

Stringy black holes and energy conditions

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The energy condition inequalities for the matter stress energy comprised of the dilaton and Maxwell fields in the dilaton-Maxwell gravity theories emerging out of string theory are examined in detail. In the simplistic $(1+1)$ -dimensional models, $R \leq 0$ (where R is the Ricci scalar) turns out to be the requirement for ensuring focusing of timelike geodesics. In $3+1$ dimensions, we outline the requirements on matter for pure dilaton theories—these in turn constrain the functional forms of the dilaton. Furthermore, in charged dilaton gravity a curious opposite behavior of the matter stress energy with respect to the violation or conservation of the weak energy condition is noted for the electric and magnetic black hole metrics written in the string frame of reference. We also investigate the matter that is necessary for creating certain specific nonasymptotically flat black holes. For the electric and magnetic black hole metrics, strangely, matter satisfies the weak energy condition in the string frame. Finally, the averaged null energy condition is evaluated along radial null geodesics for each of these black hole spacetimes. [S0556-2821(97)01206-X]

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I. INTRODUCTION

The low-energy effective theory that emerges out of full string theory by imposing quantum conformal invariance in the world-sheet σ model and thereby equating the one-loop β functions for the metric and matter couplings to zero, largely resembles general relativity (GR) with some new “matter” fields such as the dilaton, axion, etc. [1]. The field equations (which are the ones obtained by equating the one-loop β functions to zero) can, therefore, be solved with different *Ansätze* for the metrics and matter fields. These solutions thus represent allowed backgrounds for string propagation—the “allowance” being the fact that quantum conformal invariance is satisfied on the world sheet. Presently, there do exist many solutions of these equations representing black holes [2], cosmologies [3–5], etc.

An important fact about the low-energy theory is that there exists two different frames in which the features of the spacetime may look very different. These frames which are known as the “Einstein frame” and the “string frame” are related to each other by a conformal transformation ($g_{\mu\nu}^E = e^{-2\phi} g_{\mu\nu}^S$) which involves the massless dilaton field as the conformal factor. The existence of two different frames is, however, a known feature in certain modifications to Einstein’s theory. In fact, the “string frame” is actually similar to the Brans-Dicke frame in the well-known Jordan-Brans-Dicke theory. In the context of string theory one says that the string “sees” the string metric (which is the metric written in the string frame). Several of the important symmetries of string theory also rely on the choice of the string frame or the Einstein frame. For instance, the familiar T duality [6] transformation relates metrics in the string frame only, whereas

S duality [7] is a valid symmetry only if the equations are written in the Einstein frame.

It has been mentioned several times in the literature [8,9] that the metric in the string frame violates the inequality $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ and hence also the assumption of a local energy condition. Therefore, it has been argued that the singularity theorems of GR [10,11] are not valid for the low-energy theory emerging out of string theory. This is because the singularity theorems assume an energy condition and such an assumption essentially leads to the concept of geodesic focusing, a conclusion resulting out of an analysis of the Raychaudhuri equation [12,13]. The focusing theorem along with some additional assumptions essentially implies the existence of spacetime singularities.

In this paper, we explicitly examine several black hole geometries in two and four dimensions with regard to the weak energy condition (WEC), $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$. We shall point out the domains of violation of these energy conditions and also attempt to arrive at some general statements. It will turn out that there are black holes which require a violation of these conditions as well as solutions which do not violate them. Moreover, we shall demonstrate solutions which satisfy the WEC and also check the averaged null energy condition (ANEC) for several black holes. It will turn out that there are quite a few geometries for which the ANEC integral along radial null geodesics is positive definite. With all these we will try and conclude that it is somewhat premature to make statements about the nonvalidity of the singularity theorems for the class of theories emerging out of full string theory.

The paper is organized as follows. In Sec. II we analyze the energy conditions and geodesic focusing for $(1+1)$ -dimensional theories of gravity. Section III deals with the conditions for Einstein-dilaton gravity. Charged dilaton black holes are discussed in the fourth section. The ANEC integral is checked in Sec. V. Finally, Sec. VI contains a summary of the main results of the paper.

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The sign conventions followed in this paper are those of Misner, Thorne, and Wheeler [14].

II. DILATON GRAVITY IN 1+1 DIMENSIONS

We begin with the simple models of gravity in 1+1 dimensions [9,8] where the first stringy black hole was discovered. Before we explicitly relate $R_{\mu\nu}\xi^\mu\xi^\nu$ with the quantities involving the dilaton (using the one-loop β function equations), let us look at the focusing conditions emerging out of an analysis of the Raychaudhuri equation.

