Radiative 3.5 post-Newtonian ADM Hamiltonian for many-body point-mass systems

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We calculate to post$^{3.5}$-Newtonian order of approximation of general relativity the radiation-reaction part of the ADM Hamiltonian for a many-body system of nonspinning point masses. The Hamiltonian is applied to the derivation of the gravitational energy loss of a gravitationally bound two-body system in quasielliptic motion. Agreement with the known result for the time-averaged energy loss is obtained to order $(v/c)^7$. [S0556-2821(97)06408-4]

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I. INTRODUCTION AND SUMMARY

The calculation of the dynamical evolution of two-body systems in general relativity is a major challenge in the theoretical investigation of the motion of compact astrophysical binaries. Twenty years of observations of the Hulse-Taylor binary pulsar PSR 1913+16 yielded measurements of the conservative part of the dynamics as precise as to reach levels two orders of magnitude $[O((v/c)^4)]$ beyond Newtonian theory, and the leading radiation-reaction part $[O((v/c)^5)]$, the quadrupole gravitational radiation damping, was found to be in agreement with general relativity with a precision of 0.35% [1]. On the other hand, future gravitational-wave astronomy will need theoretical knowledge of the motion of binary systems to even higher post-Newtonian orders [2].

Recently, Iyer and Will [3] derived the equations of motion for nonspinning pointlike binaries at the dissipative post$^{3.5}$-Newtonian (3.5PN) order, i.e., one post-Newtonian order beyond the leading order in the damping, using a postulated balance between an instantaneous flux of energy and angular momentum in the far zone and an instantaneous loss of energy and angular momentum in the system's near zone. In a more recent paper Iyer and Will [4] improved their derivation of the radiation-reaction force by the two-body specification and evaluation of Blanchet's radiation-reaction potentials obtained by the method of asymptotic matching [5] and confirmed their previously postulated balance between instantaneous loss and instantaneous flux expressions.

In this paper we are interested in the dynamics of n-body point-mass systems. The tool we use is the Arnowitt-Deser-Misner (ADM) Hamiltonian formalism of general relativity [6]. This formalism has been proved very efficient in the calculation of the approximate general relativistic dynamics to post$^{2.5}$-Newtonian order [7,8]. The advantage of the ADM formalism, as of any other Hamiltonian formalism, is the absence of asymptotic matching by which the near field, given in a near-zone-defined coordinate system, is interrelated with the far field, given in a far-zone-defined coordinate system. In our calculations only one coordinate system is employed, which is defined globally.

In the ADM formalism, there are no natural balance relations between instantaneous losses and instantaneous fluxes, even if one assumes quasistationarity in the radiation emission (see, e.g., [9]). Instantaneous balance between losses and fluxes can be achieved by *ad hoc* constructions of appropriate expressions only. By choosing total time derivatives judiciously, the expressions for instantaneous losses and fluxes can gain the property of being invariant against (infinitesimal) coordinate transformations in, respectively, the near and far zones [4].

In the present paper, the n-body ADM Hamiltonian is calculated fully explicitly at the dissipative 3.5PN order. We apply this Hamiltonian to the derivation of the energy loss of a two-body system on quasielliptic orbits. The expression obtained for the energy loss is different from the expression used and derived by Iyer and Will in their papers; only the time-averaged expressions coincide.

We use units in which $16\pi G=c=1$, where G is the Newtonian gravitational constant and c is the velocity of light. We employ the following notation: $\mathbf{x}=(x^i)$ denotes a point in the three-dimensional Euclidean space $\mathbb{R}^3$ endowed with a standard Euclidean metric and a scalar product (denoted by a dot). Letters $a,b,\ldots$ are particles labels, and so $\mathbf{x}_a\in\mathbb{R}^3$ denotes the position of the ath particle. We also define $\mathbf{r}_a:=\mathbf{x}-\mathbf{x}_a$, $r_a:=|\mathbf{r}_a|$, $\mathbf{n}_a:=\mathbf{r}_a/r_a$; and for $a\neq b$, $\mathbf{r}_{ab}:=\mathbf{x}_a-\mathbf{x}_b$, $r_{ab}:=|\mathbf{r}_{ab}|$, $\mathbf{n}_{ab}:=\mathbf{r}_{ab}/r_{ab}$; $|\cdot|$ stands here for the length of a vector. The momentum vector of the ath particle is denoted by $\mathbf{p}_a=(p_{ai})$, and m_a denotes its mass parameter. Indices with round brackets, like in $A_{(4)ij}$, give the order of the object in inverse powers of the velocity of light, in this case, $1/c^4$. We abbreviate $\delta(\mathbf{x}-\mathbf{x}_a)$ to δ_a. An overdot, like in $\dot{\mathbf{x}}_a$, means the total time derivative. The partial differentiation with respect to x^i is denoted by ∂_i or by a comma, i.e., $\partial_i\phi\equiv\phi_{,i}$.

II. 3.5PN FIELD EQUATIONS

In the ADM formulation of general relativity, the full information about the motion of isolated bodies and the emitted gravitational radiation results from the reduced Hamil-

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tonian H, which is a functional of the independent degrees of freedom of the system [6]. The reduced Hamiltonian itself results from the solution of the four constraint equations of the Einstein theory. Four metric functions are fixed by imposing four coordinate conditions. In this paper we choose the generalized isotropic coordinate conditions often used in applications of the ADM formalism [7,8,10].

For a system of n pointlike bodies with position vectors $\mathbf{x}_a$ and momenta $\mathbf{p}_a$ $(a=1,\ldots,n)$, the Hamiltonian reads

$$H=H[\mathbf{x}_a,\mathbf{p}_a,h_{ij}^{\mathrm{TT}},\pi^{ij\mathrm{TT}}], \qquad (1)$$

where h_{ij}^{TT} denotes the independent part of the gravitational field which is transverse and traceless in the flat-space metric δ_{ij}, and $\pi^{ij\mathrm{TT}}$ is the canonical conjugate to h_{ij}^{TT}. In terms of h_{ij}^{TT}, the metric g_{ij} of the spacelike hypersurface $x^0=t$ $=$const takes the form $g_{ij}=(1+\frac{1}{8}\phi)^4\delta_{ij}+h_{ij}^{\mathrm{TT}}$. The equations of motion for the bodies read

$$\dot{\mathbf{p}}_a=-\frac{\partial H}{\partial \mathbf{x}_a}, \quad \dot{\mathbf{x}}_a=\frac{\partial H}{\partial \mathbf{p}_a}, \qquad (2)$$

and the field equations for the independent degrees of freedom take the form

$$\frac{\partial}{\partial t}\,\pi^{ij\mathrm{TT}}=-\delta_{kl}^{\mathrm{TT}ij}\,\frac{\delta H}{\delta h_{kl}^{\mathrm{TT}}}, \quad \frac{\partial}{\partial t}\,h_{ij}^{\mathrm{TT}}=\delta_{ij}^{\mathrm{TT}kl}\,\frac{\delta H}{\delta \pi^{kl\mathrm{TT}}}, \qquad (3)$$

where $(\delta\cdots)/(\delta\cdots)$ denotes the Fréchet derivative and where the TT-projection operator is defined by [see, e.g., Eq. (2.17) of [8]; Δ^{-1} is the inverse Laplacian in flat space]

$$\delta_{ij}^{\mathrm{TT}kl}:=\tfrac{1}{2}[(\delta_{il}-\Delta^{-1}\partial_i\partial_l)(\delta_{jk}-\Delta^{-1}\partial_j\partial_k)+(\delta_{ik}-\Delta^{-1}\partial_i\partial_k)\times(\delta_{jl}-\Delta^{-1}\partial_j\partial_l)-(\delta_{kl}-\Delta^{-1}\partial_k\partial_l)\times(\delta_{ij}-\Delta^{-1}\partial_i\partial_j)]. \qquad (4)$$

The Hamiltonian H can be uniquely decomposed into a matter part H^{mat}, which depends only on the matter variables, a field part H^{field}, which depends only on the field variables, and an interaction part H^{int}, which depends on both sets of variables, matter and field, and which vanishes if one of the two sets is put equal to zero. Thus the full content of the field-plus-matter dynamics up to the 3.5PN approximation is included in the Hamiltonian

$$H_{\leqslant 3.5\mathrm{PN}}=H_{\leqslant 3.5\mathrm{PN}}^{\mathrm{mat}}+H_{\leqslant 3.5\mathrm{PN}}^{\mathrm{field}}+H_{\leqslant 3.5\mathrm{PN}}^{\mathrm{int}}. \qquad (5)$$

