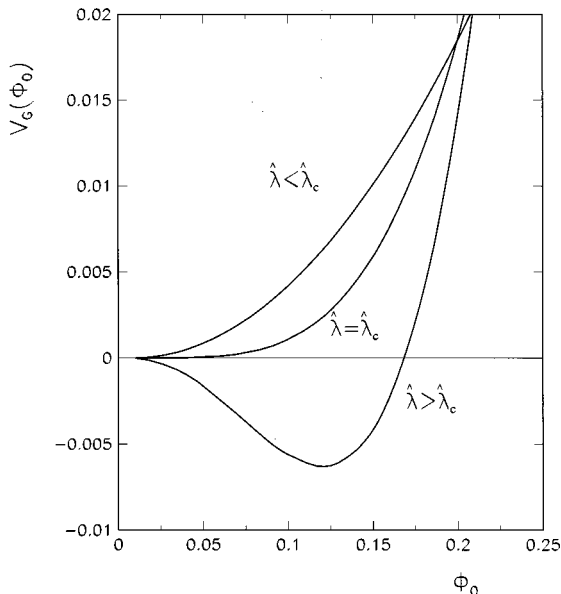


FIG. 9. The second order generalized Gaussian effective potential for $\nu=2$ and three different values of $\hat{\lambda}$.

coupling $\hat{\lambda}_c = 3.0959$ differs from that of the Gaussian effective potential by less than 1%. We feel that the only sound conclusion we can draw is the exclusion of a first order phase transition. Note that unlike what is stated in Ref. [24], the absence of a broken phase is not in contradiction with the analysis by Magruder [25] and Chang and Magruder [26].

We would like to conclude this rather technical section by stressing the most important achievements of our analysis. Our analysis of the ultraviolet divergences in two spatial dimensions showed that our renormalization procedure works up to second order. However, it is clear that our renormalization can be extended to higher orders by the usual renormalization procedure. Thus we feel that our results put the generalized Gaussian effective potential on the same level as the effective potential.

VII. THERMAL CORRECTIONS TO THE GENERALIZED GAUSSIAN EFFECTIVE POTENTIAL

The aim of this section is to study the thermal corrections to the generalized Gaussian effective potential. For the reader's convenience, let us first recall the essential points of the finite temperature effective potential [27–29] and the finite temperature Gaussian effective potential [30,31].

Following the classical paper by Dolan and Jackiw [29] the finite temperature effective potential in the one-loop approximation is given by

$$V_\beta^{1\text{loop}}(\phi_0) = \frac{1}{2\beta} \sum_n \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \ln(E^2 + \omega_n^2), \quad (7.1)$$

where $\omega_n = 2\pi\beta n$, $\beta = 1/T$, are the Matsubara's frequencies [32], and $E^2 = \vec{k}^2 + M^2(\phi_0)$, $M^2(\phi_0) = m^2 + (\lambda/2)\phi_0^2$. Performing the sum over n one finds [29]

$$V_\beta^{1\text{loop}}(\phi_0) = \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{E}{2} + \frac{1}{\beta} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \ln[1 - e^{-\beta E}]. \quad (7.2)$$

On the right-hand of Eq. (7.2) the first term is the one-loop zero temperature effective potential, while the second term gives the one-loop thermal corrections.

As concerns the Gaussian effective potential at finite temperature, we shall follow Hajj and Stevenson [30]. Let us consider a system in thermal equilibrium; this means that our system has minimized its free energy,

$$F = -\frac{1}{\beta} \ln Z, \quad (7.3)$$

where Z is the partition function:

$$Z = \text{Tr}(e^{-\beta H}). \quad (7.4)$$

In order to evaluate the thermal corrections to the Gaussian effective potential, the authors of Ref. [30] split the Hamiltonian as

$$H = H_0 + H_I, \quad (7.5)$$

where H_0 is the Hamiltonian of a scalar field with variational mass M^2 , while H_I comprises the remainder. The variational mass is fixed by minimizing the free energy, Eq. (7.3). To do

this one uses the thermodynamic perturbation theory to evaluate the free energy in the lowest order in the perturbation Hamiltonian. Writing

$$e^{-\beta(H_0+H_I)} \simeq e^{-\beta H_0}(1 - \beta H_I), \quad (7.6)$$

we get

$$Z \simeq \text{Tr} e^{-\beta H_0} [1 - \beta \langle H_I \rangle^\beta], \quad (7.7)$$

where $\langle O \rangle^\beta$ means the thermal average with respect to H_0 :

$$\langle O \rangle^\beta = \frac{\text{Tr}(e^{-\beta H_0} O)}{\text{Tr}(e^{-\beta H_0})}. \quad (7.8)$$

From Eqs. (7.3) and (7.7) we obtain

$$F = -\frac{1}{\beta} \ln \text{Tr}(e^{-\beta H_0}) + \langle H_I \rangle^\beta. \quad (7.9)$$

Now observing that the thermal average involves a summation over the eigenstates of H_0 , it is not too difficult to find [30]

$$V_{\text{GEP}}^T(\phi_0) = \frac{F}{V} = I_1 + I_1^\beta - \frac{\lambda}{8} (I_0 + I_0^\beta)^2 + \frac{m^2}{2} \phi_0^2 + \frac{\lambda}{4!} \phi_0^4, \quad (7.10)$$

where

$$I_1^\beta \equiv \frac{1}{\beta} \int \frac{d^v k}{(2\pi)^v} \ln(1 - e^{-\beta g(\vec{k})}), \quad (7.11)$$

$$I_0^\beta \equiv \int \frac{d^v k}{(2\pi)^v} \frac{1}{g(\vec{k})} \frac{1}{e^{\beta g(\vec{k})} - 1}, \quad (7.12)$$

with $g(\vec{k}) = \sqrt{\vec{k}^2 + M^2}$. It turns out that the mass M satisfies the thermal gap equation:

$$M^2 = m^2 + \frac{\lambda}{2} [I_0 + I_0^\beta + \phi_0^2]. \quad (7.13)$$

A remarkable consequence of Eq. (7.10) is that the finite temperature Gaussian effective potential can be obtained from $V_{\text{GEP}}(\phi_0)$ with the substitution rule:

$$I_0 \rightarrow I_0 + I_0^\beta, \quad (7.14)$$

$$I_1 \rightarrow I_1 + I_1^\beta. \quad (7.15)$$

The main drawback of the Hajj-Stevenson approach is that the splitting of the Hamiltonian in Eq. (7.5) is not natural, for the variational mass M , which is fixed by minimizing the free energy, depends on the approximation adopted in evaluating perturbatively the free energy. Moreover, the calculations of the thermal corrections beyond the Gaussian approximation are very difficult. On the other hand, as we have already discussed in Sec. IV, in our approach the Hamiltonian is split into two pieces, the free Hamiltonian and the interaction, in a natural manner.

As a consequence the thermal corrections to the generalized Gaussian effective potential can be evaluated easily by means of the familiar thermodynamic perturbation theory

[33]. In the remainder of this section we focus on the lowest order thermal corrections and compare with the one-loop thermal effective potential corrections and the finite temperature Gaussian effective potential. The higher order thermal corrections will be discussed in the next section.

