Post-decoherence density matrix propagator for quantum Brownian motion

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Using the path integral representation of the density matrix propagator of quantum Brownian motion, we derive its asymptotic form for times greater than the so-called localization time $(\hbar/\gamma kT)^{1/2}$, where γ is the dissipation and T the temperature of the thermal environment. The localization time is typically greater than the decoherence time, but much shorter than the relaxation time γ^{-1}. We use this result to show that the reduced density operator rapidly evolves into a state which is approximately diagonal in a set of generalized coherent states. We thus reproduce, using a completely different method, a result we previously obtained using the quantum state diffusion picture [Phys. Rev. D **52**, 7294 (1995)]. We also go beyond this earlier result, in that we derive an explicit expression for the weighting of each phase space localized state in the approximately diagonal density matrix, as a function of the initial state. For sufficiently long times it is equal to the Wigner function, and we confirm that the Wigner function is positive for times greater than the localization time (multiplied by a number of order 1). [S0556-2821(97)05008-X]

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I. INTRODUCTION

One of the simplest open systems that is amenable to straightforward analysis is the quantum Brownian motion model. This model consists of a nonrelativistic point particle, possibly in a potential, coupled to a bath of harmonic oscillators in a thermal state. The quantum Brownian motion model has been used very extensively in studies of decoherence and emergent classicality (see, for example, Refs. [1–8]).

In the simplest case of a free particle of mass m in a high-temperature bath, with negligible dissipation the master equation for the reduced density matrix $\rho(x,y)$ of the point particle is

$$\frac{\partial\rho}{\partial t}=\frac{i\hbar}{2m}\left(\frac{\partial^2\rho}{\partial x^2}-\frac{\partial^2\rho}{\partial y^2}\right)-\frac{1}{2}\,a^2(x-y)^2\rho, \tag{1.1}$$

where $a^2=4m\gamma kT/\hbar^2$. (More general forms of this equation, together with the derivations of it may be found in many places. See, for example, Refs. [2,5,9].)

One of the most important properties of Eq. (1.1) (and also its more general forms) is that the density operator tends to become approximately diagonal in both position and momentum after a short time. This has been seen in numerical solutions and in the evolution of particular types of initial states for which analytic solution is possible [6–8,10–15].

A more precise demonstration of this statement was given in Ref. [16] by appealing to an alternative description of open systems known as the quantum state diffusion picture [17–21]. In that picture, the density operator ρ satisfying Eq. (1.1) is regarded as a mean over a distribution of pure state density operators:

$$\rho=M|\psi\rangle\langle\psi|, \tag{1.2}$$

where M denotes the mean (defined below), with the pure states evolving according to a nonlinear stochastic Langevin-Ito equation, which for the model of this paper is

$$|d\psi\rangle=-\frac{i}{\hbar}H|\psi\rangle dt-\frac{1}{2}(L-\langle L\rangle)^2|\psi\rangle dt+(L-\langle L\rangle)|\psi\rangle d\xi(t) \tag{1.3}$$

for the normalized state vector $|\psi\rangle$, where $H=p^2/2m$ and $L=a\hat{x}$. Here, $d\xi$ is a complex differential random variable representing a complex Wiener process. The linear and quadratic means are

$$M[d\xi d\xi^*]=dt,\quad M[d\xi d\xi]=0,\quad M[d\xi]=0. \tag{1.4}$$

The appeal of this picture is that the solutions to the stochastic Eq. (1.3) appear to describe the expected behavior of an individual history of the system and have been seen to correspond to single runs of laboratory experiments. For example, for the quantum Brownian motion model, the solutions tend to phase space localized states of constant width whose centers undergo classical Brownian motion [16,20,22–24]. The time scale of this process, the localization time, is at slowest of order $(\hbar/\gamma kT)^{1/2}$, which is the time scale on which the thermal fluctuations overtake the quantum fluctuations [25–27]. For an initial superposition of localized states a distance l apart, localization initially proceeds on a much shorter time scale, of order $\hbar^2/(l^2m\gamma kT)$ (which is often called the decoherence time [8,14]), thereafter going over to the slower time scale above.

It is important to note, however, that the quantum state diffusion picture is a purely *phenomenological* picture of open systems. It is *not* a fundamental modification of the Schrödinger equation, since the whole system (the distinguished system together with its environment) still evolves according to the standard rules of unitary evolution. Justification of this phenomenological view is ultimately a matter of experiment, and as indicated above, experiment appears to confirm it. There is, furthermore, theoretical evidence in favor of this phenomenological view. The quantum state diffusion picture has been shown [21] to coincide, at least in its intuitive picture and physical predictions, with a more fundamental formulation of quantum theory, namely the decoherent histories approach [28–30].

