Exact solution for the metric and the motion of two bodies in (1+1)-dimensional gravity

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We present the exact solution of two-body motion in $(1+1)$-dimensional dilaton gravity by solving the constraint equations in the canonical formalism. The determining equation of the Hamiltonian is derived in a transcendental form and the Hamiltonian is expressed for the system of two identical particles in terms of the Lambert W function. The W function has two real branches which join smoothly onto each other and the Hamiltonian on the principal branch reduces to the Newtonian limit for a small coupling constant. On the other branch the Hamiltonian yields a new set of motions which cannot be understood as relativistically correcting the Newtonian motion. The explicit trajectory in the phase space (r,p) is illustrated for various values of the energy. The analysis is extended to the case of unequal masses. The full expression of metric tensor is given and the consistency between the solution of the metric and the equations of motion is rigorously proved. [S0556-2821(97)01108-9]

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I. INTRODUCTION

One of the oldest and most notoriously vexing problems in gravitational theory is that of determining the (self-consistent) motion of N bodies and the resultant metric they collectively produce under their mutual gravitational influence [1]. In general there is no exact solution to this problem (although approximation techniques exist [2]) except in the case $N=2$ for Newtonian gravity, since energy dissipation in the form of gravitational radiation obstructs attempts to obtain an exact $N=2$ solution. Only the static sector of the Hamiltonian has thus far been determined exactly [3].

Lower dimensional theories of gravity do not contain gravitational radiation and so offer the possibility of making useful progress on this problem. For example in $2+1$ dimensions, the absence of a static gravitational potential allows one to generalize the static two-body metric to that of two bodies moving with any speed [4]. In $1+1$ dimensions one must necessarily consider a dilatonic theory of gravity [5] since the Einstein tensor is topologically trivial in two dimensions. One such theory in this class has been of particular interest insofar as it has a consistent Newtonian limit [6] (a problematic issue for a generic dilaton gravity theory [7]), allowing for the formulation of a general framework for deriving a Hamiltonian for a system of particles [8]. In the slow motion, weak field limit this Hamiltonian coincides with that of Newtonian gravity in $1+1$ dimensions.

Motivated by the above, we consider here the problem of the relativistic motion of two point masses under gravity in $1+1$ dimensions. We work in the context of the dilatonic theory of gravity mentioned above [6]. Both the classical and quantum properties of this theory (referred to as $R=T$ theory) have been extensively investigated [6,9–11], and it contains the Jackiw-Teitelboim lineal gravity theory [12] as a special case [6]. The specific form of the coupling of the dilaton field Ψ to gravity is chosen so that it decouples from the classical field equations in such a way as to ensure that the evolution of the gravitational field is determined only by the matter stress energy (and reciprocally) [6,9]. It thereby captures the essence of classical general relativity (as opposed to classical scalar-tensor theories) in two spacetime dimensions, and has $(1+1)$-dimensional analogs of many of its properties [9,10]. Furthermore, this theory can be understood as the $D\rightarrow 2$ limit of general relativity [as opposed to some particular solution(s)] [13].

We consequently find that the motion of the bodies is governed entirely by their mutual gravitational influence, and that the spacetime metric is likewise fully determined by their stress energy [6,9]. Unlike the $(2+1)$-dimensional case, a Newtonian limit exists, and there is a static gravitational potential. The solution we obtain gives the exact Hamiltonian to infinite order in the gravitational coupling constant. Hence the full structure of the theory from the weak field to the strong field limits can be studied. While some of the phase-space trajectories we obtain can be viewed as relativistic extensions of Newtonian motion, we find that for sufficiently large values of the total energy a qualitatively new set of trajectories arises that cannot be viewed in this way.

The outline of our paper is as follows. In Sec. II we recapitulate the derivation of the canonically reduced N-body Hamiltonian in $1+1$ dimensions. In Secs. III and IV we solve the constraint equation and then derive an expression for the exact Hamiltonian in the two-body case. In Sec. V we analyze the motion in the case of equal masses, and in Sec. VI we consider the unequal mass case. In Sec. VII we solve for the spacetime metric and in Sec. VIII we investigate the test-particle limit of our solution. In Sec. IX we consider the the dependence of the Newtonian limit on dimensionality. We close our manuscript with some concluding remarks and directions for further work.

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II. THE CANONICALLY REDUCED N-BODY HAMILTONIAN

First we shall review the outline of the canonical reduction of $(1+1)$-dimensional dilaton gravity [8]. The action integral for the gravitational field coupled with N point particles is

$$I=\int dx^2\left[\frac{1}{2\kappa}\sqrt{-g}\left\{\Psi R+\frac{1}{2}g^{\mu\nu}\nabla_\mu\Psi\nabla_\nu\Psi\right\}-\sum_{a=1}^{N}m_a\int d\tau_a\left\{-g_{\mu\nu}(x)\frac{dz_a^\mu}{d\tau_a}\frac{dz_a^\nu}{d\tau_a}\right\}^{1/2}\delta^2(x-z_a(\tau_a))\right],\tag{1}$$

where Ψ is the dilaton field, $g_{\mu\nu}$ is the metric (with determinant g), R is the Ricci scalar, and τ_a is the proper time of ath particle, with $\kappa=8\pi G/c^4$. The symbol ∇_μ denotes the covariant derivative associated with $g_{\mu\nu}$.

The field equations derived from the action (1) are

$$R-g^{\mu\nu}\nabla_\mu\nabla_\nu\Psi=0,\tag{2}$$

$$\frac{1}{2}\nabla_\mu\Psi\nabla_\nu\Psi-\frac{1}{4}g_{\mu\nu}\nabla^\lambda\Psi\nabla_\lambda\Psi+g_{\mu\nu}\nabla^\lambda\nabla_\lambda\Psi-\nabla_\mu\nabla_\nu\Psi=\kappa T_{\mu\nu},\tag{3}$$

$$\frac{d}{d\tau_a}\left\{g_{\mu\nu}(z_a)\frac{dz_a^\nu}{d\tau_a}\right\}-\frac{1}{2}g_{\nu\lambda,\mu}(z_a)\frac{dz_a^\nu}{d\tau_a}\frac{dz_a^\lambda}{d\tau_a}=0,\tag{4}$$

where the stress energy due to the point masses is

$$T_{\mu\nu}=\sum_{a=1}^{N}m_a\int d\tau_a\frac{1}{\sqrt{-g}}g_{\mu\sigma}g_{\nu\rho}\frac{dz_a^\sigma}{d\tau_a}\frac{dz_a^\rho}{d\tau_a}\delta^2(x-z_a(\tau_a)).\tag{5}$$

Inserting the trace of Eq. (3) into Eq. (2) yields

$$R=\kappa T^\mu{}_\mu.\tag{6}$$

Particle dynamics in $R=T$ theory may therefore be described in terms of the equations (4) and (6), which forms a closed system of equations for the gravity or matter system.

At first sight it may seem that the dynamics is independent of the dilaton field, since both Eqs. (4) and Eq. (6) do not include Ψ. Note, however, that all three components of the metric tensor cannot be determined from Eq. (6), since it is only one equation. The two extra degrees of freedom are related to the choice of coordinates. If the coordinate conditions are chosen to be independent of Ψ, Eq. (6) determines the metric tensor completely. However, this need not be the case, and so more generally we need to know the dilaton field Ψ, through which the metric tensor is (indirectly) determined—it is this field that guarantees conservation of the stress-energy tensor via Eq. (3).

In the canonical formalism the action (1) is written in the form

$$I=\int dx^2\left\{\sum_a p_a\dot{z}_a\delta(x-z_a(x^0))+\pi\dot{\gamma}+\Pi\dot{\Psi}+N_0R^0+N_1R^1\right\},\tag{7}$$

where $\gamma=g_{11}$, $N_0=(-g^{00})^{-1/2}$, $N_1=g_{10}$, π and Π are conjugate momenta to γ and Ψ, respectively, and

$$R^0=-\kappa\sqrt{\gamma}\gamma\pi^2+2\kappa\sqrt{\gamma}\pi\Pi+\frac{1}{4\kappa\sqrt{\gamma}}(\Psi')^2-\frac{1}{\kappa}\left(\frac{\Psi'}{\sqrt{\gamma}}\right)'-\sum_a\sqrt{\frac{p_a^2}{\gamma}+m_a^2}\,\delta(x-z_a(x^0)),\tag{8}$$

$$R^1=\frac{\gamma'}{\gamma}\pi-\frac{1}{\gamma}\Pi\Psi'+2\pi'+\sum_a\frac{p_a}{\gamma}\delta(x-z_a(x^0)),\tag{9}$$

with the overdot and prime denoting ∂_0 and ∂_1, respectively.

The transformation from Eq. (1) to Eq. (7) is carried out by using the decomposition of the scalar curvature in terms of the extrinsic curvature K via

$$\sqrt{-g}R=-2\partial_0(\sqrt{\gamma}K)+2\partial_1[\sqrt{\gamma}(N^1K-\gamma^{-1}\partial_1N_0)],$$

where $K=(2N_0\gamma)^{-1}(2\partial_1N_1-\gamma^{-1}N_1\partial_1\gamma-\partial_0\gamma)$ and changing the particle Lagrangian into first order form.

The action (7) leads to the system of field equations:

$$\dot{\pi}+N_0\left\{\frac{3\kappa}{2}\sqrt{\gamma}\pi^2-\frac{\kappa}{\sqrt{\gamma}}\pi\Pi+\frac{1}{8\kappa\sqrt{\gamma}\gamma}(\Psi')^2-\sum_a\frac{p_a^2}{2\gamma^2\sqrt{\frac{p_a^2}{\gamma}+m_a^2}}\delta(x-z_a(x^0))\right\}+N_1\left\{-\frac{1}{\gamma^2}\Pi\Psi'+\frac{\pi'}{\gamma}+\sum_a\frac{p_a}{\gamma^2}\delta(x-z_a(x^0))\right\}+N_0'\frac{1}{2\kappa\sqrt{\gamma}\gamma}\Psi'+N_1'\frac{\pi}{\gamma}=0,\tag{10}$$

$$\dot{\gamma}-N_0(2\kappa\sqrt{\gamma}\gamma\pi-2\kappa\sqrt{\gamma}\Pi)+N_1\frac{\gamma'}{\gamma}-2N_1'=0,\tag{11}$$

$$R^0=0,\tag{12}$$

$$R^1=0,\tag{13}$$

$$\dot{\Pi}+\partial_1\left(-\frac{1}{\gamma}N_1\Pi+\frac{1}{2\kappa\sqrt{\gamma}}N_0\Psi'+\frac{1}{\kappa\sqrt{\gamma}}N_0'\right)=0,\tag{14}$$

$$\dot{\Psi}+N_0(2\kappa\sqrt{\gamma}\pi)-N_1\left(\frac{1}{\gamma}\Psi'\right)=0,\tag{15}$$

$$\dot{p}_a+\frac{\partial N_0}{\partial z_a}\sqrt{\frac{p_a^2}{\gamma}+m_a^2}-\frac{N_0}{2\sqrt{\frac{p_a^2}{\gamma}+m_a^2}}\frac{p_a^2}{\gamma^2}\frac{\partial\gamma}{\partial z_a}-\frac{\partial N_1}{\partial z_a}\frac{p_a}{\gamma}$$

$$+N_1\frac{p_a}{\gamma^2}\frac{\partial\gamma}{\partial z_a}=0, \tag{16}$$

$$\dot{z}_a-N_0\frac{\frac{p_a}{\gamma}}{\sqrt{\frac{p_a^2}{\gamma}+m_a^2}}+\frac{N_1}{\gamma}=0. \tag{17}$$

In the equations (16) and (17), all metric components (N_0, N_1, γ) are evaluated at the point $x=z_a$ and

$$\frac{\partial f}{\partial z_a}\equiv\frac{\partial f(x)}{\partial x}\bigg|_{x=z_a}.$$

This system of equations is equivalent to the set of equations (2), (3), and (4).

The action (7) also shows that N_0 and N_1 are Lagrange multipliers, and the equations (12) and (13) are constraints. We may investigate the canonical structure of the theory via the generator which arises from the variation of the action at the boundaries:

$$G=\int dx\left\{\sum_a p_a\delta(x-z_a)\,\delta z_a+\pi\,\delta h-\Psi\,\delta\Pi\right\}, \tag{18}$$

where $h\equiv 1+\gamma$. This form was obtained by adding a total time derivative $-\partial_0(\Pi\Psi)$ to the original action (7) and taking the constraints into account. Since the only linear terms in the constraints are $(\Psi'/\sqrt{\gamma})'$ and π', we may solve for these quantities in terms of the dynamical and gauge (i.e., coordinate) degrees of freedom. Bearing this fact in mind, we transform the generator (18) to

$$G=\int dx\left\{\sum_a p_a\delta(x-z_a)\,\delta z_a-\left[-\frac{1}{\kappa}\left(\frac{\Psi'}{\sqrt{h-1}}\right)'\right]\right.$$
$$\times\delta\left[-\frac{\kappa}{\triangle}\left(\sqrt{h-1}\frac{1}{\triangle}\Pi'\right)'\right]-\left[2\pi'-\left(\frac{\Psi'}{h-1}\right)'\right.$$
$$\left.\left.\times\frac{1}{\triangle}\Pi'-\frac{1}{h-1}\Pi\Psi'\right]\delta\left(\frac{1}{2\triangle}h'\right)\right\}, \tag{19}$$

where $1/\triangle$ is the inverse of the operator $\triangle=\partial^2/\partial x^2$ with appropriate boundary condition, and we have discarded surface terms.

Adopting the coordinate conditions

$$x=\frac{1}{2\triangle}h',\quad t=-\frac{\kappa}{\triangle}\left(\sqrt{h-1}\frac{1}{\triangle}\Pi'\right)' \tag{20}$$

allows the generator (19) to be expressed in the canonical form

$$G=\int dx\left\{\sum_a p_a\delta(x-z_a)\,\delta z_a-\mathcal{T}_{0\mu}\delta x^{\mu}\right\}, \tag{21}$$

where

$$\mathcal{T}_{00}=\mathcal{H}=-\frac{1}{\kappa}\triangle\Psi, \tag{22}$$

$$\mathcal{T}_{01}=2\pi'. \tag{23}$$

In deriving this expression for $\mathcal{T}_{0\mu}$, we made use of the differential forms of the coordinate conditions

$$\gamma=1,\quad \Pi=0. \tag{24}$$

As discussed in [8], consistency between the integral form (20) of the coordinate conditions and the differential form (24) is assured when one retains the appropriate boundary conditions for the integral operator $1/\triangle$ and takes a limiting procedure by introducing a regulator. Thus, $\mathcal{H}$ is the Hamiltonian density of the system and $\mathcal{T}_{01}$ is the momentum density.

The action (7) is similarly tranformed and reduces to

$$I=\int dx^2\left\{\sum_a p_a\dot{z}_a\delta(x-z_a)-\mathcal{H}\right\}. \tag{25}$$

Thus the reduced Hamiltonian for the system of particles is

$$H=\int dx\mathcal{H}=-\frac{1}{\kappa}\int dx\triangle\Psi, \tag{26}$$

where Ψ is a function of z_a and p_a and is determined by solving the constraints which are, under the coordinate conditions (24),

$$\triangle\Psi-\frac{1}{4}(\Psi')^2+\kappa^2\pi^2+\kappa\sum_a\sqrt{p_a^2+m_a^2}\,\delta(x-z_a)=0, \tag{27}$$

$$2\pi'+\sum_a p_a\delta(x-z_a)=0. \tag{28}$$

The expression of the Hamiltonian (26) is analogous to the reduced Hamiltonian in $(3+1)$-dimensional general relativity. The proof of the consistency of this canonical reduction was given in [8]: namely, the canonical equations of motion derived from the reduced Hamiltonian (26) are identical with the equations (16) and (17).