The Raychaudhuri equation for timelike geodesic congruences in a (1+1)-dimensional spacetime, discussed earlier in [15], turns out to be given as

$$\frac{d\theta}{d\lambda} + \theta^2 = -R_{\mu\nu}\xi^\mu\xi^\nu. \quad (1)$$

Note that a way to arrive at this equation without starting from first principles (as is done in [15]) is to substitute appropriate values of N (background spacetime dimensions, in this case $N=2$) and D (dimensions of the embedded geometric object, in this case $D=1$) in the generalized Raychaudhuri equation for families of D -dimensional surfaces in an N -dimensional background (for details, see [17]).

Now, recall that in 1+1 dimensions, we have

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R. \quad (2)$$

Therefore, we can rewrite the above equation as

$$\frac{d^2F}{d\lambda^2} + (\frac{1}{2}Rg_{\mu\nu}\xi^\mu\xi^\nu)F = 0, \quad (3)$$

where $\theta = F'/F$. If the ξ^μ are timelike, then we need

$$R \leq 0 \quad (4)$$

in order to have the existence of zeros in a solution of Eq. (3). Such zeros essentially imply a divergence in θ . Hence, a converging (θ negative) timelike geodesic congruence must necessarily come to a focus ($\theta \rightarrow -\infty$) within a finite value of the affine parameter λ . If $R=0$, then, of course, the right-hand side (RHS) of the Raychaudhuri equation is identically zero and we always get a focusing effect. Note that for $R=0$, the Raychaudhuri equation has solutions $F = \text{const}$ and $F = a\lambda + b$. The former results in an expansion which is everywhere zero, while with the latter, one can arrive at focusing.

In 1+1 dimensions null geodesics have a unique behavior. It can be shown that θ is identically equal to zero. This is largely due to the fact that in 1+1 dimensions all metrics are conformally flat and therefore null geodesics are the same as that for flat (1+1)-dimensional spacetime. A discussion on this can be found in [16].

Let us now go back to timelike geodesics. The question to ask now is whether the two-dimensional black hole metrics known to us satisfy the condition $R \leq 0$. To see this we have to investigate some cases explicitly. Note, however, that in 1+1 dimensions a Weyl rescaling of the metric leaves the ‘‘Brans-Dicke’’ form of the action invariant. Therefore, it is

perhaps not worth referring to the ‘‘Einstein frame’’ as such because it is essentially defined as a frame in which the theory is in the canonical form.

In pure dilaton gravity with a cosmological constant we can show from the field equations that

$$R = 4\{\lambda^2 - (\nabla\phi)^2\}. \quad (5)$$

Therefore, if λ is zero, R is always positive or negative depending on what ϕ is functionally. This implies that the focusing may or may not take place in the solutions of such a theory.

Let us now concentrate on some examples of black holes in two-dimensional dilaton gravity.

The two-dimensional black hole metric of Mandal, Sengupta, and Wadia [2] is given by the metric

$$ds^2 = -(1 - ae^{Qr})dt^2 + \frac{1}{(1 - ae^{Qr})}dr^2. \quad (6)$$

This is a solution in pure dilaton gravity in 1+1 dimensions (i.e., without a cosmological constant). The Ricci scalar for this geometry turns out to be

$$R = aQ^2e^{Qr}, \quad (7)$$

which is clearly positive. Therefore, even though there is a singularity at $r = \infty$, focusing within a finite value of the affine parameter for an initially converging timelike geodesic congruence does not occur.

We now turn to the *exact* metric (exact in the sense of full string theory) due to Dijkgraaf, Verlinde, and Verlinde [18] (see also [19]) is given as

$$ds^2 = 2(k-2)[- \beta(r)dt^2 + dr^2], \quad (8)$$

where $\beta(r) = (\coth^2 r - 2/k)^{-1}$ and the dilaton is given as

$$\phi = \phi_0 + \frac{1}{2}\ln|\sinh^2 \frac{2r}{\beta}|. \quad (9)$$

The quantity k stands for the Kac-Moody level. It is related to the central charge c of the Wess-Zumino-Witten model based on the group $SO(2,1)$ gauged by the subgroup $SO(1,1)$ (which is an exact conformal field theory description of the Witten solution quoted below) by the relation $c = 3k/(k-2) - 1$. k takes the value $\frac{9}{4}$ for a bosonic string background for which $c = 26$.

The Ricci scalar turns out to be

$$R = \frac{1}{2(k-2)} \frac{2\text{csch}^2 r}{\left(\frac{2}{\coth^2 r - \frac{2}{k}}\right)^2} \left[\frac{2}{k} + 2\coth^2 r \left(1 - \frac{3}{k}\right) \right]. \quad (10)$$

The positivity or negativity of R crucially depends on the value of k . If $k \geq 3$, then R is always positive. For k lying between 2 and 3 there is always a domain in which R is negative as is easily noticeable from the expression.