In what follows we shall only need the field and interaction parts of the Hamiltonian (5). They can, up to 3.5PN order, be written as [11]

$$H_{\leqslant 3.5\mathrm{PN}}^{\mathrm{field}}=\int d^3x\left[\frac{1}{4}\,(h_{ij,k}^{\mathrm{TT}})^2+(\pi^{ij\mathrm{TT}})^2\right], \qquad (6)$$

$$H_{\leqslant 3.5\mathrm{PN}}^{\mathrm{int}}=\int d^3x\left[h_{ij}^{\mathrm{TT}}\left(\frac{1}{2}A_{(4)ij}+B_{(6)ij}\right)-\frac{1}{8}\,(h_{ij}^{\mathrm{TT}})^2\sum_a m_a\delta_a -\frac{1}{4}\,\phi_{(2)}(h_{ij,k}^{\mathrm{TT}})^2-\phi_{(2)}\tilde{\pi}_{(3)}^{ij}\pi^{ij\mathrm{TT}}\right], \qquad (7)$$

where we used the definitions

$$A_{(4)ij}:=-\sum_a\frac{p_{ai}p_{aj}}{m_a}\,\delta_a-\frac{1}{4}\,\phi_{(2),i}\phi_{(2),j}, \qquad (8)$$

$$\begin{aligned}B_{(6)ij}:=&\frac{1}{4}\sum_a\frac{\mathbf{p}_a^2p_{ai}p_{aj}}{m_a^3}\,\delta_a+\frac{5}{8}\,\phi_{(2)}\sum_a\frac{p_{ai}p_{aj}}{m_a}\,\delta_a+\left(\frac{1}{16\pi}\right)\phi_{(2),ij}\sum_a\frac{\mathbf{p}_a^2}{m_ar_a}+\left(\frac{1}{16\pi}\right)\tilde{\pi}_{(3)}^{ij}\sum_a p_{ak}\left(\frac{1}{r_a}\right)_{,k}\\&+\frac{1}{2}\left(\frac{1}{16\pi}\right)\tilde{\pi}_{(3),k}^{ij}\left(8\sum_a p_{ak}\,\frac{1}{r_a}-\sum_a p_{al}r_{a,kl}\right)+4\left(\frac{1}{16\pi}\right)\tilde{\pi}_{(3)}^{jk}\sum_a\left[p_{ak}\left(\frac{1}{r_a}\right)_{,i}-p_{ai}\left(\frac{1}{r_a}\right)_{,k}\right]\\&-\frac{3}{4}\left(\frac{1}{16\pi}\right)^2\phi_{(2),ij}\sum_a\sum_{b\neq a}\frac{m_am_b}{r_{ab}r_a}+\frac{5}{64}\,\phi_{(2)}\phi_{(2),i}\phi_{(2),j}\,.\end{aligned} \qquad (9)$$

The functions $\phi_{(2)}$ and $\tilde{\pi}_{(3)}^{ij}$, entering Eqs. (8) and (9), are equal to

$$\phi_{(2)}=\frac{1}{4\pi}\sum_a\frac{m_a}{r_a}, \qquad (10)$$

$$\tilde{\pi}_{(3)}^{ij}=\frac{1}{16\pi}\sum_a p_{ak}\left\{-\delta_{ij}\left(\frac{1}{r_a}\right)_{,k}+2\left[\delta_{ik}\left(\frac{1}{r_a}\right)_{,j}+\delta_{jk}\left(\frac{1}{r_a}\right)_{,i}\right]-\frac{1}{2}\,r_{a,ijk}\right\}. \qquad (11)$$

The field equations for the field variables h_{ij}^{TT} and $\pi^{ij\mathrm{TT}}$, valid up to 3.5PN order, are obtained by varying the Hamiltonian (5) according to Eqs. (3). Combining Eqs. (3) and taking Eqs. (5)–(9) into account, one obtains

$$\Box h_{ij}^{\mathrm{TT}}=\delta_{ij}^{\mathrm{TT}kl}\left[A_{(4)kl}+2B_{(6)kl}-\frac{1}{2}\,h_{kl}^{\mathrm{TT}}\sum_a m_a\delta_a+(h_{kl,m}^{\mathrm{TT}}\phi_{(2)})_{,m}+\frac{d}{dt}\,(\phi_{(2)}\tilde{\pi}_{(3)}^{kl})\right], \qquad (12)$$

$$\pi^{ij\mathrm{TT}}=\frac{1}{2}\left[\frac{d}{dt}\,h^{\mathrm{TT}}_{ij}+\delta^{\mathrm{TT}ij}_{kl}(\phi_{(2)}\tilde{\pi}^{kl}_{(3)})\right], \tag{13}$$

where $\Box$ is the d'Alembertian in flat space. The expressions for $h^{\mathrm{TT}}_{(4)ij}$, $h^{\mathrm{TT}}_{(5)ij}$, $h^{\mathrm{TT}}_{(6)ij}$, and $h^{\mathrm{TT}}_{(7)ij}$ can be extracted from Eq. (12) by the aid of the near-zone expansion of the retarded solution. We only need those for $h^{\mathrm{TT}}_{(4)ij}$, $h^{\mathrm{TT}}_{(5)ij}$, and $h^{\mathrm{TT}}_{(7)ij}$. They read

$$h^{\mathrm{TT}}_{(4)ij}=\Delta^{-1}\delta^{\mathrm{TT}kl}_{ij}A_{(4)kl}, \tag{14}$$

$$h^{\mathrm{TT}}_{(5)ij}=\Box^{-1}_{\mathrm{ret}}\delta^{\mathrm{TT}kl}_{ij}A_{(4)kl}-h^{\mathrm{TT}}_{(4)ij}+O\left(\frac{1}{c^6}\right), \tag{15}$$

$$\begin{aligned}h^{\mathrm{TT}}_{(7)ij}=\Box^{-1}_{\mathrm{ret}}\delta^{\mathrm{TT}kl}_{ij}\bigg[&A_{(4)kl}+2B_{(6)kl}-\frac{1}{2}\,(h^{\mathrm{TT}}_{(4)kl}\\&+h^{\mathrm{TT}}_{(5)kl})\sum_a m_a\delta_a+(h^{\mathrm{TT}}_{(4)kl,m}\phi_{(2)})_{,m}\\&+\frac{d}{dt}\,(\phi_{(2)}\tilde{\pi}^{kl}_{(3)})\bigg]-[h^{\mathrm{TT}}_{(4)ij}+h^{\mathrm{TT}}_{(5)ij}+h^{\mathrm{TT}}_{(6)ij}]+O\left(\frac{1}{c^8}\right),\end{aligned} \tag{16}$$

where $\Box^{-1}_{\mathrm{ret}}$ is the retarded inverse d'Alembertian in flat space. The radiative 2.5PN and 3.5PN Hamiltonians can be calculated from Eq. (7). Taking into account the relation

$$\pi^{ij\mathrm{TT}}_{(6)}=\frac{1}{2}\left(\frac{d}{dt}\,h^{\mathrm{TT}}_{(5)ij}\right), \tag{17}$$

which follows from Eq. (13), one gets

$$H^{\mathrm{int}}_{2.5\mathrm{PN}}=\frac{1}{2}\int d^3x\; h^{\mathrm{TT}}_{(5)ij}A_{(4)ij}, \tag{18}$$

$$\begin{aligned}H^{\mathrm{int}}_{3.5\mathrm{PN}}=\int d^3x\bigg[&\frac{1}{2}\,h^{\mathrm{TT}}_{(7)ij}A_{(4)ij}\\&+h^{\mathrm{TT}}_{(5)ij}\left(B_{(6)ij}-\frac{1}{4}\,h^{\mathrm{TT}}_{(4)ij}\sum_a m_a\delta_a\right)\\&-\frac{1}{2}\left(\frac{d}{dt}\,h^{\mathrm{TT}}_{(5)ij}\right)\phi_{(2)}\tilde{\pi}^{ij}_{(3)}\bigg].\end{aligned} \tag{19}$$