In the lowest order in the perturbation we write [34]

$$F = -\frac{1}{\beta} \ln \text{Tr}(e^{-\beta H_0}) + \langle H_I \rangle^\beta, \quad (7.16)$$

where, now, H_0 is given by Eq. (4.18) and H_I by Eq. (4.19). Observing that the eigenstates of H_0 are the states $|n\rangle$, Eq. (4.11), with eigenvalues E_n given Eq. (4.17), we have

$$\langle H_I \rangle^\beta = \frac{\text{Tr} e^{-\beta H_0} H_I}{\text{Tr} e^{-\beta H_0}} = \frac{\sum_n e^{-\beta E_n} \langle n | H_I | n \rangle}{\sum_n e^{-\beta E_n}}. \quad (7.17)$$

According to our definition Eq. (4.3) we have $\langle n | H_I | n \rangle = 0$, and so we end up with

$$\langle H_I \rangle^\beta = 0. \quad (7.18)$$

The calculation of the partition function $Z_0 = \text{Tr} e^{-\beta H_0}$ is straightforward:

$$\begin{aligned} Z_0 &= \text{Tr} e^{-\beta H_0} = e^{-\beta E_0} \text{Tr} \exp\left(-\beta \sum_{\vec{k}} g(\vec{k}) a^\dagger \vec{k} a_{\vec{k}}\right) \\ &= e^{-\beta E_0} \prod_{\vec{k}} \frac{1}{1 - e^{-\beta g(\vec{k})}} = e^{-\beta E_0} \\ &\quad \times \exp\left(-V \int \frac{d^v k}{(2\pi)^v} \ln[1 - e^{-\beta g(\vec{k})}]\right), \end{aligned} \quad (7.19)$$

where $g(\vec{k}) = \sqrt{\vec{k}^2 + \mu^2(\phi_0)}$, $\mu^2(\phi_0)$ satisfies the zero temperature gap equation (2.19), and $E_0 = E[\phi_0, g(\vec{k})]$, Eq. (4.6). The insertion of Eqs. (7.18) and (7.19) into Eq. (7.16) leads to

$$V_G(\phi_0) = \frac{F}{V} = V_{\text{GEP}}(\phi_0) + \frac{1}{\beta} \int \frac{d^v k}{(2\pi)^v} \ln(1 - e^{-\beta g(\vec{k})}). \quad (7.20)$$

Note that Eq. (7.20) differs from the finite temperature Gaussian effective potential Eq. (7.16). The difference resides in the different use of the gap equation. In our scheme the gap equation (2.19) is fixed once and for all. In particular it does not depend on the temperature. On the other hand, in the finite temperature Gaussian effective potential approach the gap equation includes the thermal effects. The gap equation fixes the basis to sum over in the thermal average, so that different gap equations lead to inequivalent basis. In fact the discrepancy between our results, Eq. (7.20), and the finite temperature Gaussian effective potential comes from the thermal average of the interaction Hamiltonian. In our approach Eq. (7.18) holds, whereas in Ref. [30] $\langle H_I \rangle^\beta \neq 0$. Note that the possibility of a nonequivalent basis is a peculiar feature of quantum systems with an infinite number of degrees of freedom.

It is worthwhile to compare the one-loop thermal correction to the effective potential with our finite temperature gen-

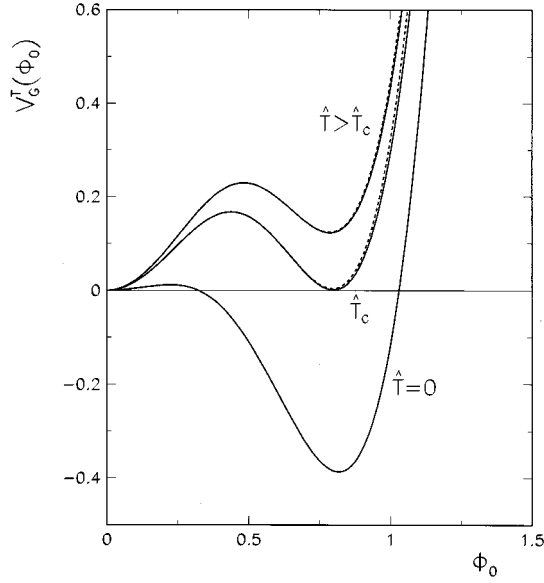


FIG. 10. The generalized Gaussian effective potential with the lowest order thermal corrections for $\nu=1$ and $\hat{\lambda}=4$. Dashed lines refer to the high temperature expansion.

erized Gaussian effective potential, Eq. (7.20). Comparing Eq. (7.2) with Eq. (7.20) we see that the former agrees with the latter if

$$M^2(\phi_0) = m^2 + \frac{\lambda}{2} \phi_0^2 \rightarrow \mu^2(\phi_0). \quad (7.21)$$

Now $\mu^2(\phi_0)$ satisfies the gap equation (2.19) which, from a diagrammatic point of view, is obtained by summing the infinite set of the superdaisy graphs in the zero temperature propagator. In other words, if in the thermal corrections of the effective potential we replace the tree level mass of the shifted theory with the mass $\mu^2(\phi_0)$ obtained by summing the superdaisy graphs at $T=0$ in the propagator, then we obtain again a free energy density. Up to now this remarkable result in thermal scalar field theories holds for the one-loop approximation. In the next section we will show that it extends to higher order thermal corrections too.

Let us analyze the lowest order thermal corrections, Eq. (7.20), in the case $\nu=1,2$ [34]. In one spatial dimension Eq. (7.20) reads

$$V_G^T(\Phi_0) = V_{\text{GEP}}^{1+1} + \frac{1}{\pi \hat{\beta}^2} \int_0^\infty dt \ln[1 - e^{-\sqrt{t^2 + \hat{\beta}^2 x}}], \quad (7.22)$$

where $\hat{\beta} = \beta \mu_0$. In Fig. 10 we show $V_G^T(\Phi_0) - V_G^T(0)$ (in units of μ_0^2) versus Φ_0 for $\hat{\lambda} > \hat{\lambda}_c$. As we can see, the symmetry broken at $T=0$ gets restored for $\hat{T} > \hat{T}_c$. Obviously the critical temperature depends on $\hat{\lambda}$. For $\hat{\lambda}=4$ we find $\hat{T}_c \approx 1.27$. It turns out that \hat{T}_c can be estimated, within a few percent, by means of the high temperature expansion of the integral in Eq. (7.22). From the results of the Appendix A [see Eq. (A16)] we find the high temperature expansion

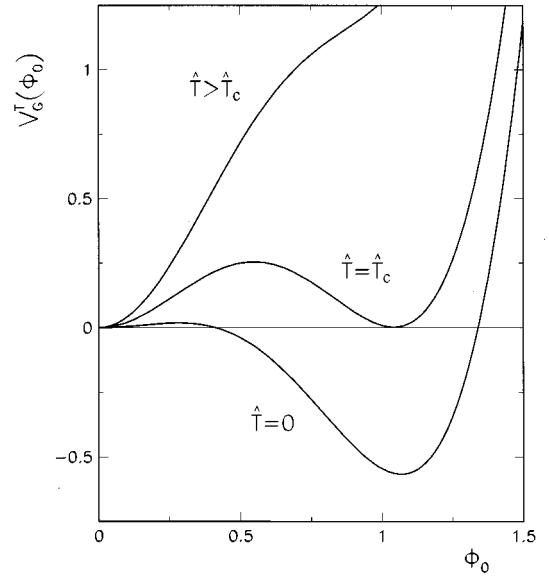


FIG. 11. The generalized Gaussian effective potential with the lowest order thermal corrections for $\nu=2$ and $\hat{\lambda}=4$. The critical temperature is $\hat{T}_c \approx 1.60$.