In the $k \rightarrow \infty$ limit the metric goes over to the Witten black hole (modulo the overall factor $k-2$). This is given by the metric

$$ds^2 = -\tanh^2 r dt^2 + dr^2, \quad (11)$$

which is related by a coordinate transformation to the metric discussed first in this sequence (the Mandal, Sengupta, Wadia black hole).

The Ricci scalar is given as

$$R = \frac{4}{\cosh^2 r} \quad (12)$$

which, once again is positive. However, note that the metric written in the above form does not have a singularity anywhere. The point $r=0$ is actually a coordinate singularity. One can, following Witten, do a Kruskal extension of the geometry and arrive at the form:

$$ds^2 = -\frac{du dv}{1-uv}, \quad (13)$$

where $2u = -e^{r'-t}$, $2v = e^{r'+t}$, and $r' = r + \ln(1 - e^{-2r})$. The denominator in the Ricci scalar now gets replaced by $1-uv$ and $uv=1$ corresponds to the divergence of R and, therefore, is a real singularity. However, even if we consider the maximally extended metric, the Ricci scalar is still positive definite and timelike geodesic congruences do not focus. This is in accordance with the discussion regarding focusing on the Mandal, Sengupta, and Wadia form of this solution presented earlier in this section.

There do exist a multitude of other solutions in 1+1 dimensions such as the ones derived in [20,21] for which one can do a similar analysis. Since the basic idea is the same we refrain from repeating the same exercise here.

III. PURE DILATON THEORIES IN 3+1 DIMENSIONS

A. Energy conditions

Before we embark on an analysis of the matter sector of dilaton gravity theories let us briefly recall the content of the various energy conditions which we shall be checking out for specific solutions. Below, $T_{\mu\nu}$ represents the energy-momentum tensor for matter and ξ^μ represents a timelike or null vector as specified in the respective conditions.

1. Local energy conditions

(a) Strong energy condition

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\xi^\mu\xi^\nu \geq 0 \quad \forall \text{ timelike } \xi^\mu. \quad (14)$$

(b) Weak energy condition

$$(T_{\mu\nu}\xi^\mu\xi^\nu) \geq 0 \quad \forall \text{ timelike } \xi^\nu. \quad (15)$$

(c) Null energy condition

$$(T_{\mu\nu}k^\mu k^\nu) \geq 0 \quad \forall \text{ null } k^\mu. \quad (16)$$

2. Global energy conditions [13]

(a) Averaged weak energy condition

$$\int_{-\infty}^{\infty} T_{\mu\nu}\xi^\mu\xi^\nu d\lambda \geq 0, \quad (17)$$

where the integration is over a complete, timelike geodesic in the spacetime.

(b) Averaged null energy condition

$$\int_{-\infty}^{\infty} T_{\mu\nu}k^\mu k^\nu d\lambda \geq 0, \quad (18)$$

where the integration is now over a complete, null geodesic in the spacetime.

B. Checking the energy conditions for dilaton gravity

We now check the various energy conditions listed above for the theory of dilaton gravity in 3+1 dimensions.

The action integral for the Einstein-dilaton-Maxwell theory is given as

$$S_{\text{EDM}} = \int d^4x \sqrt{-g} e^{-2\phi} \left[R + 4g_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{2} g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} \right]. \quad (19)$$

Varying with respect to the metric, dilaton, and Maxwell fields, we get the field equations for the theory given as

$$R_{\mu\nu} = -2\nabla_\mu \nabla_\nu \phi + 2F_{\mu\lambda} F_{\nu}^{\lambda}, \quad (20)$$

$$\nabla^\nu (e^{-2\phi} F_{\mu\nu}) = 0, \quad (21)$$

$$4\nabla^2 \phi - 4(\nabla \phi)^2 + R - F^2 = 0. \quad (22)$$

These equations are also the β function equations for a world sheet σ model obtained by imposing quantum conformal invariance and setting the β functions to zero. Note that without the Maxwell field we have essentially a Brans-Dicke-type theory with the Brans-Dicke parameter explicitly set to $\omega = -1$.

Consider the very first equation (without the Maxwell field) and recast it in the form of the Einstein equation, $G_{\mu\nu} = e^{2\phi} T_{\mu\nu}$. This is the generic form of the Einstein equation for a Brans-Dicke-type theory (for details, see Weinberg [22]). The $T_{\mu\nu}$ contains a part $T_{\mu\nu}^\phi$ and a part $T_{\mu\nu}^M$ where M stands for all matter apart from the dilaton. With this, one can now write down the energy-momentum tensor for the dilaton field which turns out to be

$$T_{\mu\nu}^\phi = e^{-2\phi} [-2\nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \nabla^2 \phi]. \quad (23)$$

Therefore, the various energy conditions turn out to be equivalent to the following inequalities.