III. DERIVATION OF $h^{\mathrm{TT}}_{(7)ij}$

The explicit solutions of Eqs. (14) and (15) are well known. We cite them here for the convenience of the reader as well as for the future reference. The solution of Eq. (14) reads (here $s_{ab}:=r_a+r_b+r_{ab}$)

$$\begin{aligned}h^{\mathrm{TT}}_{(4)ij}(\mathbf{x},\mathbf{x}_a,\mathbf{p}_a)=&\frac{1}{4}\left(\frac{1}{16\pi}\right)\sum_a\frac{1}{m_ar_a}\{[\mathbf{p}_a^2-5(\mathbf{n}_a\cdot\mathbf{p}_a)^2]\delta_{ij}+2p_{ai}p_{aj}+[3(\mathbf{n}_a\cdot\mathbf{p}_a)^2-5\mathbf{p}_a^2]n^i_an^j_a+6(\mathbf{n}_a\cdot\mathbf{p}_a)(n^i_ap_{aj}+n^j_ap_{ai})\}\\&+\frac{1}{8}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}m_am_b\bigg\{-\frac{32}{s_{ab}}\left(\frac{1}{r_{ab}}+\frac{1}{s_{ab}}\right)n^i_{ab}n^j_{ab}+2\left(\frac{r_a+r_b}{r^3_{ab}}+\frac{12}{s^2_{ab}}\right)n^i_an^j_b\\&+16\left(\frac{2}{s^2_{ab}}-\frac{1}{r^2_{ab}}\right)(n^i_an^j_{ab}+n^j_an^i_{ab})+\left[\frac{5}{r_{ab}r_a}-\frac{r_a}{r^3_{ab}}\left(\frac{r_a}{r_b}+3\right)-\frac{8}{s_{ab}}\left(\frac{1}{r_a}+\frac{1}{s_{ab}}\right)\right]n^i_an^j_a\\&+\left[5\,\frac{r_a}{r^3_{ab}}\left(\frac{r_a}{r_b}-1\right)-\frac{17}{r_{ab}r_a}+\frac{4}{r_ar_b}+\frac{8}{s_{ab}}\left(\frac{1}{r_a}+\frac{4}{r_{ab}}\right)\right]\delta_{ij}\bigg\}.\end{aligned} \tag{20}$$

The solution to Eq. (15) can be written as

$$h^{\mathrm{TT}}_{(5)ij}(t)=\frac{d}{dt}\,\chi_{(4)ij}(\mathbf{x}_a(t),\mathbf{p}_a(t)), \tag{21}$$

where

$$\chi_{(4)ij}(\mathbf{x}_a,\mathbf{p}_a):=\frac{1}{60\pi}\left[\sum_a\frac{2}{m_a}\,(\mathbf{p}_a^2\delta_{ij}-3p_{ai}p_{aj})+\frac{1}{16\pi}\sum_a\sum_{b\neq a}\frac{m_am_b}{r_{ab}}\,(3n^i_{ab}n^j_{ab}-\delta_{ij})\right]. \tag{22}$$

The explicit solution of Eq. (16) we decompose into several parts

$$h^{\mathrm{TT}}_{(7)ij}(\mathbf{x},t)=P_{1ij}(t)+P_{2ij}(t)+P_{3ij}(t)+Q_{ij}(\mathbf{x},t)+R_{ij}(\mathbf{x},t), \tag{23}$$

where the following definitions were used:

$$P_{1ij}(t):=\frac{1}{2\pi}\frac{d}{dt}\,\delta^{\mathrm{TT}kl}_{ij}\left[\int d^3x'\;B_{(6)kl}(\mathbf{x}',t)\right], \tag{24}$$

$$P_{2ij}(t):=-\frac{1}{8\pi}\frac{d}{dt}\,\delta_{ij}^{\mathrm{TT}kl}\left[\int d^3x'\;h^{\mathrm{TT}}_{(4)kl}(\mathbf{x}',t)\sum_a m_a\delta(\mathbf{x}'-\mathbf{x}_a(t))\right], \tag{25}$$

$$P_{3ij}(t):=\frac{1}{4\pi}\frac{d^2}{dt^2}\,\delta_{ij}^{\mathrm{TT}kl}\left[\int d^3x'\;\phi_{(2)}(\mathbf{x}',t)\tilde{\pi}^{kl}_{(3)}(\mathbf{x}',t)\right], \tag{26}$$

$$Q_{ij}(x,t):=\frac{1}{8\pi}\,\delta_{ij}^{\mathrm{TT}kl}\left[h^{\mathrm{TT}}_{(5)kl}(t)\int d^3x'|\mathbf{x}-\mathbf{x}'|^{-1}\sum_a m_a\delta(\mathbf{x}'-\mathbf{x}_a(t))\right], \tag{27}$$

$$R_{ij}(\mathbf{x},t):=\frac{1}{24\pi}\,\delta_{ij}^{\mathrm{TT}kl}\left[\int d^3x'|\mathbf{x}-\mathbf{x}'|^2\,\frac{\partial^3}{\partial t^3}A_{(4)kl}(\mathbf{x}',t)\right]. \tag{28}$$

To compute the integrals which enter Eqs. (24)–(28), we apply the techniques described in Appendixes A–C. After long calculations we obtain (here $s_{abc}:=r_{ab}+r_{bc}+r_{ca}$)

$$P_{1ij}(t)=\frac{d}{dt}\,\Pi_{1ij}(\mathbf{x}_a(t),\mathbf{p}_a(t)), \tag{29}$$

$$\begin{aligned}
\Pi_{1ij}(\mathbf{x}_a,\mathbf{p}_a):=&\frac{4}{15}\left(\frac{1}{16\pi}\right)\sum_a\frac{\mathbf{p}_a^2}{m_a^3}(-\mathbf{p}_a^2\delta_{ij}+3p_{ai}p_{aj})+\frac{8}{5}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}\frac{m_b}{m_ar_{ab}}(-2\mathbf{p}_a^2\delta_{ij}+5p_{ai}p_{aj}+\mathbf{p}_a^2n_{ab}^in_{ab}^j)\\
&+\frac{1}{5}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}\frac{1}{r_{ab}}\{[19(\mathbf{p}_a\cdot\mathbf{p}_b)-3(\mathbf{n}_{ab}\cdot\mathbf{p}_a)(\mathbf{n}_{ab}\cdot\mathbf{p}_b)]\delta_{ij}-42p_{ai}p_{bj}\\
&-3[5(\mathbf{p}_a\cdot\mathbf{p}_b)+(\mathbf{n}_{ab}\cdot\mathbf{p}_a)(\mathbf{n}_{ab}\cdot\mathbf{p}_b)]n_{ab}^in_{ab}^j+6(\mathbf{n}_{ab}\cdot\mathbf{p}_b)(n_{ab}^ip_{aj}+n_{ab}^jp_{ai})\}\\
&+\frac{41}{15}\left(\frac{1}{16\pi}\right)^3\sum_a\sum_{b\neq a}\frac{m_a^2m_b}{r_{ab}^2}(\delta_{ij}-3n_{ab}^in_{ab}^j)+\frac{1}{45}\left(\frac{1}{16\pi}\right)^3\sum_a\sum_{b\neq a}\sum_{c\neq a,b}m_am_bm_c\left\{\frac{18}{r_{ab}r_{ca}}(\delta_{ij}-3n_{ab}^in_{ab}^j)\right.\\
&\left.-\frac{180}{s_{abc}}\left[\left(\frac{1}{r_{ab}}+\frac{1}{s_{abc}}\right)n_{ab}^in_{ab}^j+\frac{1}{s_{abc}}n_{ab}^in_{bc}^j\right]+\frac{10}{s_{abc}}\left[4\left(\frac{1}{r_{ab}}+\frac{1}{r_{bc}}+\frac{1}{r_{ca}}\right)-\frac{r_{ab}^2+r_{bc}^2+r_{ca}^2}{r_{ab}r_{bc}r_{ca}}\right]\delta_{ij}\right\},
\end{aligned} \tag{30}$$

$$P_{2ij}(t)=\frac{d}{dt}\,\Pi_{2ij}(\mathbf{x}_a(t),\mathbf{p}_a(t)), \tag{31}$$

$$\begin{aligned}
\Pi_{2ij}(\mathbf{x}_a,\mathbf{p}_a):=&\frac{1}{5}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}\frac{m_b}{m_ar_{ab}}\{[5(\mathbf{n}_{ab}\cdot\mathbf{p}_a)^2-\mathbf{p}_a^2]\delta_{ij}-2p_{ai}p_{aj}+[5\mathbf{p}_a^2-3(\mathbf{n}_{ab}\cdot\mathbf{p}_a)^2]n_{ab}^in_{ab}^j-6(\mathbf{n}_{ab}\cdot\mathbf{p}_a)\\
&\times(n_{ab}^ip_{aj}+n_{ab}^jp_{ai})\}+\frac{6}{5}\left(\frac{1}{16\pi}\right)^3\sum_a\sum_{b\neq a}\frac{m_a^2m_b}{r_{ab}^2}(3n_{ab}^in_{ab}^j-\delta_{ij})+\frac{1}{10}\left(\frac{1}{16\pi}\right)^3\sum_a\sum_{b\neq a}\sum_{c\neq a,b}m_am_bm_c\\
&\times\left\{\left[5\frac{r_{ca}}{r_{ab}^3}\left(1-\frac{r_{ca}}{r_{bc}}\right)+\frac{13}{r_{ab}r_{ca}}-\frac{40}{r_{ab}s_{abc}}\right]\delta_{ij}+\left[3\frac{r_{ab}}{r_{ca}^3}+\frac{r_{bc}^2}{r_{ab}r_{ca}^3}-\frac{5}{r_{ab}r_{ca}}+\frac{40}{s_{abc}}\left(\frac{1}{r_{ab}}+\frac{1}{s_{abc}}\right)\right]n_{ab}^in_{ab}^j\right.\\
&\left.+\left[2\frac{(r_{ab}+r_{ca})}{r_{bc}^3}-16\left(\frac{1}{r_{ab}^2}+\frac{1}{r_{ca}^2}\right)+\frac{88}{s_{abc}^2}\right]n_{ab}^in_{ca}^j\right\},
\end{aligned} \tag{32}$$