$$V_G^T(\Phi_0) = V_{\text{GEP}}^{1+1}(\Phi_0) - \frac{\pi}{2\hat{\beta}^2} + \frac{\sqrt{x}}{2\hat{\beta}} + \frac{x}{4\pi} \ln\left(\frac{\hat{\beta}\sqrt{x}}{4\pi}\right) - \frac{\zeta(3)}{64\pi^3} x^2 \hat{\beta}^2 + \frac{\zeta(5)}{512\pi^5} \hat{\beta}^4 x^3 + O(\hat{\beta}^6), \quad (7.23)$$

where $\zeta(z)$ is the Riemann's zeta function. In Fig. 10 we also show the high temperature expansion, Eq. (7.23) (dashed lines); as we can see, the high temperature expansion is a very good approximation even near the critical temperature. Indeed, for $\hat{\lambda}=4$, Eq. (7.23) predicts a critical temperature which differs by less than 1% from the numerically estimated value.

The case of two spatial dimensions can be dealt with in a similar way. We have

$$V_G^T(\Phi_0) = V_{\text{GEP}}^{2+1}(\Phi_0) + \frac{1}{2\pi \hat{\beta}^3} \int_0^\infty dt t \ln(1 - e^{-\sqrt{t^2 + \hat{\beta}^2 x}}). \quad (7.24)$$

In Fig. 11 we display Eq. (7.24) [we subtract the temperature-dependent constant $V_G^T(0)$] for three different values of the temperature and $\hat{\lambda}=4$. Again the thermal corrections lead to the expected symmetry restoration at high temperatures. As in the previous case we performed the high temperature expansion of the integral in Eq. (7.24). We find (see Appendix A):

$$V_G^T(\Phi_0) = V_{\text{GEP}}^{2+1}(\Phi_0) + \frac{x-1}{8\pi \hat{\beta}} - \frac{x}{8\pi \hat{\beta}} \ln(\hat{\beta}^2 x) + \frac{1}{8\pi \hat{\beta}} \ln \hat{\beta}^2 + \frac{x^{3/2}-1}{12\pi} - \frac{\hat{\beta}(x^2-1)}{92\pi}. \quad (7.25)$$

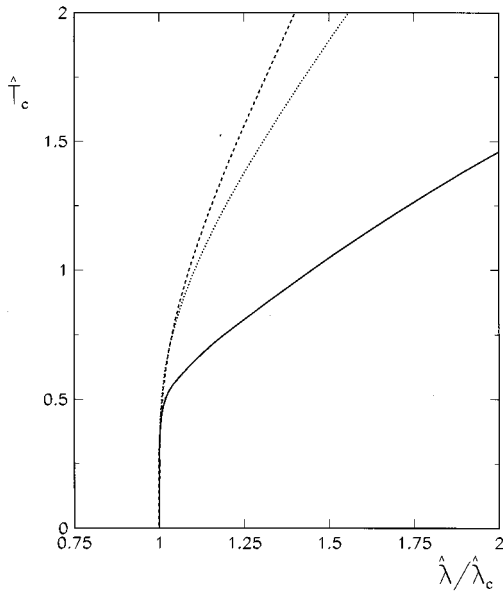


FIG. 12. The critical temperature versus the coupling $\hat{\lambda}$ for the one-loop effective potential (dotted line), the Gaussian effective potential (dashed line), and the generalized Gaussian effective potential (solid line) in two spatial dimensions.

However, we would like to stress that the expansion parameter in the above-mentioned integral is $\hat{\beta}^2 x$, so that in the region $\Phi_0 \sim 1$ where $x \gg 1$ the high temperature expansion Eq. (7.25) breaks down. In Appendix A we develop an alternative expansion which is useful in the region $\hat{\beta}^2 x \gg 1$.

To conclude this section, it is worthwhile to perform a quantitative comparison of our generalized Gaussian effective potential with the finite temperature Gaussian effective potential and the one-loop thermal effective potential. For definiteness we focus on the critical temperature as a function of the coupling constant λ in the case of two spatial dimensions. In this case the one-loop thermal effective potential reads (assuming unit mass)

$$V_{\beta}^{1\text{loop}}(\Phi_0) = \frac{1}{2}\Phi_0^2 + \hat{\lambda}\Phi_0^4 + \frac{1}{8\pi}(1 + 12\hat{\lambda}\Phi_0^2) - \frac{1}{12\pi} \times (1 - 12\hat{\lambda}\Phi_0^2)^{3/2} - \frac{1}{24\pi} + I_1^{\beta}(1 + 12\hat{\lambda}\Phi_0^2). \tag{7.26}$$

As concerns the finite temperature Gaussian effective potential, the critical temperature can be extracted from Eqs. (7.10), (7.11), and (7.12) with $\nu=2$. In Fig. 12 we compare the critical temperature as a function of $\hat{\lambda}$ (in units of $\hat{\lambda}_c$) for the three different potentials. We see that our finite temperature generalized Gaussian effective potential leads to a critical temperature which increases more slowly than for the other two potentials. This is due to our choice of the variational basis which implies $\langle H_I \rangle^{\beta} = 0$.

VIII. FINITE TEMPERATURE DIAGRAMMATIC EXPANSION

In the previous section we evaluated the lowest order thermal corrections by means of the thermodynamic pertur-

bation theory. Presently we would like to calculate the higher order thermal corrections. To do this the usual thermodynamic perturbation theory is useless. Instead we may follow Matsubara's methods [32,33]. In Matsubara's scheme one deals with scalar fields which depend on the fictitious imaginary time τ varying in the interval $(0, \beta)$. If the Hamiltonian of the system in thermal equilibrium can be written as $H = H_0 + H_I$, then one can show that the corrections to the thermodynamic potential are given by (see Appendix B)

$$\Delta\Omega = -\frac{1}{\beta} \ln \left\langle T_{\tau} \exp \left\{ - \int_0^{\beta} H_I(\tau) d\tau \right\} \right\rangle^{\beta}, \tag{8.1}$$

where $H_I(\tau)$ is the interaction Hamiltonian in Matsubara's interaction representation. T_{τ} is the τ -ordering operator and the thermal averages are done with respect to the free field partition function. Moreover, it turns out that only the connected diagrams contribute to $\Delta\Omega$:

$$\Delta\Omega = -\frac{1}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_0^{\beta} d\tau_1 \cdots d\tau_m \times \langle T_{\tau} [H_I(\tau_1) \cdots H_I(\tau_m)] \rangle_{\text{conn}}^{\beta}. \tag{8.2}$$

In our case, if we write

$$V_G^T(\phi_0) = V_{\text{GEP}}(\phi_0) + \frac{1}{\beta} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \ln(1 - e^{-\beta g(\vec{k})}) + \Delta V_G^T(\phi_0), \tag{8.3}$$

we readily get

$$\Delta V_G^T(\phi_0) = -\frac{1}{\beta V} \sum_{m=2}^{\infty} \frac{(-1)^m}{m!} \int_0^{\beta} d\tau_1 \cdots d\tau_m \times \langle T_{\tau} [H_I(\tau_1) \cdots H_I(\tau_m)] \rangle_{\text{conn}}^{\beta}. \tag{8.4}$$