III. MATCHING CONDITIONS AND THE SOLUTION TO THE CONSTRAINT EQUATIONS

The standard approach for investigating the dynamics of a system of particles is to derive an explicit expression for the Hamiltonian, in which all information on the motion of the particles is included. In this section we solve the constraints (27) and (28) for the system of two particles and determine the Hamiltonian (28).

Defining ϕ and χ by

$$\Psi=-4\ln|\phi|,\quad \pi=\chi' \tag{29}$$

the constraints (27) and (28) for a two-particle system become

$$\triangle\phi-\frac{\kappa^2}{4}(\chi')^2\phi=\frac{\kappa}{4}\{\sqrt{p_1^2+m_1^2}\phi(z_1)\,\delta(x-z_1)$$
$$+\sqrt{p_2^2+m_2^2}\phi(z_2)\,\delta(x-z_2)\}, \tag{30}$$

$$\triangle\chi=-\frac{1}{2}\{p_1\delta(x-z_1)+p_2\delta(x-z_2)\}\ . \tag{31}$$

The general solution to Eq. (31) is

$$\chi=-\frac{1}{4}\{p_1|x-z_1|+p_2|x-z_2|\}-\epsilon Xx+\epsilon C_\chi. \tag{32}$$

The factor ϵ ($\epsilon^2=1$) has been introduced in the constants X and C_χ so that the T-inversion (time reversal) properties of χ are explicitly manifest. By definition, ϵ changes sign under time reversal and so, therefore, does χ.

Consider first the case $z_2<z_1$, for which we may divide spacetime into three regions:

$$z_1<x \quad (+)\text{ region},$$

$$z_2<x<z_1 \quad (0)\text{ region},$$

$$x<z_2 \quad (-)\text{ region}.$$

In each region χ' is constant:

$$\chi'=\begin{cases}-\epsilon X-\frac{1}{4}(p_1+p_2) & (+)\text{ region},\\ -\epsilon X+\frac{1}{4}(p_1-p_2) & (0)\text{ region},\\ -\epsilon X+\frac{1}{4}(p_1+p_2) & (-)\text{ region}.\end{cases} \tag{33}$$

General solutions to the homogeneous equation $\triangle\phi-(\kappa^2/4)(\chi')^2\phi=0$ in each region are

$$\phi_+(x)=A_+\exp\left[\frac{\kappa}{2}\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)x\right]+B_+\exp\left[-\frac{\kappa}{2}\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)x\right],$$

$$\phi_0(x)=A_0\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)x\right]+B_0\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)x\right],$$

$$\phi_-(x)=A_-\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)x\right]+B_-\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)x\right]. \tag{34}$$

For these solutions to be the actual solutions to Eq. (30) with delta function source terms, they must satisfy the following matching conditions at $x=z_1,z_2$:

$$\phi_+(z_1)=\phi_0(z_1)=\phi(z_1), \tag{35a}$$

$$\phi_-(z_2)=\phi_0(z_2)=\phi(z_2), \tag{35b}$$

$$\phi'_+(z_1)-\phi'_0(z_1)=\frac{\kappa}{4}\sqrt{p_1^2+m_1^2}\,\phi(z_1), \tag{35c}$$

$$\phi'_0(z_2)-\phi'_-(z_2)=\frac{\kappa}{4}\sqrt{p_2^2+m_2^2}\,\phi(z_2). \tag{35d}$$

The conditions (35a) and (35c) lead to

$$\exp\left[\frac{\kappa}{2}\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)z_1\right]A_++\exp\left[-\frac{\kappa}{2}\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)z_1\right]B_+ =\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_1\right]A_0+\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_1\right]B_0 \tag{36}$$

and

$$\exp\left[\frac{\kappa}{2}\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)z_1\right]A_+-\exp\left[-\frac{\kappa}{2}\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)z_1\right]B_+ =\frac{\sqrt{p_1^2+m_1^2}+2\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)}{2\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)}\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_1\right]A_0+\frac{\sqrt{p_1^2+m_1^2}-2\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)}{2\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)}\times\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_1\right]B_0 \tag{37}$$

and then

$$A_+=\frac{\sqrt{p_1^2+m_1^2}+4X+\epsilon p_2}{4X+\epsilon(p_1+p_2)}\exp\left(-\frac{\kappa\epsilon}{4}p_1z_1\right)A_0+\frac{\sqrt{p_1^2+m_1^2}+\epsilon p_1}{4X+\epsilon(p_1+p_2)}\exp\left[-\kappa\left(X+\frac{\epsilon}{4}p_2\right)z_1\right]B_0, \tag{38a}$$

$$B_+=-\frac{\sqrt{p_1^2+m_1^2}-\epsilon p_1}{4X+\epsilon(p_1+p_2)}\exp\left[\kappa\left(X+\frac{\epsilon}{4}p_2\right)z_1\right]A_0-\frac{\sqrt{p_1^2+m_1^2}-4X-\epsilon p_2}{4X+\epsilon(p_1+p_2)}\exp\left(\frac{\kappa\epsilon}{4}p_1z_1\right)B_0. \tag{38b}$$

Similarly the conditions (35b) and (35d) lead to

$$\begin{aligned}&\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)z_2\right]A_-+\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)z_2\right]B_-\\&\quad=\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_2\right]A_0+\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_2\right]B_0\end{aligned} \tag{39}$$

and

$$\begin{aligned}&-\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)z_2\right]A_-+\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)z_2\right]B_-\\&\quad=\frac{\sqrt{p_2^2+m_2^2}-2\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)}{2\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)}\exp\left[\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_2\right]A_0+\frac{\sqrt{p_2^2+m_2^2}+2\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)}{2\left(X-\frac{\epsilon}{4}(p_1+p_2)\right)}\\&\quad\quad\times\exp\left[-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_2\right]B_0\end{aligned} \tag{40}$$

and then

$$A_-=-\frac{\sqrt{p_2^2+m_2^2}-4X+\epsilon p_1}{4X-\epsilon(p_1+p_2)}\exp\left(\frac{\kappa\epsilon}{4}p_2z_2\right)A_0-\frac{\sqrt{p_2^2+m_2^2}+\epsilon p_2}{4X-\epsilon(p_1+p_2)}\exp\left[-\kappa\left(X-\frac{\epsilon}{4}p_1\right)z_2\right]B_0, \tag{41a}$$

$$B_-=\frac{\sqrt{p_2^2+m_2^2}-\epsilon p_2}{4X-\epsilon(p_1+p_2)}\exp\left[\kappa\left(X-\frac{\epsilon}{4}p_1\right)z_2\right]A_0+\frac{\sqrt{p_2^2+m_2^2}+4X-\epsilon p_1}{4X-\epsilon(p_1+p_2)}\exp\left(-\frac{\kappa\epsilon}{4}p_2z_2\right)B_0. \tag{41b}$$

Since the magnitudes of both ϕ and χ increase with increasing $|x|$, it is necessary to impose a boundary condition which guarantees that the surface terms which arise in transforming the action vanish and simultaneously preserves the finiteness of the Hamiltonian.

In the iterative analysis in [8] this condition has been shown to be

$$\Psi^2-4\kappa^2\chi^2=0 \text{ in the region } z_1<x \text{ and } x<z_2\ . \tag{42}$$

Since

$$\chi=\begin{cases}-\{\epsilon X+\frac{1}{4}(p_1+p_2)\}x+\epsilon C_\chi+\frac{1}{4}(p_1z_1+p_2z_2) & (+)\text{ region}\\-\{\epsilon X-\frac{1}{4}(p_1+p_2)\}x+\epsilon C_\chi-\frac{1}{4}(p_1z_1+p_2z_2) & (-)\text{ region,}\end{cases} \tag{43}$$

the boundary condition implies

$$A_-=B_+=0, \tag{44}$$

$$-\ln A_+ - \frac{\kappa\epsilon}{8}(p_1z_1+p_2z_2)=\ln B_- + \frac{\kappa\epsilon}{8}(p_1z_1+p_2z_2)=\frac{\kappa}{2}C_\chi. \tag{45}$$

The condition (44) leads to

$$\frac{A_0}{B_0}=-\frac{\sqrt{p_2^2+m_2^2}+\epsilon p_2}{\sqrt{p_2^2+m_2^2}-4X+\epsilon p_1}\exp\left[-\kappa\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_2\right] \tag{46}$$

and

$$\frac{A_0}{B_0}=-\frac{\sqrt{p_1^2+m_1^2}-4X-\epsilon p_2}{\sqrt{p_1^2+m_1^2}-\epsilon p_1}\exp\left[-\kappa\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)z_1\right]. \tag{47}$$

From Eqs. (46) and (47) we have

$$(\sqrt{p_1^2+m_1^2}-\epsilon p_2-4X)(\sqrt{p_2^2+m_2^2}+\epsilon p_1-4X)=(\sqrt{p_1^2+m_1^2}-\epsilon p_1)(\sqrt{p_2^2+m_2^2}+\epsilon p_2)\exp\left[\kappa\left(X-\frac{\epsilon}{4}(p_1-p_2)\right)(z_1-z_2)\right]. \tag{48}$$

On the other hand the condition (45) leads to

$$\left\{\frac{\sqrt{p_1^2+m_1^2}+4X+\epsilon p_2}{4X+\epsilon(p_1+p_2)}\exp\left(-\frac{\kappa\epsilon}{4}p_1z_1\right)A_0+\frac{\sqrt{p_1^2+m_1^2}+\epsilon p_1}{4X+\epsilon(p_1+p_2)}\exp\left[-\kappa\left(X+\frac{\epsilon}{4}p_2\right)z_1\right]B_0\right\}$$
$$\times\left\{\frac{\sqrt{p_2^2+m_2^2}-\epsilon p_2}{4X-\epsilon(p_1+p_2)}\exp\left[\kappa\left(X-\frac{\epsilon}{4}p_1\right)z_2\right]A_0+\frac{\sqrt{p_2^2+m_2^2}+4X-\epsilon p_1}{4X-\epsilon(p_1+p_2)}\exp\left(-\frac{\kappa\epsilon}{4}p_2z_2\right)B_0\right\}=\exp\left(-\frac{\kappa\epsilon}{4}(p_1z_1+p_2z_2)\right). \tag{49}$$

Using the notation

$$\begin{aligned}
M_1&\equiv\sqrt{p_1^2+m_1^2}-\epsilon p_1,\\
M_2&\equiv\sqrt{p_2^2+m_2^2}+\epsilon p_2,\\
L_1&\equiv 4X-\epsilon p_1-\sqrt{p_2^2+m_2^2},\\
L_2&\equiv 4X+\epsilon p_2-\sqrt{p_1^2+m_1^2},\\
L_+&\equiv 4X+\epsilon(p_1+p_2),\\
L_0&\equiv 4X-\epsilon p_1+\epsilon p_2,\\
L_-&\equiv 4X-\epsilon(p_1+p_2),
\end{aligned} \tag{50}$$

we obtain

$$\begin{aligned}
A_0&=\frac{(M_2L_2)^{1/2}}{L_0}\exp\left(-\frac{\kappa}{8}L_0z_2\right)\\
&=\left(\frac{L_1}{M_1}\right)^{1/2}\frac{L_2}{L_0}\exp\left(-\frac{\kappa}{8}L_0z_1\right),
\end{aligned} \tag{51}$$

$$B_0=\left(\frac{L_2}{M_2}\right)^{1/2}\frac{L_1}{L_0}\exp\left(\frac{\kappa}{8}L_0z_2\right)=\frac{(M_1L_1)^{1/2}}{L_0}\exp\left(\frac{\kappa}{8}L_0z_1\right) \tag{52}$$

from Eqs. (46), (47), and (49). Substituting Eqs. (51) and (52) into Eqs. (38a) and (41b) we get

$$A_+=\left(\frac{L_1}{M_1}\right)^{1/2}\exp\left(-\frac{\kappa}{8}L_+z_1\right), \tag{53}$$

$$B_-=\left(\frac{L_2}{M_2}\right)^{1/2}\exp\left(\frac{\kappa}{8}L_-z_2\right), \tag{54}$$

and then

$$\phi_+=\left(\frac{L_1}{M_1}\right)^{1/2}\exp\left(\frac{\kappa}{8}L_+(x-z_1)\right), \tag{55}$$

$$\phi_-=\left(\frac{L_2}{M_2}\right)^{1/2}\exp\left(-\frac{\kappa}{8}L_-(x-z_2)\right), \tag{56}$$

$$\begin{aligned}
\phi_0=\frac{1}{L_0}\left(\frac{L_1L_2}{M_1M_2}\right)^{1/2}&\left\{(M_2L_2)^{1/2}\exp\left(\frac{\kappa}{8}L_0(x-z_1)\right)\right.\\
&\left.+(M_1L_1)^{1/2}\exp\left(-\frac{\kappa}{8}L_0(x-z_2)\right)\right\}.
\end{aligned} \tag{57}$$

In the case of $z_1<z_2$ we have to interchange the suffices 1 and 2. The equation (48) which determines X then generalizes to

$$(\sqrt{p_1^2+m_1^2}-\epsilon\tilde{p}_2-4X)(\sqrt{p_2^2+m_2^2}+\epsilon\tilde{p}_1-4X)$$
$$=(\sqrt{p_1^2+m_1^2}-\epsilon\tilde{p}_1)(\sqrt{p_2^2+m_2^2}+\epsilon\tilde{p}_2)$$
$$\times\exp\left[\kappa\left(X-\frac{\epsilon}{4}(\tilde{p}_1-\tilde{p}_2)\right)|r|\right], \tag{58}$$

where $r\equiv z_1-z_2$ and $\tilde{p}_a\equiv p_a\mathrm{sgn}(z_1-z_2)$.

IV. DETERMINING EQUATION FOR THE HAMILTONIAN AND THE CANONICAL EQUATIONS OF MOTION

Since the solutions of ϕ give

$$\frac{\phi_+'(x)}{\phi_+(x)}=\frac{\kappa}{2}\left(X+\frac{\epsilon}{4}(p_1+p_2)\right)$$

and (59)

$$\frac{\phi_-'(x)}{\phi_-(x)}=-\frac{\kappa}{2}\left(X-\frac{\epsilon}{4}(p_1+p_2)\right),$$

the Hamiltonian H is

$$H=-\frac{1}{\kappa}\int dx\triangle\Psi=-\frac{1}{\kappa}[\Psi']_{-\infty}^{\infty}$$
$$=\frac{4}{\kappa}\lim_{L\to\infty}\left\{\frac{\phi_+'(L)}{\phi_+(L)}-\frac{\phi_-'(-L)}{\phi_-(-L)}\right\}=4X \tag{60}$$

and so Eq. (58) becomes

$$(\sqrt{p_1^2+m_1^2}-\epsilon\tilde{p}_2-H)(\sqrt{p_2^2+m_2^2}+\epsilon\tilde{p}_1-H)$$
$$=(\sqrt{p_1^2+m_1^2}-\epsilon\tilde{p}_1)(\sqrt{p_2^2+m_2^2}+\epsilon\tilde{p}_2)$$
$$\times\exp\left(\frac{\kappa}{4}[H-\epsilon(\tilde{p}_1-\tilde{p}_2)]\ |r|\right). \tag{61}$$

Equation (61) is the determining equation for the Hamiltonian, whose solution yields H as a function of (p_1,p_2,r).