$$P_{3ij}(t)=\frac{d^2}{dt^2}\,\Pi_{3ij}(\mathbf{x}_a(t),\mathbf{p}_a(t)), \tag{33}$$

$$\Pi_{3ij}(\mathbf{x}_a,\mathbf{p}_a):=\frac{1}{5}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}m_b[-5(\mathbf{n}_{ab}\cdot\mathbf{p}_a)\delta_{ij}+(\mathbf{n}_{ab}\cdot\mathbf{p}_a)n_{ab}^in_{ab}^j+7(n_{ab}^ip_{aj}+n_{ab}^jp_{ai})], \tag{34}$$

$$Q_{ij}(\mathbf{x},t)=\frac{1}{8}\left(\frac{1}{16\pi}\right)\sum_a\frac{m_a}{r_a}[2h^{\mathrm{TT}}_{(5)ij}+6(n_a^in_a^kh^{\mathrm{TT}}_{(5)jk}+n_a^jn_a^kh^{\mathrm{TT}}_{(5)ik})-5\delta_{ij}n_a^kn_a^lh^{\mathrm{TT}}_{(5)kl}+3n_a^in_a^jn_a^kn_a^lh^{\mathrm{TT}}_{(5)kl}], \tag{35}$$

$$R_{ij}(\mathbf{x},t)=\frac{\partial^3}{\partial t^3}\Bigg\{\frac{4}{105}\left(\frac{1}{16\pi}\right)\sum_a \frac{r_a^2}{m_a}\{[5\mathbf{p}_a^2-4(\mathbf{n}_a\cdot\mathbf{p}_a)^2]\delta_{ij}-11p_{ai}p_{aj}-4\mathbf{p}_a^2n_a^in_a^j+6(\mathbf{n}_a\cdot\mathbf{p}_a)(n_a^ip_{aj}+n_a^jp_{ai})\}$$
$$+\frac{2}{105}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}m_am_b\Bigg[\left(2\frac{r_a^4}{r_{ab}^3}-2\frac{r_a^2r_b^2}{r_{ab}^3}-5\frac{r_a^2}{r_{ab}}\right)\delta_{ij}+4\frac{r_a^2}{r_{ab}}n_a^in_a^j+17\left(\frac{r_a^2}{r_{ab}}+r_{ab}\right)n_{ab}^in_{ab}^j$$
$$+\left(6\frac{r_a^3}{r_{ab}^2}+17r_a\right)(n_a^in_{ab}^j+n_a^jn_{ab}^i)\Bigg]\Bigg\}. \quad (36)$$

IV. 2.5PN AND 3.5PN INTERACTION HAMILTONIANS

The symmetric and trace-free (STF) part of a tensor T_{ij} is defined as

$$T_{ij}^{\mathrm{STF}}:=\tfrac{1}{2}(T_{ij}+T_{ji})-\tfrac{1}{3}\delta_{ij}T_{kk}. \quad (37)$$

If the tensor T_{ij} does not depend on the space variables $\mathbf{x}$ (i.e., it is a function of time t only), then from Eq. (C3) of Appendix C, with $\alpha=0$, and Eq. (37) one immediately gets

$$\delta_{ij}^{\mathrm{TT}kl}T_{kl}(t)=\tfrac{2}{5}T_{ij}^{\mathrm{STF}}(t). \quad (38)$$

To compute the 2.5PN interaction Hamiltonian (18), we must perform the integration

$$\int d^3x\, A_{(4)ij}=-\sum_a\frac{p_{ai}p_{aj}}{m_a}+\frac{1}{2}\left(\frac{1}{16\pi}\right)\sum_a\sum_{b\neq a}\frac{m_am_b}{r_{ab}}(n_{ab}^in_{ab}^j-\delta_{ij}). \quad (39)$$

Comparing Eq. (39) with Eq. (22), one can get

$$\int d^3x\, A_{(4)ij}^{\mathrm{STF}}=10\pi\chi_{(4)ij}, \quad (40)$$

which enables us to write the Hamiltonian (18) in the known form [8,12]

$$H_{2.5\mathrm{PN}}^{\mathrm{int}}(\mathbf{x}_a,\mathbf{p}_a,t)=5\pi\dot{\chi}_{(4)ij}(t)\chi_{(4)ij}(\mathbf{x}_a,\mathbf{p}_a). \quad (41)$$

Integrations involved in the 3.5PN Hamiltonian (19) are much more demanding. We start with the observation that the following integrals can be calculated immediately by means of Eq. (38) together with Eqs. (24), (26) and Eqs. (29), (33):

$$\int d^3x\, h_{(5)ij}^{\mathrm{TT}}B_{(6)ij}=\dot{\chi}_{(4)ij}(t)\int d^3x\, B_{(6)ij}^{\mathrm{STF}}=5\pi\Pi_{1ij}(\mathbf{x}_a,\mathbf{p}_a)\dot{\chi}_{(4)ij}(t), \quad (42)$$

$$\int d^3x\left(\frac{d}{dt}h_{(5)ij}^{\mathrm{TT}}\right)\phi_{(2)}\tilde{\pi}_{(3)}^{ij}=\ddot{\chi}_{(4)ij}(t)\int d^3x\,\phi_{(2)}\tilde{\pi}_{(3)}^{ij}=10\pi\Pi_{3ij}(\mathbf{x}_a,\mathbf{p}_a)\ddot{\chi}_{(4)ij}(t). \quad (43)$$

The integral over $h_{(4)ij}^{\mathrm{TT}}\,\Sigma_am_a\delta_a$ can also be easily calculated by means of Eq. (20). To do it one must remember that at this stage it is necessary to distinguish the particle positions and momenta inside and outside the TT variables. The result is

$$\int d^3x\, h_{(4)ij}^{\mathrm{TT}}(t)\sum_a m_a\delta(\mathbf{x}-\mathbf{x}_a)=-20\pi\tilde{\Pi}_{2ij}(\mathbf{x}_a,t), \quad (44)$$

where (here $s_{aa'b'}:=r_{aa'}+r_{ab'}+r_{a'b'}$ and primes denote quantities entering TT variables)

$$\widetilde{\Pi}_{2ij}(\mathbf{x}_a,t):=\frac{1}{5}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{a'}\frac{m_a}{m_{a'}r_{aa'}}\{[5(\mathbf{n}_{aa'}\cdot\mathbf{p}_{a'})^2-\mathbf{p}_{a'}^2]\delta_{ij}-2p_{a'i}p_{a'j}+[5\mathbf{p}_{a'}^2-3(\mathbf{n}_{aa'}\cdot\mathbf{p}_{a'})^2]n_{aa'}^i n_{aa'}^j-6(\mathbf{n}_{aa'}\cdot\mathbf{p}_{a'})$$
$$\times(n_{aa'}^i p_{a'j}+n_{aa'}^j p_{a'i})\}+\frac{1}{10}\left(\frac{1}{16\pi}\right)^3\sum_a\sum_{a'}\sum_{b'\neq a'}m_a m_{a'}m_{b'}\left\{\frac{32}{s_{aa'b'}}\left(\frac{1}{r_{a'b'}}+\frac{1}{s_{aa'b'}}\right)n_{a'b'}^i n_{a'b'}^j+16\left(\frac{1}{r_{a'b'}^2}\right.\right.$$
$$\left.-\frac{2}{s_{aa'b'}^2}\right)(n_{aa'}^i n_{a'b'}^j+n_{aa'}^j n_{a'b'}^i)-2\left(\frac{r_{aa'}+r_{ab'}}{r_{a'b'}^3}+\frac{12}{s_{aa'b'}^2}\right)n_{aa'}^i n_{ab'}^j+\left[\frac{r_{aa'}}{r_{a'b'}^3}\left(\frac{r_{aa'}}{r_{ab'}}+3\right)-\frac{5}{r_{a'b'}r_{aa'}}\right.$$
$$\left.\left.+\frac{8}{s_{aa'b'}}\left(\frac{1}{r_{aa'}}+\frac{1}{s_{aa'b'}}\right)\right]n_{aa'}^i n_{aa'}^j+\left[5\frac{r_{aa'}}{r_{a'b'}^3}\left(1-\frac{r_{aa'}}{r_{ab'}}\right)+\frac{17}{r_{a'b'}r_{aa'}}-\frac{4}{r_{aa'}r_{ab'}}-\frac{8}{s_{aa'b'}}\left(\frac{1}{r_{aa'}}+\frac{4}{r_{a'b'}}\right)\right]\delta_{ij}\right\}. \tag{45}$$