Note that, due to Eq. (7.18), the sum in Eq. (8.4) starts from $m=2$. The thermal average of the time-ordered products is evaluated by means of the Wick's theorem for thermal fields [35]. In this way we obtain the thermal corrections to the generalized Gaussian effective potential by means of the connected thermal vacuum diagrams. For instance, the second order thermal corrections are displayed in Fig. 13. The vertices can be extracted from the interaction Hamiltonian, Eq. (4.19). The solid lines in Fig. 13 are the thermal propagators of the free scalar fields with mass $\mu(\phi_0)$:

$$G_{\beta}(\vec{x} - \vec{y}, \tau_1 - \tau_2) = \langle T_{\tau} \eta(\vec{x}, \tau_1) \eta(\vec{y}, \tau_2) \rangle^{\beta} = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{e^{i[\vec{k} \cdot (\vec{x} - \vec{y}) - \omega_n(\tau_1 - \tau_2)]}}{\omega_n^2 + g^2(\vec{k})}, \tag{8.5}$$

where $\omega_n = 2\pi n\beta$. Note that the thermal propagator is periodic in the time variable with period $2\pi\beta$.

A distinguishing feature of the graphical expansion of Eq. (8.4) with respect to Eq. (5.19) stems from the fact that the factor $(m!)^{-1}$ coming from the m th order term is not completely canceled by the number of different Wick contrac-

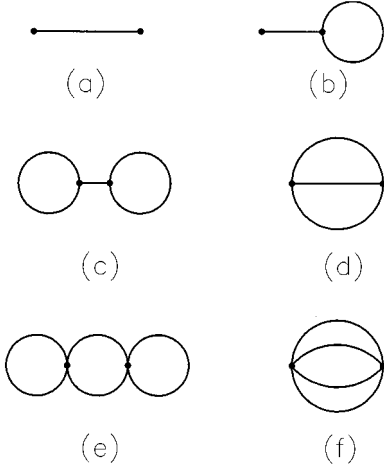


FIG. 13. Thermal Feynman diagrams contributing to the second order thermal corrections to the generalized Gaussian effective potential.

tions corresponding to a given graph. Consequently, a graph contributes to $\Delta V_G^T(\phi_0)$ in proportion to a combinatoric coefficient depending on the order of the graph. Moreover, in evaluating the contribution due to a given graph one should take care of the normal-ordering prescription in the interaction Hamiltonian. In turns out that normal ordering in H_I modifies the so-called anomalous diagrams, i.e., the diagrams which vanish at zero temperature [36]. For instance, in Fig. 13 diagrams (b), (c), and (e) are anomalous. To see this, we note that normal ordering in H_I is ineffective when we consider a thermal contraction of two scalar fields belonging to different vertices. Therefore, normal ordering comes into play when we contract two fields which belong to the same vertex. In this case we get the thermal average

$$\tilde{G}_\beta(0) = \langle T_\tau : \eta(\vec{x}, \tau) \eta(\vec{x}, \tau) : \rangle^\beta, \quad (8.6)$$

instead of $G_\beta(0)$. Taking into account canonical commutation relations between the creation and annihilation operators it is straightforward to show that

$$\tilde{G}_\beta(0) = G_\beta(0) - \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{2g(\vec{k})}. \quad (8.7)$$

Now we observe that

$$\int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{2g(\vec{k})} = \lim_{\beta \rightarrow \infty} G_\beta(0). \quad (8.8)$$

Indeed, from Eq. (8.5) it follows that

$$G_\beta(0) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{\omega_n^2 + g^2(\vec{k})}. \quad (8.9)$$

By using the well-known identity [21]

$$\coth(x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}, \quad (8.10)$$

we rewrite Eq. (8.9) as

$$G_\beta(0) = \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{2g(\vec{k})} \coth\left[\frac{\beta g(\vec{k})}{2}\right]. \quad (8.11)$$

Finally, performing the limit $\beta \rightarrow \infty$ in Eq. (8.11) we recover Eq. (8.8).

Using Eq. (8.11) we rewrite Eq. (8.7) as

$$\tilde{G}_\beta(0) = \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \frac{1}{2g(\vec{k})} \left[\coth\left(\frac{\beta}{2} g(\vec{k})\right) - 1 \right]. \quad (8.12)$$

Note that $\tilde{G}_\beta(0)$ is free from ultraviolet divergences in any spatial dimensions. This means that the ultraviolet divergences due to the tadpole $G_\beta(0)$ are cured by normal ordering the Hamiltonian at $T=0$, in accordance with the well-known result that the thermal corrections in quantum field theories are ultraviolet finite [37].

We are now in the position of extending the result implied by Eq. (7.21) to the higher order thermal corrections. To this end we observe that the higher order thermal corrections to the effective potential are given by Eq. (1.9) of Ref. [29]. Observing that in the imaginary time formalism the interaction Lagrangian agrees with the interaction Hamiltonian and that the Gaussian functional integrations with periodic boundary conditions in Ref. [29] correspond to the thermal Wick theorem, we obtain the desired result. There are, however, two further points which need to be discussed. First, our interaction Hamiltonian is normal ordered at $T=0$. However, our previous discussion tells us that the normal ordering does not affect the thermal corrections, for $\tilde{G}_\beta(0)$ and $G_\beta(0)$ differ by a temperature-independent term. Second, in Ref. [29] there is not the linear term in the shifted scalar field. This means that our substitution rule, Eq. (7.21), holds for the physically relevant on-shell thermal effective potential.

Let us now explicitly evaluate the second order thermal corrections in the case of one spatial dimension. From Eq. (8.4) we have

$$\Delta V_G^T(\phi_0) = -\frac{1}{2\beta V} \int_0^\infty d\tau_1 d\tau_2 \langle T_\tau H_I(\tau_1) H_I(\tau_2) \rangle_{\text{conn}}^\beta, \quad (8.13)$$

which gives rise to the diagrams depicted in Fig. 13.