Expanding Eq. (61) in powers of κ yields the perturbative solution

$$H=\sqrt{p_1^2+m_1^2}+\sqrt{p_2^2+m_2^2}+\frac{\kappa}{4}(\sqrt{p_1^2+m_1^2}\sqrt{p_2^2+m_2^2}-p_1p_2)|r|+\frac{\kappa\epsilon}{4}(\sqrt{p_1^2+m_1^2}\,p_2-p_1\sqrt{p_2^2+m_2^2})(z_1-z_2)$$
$$+\frac{\kappa^2}{2\times4^2}\{(\sqrt{p_1^2+m_1^2}\sqrt{p_2^2+m_2^2}-2p_1p_2)(\sqrt{p_1^2+m_1^2}+\sqrt{p_2^2+m_2^2})+\sqrt{p_1^2+m_1^2}\,p_2^2+p_1^2\sqrt{p_2^2+m_2^2}\}r^2$$
$$+\frac{\kappa^2\epsilon}{2\times4^2}\{-2(\sqrt{p_1^2+m_1^2}\sqrt{p_2^2+m_2^2}-p_1p_2)(p_1-p_2)+m_1^2p_2-p_1m_2^2\}|r|(z_1-z_2)\cdots \tag{62}$$

up to $O(\kappa^2)$. This is identical with the Hamiltonian derived in the iterative method in [8].

For the case of $z_2<z_1$, Eq. (61) is

$$L_1L_2=M_1M_2\exp\left(\frac{\kappa}{4}L_0(z_1-z_2)\right). \tag{63}$$

Differentiating Eq. (63) with respect to z_1 leads to

$$\frac{\partial H}{\partial z_1}(L_1+L_2)=\frac{\kappa}{4}L_1L_2\left(r\frac{\partial H}{\partial z_1}+L_0\right).$$

Then we have the canonical equation

$$\dot{p}_1=-\frac{\partial H}{\partial z_1}=\frac{-\frac{\kappa}{4}L_0L_1L_2}{L_1+L_2-\frac{\kappa r}{4}L_1L_2} \tag{64}$$

and similarly

$$\dot{p}_2=\frac{\frac{\kappa}{4}L_0L_1L_2}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}. \tag{65}$$

Differentiating Eq. (63) with respect to p_1 leads to

$$\left(\frac{\partial H}{\partial p_1}-\frac{p_1}{\sqrt{p_1^2+m_1^2}}\right)L_1+\left(\frac{\partial H}{\partial p_1}-\epsilon\right)L_2$$
$$=L_1L_2\left\{-\frac{\epsilon}{\sqrt{p_1^2+m_1^2}}+\frac{\kappa r}{4}\left(\frac{\partial H}{\partial p_1}-\epsilon\right)\right\}.$$

We have also the canonical equation

$$\dot{z}_1=\frac{\partial H}{\partial p_1}=\epsilon-\frac{\epsilon L_0L_1}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\frac{1}{\sqrt{p_1^2+m_1^2}} \tag{66}$$

and similarly

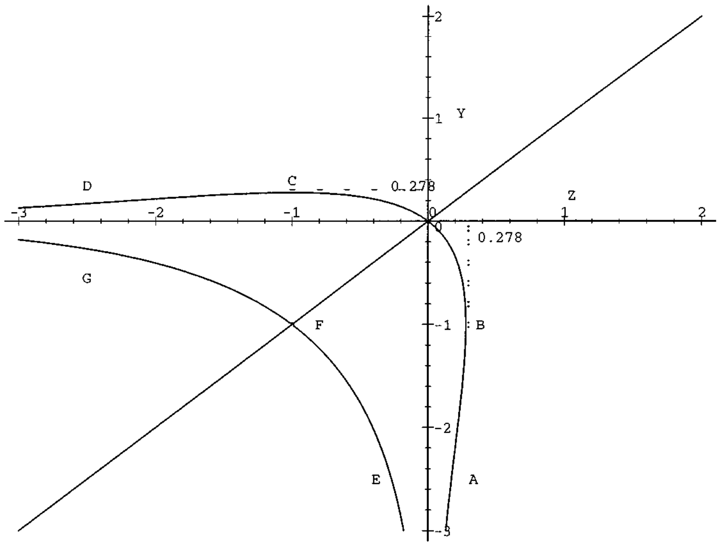

FIG. 1. Solutions to Eq. (71). The points B and C represent the extremal Z and Y values of $W^{-1}(1/e)=0.278$ on the principal branch.

$$\dot{z}_2=-\epsilon+\frac{\epsilon L_0 L_2}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\frac{1}{\sqrt{p_2^2+m_2^2}}. \qquad (67)$$

It is evident that the Hamiltonian and the total momentum $P=p_1+p_2$ are constants of motion:

$$\dot{H}=0, \quad \dot{P}=\dot{p}_1+\dot{p}_2=0\,. \qquad (68)$$

V. HAMILTONIAN OF TWO IDENTICAL PARTICLES

In this section we shall try to solve Eq. (61) for a system of two identical particles. We may choose the center of inertia frame with $p_1=-p_2=p$. Then Eq. (61) becomes

$$(H-\sqrt{p^2+m^2}-\epsilon\widetilde{p})^2=(\sqrt{p^2+m^2}-\epsilon\widetilde{p})^2 \times\exp\left(\frac{\kappa}{4}(H-2\epsilon\widetilde{p})|r|\right). \qquad (69)$$

After setting

$$H=\sqrt{p^2+m^2}+\epsilon\widetilde{p}-\frac{8}{\kappa|r|}Y(p,r), \qquad (70)$$

Eq. (69) becomes

$$Y^2e^{2Y}=Z^2e^{2Z}, \qquad (71)$$

where $Z\equiv(\kappa|r|/8)(\sqrt{p^2+m^2}-\epsilon\widetilde{p})$.

Equation (71) has three solutions shown in Fig. 1. The trivial solution, $Y=Z$, yields $H=2\epsilon\widetilde{p}$, which is unphysical because it has no interaction term. The second solution (curve A-B-O-C-D) is represented by

$$Y=W(-Ze^Z), \quad Z\leqslant z_0=W^{-1}(e^{-1}) \qquad (72)$$

and the third solution (curve E-F-G) is represented by

$$Y=W(Ze^Z), \qquad Z<0, \qquad (73)$$

where $W(x)$ is the Lambert W function defined via

$$y\cdot e^y=x\Rightarrow y=W(x). \qquad (74)$$

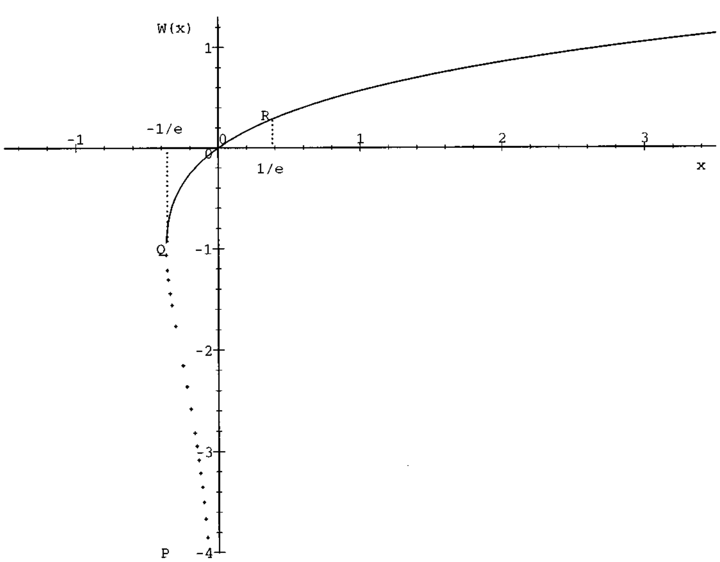

FIG. 2. The Lambert W function.

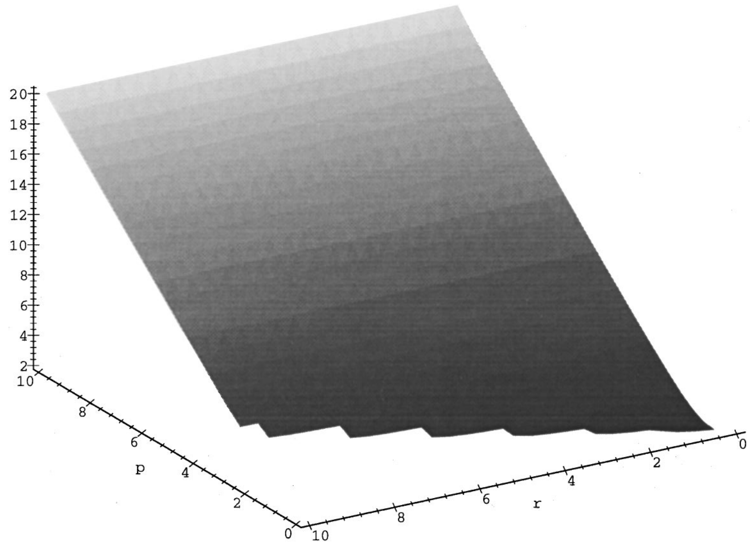

FIG. 3. Hamiltonian as a function of momentum $p>0$ and κr in units of m.

In general W is complex and multivalued. When x is real, the function has two real branches shown in Fig. 2 [14]. The branch satisfying $-1\leqslant W(x)$ (a solid line) is denoted by $W_0(x)$ and is referred to as the principal branch. The branch satisfying $W(x)\leqslant -1$ (a broken line) is denoted by $W_{-1}(x)$, and is real valued only for $-1/e\leqslant x<0$. Then for $-1/e\leqslant x<0$ the function is double valued. The principal branch is analytic at $x=0$ and has a derivative singularity at $x=-1/e$ beyond which $W(x)$ becomes complex. The series expansion of the principal branch is given by

$$W_0(x)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}x^n. \tag{75}$$

The correspondence between the curves in Fig. 1 and those in Fig. 2 is

curve A-B-O-C-D⇔curve P-Q-O-R-O,

curve E-F-G⇔curve O-Q-P.

Since the physical domain of Z is $Z\geqslant 0$, the only physical solution is Eq. (72), which yields the Hamiltonian

$$H=\sqrt{p^2+m^2}+\epsilon p\,\mathrm{sgn}(r)-8\,\frac{W\left[-\frac{\kappa}{8}(|r|\sqrt{p^2+m^2}-\epsilon pr)\exp\left(\frac{\kappa}{8}(|r|\sqrt{p^2+m^2}-\epsilon pr)\right)\right]}{\kappa|r|}. \tag{76}$$

This Hamiltonian is exact to infinite order in the gravitational coupling constant. We can thus view the whole structure of the theory from the weak field to the strong field limits.

The weak field expansion has already been given in the general case in Eq. (62). The small p expansion is

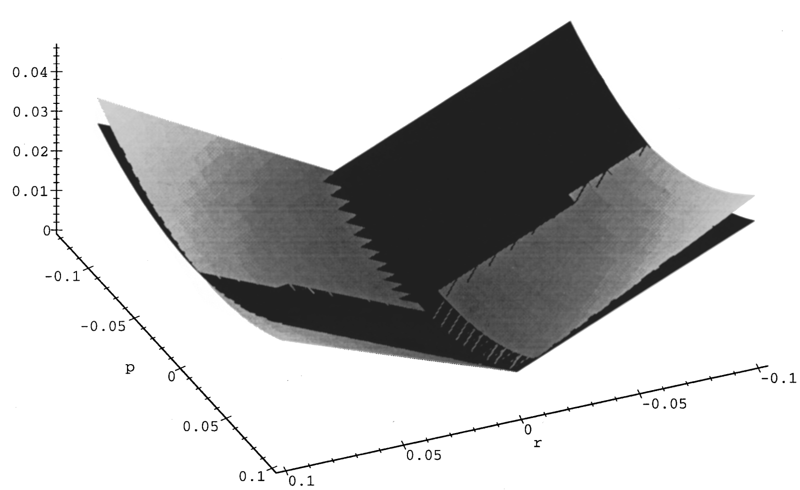

FIG. 4. H in the nonrelativistic limit compared with the Newtonian Hamiltonian.

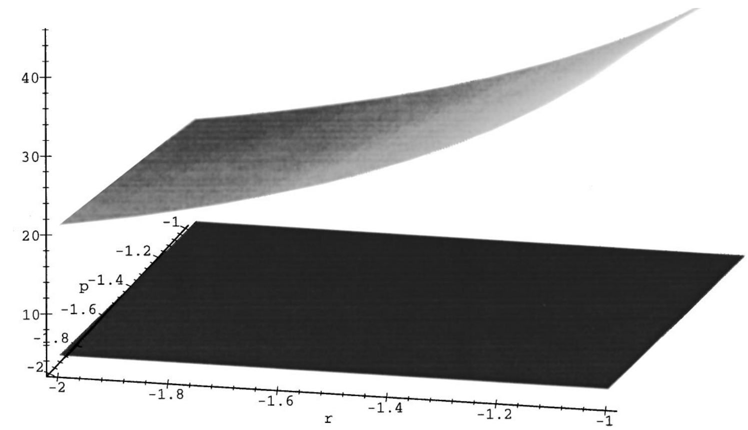

FIG. 5. Both branches of the Hamiltonian.

$$H=\frac{m\kappa|r|-8w}{\kappa|r|}+\frac{m\kappa|r|+2m\kappa|r|w+8w}{m\kappa|r|(1+w)}\epsilon\tilde{p}-\frac{1}{16}\frac{-8m\kappa|r|-8m\kappa|r|w^2+64w+m^2\kappa^2|r|^2w}{m^2\kappa|r|(1+w)^3}p^2+\cdots, \tag{77}$$

where

$$w\equiv W\left[-\frac{m\kappa|r|}{8}\exp\left(\frac{m\kappa|r|}{8}\right)\right].$$

The leading term is simply the mass plus a static gravitational correction, which is $m^2\kappa|r|$ to lowest order in κ. The term linear in p is due to purely to gravity, since it vanishes in the limit $\kappa\to 0$. The argument of the function w must be larger than $-1/e$. This translates into the limit $m\kappa r< W(1/e)$ which means that if $m\kappa r$ is sufficiently large, there is no small p expansion—i.e., there is some minimum value of p below which the Hamiltonian is no longer real. This situation is shown in Fig. 3.

The small m expansion (which is the same as the large p expansion) is also easily obtained. For example in the region $p>0$, $r>0$ with $\epsilon=1$ we find

$$H=2p+\frac{1}{p}m^2+\frac{1}{16}\frac{-4+\kappa rp}{p^3}m^4+\frac{1}{128}\frac{16-4\kappa rp+\kappa^2r^2p^2}{p^5}m^6+\cdots. \tag{78}$$

The κ-independent terms are equivalent to those obtained for two free relativistic particles of equal mass in the small mass limit. We see that the effects of gravity modify the Hamiltonian to include interaction terms whose strength grows with increasing separation, as one might expect from the basic structure of two-dimensional gravity.

The Hamiltonian (76) describes the surface in (r,p,H) space of all allowed phase-space trajectories. Since H is a constant of the motion, a trajectory in the (r,p) plane is uniquely determined by setting $H=H_0$ in Eq. (76). However, there are two distinct sets of trajectories which correspond to the two real branches of W function which join smoothly onto each other.