For us, the interesting feature of the quantum state diffusion picture is the purely mathematical connection between the stochastic Eq. (1.3) and the master equation (1.1). The solutions to Eq. (1.3) permit one to deduce some useful information about the form of the density operator on time scales greater than the localization time. Given a set of localized phase space solutions $|\psi_{pq}\rangle$, the density operator may be reconstructed via Eq. (1.2). This, it may be shown [16], may be written explicitly as

$$\rho=\int dp\,dq\,f(p,q,t)|\Psi_{pq}\rangle\langle\Psi_{pq}|. \tag{1.5}$$

Here, $f(p,q,t)$ is a non-negative, normalized solution to the Fokker-Planck equation describing the classical Brownian motion undergone by the centers of the stationary solutions. This is therefore an explicit, albeit indirect, demonstration of the approach to approximately phase space diagonal form on short time scales.

The above demonstration was described by us in detail in Ref. [16]. However, we were not able to deduce an explicit form for the function $f(p,q,t)$ using the quantum state diffusion picture. That is, we know that it is a solution to the Fokker-Planck equation, but it was not clear how to pick out the particular solution corresponding to a particular initial density operator. Intuitively, it is clear that $f(p,q,t)$ is something like the Wigner function of the initial state, coarse grained sufficiently to make it positive, evolved forwards in time, and with the interference terms thrown away. We would like to be able to show this explicitly.

The aim of the present paper is to derive the form (1.5) for times greater than the localization time directly from the path integral representation of the density matrix propagator corresponding to Eq. (1.1), without using the quantum state diffusion picture used in Ref. [16]. As we shall see, this derivation has the advantage over Ref. [16] that it gives an explicit expression for $f(p,q,t)$. In particular, we shall show that $f(p,q,t)$ coincides with the Wigner function $W_t(p,q)$ of the density operator at time t, for sufficiently large times.

We will not use the stochastic equation (1.3) in this paper, and indeed, our results could easily have been described without mentioning quantum state diffusion at all. It is mentioned here only to contrast the present paper with our previous one, Ref. [16], and to give some mathematical insight into the origin of our path integral method.

II. THE DENSITY MATRIX PROPAGATOR

The solution to the master equation (1.1) may be written in terms of the propagator J:

$$\rho_t(x,y)=\int dx_0 dy_0 J(x,y,t|x_0,y_0,0)\rho_0(x_0,y_0) \tag{2.1}$$

(see, for example, Refs. [2,27] for further details of the quantum Brownian motion model). The propagator may be given in general by a path integral expression, which for the particular case considered here is

$$J(x_f,y_f,t|x_0,y_0,0)=\int \mathcal{D}x\mathcal{D}y\ \exp\left(\frac{im}{2\hbar}\int dt(\dot{x}^2-\dot{y}^2) - \frac{a^2}{2}\int dt(x-y)^2\right). \tag{2.2}$$

This is readily evaluated, with the result

$$J(x_f,y_f,t|x_0,y_0,0)=\exp\left(\frac{im}{2\hbar t}[(x_f-x_0)^2-(y_f-y_0)^2] - \frac{a^2t}{6}[(x_f-y_f)^2+(x_f-y_f)(x_0-y_0) + (x_0-y_0)^2]\right). \tag{2.3}$$

(For convenience we will ignore prefactors in what follows. They may be recovered where required by appropriate normalizations.)

The main result of the present paper comes from the simple observation that the real part of the exponent in the path integral (2.2) may be written

$$\exp\left(-\frac{a^2}{2}\int dt(x-y)^2\right)=\int \mathcal{D}\bar{x}\exp\left(-a^2\int dt(x-\bar{x})^2 - a^2\int dt(y-\bar{x})^2\right). \tag{2.4}$$

The path integral representation of the propagator may, therefore, be written

$$J(x_f,y_f,t|x_0,y_0,0)=\int \mathcal{D}\bar{x}K_{\bar{x}}(x_f,t|x_0,0)K^*_{\bar{x}}(y_f,t|y_0,0), \tag{2.5}$$

where

$$K_{\bar{x}}(x_f,t|x_0,0)=\int \mathcal{D}x\ \exp\left(\frac{im}{2\hbar}\int dt\ \dot{x}^2 - a^2\int dt(x-\bar{x})^2\right). \tag{2.6}$$

For a pure initial state $\rho_0(x,y)=\Psi_0(x)\Psi_0^*(y)$, the density operator at time t may, therefore, be written

$$\rho_t(x,y)=\int \mathcal{D}\bar{x}\Psi_{\bar{x}}(x,t)\Psi^*_{\bar{x}}(y,t), \tag{2.7}$$

where the (unnormalized) wave function $\Psi_{\bar{x}}$ is given by

$$\Psi_{\bar{x}}(x_f,t)=\int dx_0 K_{\bar{x}}(x_f,t|x_0,0)\Psi_0(x_0). \tag{2.8}$$

(Wave functions of this type often appear in discussions of systems undergoing continuous measurement [31–34].)