It is easy to see that graph (a) is temperature independent. So it does not contribute to $\Delta V_G^T(\phi_0)$ due to the stability condition $\langle \Omega | \eta | \Omega \rangle = 0$. As concerns graph (b) we get

$$\begin{aligned} \text{(b)} &= -\frac{\lambda \phi_0}{4\beta V} \left(\mu^2 \phi_0 - \frac{\lambda}{3} \phi_0^3 \right) \int_{-\infty}^{+\infty} dx dy \\ &\times \int_0^\beta d\tau_1 d\tau_2 \langle T_\tau \eta(x, \tau_1) \eta(y, \tau_2) \rangle^\beta \\ &\times \langle T_\tau : \eta(x, \tau_2) \eta(y, \tau_2) : \rangle^\beta. \end{aligned} \quad (8.14)$$

According to our previous discussion we obtain

$$\text{(b)} = -\frac{\lambda \phi_0^2}{4} \left(1 - \frac{\lambda}{3\mu^2} \phi_0^2 \right) \tilde{G}_\beta(0). \quad (8.15)$$

In a similar way we find

$$(c) = -\frac{\lambda^2 \phi_0^2}{8} \frac{1}{\mu^2} \tilde{G}_\beta^2(0). \quad (8.16)$$

For the graph (d) we have

$$(d) = -\frac{\lambda^2 \phi_0^2}{12} \int_{-\beta/2}^{+\beta/2} d\tau \int_{-\infty}^{+\infty} dx G_\beta^3(x, \tau). \quad (8.17)$$

Using Eq. (8.5) and the result [see Appendix B, Eq. (B32)]

$$\frac{1}{\beta} \sum_n \frac{e^{i\omega_n \tau}}{\omega_n^2 + g^2(\vec{k})} = \frac{e^{-g(\vec{k})|\tau|}}{2g(\vec{k})} + \frac{1}{2g(\vec{k})} [e^{g(\vec{k})\tau} + e^{-g(\vec{k})\tau}] \frac{1}{e^{\beta g(\vec{k})} - 1}, \quad (8.18)$$

we get

$$(d) = -\frac{\lambda^2 \phi_0^2}{48(2\pi)^2} \int_0^{\beta/2} d\tau \int_{-\infty}^{+\infty} \frac{dk_1 dk_2}{g(k_1)g(k_2)g(k_3)} \times \prod_{i=1}^3 \left[e^{-g(k_i)\tau} + \frac{e^{g(k_i)\tau} + e^{-g(k_i)\tau}}{e^{\beta g(k_i)} - 1} \right], \quad (8.19)$$

where $\sum_{i=1}^3 k_i = 0$. Finally, using again Eq. (8.18) we find

$$(e) = -\frac{\lambda^2}{32(2\pi)} \tilde{G}_\beta^2(0) \int_0^{\beta/2} d\tau \int_{-\infty}^{+\infty} \frac{dk}{g^2(k)} \left[e^{-g(k)\tau} + \frac{e^{g(k)\tau} + e^{-g(k)\tau}}{e^{\beta g(k)} - 1} \right]^2, \quad (8.20)$$

$$(f) = -\frac{\lambda^2 V}{384(2\pi)^3} \int_0^{\beta/2} d\tau \times \int_{-\infty}^{+\infty} \frac{dk_1 dk_2 dk_3}{g(k_1)g(k_2)g(k_3)g(k_4)} \times \prod_{i=1}^4 \left[e^{-g(k_i)\tau} + \frac{e^{g(k_i)\tau} + e^{-g(k_i)\tau}}{e^{\beta g(k_i)} - 1} \right], \quad (8.21)$$

with $\sum_{i=1}^4 k_i = 0$.

Some comments are in order. In Eqs. (8.19)–(8.21) the τ integration can be performed explicitly, while the remaining integrations over the momenta must be handled numerically. In the limit $\beta \rightarrow \infty (T \rightarrow 0)$ the anomalous graphs (b), (c), and (e) go exponentially to zero due to the factor $\tilde{G}_\beta(0)$. On the other hand, graphs (d) and (f) reduce to the zero temperature second order generalized Gaussian effective potential. Indeed, in that limit in Eqs. (8.19)–(8.21) only the factors $e^{-g(k_i)\tau}$ survive. Performing the elementary τ integration we obtain the zero temperature contributions. As a consequence the zero temperature limit of $V_G^T(\phi_0)$ reduces to $V_G(\phi_0)$.

In the high temperature limit we find that graphs (e) and (f) dominate:

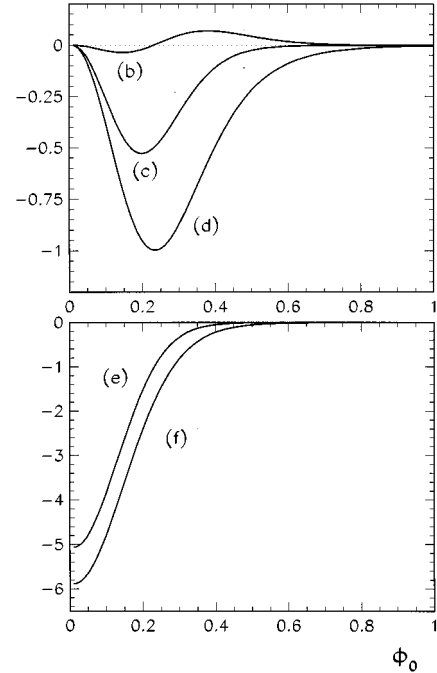


FIG. 14. Contributions due to the second order thermal corrections to the generalized Gaussian effective potential for $\nu=1$, $\hat{\lambda}=4$, and $\hat{\beta}=1$.

$$(e) \sim -\frac{\lambda^2}{256} \frac{1}{\beta^3 \mu^5}, \quad (8.22)$$

$$(f) \sim -\frac{\lambda^2}{1536} \frac{1}{\beta^3 \mu^5}. \quad (8.23)$$

Therefore, in the intermediate temperature region $\hat{\beta} \sim 1$ we expect that the main contribute to $V_G^T(\phi_0)$ comes from graphs (d), (e), and (f). This is indeed the case as shown in Fig. 14 where we display the contributions due to the second order graphs for $\hat{\beta}=1$.

In Fig. 15 we display the finite temperature generalized Gaussian effective potential (in units of μ_0^2) for three different values of \hat{T} and $\hat{\lambda} > \hat{\lambda}_c$. We see that the symmetry broken at $T=0$ gets restored by increasing the temperature through a continuous phase transition.

We have also performed the analysis of the second order thermal corrections in two spatial dimensions. The calculation are very similar to the previous case. Moreover, we find that the contributions to the second order thermal corrections behave similarly to the ones of the one-dimensional case. So we do not discuss any further this matter.

IX. CONCLUSIONS

In this paper we have developed a perturbation theory with a variational basis for self-interacting scalar quantum field theories. Our aim was to evaluate in a systematic manner the corrections to the variational Gaussian approximation. In particular we introduced the generalized Gaussian effective potential which allowed us to determine the corrections to the Gaussian effective potential [38].

Our method has been illustrated in the case of self-

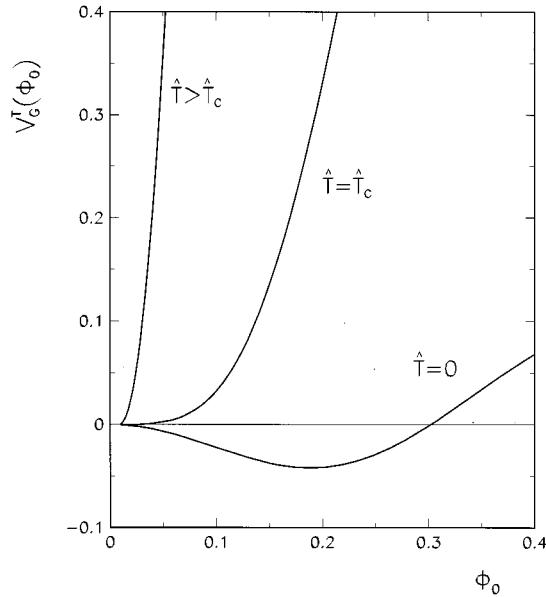


FIG. 15. The generalized Gaussian effective potential with second order thermal corrections for $\nu=1$, $\hat{\lambda}=4$, and three different values of the temperature.

interacting scalar fields. However, we feel that there are no problems in extending our method to scalar fields with continuous internal symmetry. As a matter of fact, recently our approach has been applied to scalar fields with $O(2)$ internal symmetry [39].