Now we have to perform the integration over the product $h_{(7)ij}^{\mathrm{TT}}A_{(4)ij}$. Applying Eq. (23), one can write

$$\int d^3x\; h_{(7)ij}^{\mathrm{TT}}A_{(4)ij}=[P_{1ij}(t)+P_{2ij}(t)+P_{3ij}(t)]\int d^3x\; A_{(4)ij}+\int d^3x\; Q_{ij}(\mathbf{x},t)A_{(4)ij}+\int d^3x\; R_{ij}(\mathbf{x},t)A_{(4)ij}. \tag{46}$$

The first term on the right-hand side of Eq. (46) is easy. By means of Eqs. (40), (29), (31), and (33), we have

$$[P_{1ij}(t)+P_{2ij}(t)+P_{3ij}(t)]\int d^3x\; A_{(4)ij}=10\pi[\dot{\Pi}_{1ij}(t)+\dot{\Pi}_{2ij}(t)+\ddot{\Pi}_{3ij}(t)]\chi_{(4)ij}(\mathbf{x}_a,\mathbf{p}_a). \tag{47}$$

The next two terms on the right-hand side of Eq. (46) we split using definition (8) of the $A_{(4)ij}$:

$$\int d^3x\; Q_{ij}(\mathbf{x},t)A_{(4)ij}=-\frac{1}{2}\sum_a\frac{p_{ai}p_{aj}}{m_a}\int d^3x\; Q_{ij}\delta(\mathbf{x}-\mathbf{x}_a)-\frac{1}{8}\int d^3x\; Q_{ij}\phi_{(2),i}\phi_{(2),j}, \tag{48}$$

$$\int d^3x\; R_{ij}(\mathbf{x},t)A_{(4)ij}=-\frac{1}{2}\sum_a\frac{p_{ai}p_{aj}}{m_a}\int d^3x\; R_{ij}\delta(\mathbf{x}-\mathbf{x}_a)-\frac{1}{8}\int d^3x\; R_{ij}\phi_{(2),i}\phi_{(2),j}. \tag{49}$$

These integrals in Eqs. (48) and (49) which contain δ can be computed immediately. The result is

$$-\frac{1}{2}\sum_a\frac{p_{ai}p_{aj}}{m_a}\int d^3x\; Q_{ij}(t,\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_a)=Q'_{ij}(\mathbf{x}_a,\mathbf{p}_a,t)\dot{\chi}_{ij}(t), \tag{50}$$

$$-\frac{1}{2}\sum_a\frac{p_{ai}p_{aj}}{m_a}\int d^3x\; R_{ij}(t,\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_a)=\frac{\partial^3}{\partial t^3}R'(\mathbf{x}_a,\mathbf{p}_a,t), \tag{51}$$

where

$$Q'_{ij}(\mathbf{x}_a,\mathbf{p}_a,t):=-\frac{1}{16}\left(\frac{1}{16\pi}\right)\sum_a\sum_{a'}\frac{m_{a'}}{m_a r_{aa'}}[2p_{ai}p_{aj}+12(\mathbf{n}_{aa'}\cdot\mathbf{p}_a)n_{aa'}^i p_{aj}-5\mathbf{p}_a^2 n_{aa'}^i n_{aa'}^j+3(\mathbf{n}_{aa'}\cdot\mathbf{p}_a)^2 n_{aa'}^i n_{aa'}^j], \tag{52}$$

$$R'(\mathbf{x}_a,\mathbf{p}_a,t):=\frac{2}{105}\left(\frac{1}{16\pi}\right)\sum_a\sum_{a'}\frac{r_{aa'}^2}{m_a m_{a'}}[-5\mathbf{p}_a^2\mathbf{p}_{a'}^2+11(\mathbf{p}_a\cdot\mathbf{p}_{a'})^2+4(\mathbf{n}_{aa'}\cdot\mathbf{p}_{a'})^2\mathbf{p}_a^2+4(\mathbf{n}_{aa'}\cdot\mathbf{p}_a)^2\mathbf{p}_{a'}^2$$
$$-12(\mathbf{n}_{aa'}\cdot\mathbf{p}_{a'})(\mathbf{n}_{aa'}\cdot\mathbf{p}_a)(\mathbf{p}_a\cdot\mathbf{p}_{a'})]-\frac{1}{105}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{a'}\sum_{b'\neq a'}\frac{m_{a'}m_{b'}}{m_a}\left[\left(2\frac{r_{aa'}^4}{r_{a'b'}^3}-2\frac{r_{aa'}^2 r_{ab'}^2}{r_{a'b'}^3}-5\frac{r_{aa'}^2}{r_{a'b'}}\right)\mathbf{p}_a^2\right.$$
$$\left.+4\frac{r_{aa'}^2}{r_{a'b'}}(\mathbf{n}_{aa'}\cdot\mathbf{p}_a)^2+17\left(\frac{r_{aa'}^2}{r_{a'b'}}+r_{a'b'}\right)(\mathbf{n}_{a'b'}\cdot\mathbf{p}_a)^2+2\left(6\frac{r_{aa'}^3}{r_{a'b'}^2}+17r_{aa'}\right)(\mathbf{n}_{aa'}\cdot\mathbf{p}_a)(\mathbf{n}_{a'b'}\cdot\mathbf{p}_a)\right]. \tag{53}$$

The integrals in Eqs. (48) and (49) with gradients of $\phi_{(2)}$ after integration read

$$-\frac{1}{8}\int d^3x\; Q_{ij}(\mathbf{x},t)\phi_{(2),i}\phi_{(2),j}=Q''_{ij}(\mathbf{x}_a,t)\dot{\chi}_{ij}(t), \tag{54}$$

$$-\frac{1}{8}\int d^3x\, R_{ij}(\mathbf{x},t)\,\phi_{(2),i}\phi_{(2),j}=\frac{\partial^3}{\partial t^3}\,R''(\mathbf{x}_a,t), \tag{55}$$

where (here $s_{aba'}:=r_{ab}+r_{aa'}+r_{ba'}$)

$$Q''_{ij}(\mathbf{x}_a,t):=\frac{1}{32}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}\sum_{a'} m_a m_b m_{a'}\Bigg\{\frac{32}{s_{aba'}}\left(\frac{1}{r_{ab}}+\frac{1}{s_{aba'}}\right)n^i_{ab}n^j_{ab}+\Bigg[3\,\frac{r_{aa'}}{r^3_{ab}}-\frac{5}{r_{ab}r_{aa'}}+\frac{r^2_{ba'}}{r^3_{ab}r_{aa'}}$$
$$+\frac{8}{s_{aba'}}\left(\frac{1}{r_{aa'}}+\frac{1}{s_{aba'}}\right)\Bigg]n^i_{aa'}n^j_{aa'}-2\left(\frac{r_{aa'}+r_{ba'}}{r^3_{ab}}+\frac{12}{s^2_{aba'}}\right)n^i_{aa'}n^j_{ba'}+32\left(\frac{1}{r^2_{ab}}-\frac{2}{s^2_{aba'}}\right)n^i_{ab}n^j_{aa'}\Bigg\}, \tag{56}$$

$$R''(\mathbf{x}_a,t):=\frac{1}{105}\left(\frac{1}{16\pi}\right)^2\sum_a\sum_{b\neq a}\sum_{a'}\frac{m_a m_b}{m_{a'}}\Bigg[\left(5\,\frac{r^2_{aa'}}{r_{ab}}+2\,\frac{r^2_{aa'}r^2_{ba'}}{r^3_{ab}}-2\,\frac{r^4_{aa'}}{r^3_{ab}}\right)\mathbf{p}^2_{a'}-17\left(\frac{r^2_{aa'}}{r_{ab}}+r_{ab}\right)(\mathbf{n}_{ab}\cdot\mathbf{p}_{a'})^2$$
$$-4\,\frac{r^2_{aa'}}{r_{ab}}(\mathbf{n}_{aa'}\cdot\mathbf{p}_{a'})^2-2\left(\frac{6r^3_{aa'}}{r^2_{ab}}+17r_{aa'}\right)(\mathbf{n}_{ab}\cdot\mathbf{p}_{a'})(\mathbf{n}_{aa'}\cdot\mathbf{p}_{a'})\Bigg]$$
$$+\frac{1}{210}\left(\frac{1}{16\pi}\right)^3\sum_a\sum_{b\neq a}\sum_{a'}\sum_{b'\neq a'} m_a m_b m_{a'} m_{b'}\Bigg[2\,\frac{r^2_{aa'}}{r_{ab}r^3_{a'b'}}(r^2_{aa'}-r^2_{ab'})+2\,\frac{r^2_{aa'}}{r^2_{ab}r_{a'b'}}(r^2_{aa'}-r^2_{ba'})+4\,\frac{r_{ab}r^2_{aa'}}{r^3_{a'b'}}$$
$$-5\,\frac{r^2_{aa'}}{r_{ab}r_{a'b'}}-2\left(\frac{r^3_{ab}}{r^3_{a'b'}}+\frac{r_{ab}}{r_{a'b'}}\right)-4\,\frac{r_{ab}r_{aa'}r_{bb'}}{r^3_{a'b'}}(\mathbf{n}_{aa'}\cdot\mathbf{n}_{bb'})+17\left(\frac{r_{ab}}{r_{a'b'}}+\frac{r_{a'b'}}{r_{ab}}+\frac{r^2_{aa'}}{r_{ab}r_{bb'}}\right)(\mathbf{n}_{ab}\cdot\mathbf{n}_{a'b'})^2$$
$$+6\,\frac{r^4_{aa'}}{r^2_{ab}r^2_{a'b'}}(\mathbf{n}_{ab}\cdot\mathbf{n}_{a'b'})+34r^2_{aa'}\left(\frac{1}{r^2_{ab}}+\frac{1}{r^2_{a'b'}}\right)(\mathbf{n}_{ab}\cdot\mathbf{n}_{a'b'})\Bigg]. \tag{57}$$