Our strategy is to first evaluate the quantity $K_{\bar{x}}$, examine is asymptotic form for times greater than the localization time, and then use it to reconstruct the density matrix propagator J. The reason we expect this to yield the desired result is that up to normalization factors and ignoring the fact that

$\overline{x}$ is real not complex, Eq. (2.8) is essentially the solution to the Langevin-Ito equation (1.3), so the phase space localization effect should be visible in its long time limit. Moreover, Eq. (2.7) is the analogue of Eq. (1.2) or (1.5), so by reorganizing the functional integral over $\overline{x(t)}$, we might reasonably expect to derive Eq. (1.5).

The path integral (2.6) is essentially the same as that for a harmonic oscillator coupled to an external source, with the complication that the frequency is complex. The path integral is therefore readily carried out (see Ref. [35], for example), with the result

$$K_{\overline{x}}(x_f,t|x_0,0)=N\,\exp\left(\frac{i}{\hbar}\,c_1(x_f^2+x_0^2)+\frac{i}{\hbar}\,c_2x_fx_0+c_3x_f+c_4x_0+c_5-a^2\int_0^t ds\overline{x^2}(s)\right), \tag{2.9}$$

where

$$c_1=\frac{m\omega\,\cos\omega t}{2\,\sin\omega t}, \tag{2.10}$$

$$c^2=-\frac{m\omega}{\sin\omega t}, \tag{2.11}$$

$$c_3=\frac{2a^2}{\sin\omega t}\int_0^t ds\overline{x(s)}\sin\omega s, \tag{2.12}$$

$$c_4=\frac{2a^2}{\sin\omega t}\int_0^t ds\overline{x(s)}\sin\omega(t-s), \tag{2.13}$$

$$c_5=\frac{4i\hbar a^2}{m\omega\,\sin\omega t}\int_0^t ds\int_0^s ds'\overline{x(s)}\,\overline{x(s')}\sin\omega(t-s)\sin\omega s'. \tag{2.14}$$

Here $\omega=\alpha(1-i)$ and

$$\alpha=\left(\frac{\hbar a^2}{4m}\right)^{1/2}=\left(\frac{\gamma kT}{\hbar}\right)^{1/2}. \tag{2.15}$$

The time scale of evolution according to Eq. (2.9) is, therefore, α^{-1}, which coincides with the localization time discussed in Ref. [16]. The asymptotic properties of $K_{\overline{x}}$ are now easily seen. As $t\to\infty$, $c_2\to 0$, and $c_1\to\frac{1}{2}m\alpha(1+i)$ like $e^{-\alpha t}$. Since $c_2\to 0$, the propagator $K_{\overline{x}}$ factors into a product of functions of x_0 and x_f. The wave function (2.8) therefore ''forgets'' its initial conditions and becomes proportional to a Gaussian of the form

$$\exp\left(\frac{i}{\hbar}\,c_1x_f^2+c_3x_f\right) \tag{2.16}$$

on a time scale α^{-1}. This is in complete agreement with the quantum state diffusion picture analysis of Refs. [16,20].

Now introduce

$$\overline{q}=\frac{\hbar}{m\alpha}\,\mathrm{Re}c_3,\quad \overline{p}=\hbar(\mathrm{Re}c_3+\mathrm{Im}c_3). \tag{2.17}$$

Then the Gaussian may be written

$$\exp\left(\frac{i}{\hbar}\,c_1x^2+c_3x\right)=\exp\left(-\frac{m\alpha}{2\hbar}\,(1-i)(x-\overline{q})^2+\frac{i}{\hbar}\,\overline{px}+\frac{m\alpha}{2\hbar}\,(1-i)\overline{q^2}\right)\equiv\langle x|\Psi_{\overline{pq}}\rangle\exp\left(\frac{m\alpha}{2\hbar}(1-i)\overline{q}^{\,2}\right). \tag{2.18}$$

The propagator $K_{\overline{x}}$ therefore has the form

$$K_{\overline{x}}(x_f,t|x_0,0)=N\langle x_f|\Psi_{\overline{pq}}\rangle\exp\left(\frac{m\alpha}{2\hbar}(1-i)\overline{q^2}\right)\times\exp\left(\frac{i}{\hbar}\,c_1x_0^2+c_4x_0+c_5-a^2\int_0^t ds\overline{x}^{\,2}(s)\right). \tag{2.19}$$

The generalized coherent states $|\Psi_{\overline{pq}}\rangle$ depend on $\overline{x(t)}$ only through $\overline{p}$ and $\overline{q}$, which are functionals of $\overline{x(t)}$. They are close to minimal uncertainty states, satisfying $\Delta p\Delta q=\hbar/\sqrt{2}$. [20,16].