One of the most serious problems of the variational approximation in quantum field theories is due to the appearance of the ultraviolet divergences. The variational-perturbation theory developed in the present paper offers a solution to the ultraviolet divergence problem which is similar to the well-known perturbative renormalization theory. Indeed, starting from the fact that the generalized Gaussian effective potential by definition is the vacuum energy density in the presence of a scalar condensate, we showed that the divergences are corrected by the counterterms of the underlying field theory without scalar condensate.

We would like to stress that, to our knowledge, there are no rigorous results on the problem of ultraviolet divergences in the variational approach to quantum field theories. For this reason we focused on scalar field theories in one and two spatial dimensions, where one only needs to renormalize the mass. In one spatial dimension we showed that the lowest order renormalization of the mass assures that the higher order corrections are finite. In the case of two spatial dimensions we find that our mass renormalization procedure works up to second order. However, it should be clear that our prescription can be extended to higher orders without problems.

In the second part of the paper we studied the thermal corrections to our effective potential. In particular in our method the Hamiltonian is split into a free piece and an interaction in a natural way. This allows us to directly use the well-developed thermodynamic perturbation theory to evaluate the thermodynamic potential. A remarkable consequence of our analysis is that the thermal corrections to the generalized Gaussian effective potential agree with the ones

of the effective potential provided we use, Eq. (7.21).

Let us conclude by briefly discussing the more realistic case of scalar fields in three spatial dimensions. There is growing evidence that quartic self-interacting scalar field theories are trivial in four-dimensional spacetime [40]. However, recently Consoli and Stevenson proposed that the vacuum of the $(\lambda\phi^4)_4$ theory is not trivial [41]. More precisely, within the Gaussian variational approximation they argued that the elementary excitations behave as free fields while the vacuum resembles a Bose condensate.

Recently, this triviality and spontaneous symmetry breaking scenario found some evidence in the lattice approach [42,43]. If this turns out to be the case, we expect that the symmetry broken at zero temperature gets restored by increasing the temperature. Thus our approach to the calculation of the thermal corrections may be useful to investigate the nature of the thermal phase transition. In particular it is important to ascertain if the phase transition is first order or continuous.

APPENDIX A

We are interested in the high temperature expansion $\hat{\beta} \rightarrow 0$ of the integral

$$h(a^2) = \frac{1}{\pi\hat{\beta}^2} \int_0^\infty dt \ln[1 - e^{-\sqrt{t^2+a^2}}], \quad (\text{A1})$$

where $a^2 = \hat{\beta}^2 x$. Following Ref. [29] we consider

$$h^\beta(a^2) = \frac{\partial h}{\partial a^2} = \frac{1}{2\pi\hat{\beta}^2} \int_0^\infty \frac{dt}{\sqrt{t^2+a^2}(e^{\sqrt{t^2+a^2}}-1)}. \quad (\text{A2})$$

To perform the high temperature expansion of Eq. (A2) it is useful to deal with

$$h_1(\epsilon, a^2) = \int_0^\infty dt \frac{t^{-\epsilon}}{\sqrt{t^2+a^2}(e^{\sqrt{t^2+a^2}}-1)}, \quad (\text{A3})$$

with $\epsilon \rightarrow 0^+$. Using the identity [21]

$$\sum_{n=1}^{\infty} \frac{y}{y^2+n^2} = -\frac{1}{2y} + \frac{\pi}{2} \coth(\pi y), \quad (\text{A4})$$

we rewrite Eq. (A3) as

$$\begin{aligned} h_1(\epsilon, a) &= \int_0^\infty dt \frac{t^{-\epsilon}}{\sqrt{t^2+a^2}} \left[\sum_{n=-\infty}^{+\infty} \sqrt{\frac{t^2+a^2}{t^2+a^2+4\pi^2 n^2}} - \frac{1}{2} \right] \\ &\equiv I_\epsilon^{(1)}(a^2) + I_\epsilon^{(2)}(a^2). \end{aligned} \quad (\text{A5})$$

Performing the change of variable $y = \frac{t}{\sqrt{a^2+4\pi^2 n^2}}$ we rewrite the first term on the right-hand of Eq. (A5) as

$$I_\epsilon^{(1)}(a^2) = \sum_{n=-\infty}^{+\infty} \frac{1}{(a^2+4\pi^2 n^2)^{(1+\epsilon)/2}} \int_0^\infty dy \frac{y^{-\epsilon}}{y^2+1}. \quad (\text{A6})$$

The last integration can be performed to yield [21]

$$I_\epsilon^{(1)}(a^2) = \frac{\pi}{2} \frac{1}{\cos \frac{\pi}{2} \epsilon} \left\{ \frac{1}{a^{1+\epsilon}} + 2 \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{1+\epsilon}} + 2 \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{1+\epsilon}} \left[\frac{1}{\left(1 + \frac{a^2}{4\pi^2 n^2}\right)^{(1+\epsilon)/2}} - 1 \right] \right\}. \tag{A7}$$

Using the definition of Riemann’s zeta function we get

$$I_\epsilon^{(1)}(a) = \frac{\pi}{2a} + 2^{-1-\epsilon} \pi^{-\epsilon} \zeta(1+\epsilon) + \tilde{I}(a) = \frac{1}{2\epsilon} + \frac{\pi}{2a} + \frac{1}{2}(\gamma - \ln 2\pi) + \tilde{I}(a), \tag{A8}$$

where

$$\tilde{I}(a^2) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(1 + \frac{a^2}{4\pi^2 n^2}\right)^{-1/2} - 1 \right]. \tag{A9}$$

As concern the integral $I_\epsilon^{(2)}(a^2)$ in Eq. (A5) we have [21]

$$I_\epsilon^{(2)}(a^2) = -\frac{1}{2} \int_0^\infty dx \frac{x^{-\epsilon}}{\sqrt{x^2+a^2}} = -\frac{1}{2} a^{-\epsilon} \frac{1}{2^{1-\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(\epsilon/2)}{\Gamma(1-\epsilon/2)} = -\frac{1}{2\epsilon} + \frac{1}{2} \ln a, \tag{A10}$$

so that in the limit $\epsilon \rightarrow 0^+$ we obtain

$$\lim_{\epsilon \rightarrow 0^+} h_1(\epsilon, a^2) = \frac{\pi}{2a} + \frac{1}{2} \ln a + \frac{1}{2} [\gamma - \ln 4\pi + \tilde{I}(a^2)]. \tag{A11}$$

Finally we perform the Taylor expansion of $\tilde{I}(a^2)$:

$$\tilde{I}(a^2) = -\frac{\zeta(3)}{16\pi^2} a^2 + \frac{3\zeta(5)}{256\pi^4} a^4 + O(a^6). \tag{A12}$$