Using the results (42)–(57), the 3.5PN interaction Hamiltonian (19) can be written as

$$H^{\text{int}}_{3.5\text{PN}}(\mathbf{x}_a,\mathbf{p}_a,t)=5\pi\chi_{(4)ij}(\mathbf{x}_a,\mathbf{p}_a)[\dot{\Pi}_{1ij}(t)+\dot{\Pi}_{2ij}(t)+\dot{\Pi}_{3ij}(t)]+5\pi\chi_{(4)ij}(t)[\Pi_{1ij}(\mathbf{x}_a,\mathbf{p}_a)+\widetilde{\Pi}_{2ij}(\mathbf{x}_a,t)]$$
$$-5\pi\ddot{\chi}_{(4)ij}(t)\Pi_{3ij}(\mathbf{x}_a,\mathbf{p}_a)+\dot{\chi}_{(4)ij}(t)[Q'_{ij}(\mathbf{x}_a,\mathbf{p}_a,t)+Q''_{ij}(\mathbf{x}_a,t)]+\frac{\partial^3}{\partial t^3}[R'(\mathbf{x}_a,\mathbf{p}_a,t)+R''(\mathbf{x}_a,t)]. \tag{58}$$

We stress that the Hamiltonian (58) has been obtained by the aid of a well-defined post-Newtonian near-zone expansion of the retarded solution of Eq. (12) to the needed order [Eq. (23)]. Also, the integrals which represent the Hamiltonian are well defined at spacelike infinity [Eq. (19)]. Only for calculational reasons have long-range divergent integrals been employed; see the Appendixes B and C.

V. ENERGY LOSS OF A TWO-BODY SYSTEM

Let us denote by $\widetilde{H}_{\leqslant 3.5\text{PN}}$ the Hamiltonian which coincides with the Hamiltonian $H_{\leqslant 3.5\text{PN}}$ of Eq. (5) after dropping its field part. So, by definition, we have

$$\widetilde{H}_{\leqslant 3.5\text{PN}}:=H^{\text{mat}}_{\leqslant 3.5\text{PN}}+H^{\text{int}}_{\leqslant 3.5\text{PN}}. \tag{59}$$

The Hamiltonian $\widetilde{H}_{\leqslant 3.5\text{PN}}$ can be decomposed into conservative and dissipative parts

$$\widetilde{H}_{\leqslant 3.5\text{PN}}(\mathbf{x}_a,\mathbf{p}_a,t)=H^{\text{cor}}_{\leqslant 3\text{PN}}(\mathbf{x}_a,\mathbf{p}_a)+H^{\text{diss}}_{\leqslant 3.5\text{PN}}(\mathbf{x}_a,\mathbf{p}_a,t), \tag{60}$$

where

$$\text{H}^{\text{con}}_{\leqslant 3\text{PN}}:=H^{\text{mat}}_{N}+H^{\text{mat}}_{1\text{PN}}+(H^{\text{mat}}_{2\text{PN}}+H^{\text{int}}_{2\text{PN}})+(H^{\text{mat}}_{3\text{PN}}+H^{\text{int}}_{3\text{PN}}), \tag{61}$$

$$H^{\text{diss}}_{\leqslant 3.5\text{PN}}:=H^{\text{int}}_{2.5\text{PN}}+H^{\text{int}}_{3.5\text{PN}}. \tag{62}$$

The total time derivative of $\widetilde{H}_{\leqslant 3.5\text{PN}}$ is equal to its partial time derivative, and because only the dissipative part of $\widetilde{H}_{\leqslant 3.5\text{PN}}$ depends explicitly on time, we get

$$\frac{d}{dt}\widetilde{H}_{\leqslant 3.5\text{PN}}(\mathbf{x}_a,\mathbf{p}_a,t)=\frac{\partial}{\partial t}\widetilde{H}_{\leqslant 3.5\text{PN}}(\mathbf{x}_a,\mathbf{p}_a,t)$$
$$=\frac{\partial}{\partial t}H^{\text{diss}}_{\leqslant 3.5\text{PN}}(\mathbf{x}_a,\mathbf{p}_a,t). \tag{63}$$

The instantaneous energy loss of the matter system due to the gravitational wave emission is defined as

$$\mathcal{L}_{\leqslant 3.5\text{PN}}(t):=-\frac{d}{dt}\widetilde{H}_{\leqslant 3.5\text{PN}}(\mathbf{x}_a,\mathbf{p}_a,t). \tag{64}$$

The task is now to calculate $\mathcal{L}_{\leqslant 3.5\mathrm{PN}}$ for a system of two pointlike bodies with masses m_1 and m_2. We start from rewriting the right-hand side of Eq. (64) in the form [by means of Eqs. (62) and (63)]

$$\mathcal{L}_{\leqslant 3.5\mathrm{PN}} = -\frac{\partial}{\partial t}\,(H^{\mathrm{int}}_{2.5\mathrm{PN}} + H^{\mathrm{int}}_{3.5\mathrm{PN}}). \tag{65}$$

We next substitute Eqs. (41) and (58) into Eq. (65). Then we use the following relation, valid up to 1PN order, to express the particles momenta $\mathbf{p}_a$ in terms of the particle coordinate velocities $\mathbf{v}_a$ [see, e.g., Eq. (4.1) of [13]]:

$$\mathbf{p}_1 = m_1\mathbf{v}_1 + \frac{1}{2}\,m_1\mathbf{v}_1^2\mathbf{v}_1 + \frac{m_1 m_2}{32\pi r_{12}}\,[6\mathbf{v}_1 - 7\mathbf{v}_2 - (\mathbf{n}_{12}\cdot\mathbf{v}_2)\mathbf{n}_{12}], \tag{66}$$

and the analogous relation holds for $\mathbf{p}_2$. After that, we perform the time differentiation and simultaneously we eliminate the time derivatives of the coordinate velocities by means of the 1PN equations of motion [see, e.g., Eq. (1.5) of [10]]:

$$\dot{\mathbf{v}}_1 = -\frac{m_2}{16\pi r_{12}^2}\,\mathbf{n}_{12} + \frac{m_2}{16\pi r_{12}^2}\Bigg\{[4(\mathbf{n}_{12}\cdot\mathbf{v}_1) - 3(\mathbf{n}_{12}\cdot\mathbf{v}_2)] \times(\mathbf{v}_1 - \mathbf{v}_2) + \Bigg[-\mathbf{v}_1^2 - 2\mathbf{v}_2^2 + 4(\mathbf{v}_1\cdot\mathbf{v}_2) + \frac{3}{2}\,(\mathbf{n}_{12}\cdot\mathbf{v}_2)^2 + \frac{1}{16\pi r_{12}}\,(5m_1 + 4m_2)\Bigg]\mathbf{n}_{12}\Bigg\} \tag{67}$$

(we omit the analogous equation for $\dot{\mathbf{v}}_2$). In the final step, we identify positions and velocities of particles inside and outside the TT variables (i.e., we identify primed and unprimed quantities). For this goal we must apply Hadamard's regularization procedure described in Appendix A.