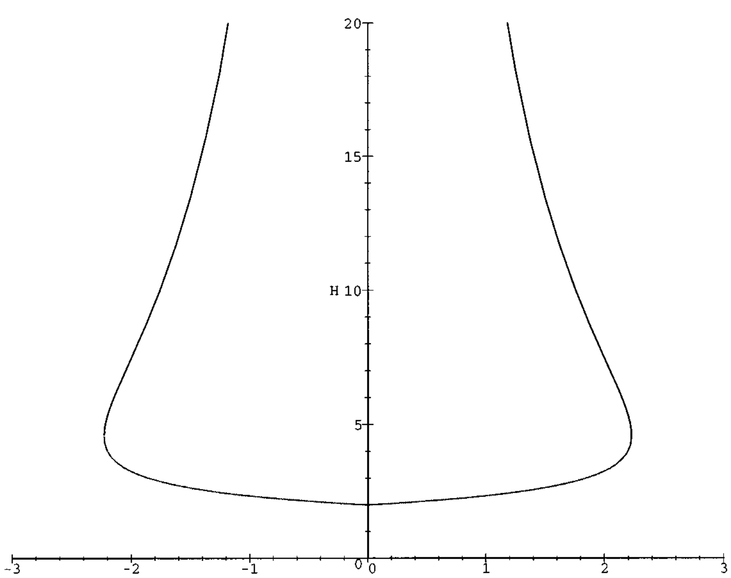

FIG. 6. A slice of H at constant p.

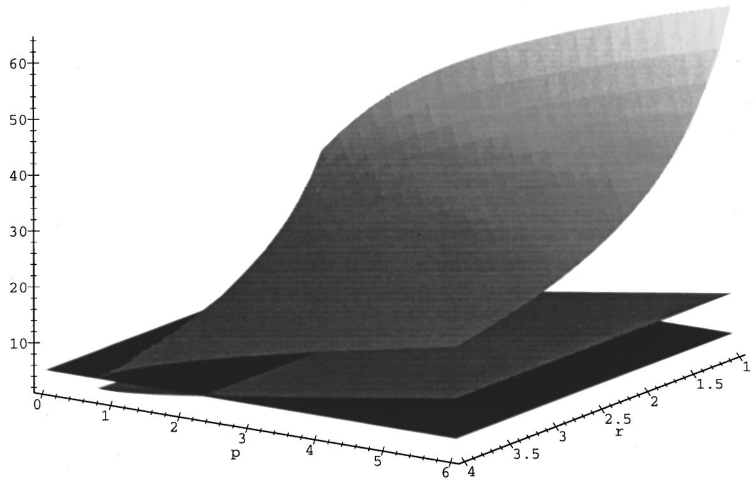

FIG. 7. A comparison of the Hamiltonian with a surface of constant energy. The flat black surface corresponds to a value of $H_0=5$. Note that it intersects both branches of the Hamiltonian.

For H_0 sufficiently small, the Hamiltonian is given by the principal branch W_0 and reduces to the Newtonian limit for small κ. Figure 4 shows the Hamiltonian for small H_0. The darker surface denotes $H(r,p)=H_0$ and the lighter surface is the Newtonian Hamiltonian.

Once H_0 becomes sufficiently large, there appears a qualitatively new set of trajectories which are not connected with the Newtonian Hamiltonian in small κ. Figure 5 demonstrates two branches of the Hamiltonian in the region $-2<r<-1$, $-2<p<-1$. The whole surface of H continues smoothly from one branch to the other. This structure is seen in Fig. 6 where the slice of H at constant p $(p=0)$ is drawn.

For H_0 in the intermediate range the constant energy surface intersects both branches shown in Fig. 7. The trajectory in (r,p) plane moves over both branches. It analytically continues from one branch to the other. Figure 8 shows also this structure from another point of view.

For a given initial condition the energy H_0 of the system is fixed and the trajectory in (r,p) plane is given as the slice of $H=H_0$ through the two-dimensional surface $H(r,p)$ in (r,p,H) phase space. Two characteristic plots are shown in Figs. 9 and 10 where the corresponding trajectories in the Newtonian theory are included for comparison. Under time reversal, the trajectory for a given value of H_0 is obtained by reflection in the $p=0$ axis.

One of the characteristics of the trajectories is that as H_0 increases the trajectory becomes more S shaped. Suppose the particles start out at the same place $(r=0)$ with positive p. r will increase and p will slowly decrease. This continues until maximum separation with some positive value of p, where the velocity $\dot{r}=0$. After that r undergoes a rapid decrease, while p is still positive. At some value of r, p becomes zero and then it goes negative. The particles continue to be pulled together and r reaches 0, where p has its maximum negative value. The particles then overshoot the mark and start the reverse motion with interchanged positions.

The main reason why the trajectories are S shaped is the appearance of the p-linear term in the Hamiltonian. The canonical equations (66) and (67) [or directly the Hamiltonian (77)] leads to

$$\dot{r}=\epsilon\ \frac{m\kappa|r|+2m\kappa|r|w+8w}{m\kappa|r|(1+w)}+(p\ \text{terms}). \tag{79}$$

The first term on right-hand side (RHS) comes from the

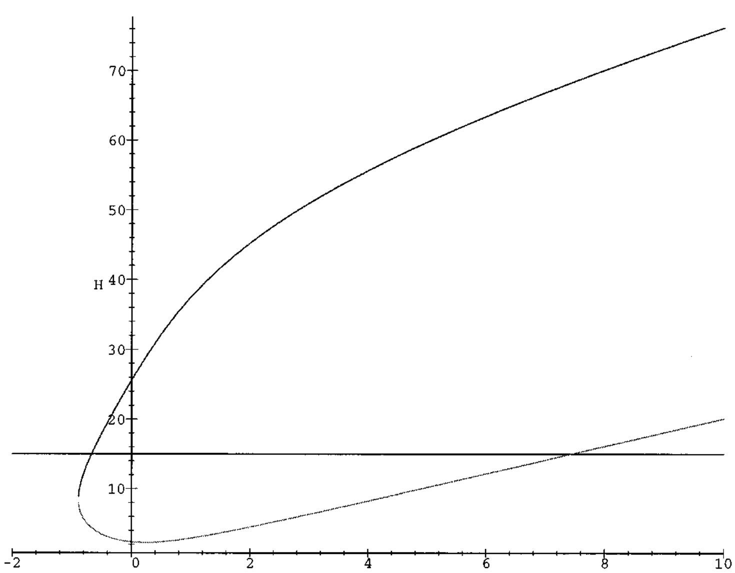

FIG. 8. A slice of both branches of H at $r=1$. The horizontal line corresponds to $H_0=15$.

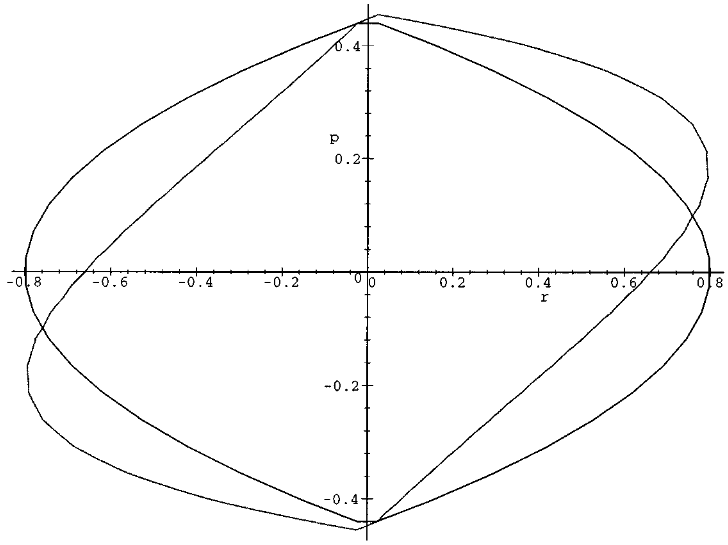

FIG. 9. Nonrelativistic (Newtonian) and relativistic trajectories for $H_0=2.2$. The undistorted oval is the nonrelativistic trajectory.

p-linear term in H and $\dot{r}=0$ does not correspond to $p=0$. This relation between $\dot{r}$ and p resembles the relation in the theory with charged particles

$$\mathbf{p}=\frac{\dot{\mathbf{r}}}{\sqrt{1-\dot{\mathbf{r}}^2}}+e\mathbf{A}.$$

In this sense the first term on RHS of Eq. (79) can be said to be purely gravitational.

VI. THE UNEQUAL MASS HAMILTONIAN

For unequal masses we set

$$H=\frac{\sqrt{p_1^2+m_1^2}+\sqrt{p_2^2+m_2^2}}{2}+\frac{\epsilon}{2}(\tilde{p}_1-\tilde{p}_2)-\frac{8}{\kappa|r|}Y \quad (80)$$

so that Eq. (61) becomes

$$(Y^2-D^2)e^{2Y}=(S^2-D^2)e^{2S}, \quad (81)$$

where $S=(\kappa|r|/16)(\tilde{M}_1+\tilde{M}_2)$ and $D=(\kappa|r|/16)(\tilde{M}_1-\tilde{M}_2)$ with $\tilde{M}_1\equiv\sqrt{p_1^2+m_1^2}-\epsilon\tilde{p}_1$ and $\tilde{M}_2\equiv\sqrt{p_2^2+m_2^2}+\epsilon\tilde{p}_2$. For equal masses, $S=Z$, $D=0$ and Eq. (81) reduces to Eq. (71). Solving Eq. (81) for Y in terms of S and D yields the Hamiltonian in the unequal mass case.

To obtain the solution, consider the equation

$$(y^2-a^2)e^{2y}=(x^2-a^2)e^{2x}, \quad a>0. \quad (82)$$

This equation also has three solutions shown in Fig. 11: the trivial solution $y=x$, the curve H-I-J-K-L denoted by $\mathcal{W}(x;a)$, and the curve S-T-U denoted by $\mathcal{W}(x;a)$. To our knowledge, discussion of the functions $\mathcal{W}(x;a)$ and $\mathcal{W}(x;a)$ have never appeared in the literature. We shall refer to $\mathcal{W}$ as the generalized Lambert function since $\lim_{a\to 0}\mathcal{W}(x;a)=W(-xe^x)$. In general $\mathcal{W}$ is also complex and multivalued, and when x is real, the function has two real branches shown in Fig. 11. The principal branch is analytic at $x=0$ and has a derivative singularity at $x=\mathcal{W}^{-1}(-\frac{1}{2}(1+\sqrt{1+4a^2}))$ beyond which it becomes complex. The other branch satisfies $\mathcal{W}<-\frac{1}{2}(1+\sqrt{1+4a^2})$ and

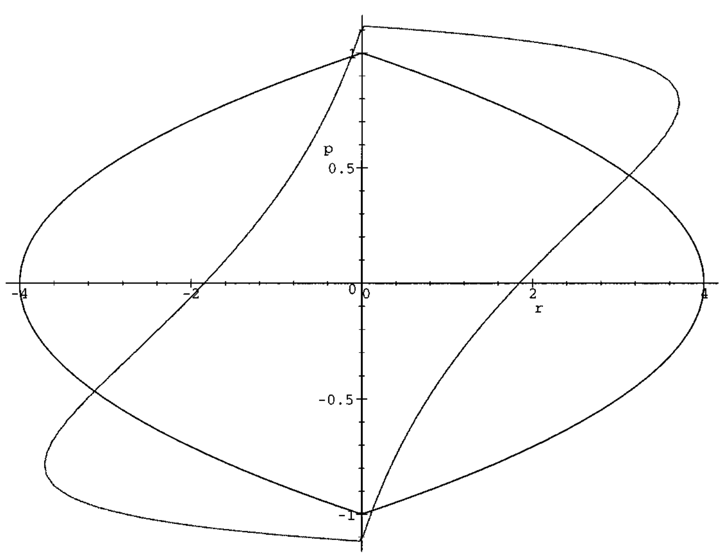

FIG. 10. Nonrelativistic (Newtonian) and relativistic trajectories for $H_0=3$. The undistorted oval is the nonrelativistic trajectory.

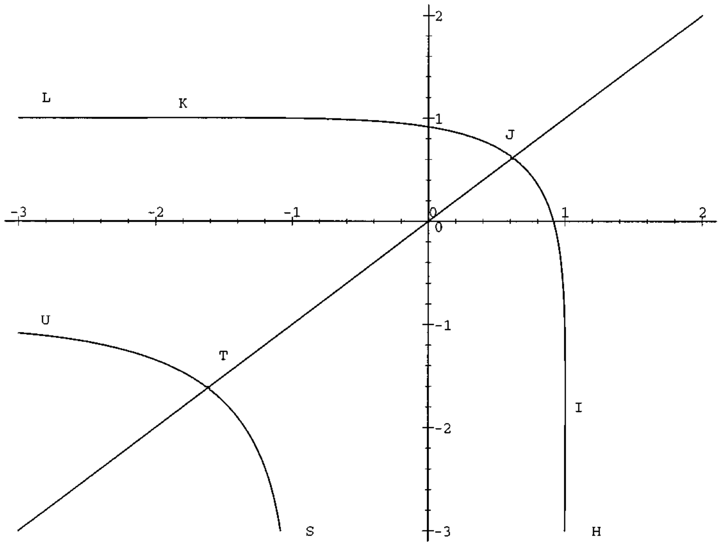

FIG. 11. A plot of solutions to Eq. (82). The curve HIJKL is the generalized Lambert function.

joins smoothly onto the first branch. The full function is double valued for $a<x<\overline{\mathcal{W}}^{-1}(-\frac{1}{2}(1+\sqrt{1+4a^2}))$. The third solution $\overline{\mathcal{W}}(x;a)$ is a generalization of $W(xe^x)$ in the region $x<0$.

As in the equal mass case, the trivial solution $Y=S$ again yields the unphysical Hamiltonian $H=\epsilon(\tilde{p}_1-\tilde{p}_2)$. Since the physical domain of S is $S\geqslant 0$, the only physical solution is

$$Y=\mathcal{W}(S;D), \tag{83}$$

which leads to the Hamiltonian

$$H=\frac{\sqrt{p_1^2+m_1^2}+\sqrt{p_2^2+m_2^2}}{2}+\frac{\epsilon}{2}(\tilde{p}_1-\tilde{p}_2) -8\frac{\kappa}{|r|}\mathcal{W}\left[\frac{\kappa|r|}{16}(\tilde{M}_1+\tilde{M}_2);\frac{\kappa|r|}{16}(\tilde{M}_1-\tilde{M}_2)\right]. \tag{84}$$

The expansion in κ for this Hamiltonian is given by Eq. (62).

Choosing also the center of inertia frame with $p_1=-p_2=p$ and setting $m=m_2/m_1$, we shall look at the trajectories. First, take a value for H_0 just above the minimal (rest-mass) value of $1+m$ and compare this to Newtonian theory in Fig. 12. The trajectory is almost exactly the same as Newtonian theory, since it is the equal mass case. For larger m the separation between particles cannot get to be very large and the trajectory becomes more compact as shown in Fig. 13. The trajectories for smaller values of m are shown in Fig. 14, where the innermost line is the $m=0.9$ case and the outermost is the $m=0.1$ case.

Finally in Fig. 15 the trajectories of different values of m both large and small are compared.

VII. SOLUTION OF THE METRIC TENSOR

To determine the Hamiltonian and derive the canonical equations of motion, we had only to solve the constraints (12) and (13) of the system of the field equations (10)–(17). In this section we shall solve the remaining equations to determine the metric and to confirm directly the consistency of Euler-Lagrange equations (16) and (17) with the canonical equations derived from the Hamiltonian, though formal proof of the consistency was already given in [8].