WEC:

$$T_{\mu\nu}^\phi \xi^\mu \xi^\nu = -e^{-2\phi} [2\xi^\mu \xi^\nu \nabla_\mu \nabla_\nu \phi + \nabla^2 \phi] \geq 0. \quad (24)$$

For ξ^μ along a geodesic curve, the first term inside the square brackets can be shown to reduce to $2(d^2\phi/ds^2)$. Since from the field equations we have $\nabla^2 \phi = 2(\nabla \phi)^2$, the WEC reduces to $2\phi'' + 2(\nabla \phi)^2 \leq 0$.

AWEC:

$$\int_{-\infty}^{\infty} e^{-2\phi} [2\phi'' + 2(\nabla \phi)^2] d\lambda \leq 0. \quad (25)$$

NEC:

$$T_{\mu\nu}^{\phi}\xi^{\mu}\xi^{\nu} = -2e^{-2\phi}k^{\mu}k^{\nu}\nabla_{\mu}\nabla_{\nu}\phi. \quad (26)$$

For k^{μ} , a tangent along a null geodesic, we have the requirement

$$\phi'' \leq 0. \quad (27)$$

ANEC:

$$\int_{-\infty}^{\infty} T_{\mu\nu}^{\phi}k^{\mu}k^{\nu}d\lambda = 2 \int_{-\infty}^{\infty} e^{-2\phi}\phi''d\lambda \leq 0. \quad (28)$$

These are the constraints which the dilaton field has to obey in order to satisfy the respective energy conditions.

In order to understand the constraints on ϕ better we write down the WEC inequalities explicitly for the general, static spherisymmetric metric given by:

$$ds^2 = -e^{2\psi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (29)$$

where $b(r)$ and $\psi(r)$ are two unknown functions.

The $\rho \geq 0$, $\rho + \tau \geq 0$, and $\rho + p \geq 0$ inequalities turn out to be

$$-2e^{-2\phi}\left(1 - \frac{b(r)}{r}\right)[\phi'^2 - \phi'\psi'] \geq 0, \quad (30)$$

$$-4e^{-2\phi}\left(1 - \frac{b(r)}{r}\right)\left[\phi'^2 - \frac{\phi'}{r}\right] \geq 0, \quad (31)$$

$$2e^{-2\phi}\left(1 - \frac{b(r)}{r}\right)\phi'\left[\psi' - \frac{1}{r}\right] \geq 0. \quad (32)$$

The above inequalities are obtained using the expression for $T_{\mu\nu}^{\phi}$ in terms of the scalar field and its derivatives [Eq. (23)]. These are the constraints on ϕ , ψ , and b which must be obeyed if we believe in the WEC. We now analyze a couple of choices of ϕ .

If ϕ is linear in r , i.e., $\phi(r) = r$ say, then we find that the requirements turn out to be $\psi' \geq 1/r$, $1/r \geq 1$, and $\psi' \geq 1$. Clearly, as $r \rightarrow \infty$, the second condition is violated. Thus, with a linear dilaton field one always will end up with a violation as $r \rightarrow \infty$. In contrast, if $\phi = \ln r$, then the first and third inequalities reduce to the requirement $\psi' \geq 1/r$, whereas the second one is identically satisfied [the left-hand side (LHS) of the second WEC inequality is zero]. Thus with a logarithmic dilaton, which means a linear string coupling, the WEC can be satisfied everywhere.

IV. ENERGY CONDITIONS FOR ELECTRIC AND MAGNETIC BLACK HOLES IN DILATON-MAXWELL GRAVITY

In this section we focus our attention on the energy condition inequalities for the electric and magnetic black hole solutions in dilaton-Maxwell gravity [23,24].

Evaluating the Einstein tensor $G_{\mu\nu}$ for the general, static, spherisymmetric metric quoted in the previous section, we can write down the expression for the energy conditions. This is equivalent to looking at the matter energy-momentum tensor because we are dealing with exact solutions of the field equations.

For a diagonal $T_{\mu\nu} = (\rho(r), \tau(r), p(r), p(r))$, we find that the WEC reduces to the inequalities

$$\rho \geq 0, \quad \rho + \tau \geq 0, \quad \rho + p \geq 0. \quad (33)$$

Note here that the NEC consists of only the second and third inequalities. From the Einstein equations we, therefore, end up with the following conditions on $b(r)$, $\psi(r)$ and its derivatives:

$$\rho(r) = \frac{b'}{r^2} \geq 0, \quad (34)$$

$$\rho(r) + \tau(r) = \frac{b'r - b}{r^3} + \frac{2\psi'}{r}\left(1 - \frac{b(r)}{r}\right) \geq 0, \quad (35)$$

$$\rho(r) + p(r) = \frac{b + b'r}{2r^3} + \left(1 - \frac{b(r)}{r}\right)\left[\psi'' + \psi'^2 + \frac{\psi'}{r} + \frac{b - b'r}{2r(r-b)}\psi'\right] \geq 0. \quad (36)$$

We have absorbed the factor $e^{2\phi}$ in a redefinition of the components of $T_{\mu\nu}$ (i.e., $\rho = e^{2\phi}\bar{\rho}$ and so on, where $\bar{\rho}$ is the actual component of the energy-momentum tensor). This factor will, however, have to be brought back when we discuss the averaged versions of the energy conditions. For a discussion of the local conditions, the overall $e^{2\phi}$ is irrelevant.