To write the final formula for the energy loss in a more compact way, we introduce the total mass $M := m_1 + m_2$ of the system, its reduced mass $\mu := m_1 m_2/M$, and the parameter $\nu := \mu/M$. Then the individual masses of the objects can be expressed in terms of the parameters μ and ν:

$$m_1 = \frac{\mu}{2\nu}\,(1 + \sqrt{1-4\nu}), \quad m_2 = \frac{\mu}{2\nu}\,(1 - \sqrt{1-4\nu}), \tag{68}$$

where we assumed $m_1 \geqslant m_2$. We also want to express the velocities of the bodies by their relative velocity $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$ and their relative position $\mathbf{r}$, $\mathbf{v} = \dot{\mathbf{r}}$. To do this we need the following relations linking the positions of the individual bodies with their relative position valid up to 1PN order [see, e.g., Eq. (2.4) of [14]]:

$$\mathbf{r}_1 = \left[\frac{\mu}{m_1} + \frac{\mu(m_1 - m_2)}{2M^2}\left(v^2 - \frac{M}{16\pi r}\right)\right]\mathbf{r}, \tag{69}$$

$$\mathbf{r}_2 = \left[-\frac{\mu}{m_2} + \frac{\mu(m_1 - m_2)}{2M^2}\left(v^2 - \frac{M}{16\pi r}\right)\right]\mathbf{r}, \tag{70}$$

where $r = |\mathbf{r}|$. We differentiate Eqs. (69) and (70) with respect to time, and then we eliminate the time derivatives using the Newtonian equation of the relative motion: $\dot{\mathbf{v}} = -M/(16\pi r^2)\mathbf{n}$, where $\mathbf{n} := \mathbf{r}/r$. The result is

$$\mathbf{v}_1 = \frac{2\nu\mathbf{v}}{1 + \sqrt{1-4\nu}} + \frac{1}{2}\,\sqrt{1-4\nu}\left[\nu v^2\mathbf{v} - \frac{\mu}{16\pi r}\,[\mathbf{v} + (\mathbf{n}\cdot\mathbf{v})\mathbf{n}]\right], \tag{71}$$

$$\mathbf{v}_2 = \frac{-2\nu\mathbf{v}}{1 - \sqrt{1-4\nu}} + \frac{1}{2}\,\sqrt{1-4\nu}\left[\nu v^2\mathbf{v} - \frac{\mu}{16\pi r}\,[\mathbf{v} + (\mathbf{n}\cdot\mathbf{v})\mathbf{n}]\right]. \tag{72}$$

After eliminating the coordinate velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ by means of Eqs. (71) and (72), the instantaneous energy loss $\mathcal{L}_{\leqslant 3.5\mathrm{PN}}$ can be written as

$$\begin{aligned}\mathcal{L}_{\leqslant 3.5\mathrm{PN}} = {} & \frac{4}{15}\left(\frac{1}{16\pi}\right)^2 \frac{M\mu^2}{r^3}\left\{2\left(\frac{M}{16\pi r}\right)^2 + [11v^2 - 9(\mathbf{n}\cdot\mathbf{v})^2]\,\frac{M}{16\pi r} + [11v^4 - 60(\mathbf{n}\cdot\mathbf{v})^2 v^2 + 45(\mathbf{n}\cdot\mathbf{v})^4]\right\} \\ & + \frac{1}{105}\left(\frac{1}{16\pi}\right)^2 \frac{M\mu^2}{r^3}\left\{2(97 - 52\nu)\left(\frac{M}{16\pi r}\right)^3 + 4[(-423 + 215\nu)v^2 + (377 + 254\nu)(\mathbf{n}\cdot\mathbf{v})^2]\left(\frac{M}{16\pi r}\right)^2\right. \\ & + [-(1378 + 1267\nu)v^4 + 8(2597 + 547\nu)(\mathbf{n}\cdot\mathbf{v})^2 v^2 - 3(6518 + 971\nu)(\mathbf{n}\cdot\mathbf{v})^4]\,\frac{M}{16\pi r} \\ & \left. + [-(206 + 1253\nu)v^6 + 3(380 + 2507\nu)(\mathbf{n}\cdot\mathbf{v})^2 v^4 - 15(198 + 413\nu)(\mathbf{n}\cdot\mathbf{v})^4 v^2 + 105(28 - \nu)(\mathbf{n}\cdot\mathbf{v})^6]\right\}.\end{aligned} \tag{73}$$

VI. CONCLUDING REMARKS

The comparison of expression (73) for the instantaneous gravitational energy loss with the known instantaneous far-zone-flux expression [see Eq. (51) of [15] corrected by an erratum and equation below Eq. (3.40) of [16]] shows that the two expressions are different. This is to be expected as general expressions for the instantaneous gravitational energy loss of a matter system are as well coordinate as representation dependent; e.g., the representation of the post$^{3.5}$-energy loss as $[H_{\leqslant 1\mathrm{PN}}, H_{\leqslant 3.5\mathrm{PN}}]$, where $[\cdot,\cdot]$ denotes the Poisson brackets, yields an expression which differs from both other expressions. Iyer and Will [4], particularly, would have obtained similar results if they would not have forced their constants $\mathcal{R}_1,\dots,\mathcal{R}_{10},\mathcal{S}_1,\dots,\mathcal{S}_{10}$ in Eqs. (2.16) of [4] to fit to those energy and angular momentum flux expressions in the far zone which are invariant against (infinitesimal) coordinate transformations. Averaging the formula (73) over one period of quasielliptic motion of the binary, using the method by Blanchet and Schäfer (see Sec. 4 of [16]), yields the known result for the averaged energy loss [see Eqs. (4.20) and (4.21) of [16]].

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APPENDIX A: HADAMARD'S REGULARIZATION

Let f be a real-valued function defined in a neighborhood of a point $\mathbf{x}_0 \in \mathbb{R}^3$, excluding this point. At $\mathbf{x}_0$, the function f is assumed to be singular. We define a family of auxiliary complex valued functions $f_{\mathbf{n}}$ (labeled by unit vectors $\mathbf{n}$) in the following way:

$$f_{\mathbf{n}}:\mathbb{C}\ni\varepsilon\mapsto f_{\mathbf{n}}(\varepsilon):=f(\mathbf{x}_0+\varepsilon\mathbf{n})\in\mathbb{C}. \tag{A1}$$

We expand $f_{\mathbf{n}}$ into Laurent series around $\varepsilon=0$:

$$f_{\mathbf{n}}(\varepsilon)=\sum_{m=-N}^{\infty} a_m(\mathbf{n})\varepsilon^m. \tag{A2}$$

Coefficients a_m of this expansion depend on the unit vector $\mathbf{n}$. We define the regularized value of the function f at $\mathbf{x}_0$ as the coefficient at ε^0 in the expansion (A2) averaged over all unit vectors $\mathbf{n}$:

$$f_{\mathrm{reg}}(\mathbf{x}_0):=\frac{1}{4\pi}\oint d\Omega\; a_0(\mathbf{n}). \tag{A3}$$

We use Eq. (A3) to compute all integrals which contain Dirac δ distribution. It means that we define

$$\int d^3x\; f(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}_a):=f_{\mathrm{reg}}(\mathbf{x}_a). \tag{A4}$$

APPENDIX B: RIESZ'S FORMULA

The following formula, first derived by Riesz [see Eqs. (7) and (10) in Chap. 2 of [17]], can serve as a tool to regularize a class of divergent integrals using the analytic continuation arguments (here $a\neq b$; α and β are complex numbers):

$$\int d^3x\; r_a^{\alpha} r_b^{\beta} = \pi^{3/2}\,\frac{\Gamma((\alpha+3)/2)\Gamma((\beta+3)/2)\Gamma(-(\alpha+\beta+3)/2)}{\Gamma(-\alpha/2)\Gamma(-\beta/2)\Gamma((\alpha+\beta+6)/2)} \times r_{ab}^{\alpha+\beta+3}. \tag{B1}$$

In the calculation of $h^{\mathrm{TT}}_{(7)ij}$ in Sec. III we used the following regularized values of integrals which follows directly from the above formula:

$$\int d^3x\;\frac{1}{r_a r_b}=-2\pi r_{ab},\qquad \int d^3x\;\frac{r_a}{r_b}=-\frac{\pi}{3}\,r_{ab}^3,$$

$$\int d^3x\; r_a r_b=-\frac{\pi}{45}\,r_{ab}^5.$$

APPENDIX C: INVERSE LAPLACIAN TECHNIQUE

The inverse Laplacian operator Δ^{-1} is an integral operator which is defined as

$$(\Delta^{-1}f\,)(\mathbf{x}):=-\frac{1}{4\pi}\int d^3x'\;\frac{f(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{C1}$$

Inverting the obvious identity $\Delta r_a^{\alpha}=\alpha(\alpha+1)r_a^{\alpha-2}$, one can write ($r_a^{\alpha}$ is treated here as analytic function in the complex α plane)

$$\Delta^{-1}r_a^{\alpha}=\frac{1}{(\alpha+2)(\alpha+3)}\,r_a^{\alpha+2}. \tag{C2}$$