Under the coordinate conditions (24) the field equations

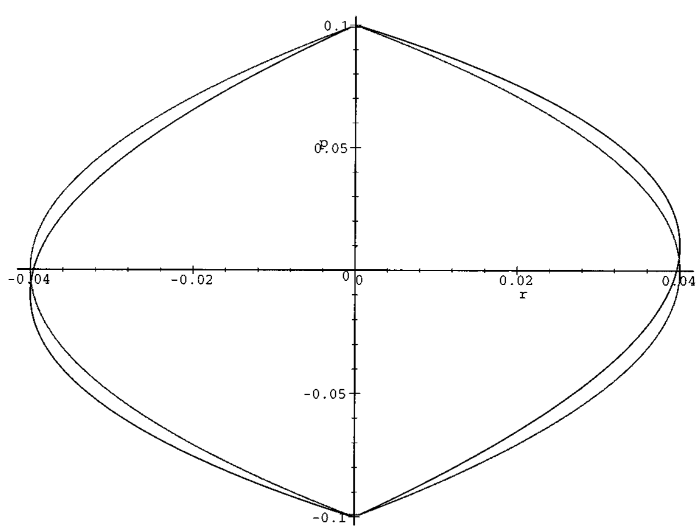

FIG. 12. Nonrelativistic (Newtonian) and relativistic trajectories for $H_0=2.01$ in the unequal mass case with $m=1$. The undistorted oval is the nonrelativistic trajectory.

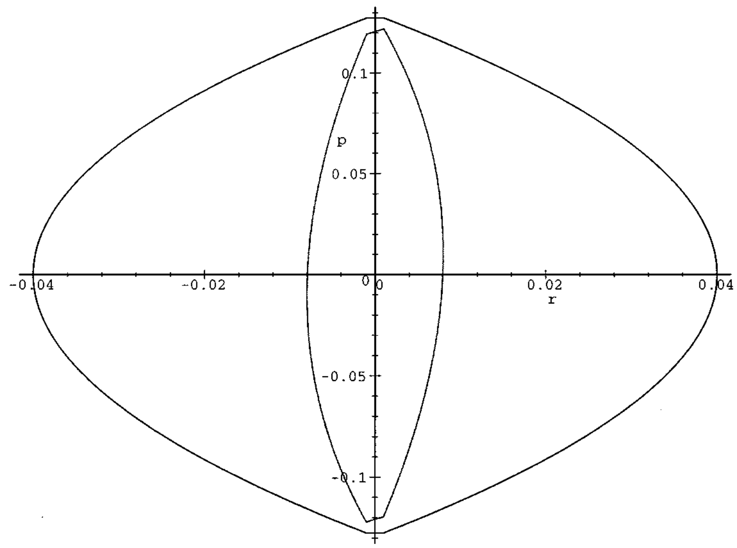

FIG. 13. Nonrelativistic (Newtonian) and relativistic trajectories for $H_0=6.01$ in the unequal mass case with $m=5$. The narrow oval in the middle is the relativistic trajectory.

(10), (11), (14), and (15) become

$$\dot{\pi}+N_0\left\{\frac{3\kappa}{2}\pi^2+\frac{1}{8\kappa}(\Psi')^2-\sum_a \frac{p_a^2}{2\sqrt{p_a^2+m_a^2}}\delta(x-z_a(x^0))\right\}+N_1\left\{\pi'+\sum_a p_a\delta(x-z_a(x^0))\right\}+\frac{1}{2\kappa}N_0'\Psi'+N_1'\pi=0, \tag{85}$$

$$\kappa\pi N_0+N_1'=0, \tag{86}$$

$$\partial_1\left(\frac{1}{2}N_0\Psi'+N_0'\right)=0, \tag{87}$$

$$\dot{\Psi}=-2\kappa\pi N_0+N_1\Psi'. \tag{88}$$

In the following we shall carry out our calculations assuming $z_2<z_1$—the case $z_1<z_2$ is completely analogous and will not be presented. The solution to Eq. (87) is

$$N_0=Ae^{-(1/2)\Psi}=A\phi^2=\begin{cases}A\phi_+^2 & (+)\text{ region}\\ A\phi_0^2 & (0)\text{ region}\\ A\phi_-^2 & (-)\text{ region,}\end{cases} \tag{89}$$

where A is an integration constant and $\phi_\pm$ and ϕ_0 are given in Eqs. (55), (56), and (57). Equation (86) becomes

$$N_1'=-\kappa A\chi'\phi^2. \tag{90}$$

The solution in each region is

$(+)$ region:

$$N_{1(+)}=\epsilon\left\{A\frac{L_1}{M_1}\exp\left(\frac{\kappa}{4}L_+(x-z_1)\right)-1\right\}=\epsilon(A\phi_+^2-1), \tag{91}$$

$(-)$ region:

$$N_{1(-)}=-\epsilon\left\{A\frac{L_2}{M_2}\exp\left(-\frac{\kappa}{4}L_-(x-z_2)\right)-1\right\}=-\epsilon(A\phi_-^2-1), \tag{92}$$

(0) region:

$$N_{1(0)}=\epsilon A\frac{L_1L_2}{L_0^2}\left\{\frac{L_2}{M_1}\exp\left(\frac{\kappa}{4}L_0(x-z_1)\right)-\frac{L_1}{M_2}\exp\left(-\frac{\kappa}{4}L_0(x-z_2)\right)\right\}+\frac{\kappa\epsilon}{2}A\frac{L_1L_2}{L_0}x+\epsilon C_0, \tag{93}$$

where we chose the integration constants in $N_{1(+)}$ and $N_{1(-)}$ to be $(-\epsilon)$ and ϵ, respectively. In general we obtain arbitrary constants C_+ and C_- in the expressions for Eqs. (91) and (92). However, a lengthy calculation reveals that $C_\pm=\mp\epsilon$ and so for simplicity we shall set $C_+=-\epsilon$ and $C_-=\epsilon$ from the outset. In deriving $N_{1(0)}$ we used Eq. (63).

The continuity condition at $x=z_1$,

$$N_{1(+)}(z_1)=N_{1(0)}(z_1), \tag{94}$$

leads to

$$C_0=-1+A\left\{\frac{L_1}{M_1}-\frac{L_1(L_2-M_1)}{L_0M_1}-\frac{\kappa}{2}\frac{L_1L_2}{L_0}z_1\right\}, \tag{95}$$

where Eq. (63) and the relation $L_2+M_1=L_0$ are used. The continuity condition at $x=z_2$,

$$N_{1(-)}(z_2)=N_{1(0)}(z_2), \tag{96}$$

similarly leads to

$$C_0=1-A\left\{\frac{L_2}{M_2}-\frac{L_2(L_1-M_2)}{L_0M_2}+\frac{\kappa}{2}\frac{L_1L_2}{L_0}z_2\right\}. \tag{97}$$

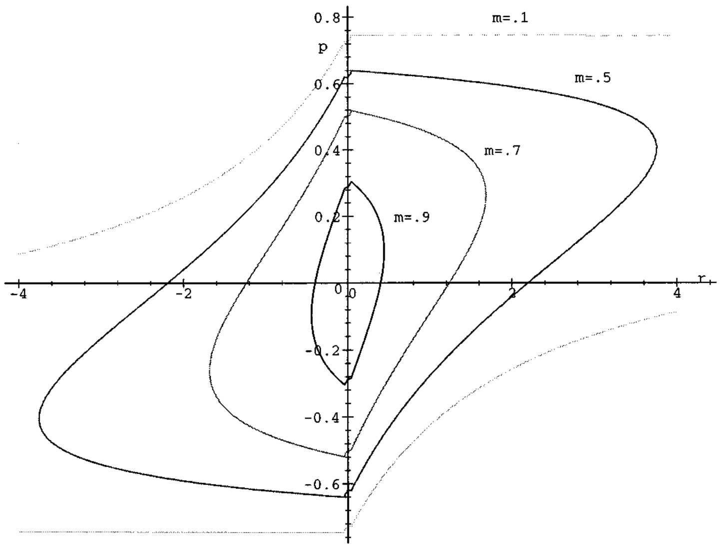

FIG. 14. Relativistic trajectories for several values of m, where $H_0=2$.

From the consistency of Eqs. (95) and (97) the constant A is determined as

$$A=\frac{L_0}{L_1+L_2-\frac{\kappa r}{4}L_1L_2} \tag{98}$$

and

$$C_0=\frac{M_1-M_2-\frac{\kappa}{4}(z_1+z_2)L_1L_2}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\,. \tag{99}$$

Now we are ready to check Eq. (85). First we treat three regions $(+)$, (0), and $(-)$ separately, and next consider the matching conditions at $x=z_1,z_2$.

For $(+)$ and $(-)$ regions it is straightforward to show that the LHS of Eq. (85) vanishes, by substituting the explicit solutions of $\pi=\chi',\Psi'=-4\phi'/\phi,N_0$, and N_1. For the (0) region the solutions of the metric and the dilaton field are

$$\pi=-\frac{\epsilon}{4}L_0,\quad \pi'=0,$$

$$\dot{\pi}=-\frac{\epsilon}{4}\dot{L}_0=-\frac{\epsilon}{4}(4\dot{X}-\epsilon\dot{p}_1+\epsilon\dot{p}_2)=\frac{1}{2}\dot{p}_1,$$

$$\Psi'=-4\frac{\phi_0'}{\phi_0},\quad N_0=A\phi_0^2,$$

$$N_0'=2A\phi_0\phi_0',\quad N_1'=\frac{\kappa\epsilon}{4}L_0A\phi_0^2.$$

The LHS of Eq. (85) becomes

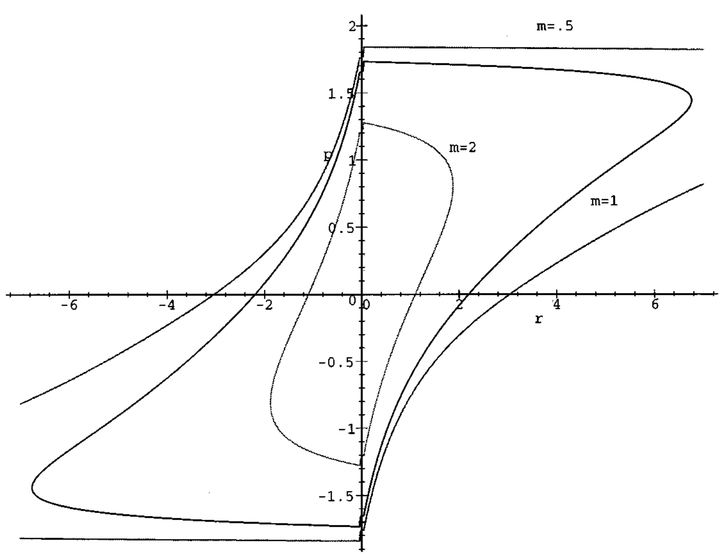

FIG. 15. Relativistic trajectories for several values of m, where $H_0=4$.

$$\text{LHS of Eq. (85)}=\frac{1}{2}\dot{p}_1+\frac{\kappa A}{32}(L_0\phi_0)^2-\frac{2A}{\kappa}(\phi_0')^2$$

$$=\frac{1}{2}\dot{p}_1+\frac{\kappa}{8}\frac{L_0L_1L_2}{L_1+L_2-\frac{\kappa r}{8}L_1L_2},$$

which vanishes due to Eq. (64).

Consider next the δ-function part at $x=z_1$. Since

$$\pi=\chi'=-\frac{1}{4}\{p_1|x-z_1|'+p_2|x-z_2|'\}-\epsilon X,$$

$$\pi'=-\frac{1}{2}\{p_1\delta(x-z_1)+p_2\delta(x-z_2)\},$$

$$\dot{\pi}=\frac{1}{2}\{p_1\dot{z}_1\delta(x-z_1)+p_2\dot{z}_2\delta(x-z_2)\}-\frac{1}{4}\{\dot{p}_1|x-z_1|'+\dot{p}_2|x-z_2|'\},$$

we have

$$[\delta\text{-function part of Eq. (11) at } x=z_1]=\frac{1}{2}p_1\dot{z}_1\delta(x-z_1)-\frac{1}{2}N_0(z_1)\frac{p_1^2}{\sqrt{p_1^2+m_1^2}}\delta(x-z_1)+\frac{1}{2}N_1(z_1)p_1\delta(x-z_1)$$

$$=\frac{1}{2}p_1\delta(x-z_1)\left\{\dot{z}_1+\frac{\epsilon L_0L_1}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\frac{1}{\sqrt{p_1^2+m_1^2}}-\epsilon\right\},$$

which also vanishes due to the canonical equation (66). Similarly, the δ-function part of Eq. (11) at $x=z_2$ is zero. We thus conclude that Eq. (85) is satisfied exactly.

As we investigated in the iterative method [8], for the consistency of Eq. (88) we need to introduce an x-independent function $f(t)$ into Ψ:

$$\Psi=-4\ln|\phi|+f(t)\ . \tag{100}$$

Since in the system of the original equations (10)–(17) all other equations except Eq. (14) contain only spatial derivatives of Ψ, $f(t)$ does not contribute to either the Hamiltonian or to the equations of motion. Equation (88) becomes

$$-4\frac{\dot{\phi}}{\phi}+\dot{f}(t)+2\kappa\pi N_0+4\frac{\phi'}{\phi}N_1=0. \tag{101}$$

We must check this equation in the three regions separately, with $f(t)$ common to all regions.

For the $(+)$ region, after substituting the solutions of $\phi_+,\pi,N_{0(+)}$, and $N_{1(+)}$, Eq. (101) yields

$$\dot{f}_+(t)=2\frac{\dot{L}_1M_1-L_1\dot{M}_1}{L_1M_1}-\frac{\kappa}{2}L_+(\dot{z}_1-\epsilon). \tag{102}$$

This ensures that f_+ is x independent. Using the canonical equations (64), (65), (66), and (67) we get

$$\dot{f}_+(t)=\frac{\frac{\kappa}{2}L_0}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\left\{L_1\left(\epsilon+\frac{p_1}{\sqrt{p_1^2+m_1^2}}\right)+L_2\left(\epsilon-\frac{p_2}{\sqrt{p_2^2+m_2^2}}\right)\right\}. \tag{103}$$

For the $(-)$ region the calculation is quite analogous to the above and leads to $\dot{f}_-(t)=\dot{f}_+(t)$. For the (0) region the calculation is rather lengthy and complicated, especially for $\dot{\phi}_0$, which is expressed as

$$\dot{\phi}_0=\left(-\frac{\dot{L}_0}{L_0}+\frac{\dot{L}_1}{2L_1}+\frac{\dot{L}_2}{2L_2}+\frac{\kappa\epsilon}{8}\frac{\frac{\kappa r}{4}L_0L_1L_2}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\right)\phi_0+\frac{\dot{L}_0}{L_0}x\phi_0'+\frac{\epsilon(M_1-M_2)}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\phi_0'$$
$$-\frac{\kappa}{8}\frac{\dot{L}_0}{L_0}(L_1L_2)^{1/2}\left\{\left(\frac{L_2}{M_1}\right)^{1/2}z_1\exp\left(\frac{\kappa}{8}L_0(x-z_1)\right)-\left(\frac{L_1}{M_2}\right)^{1/2}z_2\exp\left(-\frac{\kappa}{8}L_0(x-z_2)\right)\right\}.$$

Substitution of the expressions of $\phi_0,\phi_0',\dot{\phi}_0,\pi,N_0$, and $N_{1(0)}$ into Eq. (101) in the (0) region leads to

$$\dot{f}_0(t)=4\left(-\frac{\dot{L}_0}{L_0}+\frac{\dot{L}_1}{2L_1}+\frac{\dot{L}_2}{2L_2}\right)+2\kappa\epsilon A\frac{L_1L_2}{L_0}=\frac{\frac{\kappa}{2}L_0}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\left\{L_1\left(\epsilon+\frac{p_1}{\sqrt{p_1^2+m_1^2}}\right)+L_2\left(\epsilon-\frac{p_2}{\sqrt{p_2^2+m_2^2}}\right)\right\}, \quad (104)$$

which is equivalent to Eq. (103). Hence $f(t)$ is common in all regions

$$\dot{f}_+(t)=\dot{f}_-(t)=\dot{f}_0(t) \quad (105)$$

and the solution is self-consistent.