Putting it all together we obtain

$$h_1(a^2) = \frac{\pi}{2a} + \frac{1}{2} \ln a + \frac{1}{2} (\gamma - \ln 4\pi) - \frac{\zeta(3)}{16\pi^2} a^2 + \frac{3\zeta(5)}{256\pi^4} a^4 + O(a^6), \tag{A13}$$

whence

$$h^\beta(a^2) = \frac{1}{2\pi\hat{\beta}^2} \left\{ \frac{\pi}{2a} + \frac{1}{2} \ln a + \frac{1}{2} (\gamma - \ln 4\pi) - \frac{\zeta(3)}{16\pi^2} a^2 + \frac{3\zeta(5)}{256\pi^4} a^4 \right\} + O(a^6). \tag{A14}$$

In order to recover $h(a^2)$ we integrate $h^\beta(a^2)$ in a^2 with the boundary condition

$$h(0) = \frac{1}{\pi\hat{\beta}^2} \int_0^\infty dt \ln(1 - e^{-t}) = \frac{\pi}{6\hat{\beta}^2}. \tag{A15}$$

We get

$$h(a^2) = \frac{1}{2\pi\hat{\beta}^2} \left\{ \frac{\pi^2}{3} + \pi a - \frac{a^2}{4} + \frac{a^2}{2} \ln\left(\frac{a}{4\pi}\right) + \frac{\gamma}{2} a^2 - \frac{1}{32\pi^2} \zeta(3) a^4 + \frac{1}{256\pi^4} \zeta(5) a^6 + O(a^8) \right\}. \tag{A16}$$

Let us now evaluate the high temperature expansion ($a \rightarrow 0$) of the integral

$$J(a^2) = \int_0^\infty dt t \ln(1 - e^{-\sqrt{t^2+a^2}}) = \frac{1}{2} \int_0^\infty dy \ln(1 - e^{-\sqrt{y+a^2}}). \tag{A17}$$

To this end, we evaluate

$$J'(a^2) = \frac{dJ}{da^2} = \frac{1}{4} \int_0^\infty dy \frac{1}{\sqrt{y+a^2}} \frac{1}{e^{-\sqrt{y+a^2}} - 1}. \tag{A18}$$

Using the identity, Eq. (A4), we write

$$J'(a^2) = \lim_{\epsilon \rightarrow 0^+} [K_\epsilon^1(a^2) + K_\epsilon^2(a^2)], \tag{A19}$$

where

$$K_\epsilon^1 = \frac{1}{4} \sum_{n=-\infty}^{+\infty} \int_0^\infty dy \frac{y^{-\epsilon}}{y+a^2+4\pi^2 n^2}, \tag{A20}$$

$$K_\epsilon^2(a^2) = -\frac{1}{8} \int_0^\infty dy \frac{y^{-\epsilon}}{\sqrt{y+a^2}}. \tag{A21}$$

To evaluate $K_\epsilon^{(1)}(a^2)$ we proceed as we did for $I_\epsilon^{(1)}(a^2)$. We obtain

$$K_\epsilon^{(1)}(a^2) = \frac{1}{4\epsilon} \left[\frac{1}{a^{2\epsilon}} + \frac{2}{(4\pi^2)^\epsilon} \zeta(2\epsilon) - \frac{2a^2}{(4\pi^2)^{1+\epsilon}} \epsilon \zeta(2) \right] = -\frac{1}{2} \left[\ln a + \frac{a^2}{4\pi^2} \zeta(2) \right] + O(\epsilon). \tag{A22}$$

As concerns $K_\epsilon^{(2)}(a^2)$ we find

$$K_\epsilon^{(2)}(a^2) = -\frac{a^{1-2\epsilon}}{8} B(1-\epsilon, \epsilon - \frac{1}{2}) = \frac{a}{4} + O(\epsilon). \tag{A23}$$

Using $\zeta(2) = \pi^2/6$ we obtain

$$J'(a^2) = -\frac{1}{2} \ln a + \frac{a}{4} - \frac{a^2}{48}. \tag{A24}$$

Integrating in a^2 we are led to

$$J(a^2) = J(0) + \frac{1}{4} a^2 - \frac{1}{4} a^2 \ln a^2 + \frac{a^3}{6} - \frac{a^4}{96}, \tag{A25}$$

where

$$J(0) = \int_0^\infty dt t \ln(1 - e^{-t}) = -\zeta(3). \quad (\text{A26})$$

It is useful to perform also the low temperature expansion ($a \rightarrow \infty$) of $J(a^2)$. To do this we note that

$$J(a^2) = -\frac{1}{2} \sum_{n=1}^{\infty} \int_0^\infty dy \frac{e^{-n\sqrt{y+a^2}}}{n}. \quad (\text{A27})$$

Changing the integration variable we get

$$\begin{aligned} J(a^2) &= -\frac{a^2}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty dt e^{-na\sqrt{t+1}} \\ &= -a \sum_{n=1}^{\infty} \frac{e^{-na}}{n^2} - \sum_{n=1}^{\infty} \frac{e^{-na}}{n^3}. \end{aligned} \quad (\text{A28})$$

This last expression can be used to approximate $J(a^2)$ for $a \gg 1$.

APPENDIX B

For the reader's convenience we briefly discuss the thermodynamic perturbation theory in Matsubara's scheme [11,35]. Let us suppose that the Hamiltonian of our thermodynamic system can be written as

$$H = H_0 + H_I. \quad (\text{B1})$$

We are interested in evaluating the thermodynamic potentials perturbatively in H_I . To this end we introduce the \mathcal{S} matrix

$$e^{H\tau} = \mathcal{S}^{-1}(\tau) e^{H_0\tau}, \quad 0 \leq \tau \leq \beta. \quad (\text{B2})$$

Let us consider the field operators in the Matsubara's interaction representation:

$$\phi(\vec{x}, \tau) = e^{H_0\tau} \phi(\vec{x}) e^{-H_0\tau}. \quad (\text{B3})$$

In this representation the interaction Hamiltonian reads

$$H_I(\tau) = e^{H_0\tau} H_I e^{-H_0\tau}, \quad (\text{B4})$$

while

$$H_0(\tau) = H_0. \quad (\text{B5})$$

The solution of Eq. (B2) is well known:

$$\mathcal{S}(\tau) = T_\tau \exp \left[- \int_0^\tau H_I(\tau') d\tau' \right]. \quad (\text{B6})$$

Let us evaluate the thermodynamic potential Ω :

$$e^{-\beta\Omega} = \text{Tr}(e^{\beta H}). \quad (\text{B7})$$

From Eqs. (B2) and (B7) we get

$$\Omega = -\frac{1}{\beta} \ln \text{Tr} [e^{-\beta H_0} \mathcal{S}(\beta)]. \quad (\text{B8})$$

Defining

$$\Omega_0 = -\frac{1}{\beta} \ln \text{Tr}(e^{-\beta H_0}), \quad (\text{B9})$$

we get

$$\Omega - \Omega_0 = -\frac{1}{\beta} \ln \frac{\text{Tr} e^{-\beta H_0} \mathcal{S}(\beta)}{\text{Tr} e^{-\beta H_0}}, \quad (\text{B10})$$

whence

$$\Delta\Omega = -\frac{1}{\beta} \ln \langle \mathcal{S}(\beta) \rangle^\beta. \quad (\text{B11})$$

Using Eq. (B6) we rewrite (B11) as

$$\begin{aligned} \Delta\Omega &= -\frac{1}{\beta} \left\{ \ln \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_0^\beta d\tau_1 \cdots d\tau_m \right. \\ &\quad \left. \times \langle T_\tau [H_I(\tau_1) \cdots H_I(\tau_m)] \rangle^\beta \right\}. \end{aligned} \quad (\text{B12})$$

One can show that [35]

$$\begin{aligned} \Delta\Omega &= -\frac{1}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_0^\beta d\tau_1 \cdots d\tau_m \\ &\quad \times \langle T_\tau [H_I(\tau_1) \cdots H_I(\tau_m)] \rangle_{\text{conn}}^\beta. \end{aligned} \quad (\text{B13})$$

This last equation has been used in Sec. VIII.