Using the above formula and definition (4) of the TT-projection operator, we calculate the TT part of r_a^{α}:

$$\delta^{\mathrm{TT}kl}_{ij} r_a^{\alpha}=\frac{r_a^{\alpha}}{2(\alpha+3)(\alpha+5)}\{\alpha(\alpha-2)n_a^i n_a^j n_a^k n_a^l+\alpha(\alpha+6)(\delta_{ij}n_a^k n_a^l+\delta_{kl}n_a^i n_a^j)-\alpha(\alpha+4)(\delta_{ik}n_a^j n_a^l+\delta_{il}n_a^j n_a^k+\delta_{jk}n_a^i n_a^l+\delta_{jl}n_a^i n_a^k)+[(\alpha+1)(\alpha+5)+1](\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})-[(\alpha+1)(\alpha+5)-1]\delta_{ij}\delta_{kl}\}. \tag{C3}$$

Because $\Delta(1/r_a)=-4\pi\delta(\mathbf{x}-\mathbf{x}_a)$, applying several times Eq. (C2), one can easily derive that, for any positive integer n,

$$\Delta^{-n}\delta(\mathbf{x}-\mathbf{x}_a)=-\frac{1}{4\pi(2n-2)!}\,r_a^{2n-3}. \tag{C4}$$

Now we show two families of inverse Laplacians which we used to compute some integrals. The first family is

$$\Delta^{-1}\frac{1}{r_a r_b}=\ln(r_a+r_b+r_{ab}), \tag{C5}$$

$$\Delta^{-2}\frac{1}{r_a r_b}=\frac{1}{36}(-r_a^2+3r_a r_{ab}+r_{ab}^2-3r_a r_b+3r_{ab}r_b-r_b^2)+\frac{1}{12}(r_a^2-r_{ab}^2+r_b^2)\ln(r_a+r_{ab}+r_b), \tag{C6}$$

$$\begin{aligned}\Delta^{-3}\frac{1}{r_a r_b}=&\frac{1}{28800}(-63r_a^4+150r_a^3 r_{ab}+126r_a^2 r_{ab}^2-90r_a r_{ab}^3-63r_{ab}^4-90r_a^3 r_b+90r_a^2 r_{ab}r_b+90r_a r_{ab}^2 r_b-90r_{ab}^3 r_b-2r_a^2 r_b^2\\&+90r_a r_{ab}r_b^2+126r_{ab}^2 r_b^2-90r_a r_b^3+150r_{ab}r_b^3-63r_b^4)+\frac{1}{960}(3r_a^4-6r_a^2 r_{ab}^2+3r_{ab}^4+2r_a^2 r_b^2-6r_{ab}^2 r_b^2+3r_b^4)\\&\times\ln(r_a+r_{ab}+r_b),\end{aligned} \tag{C7}$$

and the second one reads

$$\Delta^{-1}\frac{r_a^2}{r_b}=-\frac{1}{6}r_b^3+\frac{1}{4}(r_a^2+r_{ab}^2)r_b, \tag{C8}$$

$$\Delta^{-1}\frac{r_a^4}{r_b}=\frac{4}{45}r_b^5-\frac{2}{9}(r_a^2+r_{ab}^2)r_b^3+\frac{1}{6}(r_a^4+r_a^2 r_{ab}^2+r_{ab}^4)r_b, \tag{C9}$$

$$\Delta^{-1}(r_a^2 r_b)=\frac{1}{180}(10r_a^2+5r_{ab}^2-4r_b^2)r_b^3. \tag{C10}$$

Applying a Laplacian operator (maybe several times) to the right-hand sides of Eqs. (C5)–(C10), one can immediately check that we obtain the correct arguments of the inverse Laplacian operators from the left-hand sides of these equations. Moreover, the right-hand sides of Eqs. (C5)–(C10) are unique in the following sense. Let us, for example, look for the double inverse Laplacian of $1/(r_a r_b)$, Eq. (C6). It means that we are looking for a solution f of the partial differential equation $\Delta^2 f=1/(r_a r_b)$. Let us now restrict ourselves to the f belonging only to the family of functions of type $W_1(r_a,r_b,r_{ab})+W_2(r_a,r_b,r_{ab})\ln(r_a+r_b+r_{ab})$, where W_1 and W_2 are polynomials of indicated variables consisting only of quadratic terms in these variables (i.e., the most general polynomial of this type is $a_1 r_a^2+a_2 r_b^2+a_3 r_{ab}^2+a_4 r_a r_b+a_5 r_a r_{ab}+a_6 r_b r_{ab}$). This particular form can be guessed easily from the dimensional reasons if we already know the first inverse Laplacian of $1/(r_a r_b)$, Eq. (C5). So we assume we know the general form of f, all we need is to fix the values of 12 coefficients (in fact, less because of symmetry with respect to the interchanging the labels a and b of the particles), defining the polynomials W_1 and W_2. To do this we require that the following equations must be satisfied: (1) $\Delta^2 f=1/(r_a r_b)$, (2) $\Delta f=\ln(r_a+r_b+r_{ab})$, and (3) $\Delta_a f=r_a/(2r_{ab})$, where Δ_a contains differentiations with respect to $\mathbf{x}_a$. The first two requirements are obvious. The third one follows from applying the operator $\Delta^{-2}\Delta_a$ to both sides of the equation $\Delta^2 f=1/(r_a r_b)$ (in the general, nonsymmetric case, e.g., that of Eqs. (C8)–(C10), we also need the requirement which follows from applying the operator $\Delta^{-n}\Delta_b$). The requirements (1)–(3) fix the 12 coefficients of polynomials W_1 and W_2 uniquely. A similar reasoning can be applied to all inverse Laplacians from Eqs. (C5)–(C10).

As an example of how inverse Laplacians can be used to compute divergent integrals, let us consider the following integral, needed to compute, e.g., the integral on the left-hand side of Eq. (55) (here $b\neq a$ and $c\neq a,b$):

$$\begin{aligned}\int d^3x\, r_a^2\left(\frac{1}{r_b}\right)_{,i}\left(\frac{1}{r_c}\right)_{,j}&=\partial_i^{(b)}\partial_j^{(c)}\int d^3x\,\frac{r_a^2}{r_b r_c}\\&=-4\pi\partial_i^{(b)}\partial_j^{(c)}\left(\Delta_c^{-1}\frac{r_a^2}{r_b}\right)\\&=-4\pi\partial_i^{(b)}\partial_j^{(c)}\left[-\frac{1}{6}r_{bc}^3+\frac{1}{4}(r_{ac}^2+r_{ab}^2)r_{bc}\right],\end{aligned}$$

where Eq. (C8) was employed and $\partial_i^{(b)}$ stands for differentiation with respect to x_b^i.

[1] J. H. Taylor, Class. Quantum Grav. **10**, S167 (1993); A. Wolszczan, *ibid.* **11**, A227 (1994).
[2] C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, É. E. Flanagan, D. Kennefick, D. M. Marković, A. Ori, E. Poisson, G. J. Sussman, and K. S. Thorne, Phys. Rev. Lett. **70**, 2984 (1993).
[3] B. R. Iyer and C. M. Will, Phys. Rev. Lett. **70**, 113 (1993).
[4] B. R. Iyer and C. M. Will, Phys. Rev. D **52**, 6882 (1995).
[5] L. Blanchet, Phys. Rev. D **47**, 4392 (1993).
[6] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962), p. 227.
[7] T. Ohta, H. Okamura, T. Kimura, and K. Hiida, Prog. Theor. Phys. **51**, 1598 (1974).

[8] G. Schäfer, Ann. Phys. (N.Y.) **B161**, 81 (1985).
[9] G. Schäfer, ''Post-Newtonian approximations and equations of motion of general relativity,'' report, Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1997 (unpublished).
[10] T. Damour and G. Schäfer, Nuovo Cimento B **101**, 127 (1988).
[11] G. Schäfer, in *5th Marcel Grossmann Meeting on General Relativity*, Proceedings of the Meeting held at the University of Western Australia, Perth, edited by D. G. Blair and M. J. Buckingham (World Scientific, Singapore, 1989), Pt. A, p. 467.
[12] G. Schäfer, in *Symposia Gaussiana*, Proceedings of the 2nd Gauss Symposium, Conference A: Mathematical and Theoretical Physics, Munich, edited by M. Behara, R. Fritsch, and R. G. Lintz (Walter de Gruyter, Berlin, 1995), p. 667.
[13] T. Ohta, H. Okamura, T. Kimura, and K. Hiida, Prog. Theor. Phys. **51**, 1220 (1974).
[14] T. Damour and N. Deruelle, Ann. Inst. Henri Poincaré Phys. Théor. **43**, 107 (1985).
[15] R. V. Wagoner and C. M. Will, Astrophys. J. **210**, 764 (1976); **215**, 984(E) (1977).
[16] L. Blanchet and G. Schäfer, Mon. Not. R. Astron. Soc. **239**, 845 (1989).
[17] M. Riesz, Acta Math. **81**, 1 (1949).