The desired form of the propagator is now obtained by inserting Eq. (2.19) in Eq. (2.5), but reorganizing the functional integral over $\overline{x(t)}$ into ordinary integrations over $\overline{p}$ and $\overline{q}$ and functional integrations over remaining parts of $\overline{x(t)}$. This may be achieved by writing the functional integral over $\overline{x(t)}$ as

$$\int \mathcal{D}\overline{x}=\int dpdq\int \mathcal{D}\overline{x}\,\delta(p-\overline{p})\,\delta(q-\overline{q}) \tag{2.20}$$

with $\overline{p}$ and $\overline{q}$ given in terms of $\overline{x}$ by Eq. (2.17). We thus obtain

$$\begin{aligned}J(x_f,y_f,t|x_0,y_0,0)&=\int dpdq\int \mathcal{D}\overline{x}\,\delta(p-\overline{p})\,\delta(q-\overline{q})\langle x_f|\Psi_{pq}\rangle\langle\Psi_{pq}|y_f\rangle\\&\quad\times\exp\left(\frac{m\alpha}{\hbar}\,q^2\right)\exp\left(\frac{i}{\hbar}\,c_1x_0^2-\frac{i}{\hbar}\,c_1^*y_0^2+c_4x_0+c_4^*y_0\right)\\&\quad\times\exp\left(c_5+c_5^*-2a^2\int_0^t ds\overline{x}^{\,2}(s)\right).\end{aligned} \tag{2.21}$$

This may be written

$$J(x_f,y_f,t|x_0,y_0,0)=\int dpdqf(p,q,t|x_0,y_0)\langle x_f|\Psi_{pq}\rangle\times\langle\Psi_{pq}|y_f\rangle, \tag{2.22}$$

where

$$\begin{aligned}f(p,q,t|x_0,y_0)&=\int \mathcal{D}\overline{x}\,\delta(p-\overline{p})\,\delta(q-\overline{q})\exp\left(\frac{m\alpha}{\hbar}\,q^2\right)\\&\quad\times\exp\left(\frac{i}{\hbar}\,c_1x_0^2-\frac{i}{\hbar}\,c_1^*y_0^2+c_4x_0+c_4^*y_0\right)\\&\quad\times\exp\left(c_5+c_5^*-2a^2\int_0^t ds\overline{x^2}(s)\right).\end{aligned} \tag{2.23}$$

We have clearly cast the result in the desired form. Folding an arbitrary initial state into the expression for the density matrix propagator (2.22), we obtain an expression of the desired form (1.5), where $f(p,q,t)$ is given explicitly by

$$f(p,q,t)=\int dx_0 dy_0 f(p,q,t|x_0,y_0)\rho_0(x_0,y_0). \quad (2.24)$$

This is our first main result.

III. THE PHASE SPACE DISTRIBUTION FUNCTION

It remains to evaluate the path integral expression (2.23). To do this first notice that Eq. (2.23) may be written

$$\begin{aligned} f(p,q,t|x_0,y_0)=&\exp\left(\frac{i}{\hbar}c_1 x_0^2-\frac{i}{\hbar}c_1^* y_0^2+\frac{m\alpha}{\hbar}q^2\right)\int dk dk' \\ &\times\exp\left(\frac{i}{\hbar}kp+\frac{i}{\hbar}k'q\right)\int \mathcal{D}\overline{x} \\ &\times\exp\left(-\frac{i}{\hbar}k\overline{p}-\frac{i}{\hbar}k'\overline{q}+c_4 x_0+c_4^* y_0\right) \\ &\times\exp\left(c_5+c_5^*-2a^2\int_0^t ds\overline{x^2}(s)\right). \end{aligned} \quad (3.1)$$

The functional integral over $\overline{x}$ is a Gaussian, since c_5 is quadratic in $\overline{x}$ and $\overline{p}$, $\overline{q}$, and c_4 are linear in $\overline{x}$, but it involves inverting the functional matrix contain in the last exponential in Eq. (3.1), which does not look particularly easy. However, we are saved from having to do this calculation by the following observation. From Eq. (2.5) and Eq. (2.9) (for $\alpha t \gg 1$), we see that

$$\begin{aligned} &J(x_f,y_f,t|x_0,y_0,0) \\ &=\exp\left(\frac{i}{\hbar}c_1(x_f^2+x_0^2)-\frac{i}{\hbar}c_1^*(y_f^2+y_0^2)\right) \\ &\quad\times\int \mathcal{D}\overline{x}\exp(c_3x_f+c_3^*y_f+c_4x_0+c_4^*y_0) \\ &\quad\times\exp\left(c_5+c_5^*-2a^2\int_0^t ds\overline{x^2}(s)\right). \end{aligned} \quad (3.2)$$