We will now choose the explicit functional forms of $b(r)$ and $\psi(r)$ which correspond to the well-known black hole solutions in string theory and thereby check out the WEC inequalities for each of them.

A. Electric black hole

The metric (in the string frame) and matter fields which solve the dilaton-Maxwell-Einstein field equations [Eqs. (19)–(21)] to yield the electric black hole, are given as

$$ds^2 = -\left(1 - \frac{2m}{\hat{r}}\right)\left(1 + \frac{2m\sinh^2\alpha}{\hat{r}}\right)^{-2} dt^2 + \frac{d\hat{r}^2}{1 - \frac{2m}{\hat{r}}} + \hat{r}^2 d\Omega^2, \quad (37)$$

$$A_t = -\frac{m\sinh 2\alpha}{\sqrt{2}[\hat{r} + 2m\sinh^2\alpha]}, \quad (38)$$

$$e^{-2\phi} = 1 + \frac{2m\sinh^2\alpha}{\hat{r}}. \quad (39)$$

The geometry has a horizon at $\hat{r} = 2m$ and a singularity at $r = 0$. Identifying the functions ψ and b with the metric co-

efficients in the above expressions we can now write down the WEC inequalities for this black hole geometry.

First, note that $\rho=0$ because $b(r)=2m$ which is a constant. The other two inequalities turn out to be

$$\rho + \tau = \frac{4m}{\hat{r}^3} \frac{\left(1 - \frac{2m}{\hat{r}}\right) \sinh^2 \alpha}{1 + \frac{2m}{\hat{r}} \sinh^2 \alpha} \geq 0, \quad (40)$$

$$\rho + p = \frac{1}{m^2} \frac{2x^3 \sinh^2 \alpha}{(1 + 2x \sinh^2 \alpha)^2} \{(5x - 1) + \sinh^2 \alpha 2[x(1+x)]\} \geq 0, \quad (41)$$

where in the last equation $x = m/\hat{r}$. We now have to check whether these inequalities are satisfied or are violated. It is easy to comment on the first one, for all $\hat{r} \geq 2m$ it is satisfied. The nature of the second inequality can be understood as follows.

First, the prefactor outside the curly brackets is positive. Therefore, the sign of the full expression depends entirely on the sign of the term in curly brackets. Note that this is a quadratic form in x . It can be written as

$$(x - x_1)(x - x_2), \quad (42)$$

where x_1, x_2 are the two roots of the function equated to zero. The explicit forms of the roots are

$$x_{1,2} = \frac{-2 \sinh^2 \alpha - 5 \pm \sqrt{(2 \sinh^2 \alpha + 5)^2 + 8 \sinh^2 \alpha}}{4 \sinh^2 \alpha}. \quad (43)$$

Therefore, x_1 (+sign) is always positive while x_2 is entirely negative. Hence, in order to have $(x - x_1)(x - x_2) \geq 0$, we require

$$x \geq x_1 = \frac{-2 \sinh^2 \alpha - 5 + \sqrt{(2 \sinh^2 \alpha + 5)^2 + 8 \sinh^2 \alpha}}{4 \sinh^2 \alpha}. \quad (44)$$

We also note the following:

(i) Violation occurs in the region of small x (i.e., large \hat{r}). However, as one approaches $\hat{r} \rightarrow \infty$, we find that the amount of violation becomes smaller. In fact, between a certain $x = x_0$ and $x = 0$, there is a point where the LHS of the full WEC inequality (including the prefactor) has a minimum. This indicates the maximum negative value it can take.

(ii) For α increasing, we note that the domain over which violation occurs (in x) becomes smaller, therefore, for α very large it can actually become miniscule in x (and, therefore, very large in \hat{r}).

(iii) The string coupling which is given by e^ϕ is of the form

$$e^\phi = \frac{1}{\sqrt{1 + 2x \sinh^2 \alpha}}. \quad (45)$$

This is a monotonically decreasing function of x . The string coupling is large where WEC violation occurs and it is small in the region where the WEC is satisfied. One cannot, however, conclude from this that the strength of the coupling is a sort of measure for WEC violation or satisfaction.

We now turn to an analysis of the WEC inequalities for the extremal limit of the electric black hole.