Finally we shall directly check the Euler-Lagrange equations (16) and (17) which under the coordinate conditions (24) become

$$\dot{z}_a-N_0(z_a)\frac{p_a}{\sqrt{p_a^2+m_a^2}}+N_1(z_a)=0, \quad (106)$$

$$\dot{p}_a+\frac{\partial N_0}{\partial z_a}\sqrt{p_a^2+m_a^2}-\frac{\partial N_1}{\partial z_a}p_a=0 \ . \quad (107)$$

Since N_0 and N_1 are continuous at $x=z_1,z_2$, we have

$$N_0(z_1)=A\phi_+^2(z_1)=\frac{L_0}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\frac{L_1}{M_1},$$

$$N_1(z_1)=\epsilon\left(\frac{L_0}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\cdot\frac{L_1}{M_1}-1\right) \ .$$

Then for particle 1, say, Eq. (106) is

$$\dot{z}_1=\epsilon-\frac{\epsilon L_0L_1}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}\frac{1}{\sqrt{p_1^2+m_1^2}}. \quad (108)$$

This is identical with the canonical equation (66).

On the other hand, $\partial N_0/\partial x$ and $\partial N_1/\partial x$ are discontinuous at $x=z_1,z_2$. The natural definition of $\partial N_0/\partial z_1$ is

$$\frac{\partial N_0}{\partial z_1}\equiv\frac{1}{2}\left\{\left.\frac{\partial N_0}{\partial x}\right|_{x=z_1+0}+\left.\frac{\partial N_0}{\partial x}\right|_{x=z_1-0}\right\} \quad (109)$$

$$=\frac{\kappa}{8}A\frac{L_1}{M_1}\{L_++L_2-M_1\} \quad (110)$$

and similarly

$$\frac{\partial N_1}{\partial z_1}\equiv\frac{1}{2}\left\{\left.\frac{\partial N_1}{\partial x}\right|_{x=z_1+0}+\left.\frac{\partial N_1}{\partial x}\right|_{x=z_1-0}\right\} \quad (111)$$

$$=\frac{\kappa\epsilon}{8}A\frac{L_1}{M_1}(L_++L_0). \quad (112)$$

For particle 1, Eq. (107) is

$$\dot{p}_1=\frac{-\frac{\kappa}{4}L_0L_1L_2}{L_1+L_2-\frac{\kappa r}{4}L_1L_2}, \quad (113)$$

which is identical to Eq. (64). For particle 2, Eqs. (106) and (107) also reproduce the canonical equations (65) and (67).

Thus the consistency of the solution has been completely proved.

VIII. TEST-PARTICLE APPROXIMATION

As an interesting limiting case of Eq. (61) let us try to get the Hamiltonian in the test-particle approximation.

Setting particle 1 to be a test particle μ and particle 2 to be a static source m at the origin, namely,

$$z_1=z,\quad m_1=\mu,\quad p_1=p,\quad \tilde{p}_1=\tilde{p}=p\frac{z}{|z|},$$

$$z_2=0,\quad m_2=m,\quad p_2=0,\quad \tilde{p}_2=0,$$

the defining equation (61) for the Hamiltonian becomes

$$(\sqrt{p^2+\mu^2}-H)(m+\epsilon\tilde{p}-H)=(\sqrt{p^2+\mu^2}-\epsilon\tilde{p})m \times\exp\left(\frac{\kappa}{4}(H-\epsilon\tilde{p})|z|\right). \quad (114)$$

Expanding H in a power series in $\sqrt{p^2+\mu^2}$ and $\epsilon\tilde{p}$ and taking only the linear terms we obtain

$$H=m+\sqrt{p^2+\mu^2}\ \exp\left(\frac{\kappa m}{4}|z|\right)-\epsilon p\ \frac{z}{|z|}\left[\exp\left(\frac{\kappa m}{4}|z|\right)-1\right] \quad (115)$$

for the Hamiltonian in the test-particle approximation. This Hamiltonian is expressed in terms of the metric tensor of the static source as

$$H=m+\sqrt{p^2+\mu^2}\ N_0(z)-pN_1(z), \quad (116)$$

where

$$N_0=\exp\left(\frac{\kappa m}{4}|z|\right),\quad N_1=\epsilon\ \frac{z}{|z|}\left[\exp\left(\frac{\kappa m}{4}|z|\right)-1\right]. \quad (117)$$

The canonical equations are

$$\dot{z}=\frac{p}{\sqrt{p^2+\mu^2}}N_0-N_1 \quad (118)$$

and

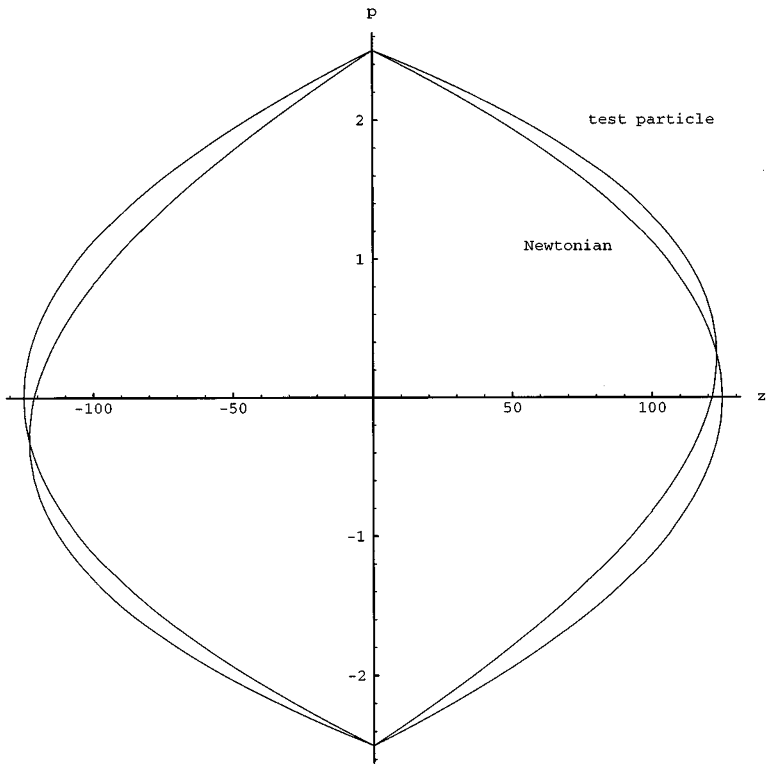

FIG. 16. Relativistic trajectory for a test particle compared to the Newtonian case.

$$\dot{p}=-\sqrt{p^2+\mu^2}\frac{\partial N_0}{\partial z}-p\frac{\partial N_1}{\partial z}=0. \tag{119}$$

Equation (118) is inversely solved as

$$p=\frac{\mu(N_1+\dot{z})}{[N_0^2-(N_1+\dot{z})^2]^{1/2}}. \tag{120}$$

We set the initial condition

$$z=0,\quad \dot{z}=v_0\quad \text{at}\ \ t=0. \tag{121}$$

Then the initial momentum $p(0)$ and the total energy H_0 are

$$p(0)=\frac{\mu v_0}{\sqrt{1-v_0^2}},\quad H_0=m+\sqrt{p(0)^2+\mu^2}. \tag{122}$$

From Eqs. (116) and (122) p is given by

$$p=\frac{1}{N_0^2-N_1^2}\{\sqrt{p(0)^2+\mu^2}N_1 \pm N_0\sqrt{p(0)^2+\mu^2-(N_0^2-N_1^2)\mu^2}\}. \tag{123}$$

We can draw a trajectory in phase space, an example of which is shown in Fig. 16. The trajectory is again S shaped due to relativistic gravitational effects.

From Eqs. (120) and (123) we get

$$\dot{z}=\frac{N_0[N_1+N_0\sqrt{1-(N_0^2-N_1^2)(1-v_0^2)}]}{\sqrt{(N_0^2-N_1^2)^2(1-v_0^2)+[N_1+N_0\sqrt{1-(N_0^2-N_1^2)(1-v_0^2)}]^2}}-N_1. \tag{124}$$

Denoting the RHS as $G(z)$, the solution is given by

$$t=\int_0^z\frac{dz}{G(z)}. \tag{125}$$

To lowest order in κ we obtain

$$G(z)=\sqrt{v_0^2-\frac{\kappa m}{2}|z|} \tag{126}$$

and

$$z=-\frac{\kappa m}{8}t^2+v_0 t \tag{127}$$

which is the Newtonian motion of a test body in 1+1 dimensions.

We shall add one comment on the form of the line element. In our canonical reduction we chose the coordinate conditions (24), under which the line element of space-time is

$$ds^2=-(N_0^2-N_1^2)dt^2+2N_1 dt dx+dx^2. \tag{128}$$

For the case of a single static source with N_0 and N_1 given by Eq. (117), we find the coordinate transformations

$$\tilde{t}=t-\epsilon|x|+\frac{2\epsilon}{\kappa m}\ln\left|2\exp\left(\frac{\kappa m}{4}|x|\right)-1\right|, \tag{129}$$

$$\tilde{x}=\frac{4}{\kappa m}\frac{x}{|x|}\left[\exp\left(\frac{\kappa m}{4}|x|\right)-1\right], \tag{130}$$

which leads to the line element [15]

$$ds^2=-\alpha(\tilde{x})d\tilde{t}^2+\frac{1}{\alpha(\tilde{x})}d\tilde{x}^2, \tag{131}$$

with

$$\alpha(\tilde{x})=1+\frac{\kappa m}{2}|\tilde{x}|. \tag{132}$$

In this coordinate frame the Hamiltonian for the test particle is

$$H(\tilde{z},\tilde{p})=\sqrt{\alpha(\tilde{z})\mu^2+\alpha(\tilde{z})^2p^2}+m\ . \tag{133}$$

IX. CORRESPONDENCE WITH NEWTONIAN GRAVITY IN $(d+1)$ DIMENSIONS

In this section we illustrate how a Newtonian limit generically arises in the $(1+1)$-dimensional theory we consider. We compare this with the emergence of a Newtonian limit in $(d+1)$ dimensions. We shall compute the Newtonian limit(s) by considering the one graviton exchange potential (keeping in mind that there are no propagating gravitons in two spacetime dimensions).

We begin by extending the theory in Eq. (1) to $d+1=n$ dimensions and coupling N scalar fields, which yields

$$L=\frac{2}{\kappa^2}\sqrt{-g}\left\{\Psi R+\frac{1}{2}g^{\mu\nu}\nabla_\mu\Psi\nabla_\nu\Psi\right\}-\frac{1}{2}\sum_a\sqrt{-g}(g^{\mu\nu}\phi_{a,\mu}\phi_{a,\nu}+m_a^2\phi_a^2), \tag{134}$$

where $\kappa^2=32\pi G$. Defining the graviton field $h_{\mu\nu}$ and the dilaton field ψ via

$$g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu},\quad \Psi=1+\kappa\psi \tag{135}$$

gives

$$L_0=-\frac{1}{2}\{\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}-\partial^\lambda h^\mu{}_\mu\partial_\lambda h^\nu{}_\nu-2\partial_\mu h^{\mu\nu}\partial^\lambda h_{\lambda\nu}+2\partial_\mu h^{\mu\nu}\partial_\nu h^\lambda{}_\lambda\}-2(\partial^\nu h_{\mu\nu}-\partial_\mu h^\nu{}_\nu)\partial^\mu\psi+\partial^\mu\psi\partial_\mu\psi-\frac{1}{2}\sum_a(\phi_a^{;\mu}\phi_{a,\mu}+m_a^2\phi_a^2) \tag{136}$$

for the free Lagrangian density following from Eq. (134). Redefining the dilaton field

$$\tilde{\psi}\equiv\psi+h^\mu{}_\mu-\frac{\partial^\mu\partial^\nu}{\Box}h_{\mu\nu} \tag{137}$$

allows us to express Eq. (136) as

$$L_0=-\frac{1}{2}\{\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}-\partial^\lambda h^\mu{}_\mu\partial_\lambda h^\nu{}_\nu-2\partial_\mu h^{\mu\nu}\partial^\lambda h_{\lambda\nu}+2\partial_\mu h^{\mu\nu}\partial_\nu h^\lambda{}_\lambda\}-(\partial^\nu h_{\mu\nu}-\partial_\mu h^\nu{}_\nu)(\partial^\lambda h^\mu{}_\lambda-\partial^\mu h^\lambda{}_\lambda)+\partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi}-\frac{1}{2}\sum_a(\phi_a^{;\mu}\phi_{a,\mu}+m_a^2\phi_a^2). \tag{138}$$

The field $\tilde{\psi}$ decouples from the Lagrangian and we shall not consider it further.

The free Lagrangian density of the graviton is obtained by simplifying the first two terms above

$$L_{0g}=-\frac{1}{2}\{\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}+\partial^\lambda h^\mu{}_\mu\partial_\lambda h^\nu{}_\nu-2\partial_\mu h^{\mu\nu}\partial_\nu h^\lambda{}_\lambda\}+\partial^\nu h_{\mu\nu}B^\mu+\frac{1}{4}B_\mu B^\mu, \tag{139}$$

where we added gauge fixing terms in the form of a Lagrange multiplier field B_μ.

Eliminating B_μ from its field equation leaves us with the Lagrangian

$$\tilde{L}_{0g}=-\frac{1}{2}\{\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}+\partial^\lambda h^\mu{}_\mu\partial_\lambda h^\nu{}_\nu-2\partial_\mu h^{\mu\nu}\partial_\nu h^\lambda{}_\lambda\}-\partial^\nu h_{\mu\nu}\partial_\lambda h^{\mu\lambda}, \tag{140}$$

whose canonical quantization we shall now undertake.

Temporarily setting the scalar fields to zero, we obtain

$$\Box h_{\mu\nu}+\eta_{\mu\nu}\Box h^\lambda{}_\lambda-\eta_{\mu\nu}\partial_\lambda\partial_\rho h^{\lambda\rho}-\partial_\mu\partial_\nu h^\lambda{}_\lambda+\partial_\mu\partial^\lambda h_{\nu\lambda}+\partial_\nu\partial^\lambda h_{\mu\lambda}=0 \tag{141}$$

for the graviton field equation. Its trace is

$$\Box h^\lambda{}_\lambda=\frac{n-2}{n}\partial_\lambda\partial_\rho h^{\lambda\rho} \tag{142}$$

implying that Eq. (141) becomes

$$\Box h_{\mu\nu}-\frac{2}{n}\eta_{\mu\nu}\partial_\lambda\partial_\rho h^{\lambda\rho}-\partial_\mu\partial_\nu h^\lambda{}_\lambda+\partial_\mu\partial^\lambda h_{\nu\lambda}+\partial_\nu\partial^\lambda h_{\mu\lambda}=0\ . \tag{143}$$

Taking the ∂^ν derivative of Eq. (143) leads to

$$\Box\partial^\nu h_{\mu\nu}=0\,. \tag{144}$$

This, along with the D'Alembertians of Eqs. (142) and (143), respectively, lead to

$$\Box^2 h_{\mu\nu}-\partial_\mu\partial_\nu\Box h^\lambda{}_\lambda=0 \quad\text{and}\quad \Box^2 h^\lambda{}_\lambda=0 \tag{145}$$

which finally implies

$$\Box^3 h_{\mu\nu}=0 \tag{146}$$

a relation characteristic of $n\geqslant 3$ Lagrangians.