This functional integral over $\overline{x}$ in this expression is very similar in form to Eq. (3.1) but we already know what the answer is: it is Eq. (2.3). In particular, equating (3.2) and (2.3), we obtain

$$\begin{aligned} &\int \mathcal{D}\overline{x}\exp\Big(c_3x_f+c_3^*y_f+c_4x_0+c_4^*y_0+c_5+c_5^* \\ &\quad -2a^2\int_0^t ds\overline{x^2}(s)\Big) \\ &=\exp\Big(\frac{im}{2\hbar t}[(x_f-x_0)^2-(y_f-y_0)^2] \\ &\quad -\frac{a^2t}{6}[(x_f-y_f)^2+(x_f-y_f)(x_0-y_0)+(x_0-y_0)^2]\Big) \\ &\quad\times\exp\left(-\frac{i}{\hbar}c_1(x_f^2+x_0^2)+\frac{i}{\hbar}c_1^*(y_f^2+y_0^2)\right). \end{aligned} \quad (3.3)$$

Now the point is that the formula (3.3) is true for *arbitrary*, x_f,y_f. In particular, using Eq. (2.17), we see that

$$c_3x_f+c_3^*y_f=\frac{m\alpha}{\hbar}[x_f+y_f-i(x_f-y_f)]\overline{q}+\frac{i}{\hbar}\overline{p}(x_f-y_f). \quad (3.4)$$

Hence the functional integral (3.3) is exactly the same as the one appearing in Eq. (3.1) if, in Eq. (3.3), we make the substitutions

$$(x_f-y_f)\rightarrow -k, \quad \frac{m\alpha}{\hbar}[x_f+y_f-i(x_f-y_f)]\rightarrow -\frac{i}{\hbar}k'. \quad (3.5)$$

Inverting for x_f and y_f, we therefore find that the functional integral over $\overline{x}(t)$ in Eq. (3.1) is equal to the right-hand side of Eq. (3.3) with

$$x_f=-\frac{(1+i)}{2}k-\frac{i}{2m\alpha}k', \quad (3.6)$$

$$y_f=\frac{(1-i)}{2}k-\frac{i}{2m\alpha}k'. \quad (3.7)$$

Using this result and changing variables from k' to $K=k+k'/m\alpha$ in Eq. (3.1), we obtain

$$\begin{aligned} f(p,q,t|x_0,y_0)=&\exp\left(\frac{m\alpha}{\hbar}q^2+i\frac{mX_0\xi_0}{\hbar t}-\frac{a^2t}{6}\xi_0^2\right) \\ &\times\int dk dK\exp\Big[-\left(\frac{a^2t}{6}-\frac{m\alpha}{4\hbar}\right)k^2 \\ &-\frac{m\alpha}{4\hbar}K^2+\left(\frac{m\alpha}{2\hbar}-\frac{m}{2\hbar t}\right)kK\Big] \\ &\times\exp\left[\frac{i}{\hbar}k\left(p-m\alpha q+\frac{mX_0}{t}-i\frac{\hbar a^2t}{6}\xi_0\right)\right] \\ &\times\exp\left[\frac{i}{\hbar}K\left(m\alpha q+i\frac{m\xi_0}{2t}\right)\right], \end{aligned} \quad (3.8)$$

where $X_0=\frac{1}{2}(x_0+y_0)$, $\xi_0=x_0-y_0$. This may now be evaluated.

An alternative way of writing Eq. (3.8) is to carry out the same steps, but to change variables in Eq. (3.1) from k,k' to x_f,y_f, with the formal result

$$\begin{aligned} f(p,q,t|x_0,y_0)=&\int dx_f dy_f \\ &\times\exp\left(\frac{m\alpha}{2\hbar}(1-i)(x_f-q)^2-\frac{i}{\hbar}px_f\right) \\ &\times\exp\left(\frac{m\alpha}{2\hbar}(1+i)(y_f-q)^2+\frac{i}{\hbar}py_f\right) \\ &\times J(x_f,y_f,t|x_0,y_0,0). \end{aligned} \quad (3.9)$$

Folding in the initial state via Eq. (2.24), we obtain

$$f(p,q,t)=\int dx_f dy_f \times\exp\left(\frac{m\alpha}{2\hbar}(1-i)(x_f-q)^2-\frac{i}{\hbar}px_f\right) \times\exp\left(\frac{m\alpha}{2\hbar}(1+i)(y_f-q)^2+\frac{i}{\hbar}py_f\right) \times\rho_t(x_f,y_f), \tag{3.10}$$

which has the appearance of a formal inversion of the relation (1.5).