The extremal limit is obtained by taking the following limit for the parameters appearing in the electric black hole:

$$m \rightarrow 0, \quad \alpha \rightarrow \infty \quad \text{but } m \cosh^2 \alpha \text{ fixed.} \quad (46)$$

The line element turns out to be

$$ds^2 = - \left(1 + \frac{2M}{\hat{r}}\right)^{-2} dt^2 + d\hat{r}^2 + \hat{r}^2 d\Omega^2, \quad (47)$$

with $m \cosh^2 \alpha = M$.

The WEC inequalities (only $\rho + \tau \geq 0$, $\rho + p \geq 0$ because $\rho = 0$) turn out to be equivalent to

$$\rho + \tau = \frac{4M}{\hat{r}^3} \frac{1}{1 + \frac{2M}{\hat{r}}} \geq 0, \quad (48)$$

$$\rho + p = - \frac{2M}{\hat{r}^3} \frac{1 - \frac{2M}{\hat{r}}}{\left(1 + \frac{2M}{\hat{r}}\right)^2}. \quad (49)$$

Notice that the third inequality is violated everywhere (i.e., $\forall \hat{r} > 2M$), whereas the second one is satisfied everywhere. The lower bound of the domain of WEC violation has now shifted from $r = r_0 > 2M$ to $r = 2M$.

B. Magnetic black hole

In the string frame the dual solution known as the magnetic black hole is obtained by multiplying the electric metric in the Einstein frame by a factor $e^{-2\phi}$ (note the sign of ϕ). In a more generalized sense, this is the S -duality transformation which changes $\phi \rightarrow -\phi$ and thereby inverts the strength of the string coupling. Also recall that the magnetic and electric solutions are the same if looked at from the Einstein frame.

Therefore, the magnetic black hole metric is given by

$$ds^2 = - \frac{1 - \frac{2M}{r}}{1 - \frac{Q^2}{Mr}} dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right) \left(1 - \frac{Q^2}{Mr}\right)} + r^2 d\Omega^2. \quad (50)$$

Using the same methods as before, we write down the three energy condition inequalities for the metric given above. These turn out to be

$$\rho = \frac{2Q^2}{r^4} \geq 0, \quad (51)$$

TABLE I. WEC and electric, magnetic solutions.

Black hole	WEC2	WEC3	e^ϕ at $r=0$	e^ϕ at $r \rightarrow \infty$
Electric	Satisfied	Violated $\forall r=r_0 \geq 2m$	Weak	Strong
Magnetic	Violated	Satisfied	Strong	Weak

$$\rho + \tau = \frac{2Q^2}{Mr^3} \left(\frac{2M}{r} - 1 \right) \geq 0, \quad (52)$$

$$\rho + p = \frac{Q^2}{2Mr^3} \left(\frac{1 - \frac{2M}{r}}{1 - \frac{Q^2}{Mr}} + 1 \right) \geq 0. \quad (53)$$

It is easily seen that the second inequality is now violated for all $r > 2M$, whereas the third is satisfied for all $r > 2M$. Note that this is exactly opposite to what happened to the energy conditions for the electric black hole. This opposite behavior is shown in Table I given below.

However, one cannot actually make a general statement about the relation between electric-magnetic solutions (which are dual to each other), the violation of the energy condition inequalities, and the strength of the string coupling.

Finally, before we move on to other black hole geometries, let us look at the extremal limit of the magnetic black hole solution.

The metric for the solution in the extremal limit is given as

$$ds^2 = -dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} + r^2 d\Omega^2. \quad (54)$$

The energy condition inequalities turn out to be

$$\rho = \frac{4M^2}{r^4} \geq 0, \quad (55)$$

$$\rho + \tau = \frac{4M}{r^3} \left(\frac{2M}{r} - 1 \right) \geq 0, \quad (56)$$

$$\rho + p = \frac{2M}{r^3} \geq 0. \quad (57)$$

Therefore, the first and the third inequalities are satisfied for all values of r , whereas the second one is violated only if $r > 2M$. The geometry has spacelike slices resembling an infinite horn extending from infinity to $r = 2M$. There is no singularity here.

C. Other black hole metrics

We now move on towards analyzing the energy condition inequalities for certain recently derived nonasymptotically flat black hole solutions in dilaton-Maxwell gravity due to Chan, Mann, and Horne [25]. Here, we have a surprise in store for us. We will see that for the electric solution the

energy condition inequalities are satisfied in both the string as well as the Einstein frame of reference.