The conjugate momentum is

$$\pi^{\mu\nu}=\partial_0 h^{\mu\nu}+\eta^{\mu\nu}(\partial_0 h^\lambda{}_\lambda+\partial_\lambda h^\lambda{}_0) + \eta^{\mu 0}\left(\frac{1}{2}\partial^\nu h^\lambda{}_\lambda-\partial_\lambda h^{\nu\lambda}\right)+\eta^{\nu 0}\left(\frac{1}{2}\partial^\mu h^\lambda{}_\lambda-\partial_\lambda h^{\mu\lambda}\right), \tag{147}$$

which implies

$$\partial_0 h_{00}=\frac{1}{2}\pi^{00}+\frac{1}{2}\partial_i h_{0i}\,,$$

$$\partial_0 h_{0i}=-\frac{1}{2}\pi^{0i}+\frac{1}{4}\partial_i h_{00}-\frac{1}{4}\partial_i h_{jj}+\frac{1}{2}\partial_j h_{ij}\,, \tag{148}$$

$$\partial_0 h_{ij}=\pi^{ij}-\frac{\delta_{ij}}{n}\pi^{kk}+\frac{\delta_{ij}}{n}\partial_k h_{0k}\,.$$

The equal-time commutation relations are

$$[h_{\mu\nu}(x),\pi^{\lambda\rho}(y)]_{\rm eq}=\frac{i}{2}(\delta^\lambda_\mu\delta^\rho_\nu+\delta^\rho_\mu\delta^\lambda_\nu)\delta^{(n-1)}(x-y),$$

$$[h_{\mu\nu}(x),h_{\lambda\rho}(y)]_{\rm eq}=[\pi^{\mu\nu}(x),\pi^{\lambda\rho}(y)]_{\rm eq}=0, \tag{149}$$

implying that the commutators between $h_{\mu\nu}$ and $\partial_0 h_{\lambda\rho}$ become

$$[h_{\mu\nu},\partial_0 h_{\kappa\sigma}]_{\rm eq}=\frac{i}{2}\left\{\frac{1}{2}(\eta_{\mu\kappa}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\kappa}) +\frac{1}{2}(\overline{\eta_{\mu\kappa}}\,\overline{\eta_{\nu\sigma}}+\overline{\eta_{\mu\sigma}}\,\overline{\eta_{\nu\kappa}})-\frac{2}{n}\overline{\eta_{\mu\nu}}\,\overline{\eta_{\kappa\sigma}}\right\} \times\delta^{(n-1)}(x-y), \tag{150}$$

where

$$\overline{\eta_{\mu\nu}}\equiv\eta_{\mu\nu}+\eta_{\mu 0}\eta_{\nu 0}. \tag{151}$$

The proof of this relation is given in the Appendix.

The solution to Eq. (146) is

$$h_{\mu\nu}(x)=-\int d^{n-1}z D^{(n)}(x-z)\overline{\partial_0^z}h_{\mu\nu}(z) -\int d^{n-1}z\widetilde{D}^{(n)}(x-z)\overline{\partial_0^z}\Box h_{\mu\nu}(z) -\int d^{n-1}z\tilde{\tilde{D}}^{(n)}(x-z)\overline{\partial_0^z}\Box^2 h_{\mu\nu}(z), \tag{152}$$

where $D^{(n)}$, $\widetilde{D}^{(n)}$, and $\tilde{\tilde{D}}^{(n)}$ are defined via

$$D^{(n)}(x)=-\frac{i}{(2\pi)^{n-1}}\int d^n k\,\epsilon(k_0)\,\delta(k^2)e^{ikx},$$

$$\widetilde{D}^{(n)}(x)=-\frac{i}{(2\pi)^{n-1}}\int d^n k\,\epsilon(k_0)\,\delta'(k^2)e^{ikx},$$

$$\tilde{\tilde{D}}^{(n)}(x)=-\frac{i}{(2\pi)^{n-1}}\int d^n k\,\epsilon(k_0)\,\delta''(k^2)e^{ikx},$$

and the symbol $\overline{\partial_0^z}$ denotes

$$f\overline{\partial_0}^z g=f\frac{\partial g}{\partial z^0}-\frac{\partial f}{\partial z^0}g\,.$$

We next need to express all of $h_{\mu\nu}$, $\partial_0 h_{\mu\nu}$, $\Box\partial_0 h_{\mu\nu}$, $\Box^2 h_{\mu\nu}$, and $\Box^2\partial_0 h_{\mu\nu}$ in terms of the canonical variables and calculate commutators at equal time. This rather lengthy and complicated calculation is given in the Appendix.

From Eq. (152) and the equal-time commutators, the commutator among the components of $h_{\mu\nu}$ at two arbitrary space-time points can be calculated:

$$[h_{\mu\nu}(x),h_{\lambda\rho}(y)]=\frac{i}{2}\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{2}{n}\eta_{\mu\nu}\eta_{\lambda\rho}\right)D^{(n)}(x-y)+\frac{i}{4}\left\{-\eta_{\mu\lambda}\partial_\nu\partial_\rho-\eta_{\mu\rho}\partial_\nu\partial_\lambda-\eta_{\nu\lambda}\partial_\mu\partial_\rho-\eta_{\nu\rho}\partial_\mu\partial_\lambda +\frac{4}{n}(\eta_{\mu\nu}\partial_\lambda\partial_\rho+\eta_{\lambda\rho}\partial_\mu\partial_\nu)\right\}\widetilde{D}^{(n)}(x-y)+\frac{i}{2}\left(1-\frac{2}{n}\right)\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\tilde{\tilde{D}}^{(n)}(x-y)\,. \tag{153}$$

This expression is valid even when $n=2$. The graviton propagator is

$$\langle 0|T[h_{\mu\nu}(x)h_{\lambda\rho}(y)]|0\rangle = -\frac{i}{2(2\pi)^n}\int d^nk e^{ik(x-y)}\frac{X_{\mu\nu,\lambda\rho}}{k^2-i\epsilon}, \tag{154}$$

where

$$X_{\mu\nu,\lambda\rho} = \eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{2}{n}\eta_{\mu\nu}\eta_{\lambda\rho}+\frac{1}{2k^2}\left\{-\eta_{\mu\lambda}k_\nu k_\rho-\eta_{\mu\rho}k_\nu k_\lambda-\eta_{\nu\lambda}k_\mu k_\rho-\eta_{\nu\rho}k_\mu k_\lambda+\frac{4}{n}(\eta_{\mu\nu}k_\lambda k_\rho+\eta_{\lambda\rho}k_\mu k_\nu)\right\}$$
$$+\left(1-\frac{2}{n}\right)\frac{k_\mu k_\nu k_\lambda k_\rho}{(k^2)^2}. \tag{155}$$

We turn now to the scalar fields, whose Lagrangian density to lowest order in the graviton coupling is

$$L_{\text{int}} = -\frac{1}{2}\left\{\frac{1}{2}\eta^{\mu\nu}(\phi^{,\alpha}\phi_{,\alpha}+m^2\phi^2)-\phi^{,\mu}\phi^{,\nu}\right\}h_{\mu\nu}. \tag{156}$$

The one graviton exchange diagram yields the S-matrix element

$$S=\frac{4\pi iG_n}{(2\pi)^{n-2}}(p_1^0p_2^0q_1^0q_2^0)^{-1/2}\left[p_1^\mu q_1^\nu-\frac{1}{2}\eta^{\mu\nu}(p_1\cdot q_1+m_1^2)\right]\left[p_2^\lambda q_2^\rho-\frac{1}{2}\eta^{\lambda\rho}(p_2\cdot q_2+m_2^2)\right]\frac{X_{\mu\nu,\lambda\rho}}{k^2}\delta^{(n)}(p_1+p_2-q_1-q_2),$$

where p_a^μ, q_a^μ, and k^μ are the four-momenta of the initial particles, the final particles, and the transferred graviton, respectively. This result is valid for $n=2$ also. In the lowest order and the static approximation T-matrix element is

$$T_n = -4\left(1-\frac{1}{n}\right)\frac{G_n}{(2\pi)^{n-2}}\frac{m_1m_2}{\mathbf{k}^2}, \tag{157}$$

whose associated potential is $V=\int d^nke^{-ikx}T(k)$ in n dimensions.

The T-matrix elements for $n=2$, 3, and 4 are

$$T_2=-\frac{2G_2m_1m_2}{\mathbf{k}^2}, \tag{158}$$

$$T_3=-\frac{8}{3}\frac{G_3}{(2\pi)}\frac{m_1m_2}{\mathbf{k}^2}, \tag{159}$$

$$T_4=-\frac{3G_4}{(2\pi)^2}\frac{m_1m_2}{\mathbf{k}^2} \tag{160}$$

and the corresponding potentials are

$$V_2=2\pi G_2m_1m_2r, \tag{161}$$

$$V_3=2\left(\frac{4}{3}G_3\right)m_1m_2\ln r, \tag{162}$$

$$V_4=-\frac{3}{2}\frac{G_4m_1m_2}{r}. \tag{163}$$

By identifying the gravitational constants as

$$G_{N,2}\equiv G_2,\quad G_{N,3}\equiv\frac{4}{3}G_3,\quad G_{N,4}\equiv\frac{3}{2}G_4, \tag{164}$$

we get the correct Newtonian potentials in each dimension.

The above results are in strong contrast with $(d+1)$-dimensional general relativity, whose free Lagrangian density is

$$L_{0g}=-\frac{1}{2}\{\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}-\partial^\lambda h^\mu{}_\mu\partial_\lambda h^\nu{}_\nu-2\partial_\mu h^{\mu\nu}\partial^\lambda h_{\lambda\nu}$$
$$+2\partial_\mu h^{\mu\nu}\partial_\nu h^\lambda{}_\lambda\}+\left(\partial^\nu h_{\nu\mu}-\frac{1}{2}\partial_\mu h^\lambda{}_\lambda\right)B^\mu+\frac{1}{4}B_\mu B^\mu,$$

where gauge fixing terms have been added. A computation analogous to the one above gives

$$\langle 0|Th_{\mu\nu}(x)h_{\lambda\rho}(y)|0\rangle=-\frac{i}{2(2\pi)^n}\int d^nk\;\; e^{ik(x-y)}\frac{X_{\mu\nu,\lambda\rho}}{k^2-i\epsilon},$$

where

$$X_{\mu\nu,\lambda\rho}=\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{2}{n-2}\eta_{\mu\nu}\eta_{\lambda\rho}.$$

The S-matrix element of one graviton exchange diagram is

$$S=\frac{4\pi iG_n}{(2\pi)^{n-2}}(p_1^0p_2^0q_1^0q_2^0)^{-1/2}\left[p_1^\mu q_1^\nu-\frac{1}{2}\eta^{\mu\nu}(p_1\cdot q_1+m_1^2)\right]\left[p_2^\alpha q_2^\beta-\frac{1}{2}\eta^{\alpha\beta}(p_2\cdot q_2+m_2^2)\right]\frac{X_{\mu\nu,\alpha\beta}}{k^2}\delta^{(n)}(p_1+p_2-q_1-q_2),$$

which in turn yields the T-matrix element

$$T=-\frac{4G_n}{(2\pi)^{n-2}}\frac{n-3}{n-2}\frac{m_1m_2}{k^2} \tag{165}$$

in the static appoximation in n dimensions.

The potential for $n=4$ is

$$V^{(4)}=-\frac{G_4m_1m_2}{r} \tag{166}$$

in agreement with Eq. (163). However the potential for $n=3$ vanishes, and the potential for $n=2$ diverges. This latter situation can be dealt with by setting $G_n=(1-n/2)G_2$ and taking the $n\to 2$ limit [13], which yields the two-dimensional T-matix element

$$T=-\frac{2G_2m_1m_2}{k^2} \tag{167}$$

and potential

$$V^{(2)}=2\pi G_2m_1m_2r. \tag{168}$$

Unlike general relativity, three-dimensional dilaton gravity includes the Newtonian potential in any dimension, once the gravitational constant is appropriately rescaled. In this sense the theory of gravity (1) we consider is a relativistic extension of Newtonian gravity in $d+1$ dimensions. General relativity, on the other hand, does not include Newtonian gravity in $2+1$ dimensions and is empty in $1+1$ dimensions. In the latter case an appropriate rescaling of Newton's constant yields the theory (1) in the $n\to 2$ limit [13].

X. CONCLUSIONS

We have obtained an exact self-consistent solution to the two-body problem in a $(1+1)$-dimensional theory of gravity with a Newtonian limit. To our knowledge, this is the only exact relativistic two-body solution of this type. We are able to explore all possible limits of this solution, including large and small gravitational coupling and/or mass and/or momenta.

A natural extension of what we have done would be to attempt to solve the N body problem. It would also be of interest to couple other matter fields (e.g., electromagnetism), and to investigate the extent to which our methods are applicable to other dilaton theories of gravity.

Finally, and what is perhaps most interesting, is to quantize the degrees of freedom of the two-body system we consider based on the Hamiltonian given in Eq. (61). The quantum theory based on Eq. (61) is a quantum theory of gravity coupled to matter whose slow-motion weak field limits should be straightforwardly comparable to that of the nonrelativistic mechanics of two particles in a linear confining potential. As such it should offer interesting insights into the behavior of quantum gravity.

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APPENDIX: COMMUTATION RELATIONS

We use the notation

$$\overline{\eta_{\mu\nu}}=\eta_{\mu\nu}+\eta_{\mu 0}\eta_{\nu 0}, \quad \overline{\partial_\mu}=\partial_\mu+\eta_{\mu 0}\partial_0 \; .$$

The general form of the equal time commutation relations is

$$\begin{aligned}[][h_{\mu\nu}(x),\pi^{\lambda\rho}(y)]_{\rm eq}&=\Big[h_{\mu\nu}(x),\partial_0 h^{\mu\nu}+\eta^{\mu\nu}(\partial_0 h^\lambda{}_\lambda+\partial_\lambda h^{\lambda 0})\\&\quad+\eta^{\mu 0}\left(\frac{1}{2}\partial^\nu h^\lambda{}_\lambda-\partial_\lambda h^{\nu\lambda}\right)\\&\quad+\eta^{\nu 0}\left(\frac{1}{2}\partial^\mu h^\lambda{}_\lambda-\partial_\lambda h^{\mu\lambda}\right)\Big]_{\rm eq}\\&=\frac{i}{2}(\delta^\lambda_\mu\delta^\rho_\nu+\delta^\rho_\mu\delta^\lambda_\nu)\delta^{(n-1)}(x-y)\end{aligned} \tag{A1}$$

from which we shall now deduce various commutators of interest.