Because the coordinate transformation (3.6), (3.7) is complex some attention to the integration contour is necessary. In particular, k and k' are integrated along the real axis, therefore x_f+y_f is integrated along a purely imaginary contour and x_f-y_f along a real contour. More precisely, let $X=\frac{1}{2}(x_f+y_f)$ and $\xi=x_f-y_f$. Then Eq. (3.9) becomes

$$f(p,q,t|x_0,y_0)=\int_{-i\infty}^{i\infty}dX\int_{-\infty}^{+\infty}d\xi\,\exp\left[\frac{m\alpha}{\hbar}\left((X-q)^2+\frac{\xi^2}{4}\right)-\frac{i}{\hbar}m\alpha\xi(X-q)-\frac{i}{\hbar}p\xi\right]\times J\left(X+\frac{\xi}{2},X-\frac{\xi}{2},t\middle|x_0,y_0,0\right). \tag{3.11}$$

Explicitly, this integral reads

$$f(p,q,t|x_0,y_0)=\int_{-i\infty}^{i\infty}dX\int_{-\infty}^{+\infty}d\xi\,\exp\left[\frac{m\alpha}{\hbar}\left((X-q)^2+\frac{\xi^2}{4}\right)-\frac{i}{\hbar}m\alpha\xi(X-q)-\frac{i}{\hbar}p\xi\right]\times\exp\left(\frac{im}{\hbar t}(X-X_0)(\xi-\xi_0)-\frac{2m\alpha^2t}{3\hbar}(\xi^2+\xi\xi_0+\xi_0^2)\right), \tag{3.12}$$

where X_0 and ξ_0 defined in the same way as X and ξ. The X integral will clearly converge since the contour is along the imaginary axis, and the ξ integral will converge for sufficiently large αt.

Letting $X\to X+q$, the integral over X is readily carried out, with the result

$$f(p,q,t|x_0,y_0)=\int d\xi\,\exp\left(-\frac{i}{\hbar}p\xi+\frac{im}{\hbar t}(q-X_0)(\xi-\xi_0)-\frac{2m\alpha^2t}{3\hbar}(\xi^2+\xi\xi_0+\xi_0^2)\right)\times\exp\left\{\frac{m\alpha}{4\hbar}\left[\xi^2+\left(\xi-\frac{(\xi-\xi_0)}{\alpha t}\right)^2\right]\right\}. \tag{3.13}$$

The integral over ξ may now be evaluated but it is not necessary to do this, since the form of the answer is now clear. For $\alpha t\gg 1$, the terms in the second exponential are negligible compared to the similar terms in the first. Furthermore, the remaining terms have the form of the Wigner transform of the propagator [27,36]. We thus have the simple result

$$f(p,q,t|x_0,y_0)\approx\int d\xi\,\exp[-(i/\hbar)p\xi]J(q,\xi,t|X_0,\xi_0,0). \tag{3.14}$$

Attaching an arbitrary initial density matrix, it then follows from Eq. (2.24) that

$$f(p,q,t)\approx\int d\xi\,\exp[-(i/\hbar)p\xi]\rho_t\left(q+\frac{1}{2}\xi,q-\frac{1}{2}\xi\right)=W_t(p,q). \tag{3.15}$$

That is, for $\alpha t\gg 1$, $f(p,q,t)$ is the Wigner function of the density operator at time t. This is the second main result of this paper.

From any of the above representations of $f(p,q,t)$ [other than Eq. (3.14)], or from Ref. [16], it is straightforward to show that $f(p,q,t)$ obeys the Fokker-Planck equation

$$\frac{\partial f}{\partial t}=-\frac{p}{m}\frac{\partial f}{\partial q}+2m\gamma kT\frac{\partial^2 f}{\partial p^2}+(2\hbar\gamma kT)^{1/2}\frac{\partial^2 f}{\partial p\partial q}+\frac{\hbar}{2m}\frac{\partial^2 f}{\partial q^2}. \tag{3.16}$$

As we have seen, $f(p,q,t)$ approaches the Wigner function $W_t(p,q)$ for $\alpha t\gg 1$, which obeys the Fokker-Planck equation of classical Brownian motion:

$$\frac{\partial W}{\partial t}=-\frac{p}{m}\frac{\partial W}{\partial q}+2m\gamma kT\frac{\partial^2 W}{\partial p^2}. \tag{3.17}$$

What happens is that the last two terms in Eq. (3.16) become negligible for large αt, as may be seen by studying the Wigner function propagator (below).

IV. THE POSITIVITY OF THE WIGNER FUNCTION

We have shown that the density operator approaches the form (1.5), where $f(p,q,t)$ is given by the Wigner function. However, $f(p,q,t)$ is by construction positive, yet the Wigner function is not guaranteed to be positive in general [36]. What happens is that the Wigner function becomes strictly non-negative after a period of time, under evolution according to (the Wigner transform of) Eq. (1.1), as we now show.