Let us first look at the solution in the Einstein frame. The metric is given as

$$ds^2 = -\frac{1}{\gamma^4} (r^2 - 4\gamma^2 M) dt^2 + \frac{4r^2}{r^2 - 4\gamma^2 M} dr^2 + r^2 d\Omega^2, \quad (58)$$

with the dilaton and Maxwell fields as

$$\phi(r) = -\frac{1}{2} \ln 2Q^2 + \ln r, \quad (59)$$

$$F_{ir} = \frac{Qe^{2\phi}}{\gamma^2 r}. \quad (60)$$

Note that the dilaton rolls from $-\infty$ to $+\infty$ as r changes its value from 0 to ∞ .

As in the previous cases, we first write down the string metric by performing the usual conformal transformation on the metric. This turns out to be

$$ds^2 = -\frac{r^2}{\gamma^4} \left(1 - \frac{2\sqrt{2}\gamma^2 M}{Qr} \right) dt^2 + \left(1 - \frac{2\sqrt{2}\gamma^2 M}{Qr} \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (61)$$

For the Einstein metric one can check that the energy conditions are satisfied. What about the inequalities for the string metric? Note that since $b(r)$ is a constant we have $\rho = 0$ straightaway. The other two inequalities turn out to be

$$\rho + \tau = \frac{2}{r^2} \left(1 - \frac{A}{r} \right) \geq 0, \quad (62)$$

$$\rho + p = \frac{A}{2r^3} + \frac{1}{r^2} \geq 0, \quad (63)$$

where $A = 2\sqrt{2}\gamma^2 M/Q$.

Surprisingly, for all $r \geq A$, all three inequalities are satisfied. The string coupling e^ϕ changes from 0 to ∞ as one varies r from 0 to ∞ .

In a similar way, let us look at the string metric for the magnetic black hole which is given as

$$ds^2 = -\frac{2Q^2}{\gamma^4} \left(1 - \frac{4M}{r} \right) dt^2 + \frac{2Q^2}{r^2} \left(1 - \frac{4M}{r} \right)^{-1} dr^2 + 2Q^2 d\Omega^2. \quad (64)$$

Since the coefficient of $d\Omega^2$ is a constant, here we cannot straightaway use the formulas for the WEC in terms of the $b(r)$ and $\psi(r)$. After some simple algebra, we find that the WEC inequalities reduce to

TABLE II. WEC and nonasymptotically flat electric, magnetic solutions.

Black hole	WEC1	WEC2	WEC3
Electric	Satisfied	Satisfied $\forall r \geq A$	Satisfied $\forall r$
Magnetic	Satisfied ($\forall r$)	Satisfied	Satisfied ($\forall r \geq 2M$)

$$\rho = \frac{1}{2Q^2} \geq 0, \quad (65)$$

$$\rho + \tau = 0, \quad (66)$$

$$\rho + p = \frac{\left(1 - \frac{2M}{r}\right)}{2Q^2} \geq 0. \quad (67)$$

Note that the first inequality is satisfied for all values of r where $r = 4M$ is the location of the horizon. In contrast, the third inequality is satisfied for all $r > 2M$. Thus the WEC is satisfied for all values of $r \geq 2M$. These conclusions are shown in a tabular form in Table II.

V. THE STATUS OF THE ANEC FOR STRINGY BLACK HOLES

For each of the geometries discussed in the previous section we shall now evaluate the ANEC integral along radial null geodesics. To do this we need to know the tangent vectors along radial null geodesics in the general, static, spherically-symmetric metric quoted in Sec. III. A choice for k^μ in the coordinate frame is

$$k^\mu \equiv \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, 0 \right) = \left(e^{-2\psi}, e^{-\psi} \sqrt{1 - \frac{b(r)}{r}}, 0, 0 \right). \quad (68)$$

Note that these choices for $dt/d\lambda$ and $dr/d\lambda$ satisfy the geodesic equations and also maintain the null character of the geodesic. In the proper frame (we need to go to the proper frame because the $G_{\mu\nu}$ used earlier in this paper are evaluated in this frame, it is entirely a matter of choice), this tangent vector is transformed to

$$\hat{k}^\mu \equiv (e^{-\psi}, e^{-\psi}, 0, 0), \quad (69)$$

where the hat is used to distinguish between the two frames.

The ANEC integral, therefore, becomes

$$I^{\text{ANEC}} = \int_{-\infty}^{\infty} e^{-2\phi} G_{\hat{\mu}\hat{\nu}} \hat{k}^\mu \hat{k}^\nu d\lambda = \int_{r_H}^{\infty} (\rho + \tau) e^{-2\psi - 2\phi} \frac{d\lambda}{dr} dr, \quad (70)$$

where r_H is the horizon radius.

All we need to do now is to use the expressions for $\rho + \tau$ (implicitly assuming that the $e^{2\phi}$ factor is removed by a further redefinition) and the corresponding functional forms for the dilaton for each of the black holes discussed previously and evaluate the integral.