We would like now to discuss the thermal propagator of a free scalar field with mass m . From the well-known expansion

$$\phi(\vec{x}, 0) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} [e^{+i\vec{p}\cdot\vec{x}} a_{\vec{p}} + e^{-i\vec{p}\cdot\vec{x}} a_{\vec{p}}^\dagger] \quad (\text{B14})$$

we readily obtain

$$\phi(\vec{x}, \tau) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} [a_{\vec{p}} e^{i\vec{p}\cdot\vec{x} - E_p\tau} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x} + E_p\tau}], \quad (\text{B15})$$

where $E_p = \sqrt{\vec{p}^2 + m^2}$, and we used

$$e^{H_0\tau} a_{\vec{p}} e^{-H_0\tau} = a_{\vec{p}} e^{-E_p\tau}, \quad (\text{B16})$$

$$e^{H_0\tau} a_{\vec{p}}^\dagger e^{-H_0\tau} = a_{\vec{p}}^\dagger e^{-E_p\tau}. \quad (\text{B17})$$

We are interested in the thermal propagator

$$\langle T_\tau \phi(\vec{x}, \tau) \phi(0) \rangle^\beta = G_\beta(\vec{x}, \tau). \quad (\text{B18})$$

Using Eq. (B15) and

$$\langle a_{\vec{p}_1} a_{\vec{p}_2}^\dagger \rangle^\beta = \frac{\delta_{\vec{p}_1 \vec{p}_2}}{1 - e^{-\beta E_{p_1}}}, \quad (\text{B19})$$

$$\langle a_{\vec{p}_1}^\dagger a_{\vec{p}_2} \rangle^\beta = \frac{\delta_{\vec{p}_1 \vec{p}_2}}{e^{\beta E_{p_1}} - 1}, \quad (\text{B20})$$

we obtain

$$G_\beta(\vec{x}, \tau) = \frac{1}{V} \sum_{\vec{p}} \frac{1}{2E_p} \left[\frac{e^{i\vec{p}\cdot\vec{x} - E_p\tau}}{1 - e^{-\beta E_p}} + \frac{e^{-i\vec{p}\cdot\vec{x} - E_p\tau}}{e^{\beta E_p} - 1} \right]. \quad (\text{B21})$$

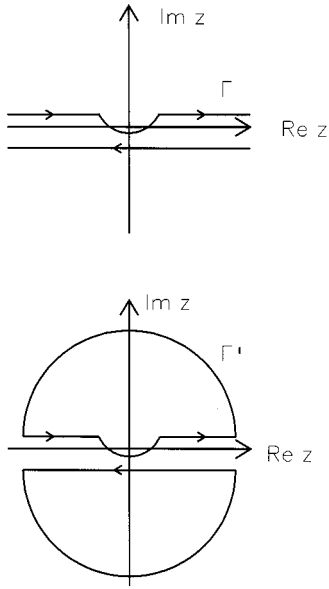


FIG. 16. The contours Γ and Γ' in the complex z plane.

Now we observe that

$$\frac{e^{-E_p\tau}}{1-e^{-\beta E_p}} = - \int_{\Gamma} \frac{dz}{2\pi i} \frac{e^{-z\tau}}{(1-e^{-\beta z})(z-E_p)}, \quad (\text{B22})$$

where the integral in the complex z plane is on the contour Γ shown in Fig. 16. The integrand in Eq. (B22) goes to zero exponential when $|z| \rightarrow \infty$. Thus we deform the contour Γ in Γ' (see Fig. 16). Applying the Cauchy's integral theorem we get

$$\frac{e^{-E_p\tau}}{1-e^{-\beta E_p}} = \frac{1}{\beta} \sum_n \frac{e^{-i\omega_n\tau}}{E_p - i\omega_n}, \quad (\text{B23})$$

where $\omega_n = 2\pi n\beta$. The other term in Eq. (B21) can be dealt with in a similar way. Thus we obtain:

$$G_{\beta}(\vec{x}, \tau) = \frac{1}{\beta} \sum_n \frac{1}{V} \sum_{\vec{p}} \frac{1}{2E_p} \left[\frac{e^{i\vec{p}\cdot\vec{x} - i\omega_n\tau}}{E_p - i\omega_n} + \frac{e^{-i\vec{p}\cdot\vec{x} + i\omega_n\tau}}{E_p - i\omega_n} \right]. \quad (\text{B24})$$

Finally, observing that

$$\frac{1}{V} \sum_{\vec{p}} \rightarrow \int \frac{d^3 p}{(2\pi)^3}$$

and performing the change of variables $\vec{p} \rightarrow -\vec{p}$, $\omega_n \rightarrow -\omega_n$ in the second term on the right-hand side of Eq. (B24), we get

$$G_{\beta}(\vec{x}, \tau) = \frac{1}{\beta} \sum_n \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-i\omega_n\tau + i\vec{p}\cdot\vec{x}}}{E_p^2 + \omega_n^2}. \quad (\text{B25})$$

In the following we need to evaluate the sum

$$\frac{1}{\beta} \sum_n \frac{e^{i\omega_n\tau}}{\omega_n^2 + E_p^2}. \quad (\text{B26})$$

To do this, we use the Sommerfeld-Watson transform [44]

$$\begin{aligned} \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} f(z=i\omega_n) &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz f(z) \\ &+ \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} [f(z) + f(-z)] \frac{1}{e^{\beta z} - 1} \\ &\equiv A_1 + A_2. \end{aligned} \quad (\text{B27})$$

In our case

$$f(z) = - \frac{e^{z\tau}}{z^2 - E_p^2}. \quad (\text{B28})$$

Let us consider, first, A_1 . We have

$$A_1 = - \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{e^{z\tau}}{z^2 - E_p^2}. \quad (\text{B29})$$

If $\tau > 0$, we close the integration contour in the semiplane $\text{Re}z < 0$, while for $\tau < 0$ the contour is closed in the semiplane $\text{Re}z > 0$. In this way, by applying the residue theorem we obtain

$$A_1 = \frac{e^{-E_p|\tau|}}{2E_p}. \quad (\text{B30})$$

In the same way we get

$$A_2 = \frac{e^{E_p\tau} + e^{-E_p\tau}}{2E_p} \frac{1}{e^{\beta E_p} - 1}. \quad (\text{B31})$$

Combining Eqs. (B30) and (B31) we obtain the desired result

$$\frac{1}{\beta} \sum_n \frac{e^{i\omega_n\tau}}{\omega_n^2 + E_p^2} = \frac{e^{-E_p|\tau|}}{2E_p} + \frac{e^{E_p\tau} + e^{-E_p\tau}}{2E_p} \frac{1}{e^{\beta E_p} - 1}. \quad (\text{B32})$$

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