The Wigner transform of the relation (2.1) yields

$$W_t(p,q)=\int dp_0dq_0K(p,q,t|p_0,q_0,0)W_0(p_0,q_0), \tag{4.1}$$

where $K(p,q,t|p_0,q_0,0)$ is the Wigner function propagator, and is given by [27]

$$K(p,q,t|p_0,q_0,0)=\exp\left[-\mu(p-p_0)^2-\nu\left(q-q_0-\frac{p_0t}{m}\right)^2+\sigma(p-p_0)\left(q-q_0-\frac{p_0t}{m}\right)\right], \tag{4.2}$$

where, introducing $D=2m\gamma kT$,

$$\mu=\frac{1}{Dt},\quad \nu=\frac{3m^2}{Dt^3},\quad \sigma=\frac{3m}{Dt^2}. \tag{4.3}$$

It is well-known that the Wigner function may take negative values only through oscillations in $\hbar$-sized regions of phase space, and that it may be rendered positive by coarse graining of such a region. Considered, for example, the smeared Wigner function

$$W_H(p,q)=2\int dp'dq' \times\exp\left(-\frac{2\sigma_q^2(p-p')^2}{\hbar^2}-\frac{(q-q')^2}{2\sigma_q^2}\right) \times W_\rho(p',q'). \tag{4.4}$$

This object is called the Husimi function [37]. It is equal to the expectation value of the corresponding density operator in a coherent state (of position width σ_q) $\langle p,q|\rho|p,q\rangle$, so is non-negative.

Loosely speaking, what happens during time evolution according to Eq. (4.1) is that, after a certain amount of time, the propagator effectively smears the Wigner function over a region of phase space greater than $\hbar$, and it becomes positive, in the manner of Eq. (4.4). We will now show this explicitly.

Letting $p_0\rightarrow p_0+p$ and $q_0\rightarrow q_0+q-p_0t/m$ in Eq. (4.1) yields

$$W_t(p,q)=\int dp_0dq_0\exp(-\mu p_0^2-\nu q_0^2+\sigma p_0q_0) \times W_0\left(p_0+p,q_0+q-\frac{p_0t}{m}\right). \tag{4.5}$$

The further transformation $p_0\rightarrow p_0+(\sigma/2\mu)q_0$ yields

$$W_t(p,q)=\int dp_0dq_0\ \exp(-\mu p_0^2-\beta q_0^2) \times W_0\left(p_0+\frac{\sigma}{2\mu}q_0+p,q_0+q-\frac{p_0t}{m}\right), \tag{4.6}$$

where $\beta=[\nu-(\sigma^2/4\mu)]$. These two transformations are canonical and therefore the transformed Wigner function appearing in the integrand of Eq. (4.6) is still the Wigner function of some state (unitarily related to the original one). Hence,

$$W_t(p,q)=\int dp_0dq_0\exp(-\mu p_0^2-\beta q_0^2)\tilde{W}_{pq}(p_0,q_0) \tag{4.7}$$

for some Wigner function $\tilde{W}_{pq}$ depending on p,q. This may now be recast as the smearing of a Husimi function:

$$W_t(p,q)=\int dp'\exp\left(-\frac{p'^2}{(\mu^{-1}-\hbar^2\beta)}\right)\int dp_0dq_0 \times\exp\left(-\frac{(p'-p_0)^2}{\hbar^2\beta}-\beta q_0^2\right)\tilde{W}_{pq}(p_0,q_0). \tag{4.8}$$

The integral over p_0,q_0 is a Husimi function with $\sigma_q^2=1/(2\beta)$. Hence $W_t(p,q)\geqslant 0$ provided the integral over p' in Eq. (4.8) exists. This will be the case if $\mu^{-1}>\hbar^2\beta$, that is, if

$$t>\left(\frac{\sqrt{3}}{2}\right)^{1/2}\left(\frac{\hbar}{\gamma kT}\right)^{1/2}. \tag{4.9}$$

The Wigner function will therefore be non-negative for times greater than the localization time (multiplied by a number of order 1).

V. DISCUSSION

We have shown that for times greater than the localization time $(\hbar/\gamma kT)^{1/2}$, the density operator satisfying Eq. (1.1) approaches the form

$$\rho=\int dpdqW_t(p,q)|\Psi_{pq}\rangle\langle\Psi_{pq}|, \tag{5.1}$$

where $W_t(p,q)$ is the Wigner function and the $|\Psi_{pq}\rangle$ are close to minimum uncertainty generalized coherent states. The Wigner function is strictly non-negative for times greater than the localization time (times a number of order 1).