Let us look at the values of the ANEC integrals for the electric and magnetic asymptotically flat black holes. These are

$$I_{\text{elec}}^{\text{ANEC}} = \frac{\sinh^2 \alpha}{m} \left(\frac{1}{2} + \frac{1}{3} \sinh^2 \alpha \right), \quad (71)$$

$$I_{\text{mag}}^{\text{ANEC}} = \frac{Q^2}{4M^3} \left(\frac{Q^2}{6M^2} - 1 \right). \quad (72)$$

Notice that the first of these is a positive quantity while the other one is negative for all $Q^2 \leq 4M^2$. The extremal limits of both these black holes exhibit the same behavior as their nonextremal counterparts.

The values of I^{ANEC} for all these black holes are quoted in Table III.

The fact to note here is that apart from the magnetic, asymptotically flat black holes and their extremal limit, all the other solutions satisfy the ANEC.

VI. SUMMARY AND OUTLOOK

The above analysis of the WEC inequalities for the well-known black hole metrics in low-energy stringy gravity indicate a few essential facts. We now list them here.

(i) In (1+1)-dimensional dilaton gravity theories, the

TABLE III. Stringy black holes and the ANEC.

Black hole	Value of ANEC integral	Status of ANEC
Electric, AF	$(\sinh^2 \alpha / m) \left(\frac{1}{2} + \frac{1}{3} \sinh^2 \alpha \right)$	Satisfied
Extremal, electric	∞	Satisfied
Magnetic, AF	$(Q^2 / 4M^3) \left(\frac{Q^2}{6M^2} - 1 \right)$	Violated
Extremal, magnetic	$-\frac{1}{3} \mu$	Violated
Electric, N-AF	$Q^2 \gamma^2 / A^4$	Satisfied
Magnetic, N-AF	0	Satisfied

$R \leq 0$ condition on the Ricci curvature is the analogue of the usual energy condition which has to be obeyed in order to ensure focusing of timelike geodesics. We have illustrated this with a couple of well-known black hole metrics in $(1+1)$ -dimensional theories which includes the exact metric due to Dijkgraaf, Verlinde, and Verlinde.

(ii) For $(3+1)$ -dimensional theories with just a dilaton field we have outlined the conditions on ϕ which must be satisfied if matter has to satisfy an energy condition. Specific choices of the dilaton are used to illustrate the violation or conservation of the WEC.

(iii) In charged dilaton gravity in $3+1$ dimensions, an explicit evaluation of these inequalities reveals interesting features. The electric black hole and its magnetic counterpart (in the string frame of reference) exhibit quite an opposite behavior as far as violation or conservation of the WEC is concerned. For the electric solution, violation occurs away from the horizon and extends up to infinity, in this region the string coupling is, of course, strong. Moreover, it is the $\rho + p$ inequality which is violated. On the contrary, for the magnetic hole, the violation is present near the horizon and occurs only for the $\rho + \tau$ inequality. These features persist if one takes the extremal limit in these metrics.

(iv) For certain nonasymptotically flat black hole solutions discovered recently, the electric black holes do not violate the WEC even in the string metric. Their magnetic counterparts also satisfy the weak energy condition.

(v) Several of the stringy black holes seem to satisfy the ANEC evaluated along radial null geodesics in the space-time. Only the asymptotically flat magnetic black hole and its extremal limit are the two exceptions.

Our aim in this paper has been to analyze in some detail the nature of the ‘‘matter’’ that is required to create the well-

known dilaton-Maxwell black hole solutions in low-energy effective string theory. Classical matter which may collapse to form such black holes must necessarily satisfy the WEC or some other energy condition. But, if matter violates the energy conditions, we cannot conclude that singularities do not exist. In fact, examples of singular metrics with WEC-violating matter do exist [26].

How does one justify the presence of singularities with the violation of the energy conditions within the context of the singularity theorems? One can take the attitude that these theorems are not valid. This is perhaps not entirely correct because there have been attempts to *extend* the singularity theorems by further *weakening or changing* the assumptions on matter. A step towards this is the proposal for global energy conditions. Singularity theorems with such global conditions have been proved by Roman [27] and Borde [28]. Some of the geometries discussed in this paper do satisfy the ANEC along radial null geodesics. However, for those stringy black holes which violate the local as well as the global energy conditions, one should try and extend the singularity theorems with some assumption on matter which is different from the known ones.

On the other hand, we also seem to have some solutions which satisfy the WEC without any problems. For such solutions the original singularity theorems are obviously valid.

Therefore, we might perhaps conclude by saying that it is somewhat premature to arrive at a general statement on the validity or nonvalidity of the singularity theorems in string-inspired gravity theories. Until we can prove a no-go theorem stating that there is *no* general assumption on matter which can be used to arrive at the existence of singularities in stringy generalizations of GR, we should leave this as an open issue worth future investigation.

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