Taking the trace of Eq. (A1) implies the relations

$$\begin{aligned}[][h_{\mu\nu},n\partial_0 h^\alpha{}_\alpha-(n-2)\partial_\alpha h_0{}^\alpha]_{\rm eq}&=i\,\eta_{\mu\nu}\delta^{(n-1)}(x-y),\\ [h_{\mu\nu},\partial_0 h^\alpha{}_\alpha-\partial_\alpha h_0{}^\alpha]_{\rm eq}&=\frac{2}{n}[h_{\mu\nu},\partial_0 h_{00}]_{\rm eq}\\&\quad+\frac{i}{n}\eta_{\mu\nu}\delta^{(n-1)}(x-y),\end{aligned} \tag{A2}$$

$$[h_{\mu\nu},\partial_0 h^\alpha{}_\alpha]_{\rm eq}=\frac{2-n}{n}[h_{\mu\nu},\partial_0 h_{00}]_{\rm eq}+\frac{i}{n}\eta_{\mu\nu}\delta^{(n-1)}(x-y),$$

from which it follows that

$$\begin{aligned}&\Big[h_{\mu\nu},\partial_0 h_{\kappa\sigma}+\left(\frac{2}{n}\overline{\eta_{\kappa\sigma}}-\eta_{\kappa 0}\eta_{\sigma 0}\right)\partial_0 h_{00}-\eta_{\kappa 0}\partial_0 h_{0\sigma}\\&\quad-\eta_{\sigma 0}\partial_0 h_{0\kappa}\Big]_{\rm eq}\\&=\frac{i}{2}\left(\eta_{\mu\kappa}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\kappa}-\frac{2}{n}\eta_{\mu\nu}\overline{\eta_{\kappa\sigma}}\right)\delta^{(n-1)}(x-y).\end{aligned} \tag{A3}$$

The $(\kappa,\sigma)=(0,\sigma)$ component gives

$$[h_{\mu\nu},\partial_0 h_{0\sigma}]_{\rm eq}=\frac{i}{4}(\eta_{\mu 0}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu 0})\delta^{(n-1)}(x-y), \tag{A4}$$

which yields in turn the commutation relation

$$\begin{aligned}[][h_{\mu\nu},\partial_0 h_{\kappa\sigma}]_{\rm eq}&=\frac{i}{2}\Big\{\frac{1}{2}(\eta_{\mu\kappa}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\kappa})\\&\quad+\frac{1}{2}(\overline{\eta_{\mu\kappa}}\,\overline{\eta_{\nu\sigma}}+\overline{\eta_{\mu\sigma}}\,\overline{\eta_{\nu\kappa}})\\&\quad-\frac{2}{n}\overline{\eta_{\mu\nu}}\,\overline{\eta_{\kappa\sigma}}\Big\}\delta^{(n-1)}(x-y).\end{aligned} \tag{A5}$$

We now consider the problem of expressing $\Box h_{\mu\nu}$, $\Box\partial_0 h_{\mu\nu}$, $\Box^2 h_{\mu\nu}$, and $\Box^2\partial_0 h_{\mu\nu}$ in terms of $h_{\mu\nu}$ and

$\partial_0 h_{\mu\nu}$ (or equivalently the canonical variables). The components of the first-order field equations (141) give

$$\Box h_{00}=\frac{1}{2}\triangle h_{ii}-\frac{1}{2}\partial_i\partial_j h_{ij},$$

$$\Box h_{0i}=\frac{1}{2}\partial_i\partial_0 h_{jj}-\frac{1}{2}\partial_j\partial_0 h_{ij}-\frac{1}{2}\partial_i\partial_j h_{0j}+\frac{1}{2}\triangle h_{0i},$$

$$\Box h_{ij}=-\frac{4}{n}\delta_{ij}\partial_k\partial_0 h_{0k}+\partial_i\partial_0 h_{j0}+\partial_j\partial_0 h_{i0}$$
$$+\frac{1}{n}\delta_{ij}(3\partial_k\partial_l h_{kl}+2\triangle h_{00}-\triangle h_{kk})$$
$$-\partial_i\partial_j h_{00}+\partial_i\partial_j h_{kk}-\partial_i\partial_k h_{jk}-\partial_j\partial_k h_{ik}, \quad \text{(A6)}$$

and

$$\partial_0^2 h_{00}=\frac{1}{2}\partial_i\partial_j h_{ij}+\triangle h_{00}-\frac{1}{2}\triangle h_{ii},$$

$$\partial_0^2 h_{0i}=-\frac{1}{2}\partial_i\partial_0 h_{jj}+\frac{1}{2}\partial_j\partial_0 h_{ij}+\partial_i\partial_j h_{0j}+\frac{1}{2}\triangle h_{0i},$$

$$\partial_0^2 h_{ij}=\frac{4}{n}\delta_{ij}\partial_k\partial_0 h_{0k}-\partial_i\partial_0 h_{j0}-\partial_j\partial_0 h_{i0}$$
$$-\frac{1}{n}\delta_{ij}(3\partial_k\partial_l h_{kl}+2\triangle h_{00}-\triangle h_{kk})+\partial_i\partial_j h_{00}$$
$$-\partial_i\partial_j h_{kk}+\partial_i\partial_k h_{jk}+\partial_j\partial_k h_{ik}+\triangle h_{ij}. \quad \text{(A7)}$$

From Eqs. (A6) and (A7) we get

$$\Box\partial_0 h_{00}=\frac{1}{2}\triangle\partial_0 h_{ii}-\frac{1}{2}\partial_i\partial_j\partial_0 h_{ij},$$

$$\Box\partial_0 h_{0i}=\triangle\partial_0 h_{0i}+\left(1-\frac{4}{n}\right)\partial_i\partial_j\partial_0 h_{0j}+\left(\frac{2}{n}-1\right)\triangle\partial_i h_{00}+\left(1-\frac{1}{n}\right)\triangle\partial_i h_{jj}-\triangle\partial_j h_{ij}+\left(\frac{3}{n}-1\right)\partial_i\partial_j\partial_k h_{jk},$$

$$\Box\partial_0 h_{ij}=\frac{1}{n}\delta_{ij}(\triangle\partial_0 h_{kk}+\partial_k\partial_l\partial_0 h_{kl}+2\triangle\partial_0 h_{00}-4\triangle\partial_k h_{0k})-\partial_i\partial_j\partial_0 h_{00}-\frac{1}{2}\partial_i\partial_k\partial_0 h_{jk}-\frac{1}{2}\partial_j\partial_k\partial_0 h_{ik}$$
$$+\partial_i\partial_j\partial_k h_{0k}+\frac{1}{2}\triangle\partial_i h_{0j}+\frac{1}{2}\triangle\partial_j h_{0i}, \quad \text{(A8)}$$

whereas Eq. (145) implies

$$\Box^2 h_{00}=\left(1-\frac{2}{n}\right)\left\{-2\triangle\partial_i\partial_0 h_{0i}+\triangle^2 h_{00}-\frac{1}{2}\triangle^2 h_{ii}+\frac{3}{2}\triangle\partial_i\partial_j h_{ij}\right\},$$

$$\Box^2 h_{0i}=\left(1-\frac{2}{n}\right)\partial_i\left\{\triangle\partial_0 h_{00}+\frac{1}{2}\triangle\partial_0 h_{jj}-2\triangle\partial_j h_{0j}+\frac{1}{2}\partial_j\partial_k\partial_0 h_{jk}\right\},$$

$$\Box^2 h_{ij}=\left(1-\frac{2}{n}\right)\partial_i\partial_j\left\{-2\partial_k\partial_0 h_{0k}+\triangle h_{00}-\frac{1}{2}\triangle h_{kk}+\frac{3}{2}\partial_k\partial_l h_{kl}\right\}, \quad \text{(A9)}$$

and

$$\Box^2\partial_0 h_{00}=\left(1-\frac{2}{n}\right)\left\{\triangle^2\partial_0 h_{00}+\frac{1}{2}\triangle^2\partial_0 h_{ii}+\frac{1}{2}\partial_i\partial_j\partial_0 h_{ij}-2\triangle^2\partial_i h_{0i}\right\},$$

$$\Box^2\partial_0 h_{0i}=\left(1-\frac{2}{n}\right)\partial_i\left\{-2\triangle\partial_j\partial_0 h_{0j}+\triangle^2 h_{00}-\frac{1}{2}\triangle^2 h_{jj}+\frac{3}{2}\triangle\partial_j\partial_k h_{jk}\right\},$$

$$\Box^2\partial_0 h_{ij}=\left(1-\frac{2}{n}\right)\partial_i\partial_j\left\{\triangle\partial_0 h_{00}+\frac{1}{2}\triangle\partial_0 h_{kk}+\frac{1}{2}\partial_k\partial_l\partial_0 h_{kl}-2\triangle\partial_k h_{0k}\right\}. \quad \text{(A10)}$$

To calculate the commutator $[h_{\mu\nu}(x),h_{\lambda\rho}(y)]$ at two arbitrary space-time points, we first express $h_{\lambda\rho}(y)$ as

$$h_{\lambda\rho}(y)=-\int d^{n-1}zD^{(n)}(y-z)\overleftrightarrow{\partial_0^z}h_{\lambda\rho}(z)-\int d^{n-1}z\tilde{D}^{(n)}(y-z)\overleftrightarrow{\partial_0^z}\Box h_{\lambda\rho}(z)-\int d^{n-1}z\tilde{\tilde{D}}^{(n)}(y-z)\overleftrightarrow{\partial_0^z}\Box^2 h_{\lambda\rho}(z), \quad \text{(A11)}$$

$$[h_{\mu\nu}(x),h_{\lambda\rho}(y)]=-\int d^{n-1}zD^{(n)}(y-z)[h_{\mu\nu}(x),\partial_0 h_{\lambda\rho}(z)]_{\rm eq}-\partial_0 D^{(n)}(y-z)[h_{\mu\nu}(x),h_{\lambda\rho}(z)]_{\rm eq}+\tilde{D}^{(n)}(y-z)$$
$$\times[h_{\mu\nu}(x),\partial_0\Box h_{\lambda\rho}(z)]_{\rm eq}-\partial_0\tilde{D}^{(n)}(y-z)[h_{\mu\nu}(x),\Box h_{\lambda\rho}]_{\rm eq}+\tilde{\tilde{D}}^{(n)}(y-z)[h_{\mu\nu}(x),\partial_0\Box^2 h_{\lambda\rho}(z)]_{\rm eq}$$
$$-\partial_0\tilde{\tilde{D}}^{(n)}(y-z)[h_{\mu\nu},\Box^2 h_{\lambda\rho}(z)]_{\rm eq}. \tag{A12}$$

We shall refer to the six terms on the RHS as term 1, term 2, . . . , term 6, respectively. From the commutator (150), terms 1 and 2 are

$$\langle\text{term 1}\rangle=\frac{i}{4}\left\{\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}+\overline{\eta_{\mu\lambda}}\,\overline{\eta_{\nu\rho}}+\overline{\eta_{\mu\rho}}\,\overline{\eta_{\nu\lambda}}-\frac{4}{n}\overline{\eta_{\mu\nu}}\,\overline{\eta_{\lambda\rho}}\right\}D^{(n)}(x-y),\quad\langle\text{ term 2}\rangle=0. \tag{A13}$$

For terms 3 and 4 we shall calculate the equal-time commutators $[h_{\mu\nu}(x),\partial_0\Box h_{\lambda\rho}]_{\rm eq}$ and $[h_{\mu\nu}(x),\Box h_{\lambda\rho}(z)]_{\rm eq}$ by using the expressions (A6) and (A8). Putting terms 3 and 4 together gives

$$\langle\text{term 3+ term 4}\rangle=\frac{i}{4}\left\{-\overline{\eta_{\mu\lambda}}\partial_\nu\partial_\rho-\overline{\eta_{\mu\rho}}\partial_\nu\partial_\lambda-\overline{\eta_{\nu\lambda}}\partial_\mu\partial_\rho-\overline{\eta_{\nu\rho}}\partial_\mu\partial_\lambda+\frac{4}{n}(\overline{\eta_{\mu\nu}}\partial_\lambda\partial_\rho+\overline{\eta_{\lambda\rho}}\partial_\mu\partial_\nu)-2\eta_{\mu0}\eta_{\nu0}\overline{\partial_\lambda\partial_\rho}-2\eta_{\lambda0}\eta_{\rho0}\overline{\partial_\mu\partial_\nu}\right.$$
$$\left.-\left(1-\frac{4}{n}\right)(\eta_{\mu0}\eta_{\lambda0}\overline{\partial_\nu\partial_\rho}+\eta_{\mu0}\eta_{\rho0}\overline{\partial_\nu\partial_\lambda}+\eta_{\nu0}\eta_{\lambda0}\overline{\partial_\mu\partial_\rho}+\eta_{\nu0}\eta_{\rho0}\overline{\partial_\mu\partial_\lambda})\right\}\tilde{D}^{(n)}(x-y)$$
$$+\frac{i}{4}\left\{\frac{4}{n}(\eta_{\mu0}\eta_{\nu0}\overline{\eta_{\lambda\rho}}+\eta_{\lambda0}\eta_{\rho0}\overline{\eta_{\mu\nu}})-(\eta_{\mu0}\eta_{\lambda0}\overline{\eta_{\nu\rho}}+\eta_{\mu0}\eta_{\rho0}\overline{\eta_{\nu\lambda}}+\eta_{\nu0}\eta_{\lambda0}\overline{\eta_{\mu\rho}}+\eta_{\nu0}\eta_{\rho0}\overline{\eta_{\mu\lambda}})\right\}$$
$$\times D^{(n)}(x-y), \tag{A14}$$

where we used the relation

$$\triangle\tilde{D}^{(n)}=\partial_0^2\tilde{D}^{(n)}+D^{(n)}.$$

Similarly by calculating the equal-time commutators $[h_{\mu\nu},\partial_0\Box^2 h_{\lambda\rho}]_{\rm eq}$ and $[h_{\mu\nu}(x),\Box^2 h_{\lambda\rho}(z)]_{\rm eq}$ [making use of the expressions (A9) and (A10)], we have

$$\langle\text{term 5+term 6}\rangle=\frac{i}{2}\left(1-\frac{2}{n}\right)\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\tilde{\tilde{D}}^{(n)}(x-y)+\frac{i}{2}\left(1-\frac{2}{n}\right)\{\eta_{\mu0}\eta_{\lambda0}\partial_\nu\partial_\rho+\eta_{\mu0}\eta_{\rho0}\partial_\nu\partial_\lambda+\eta_{\nu0}\eta_{\lambda0}\partial_\mu\partial_\rho+\eta_{\nu0}\eta_{\rho0}\partial_\mu\partial_\lambda$$
$$+\eta_{\mu0}\eta_{\nu0}\partial_\lambda\partial_\rho+\eta_{\lambda0}\eta_{\rho0}\partial_\mu\partial_\nu+2(\eta_{\mu0}\eta_{\nu0}\eta_{\lambda0}\partial_0\partial_\rho+\eta_{\mu0}\eta_{\nu0}\eta_{\rho0}\partial_0\partial_\lambda+\eta_{\mu0}\eta_{\lambda0}\eta_{\rho0}\partial_0\partial_\nu+\eta_{\nu0}\eta_{\lambda0}\eta_{\rho0}\partial_0\partial_\mu)$$
$$+4\eta_{\mu0}\eta_{\nu0}\eta_{\lambda0}\eta_{\rho0}\partial_0^2\}\tilde{D}^{(n)}(x-y)+\frac{i}{2}\left(1-\frac{2}{n}\right)\eta_{\mu0}\eta_{\nu0}\eta_{\lambda0}\eta_{\rho0}D^{(n)}(x-y), \tag{A15}$$

where we used

$$\triangle\tilde{\tilde{D}}^{(n)}=\partial_0^2\tilde{\tilde{D}}^{(n)}+\tilde{D}^{(n)}\qquad.$$

Summing up all the terms we get

$$[h_{\mu\nu}(x),h_{\lambda\rho}(y)]=\frac{i}{2}\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{2}{n}\eta_{\mu\nu}\eta_{\lambda\rho}\right)D^{(n)}(x-y)+\frac{i}{4}\left\{-\eta_{\mu\lambda}\partial_\nu\partial_\rho-\eta_{\mu\rho}\partial_\nu\partial_\lambda-\eta_{\nu\lambda}\partial_\mu\partial_\rho-\eta_{\nu\rho}\partial_\mu\partial_\lambda\right.$$
$$\left.+\frac{4}{n}(\eta_{\mu\nu}\partial_\lambda\partial_\rho+\eta_{\lambda\rho}\partial_\mu\partial_\nu)\right\}\tilde{D}^{(n)}(x-y)+\frac{i}{2}\left(1-\frac{2}{n}\right)\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\tilde{\tilde{D}}^{(n)}(x-y). \tag{A16}$$

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