Dióśi has also discussed the possibility of the phase space diagonal form (1.5) under evolution according to the master equation (1.1) [38]. His method was very different to ours in that he used the properties of the coherent states to regard Eq. (1.5) as an expansion of the density operator. He found that such an expansion is possible for times greater than the localization time, times a number of order 1, in tune with our results.

Intuitively, Eq. (5.1) indicates that the system is in a statistical mixture of quasiclassical trajectories, with the probability for each trajectory that indicated by Wigner function (regarded as a true phase space distribution function). This interpretation is not immediately clear, however, since the representation of the density matrix as diagonal in coherent states is not unique, because of the overcompleteness of the coherent states. Furthermore, there will be another basis of complete orthogonal states in which the density matrix is exactly diagonal and these states will be quite different from coherent states.

These various issues were discussed in Ref. [16]. There it was shown, using the above results, that histories consisting of time-ordered sequences of quasiprojections onto cells of phase space are approximately decoherent, and that the probabilities for them are those indicated by the classical stochastic process described by the Fokker-Planck equation (3.17), for sufficiently large phase space cells. This substantiates the interpretation in terms of a statistical ensemble of trajectories. Potential ambiguities arising from the overcompleteness of the coherent states become insignificant because the phase space cells projected onto are large. Furthermore, the exact diagonality of the density operator in a different basis corresponds to the well-known fact that a given physical system typically admits more than one decoherent set of histories [29,30]. This in turn is related to the fact that corresponding to a given master equation there can be many unravelings in terms of stochastic Schrödinger equations [16,39,40]. We stress once again that this paper is concerned with the mathematical question of deriving the form (5.1), and we do not have anything original to say about questions of its significance and interpretation.

An advantage of deriving Eq. (5.1) using path integral methods rather than quantum state diffusion, is that it yields and explicit expression for the phase space distribution function $f(p,q,t)$. Another advantage is that it is not obviously restricted to Markovian master equations. The quantum state diffusion picture, in its current state of development, exists only for systems described by a Markovian master equation. It may exist in the non-Markovian case, but is yet to be developed. The exact propagator for quantum Brownian motion, for quadratic potentials, can be given in terms of a path integral [5] and is (mildly) non-Markovian. Since the method described here utilizes path integrals, rather than the quantum state diffusion picture, there is a chance that our method may be valid in the non-Markovian case also, but this is still to be investigated.

We have concentrated in this paper on the simplest possible model of quantum Brownian motion: the free particle in a high-temperature environment with negligible dissipation γ. It is clear, however, that remaining in the context of a Markovian master equation, it would be straightforward (although perhaps tedious) to extend our considerations to the case of a harmonic oscillator with nontrivial dissipation. In the quantum state diffusion picture analysis, this case was covered in Ref. [16] and we expect the path integral treatment of the present paper to yield comparable results.

It is perhaps enlightening to comment on the various time scales involved in a more general quantum Brownian motion model and sketch the expected general physical picture part of which is described by the results of this paper. In this paper, we have largely been concerned with the localization time $(\hbar/\gamma kT)^{1/2}$, which is the time scale on which an arbitrary initial density operator approaches the form (1.5). The nomenclature ''localization time'' comes from the quantum state diffusion picture, which was the picture first used to derive some of the results described in this paper. It is so named because it is the time scale on which an arbitrary initial wave function becomes localized in phase space under evolution according to Eq. (1.3) [16,20].

Also relevant is the decoherence time $\hbar^2/(l^2 m\gamma kT)$, which is the time scale on which the off-diagonal terms of the density matrix are suppressed (in the position representation) [14]. The decoherence time necessarily involves a length scale l, which comes from the initial state. It could, for example, be the separation of a superposition of localized wave packets, and the decoherence time is then the time scale on which the interference between these packets is suppressed.

If one is interested in emergent classicality for macroscopic systems, it is appropriate to choose values order 1 in cgs units, for l, T, m, and γ. The decoherence time is then typically much shorter than the localization time. This is in turn typically much shorter than the relaxation time γ^{-1}, which is the time scale on which the system approaches thermal equilibrium (when this is possible).

Hence the general picture we have is as follows. Suppose the initial state of the system is a superposition of localized wave packets. Then the interference terms between these wave packets is destroyed on the decoherence time scale. After a few localization times the density matrix subsequently approaches the phase space diagonal form (1.5). After a much longer time of order in the relaxation time, the system reaches thermal equilibrium. Discussions of emergent classicality usually concern times between the decoherence time and the relaxation time, and it is this range of time which has been the primary concern of this paper.

Note added in proof. A stochastic decoupling of the density matrix propagator [of the form Eq. (2.5)] valid for certain non-Markovian systems has been proposed by Strunz [41].

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