Electromagnetic modes and energy production in the formation of the half Einstein universe

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The purpose of this paper is twofold. First, the electromagnetic modes in the static half Einstein universe are calculated, assuming that the spherical boundary lying at the position $\chi=\pi/2$ is perfectly conducting (χ is the radial parameter). The half Einstein universe, in contradistinction to the full Einstein universe, enables us to impose this perfect conductivity condition, which is the natural analogue to the Dirichlet condition conventionally adopted in studies of the scalar field. Second, we calculate the average photon number N and the corresponding energy produced in the lowest mode if the half Einstein universe is ''suddenly'' formed from an initial Minkowski universe. Here we use the same kind of formalism as previously used by Parker and others [Phys. Rev. Lett. **59**, 1369 (1987)] when studying the sudden formation of cosmic strings. We find that the photon number is very small, $N\simeq 0.01$, and that it is independent of the magnitude of the cosmic scale factor a_0. [S0556-2821(97)05408-8]

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I. INTRODUCTION

The static Einstein universe [1–3] continues to attract interest. The reason, of course, is the simplicity of this model, which makes it much more easy to handle, especially from a quantum-mechanical point of view, than the nonstationary Robertson-Walker model. It is also worth noticing that all closed Robertson-Walker metrics are conformally equivalent to the Einstein universe. Studies of the Einstein universe may thus be physically rewarding, although the model as such, of course, cannot be regarded to be a realistic model of our universe.

In the present paper we will first study, in Secs. II and III, the classical electromagnetic radiation field in the Einstein metric. The line element is given in Eq. (1) below, where a_0 is the constant scale factor. The parameter χ can be regarded as the radial coordinate in a three-dimensional space spanned by the axes of χ, θ, and ϕ. In the full Einstein universe, χ runs from 0 to π. In the *half* Einstein universe, by contrast, χ runs only from 0 to $\pi/2$. The fact that the half Einstein universe stands out as an interesting variant of the original Einstein model, was noted by Kennedy and Unwin [4] and also by Bayin and Ozcan [5,6]. Especially, the two last-mentioned references will be key references for us. The first part of our work is to develop the expressions for the magnetic [transverse electric (TE)] and electric [transverse magnetic (TM)] modes in the half Einstein universe, when the spherical surface at $\chi=\pi/2$ is assumed to be perfectly conducting. Now, one may object that from a physical point of view there is hardly any reason for saying that the outer surface is perfectly conducting. However, on the basis of analytical tractability of the electromagnetic theory, the boundary conditions that we adopt are by far the most natural ones; it turns out that they fit nicely into the electromagnetic mode formalism, basically in the same way as in the case of an electromagnetic radiation field in a spherical cavity in Minkowski space [7]. Our method, in fact, is just a parallel to the method used in conventional studies of the scalar field. Thus Bayin and Ozcan, in their studies of the scalar field in the half Einstein universe [5,6], adopted the completely analogous Dirichlet boundary condition at the surface $\chi=\pi/2$.

Our formulation of Maxwell's equations in the curvilinear Einstein metric is in accordance with that given in Møller's book (see Ref. [1]), and also with Refs. [8–10]. It ought to be mentioned here that the three-dimensional Maxwell theory in the form used in the frequently quoted paper of Mashhoon [11] is somewhat different; it is based upon the lines drawn up in prior works of Skrotskii [12], Plebanski [13], and Volkov and Kiselev [14]. These matters are discussed more closely in Ref. [10].

The second objective of our work, covered in Sec. IV, is to calculate the average number of photons produced if the half Einstein universe is ''suddenly'' formed from an initial Minkowski universe, having the same proper radius. We use the Bogoliubov transformation to relate the mode functions, and the creation and annihilation operators, in the two universes. The analytical technique that we use has been used earlier in a cosmological context, by Parker [15], in his analysis of the scalar field energy produced in the sudden formation of a cosmic string (an analogous analysis of the electromagnetic case was later given in Ref. [16]). The produced photon number N turns out to be remarkably small; in fact, for the lowest mode we calculate $N\simeq 0.01$. Moreover, the expression for N turns out to be independent of the value of the scale factor a_0.

II. MAXWELL'S EQUATIONS: THE FUNDAMENTAL SOLUTION

We begin by writing down the line element. In standard notation,

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$$ds^2=-dt^2+a_0^2[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\phi^2)] \qquad (1)$$

$(c=1)$, where $\theta\in[0,\pi]$ and $\phi\in[0,2\pi]$. As already mentioned, for full Einstein universe, $\chi\in[0,\pi]$, whereas for the half Einstein universe, $\chi\in[0,\pi/2]$. The value of the scale factor a_0 is determined as follows: the energy density ρ consists of two parts, one matter (dust) part ρ_0, and one vacuum part $\Lambda/8\pi G$, Λ being the cosmological constant and G the gravitational constant. The pressure is $p=-\Lambda/8\pi G$; i.e., there is a field tension. The Friedmann equations yield $\rho_0=\Lambda/4\pi G$, and finally $a_0=\Lambda^{-1/2}=(4\pi G\rho_0)^{-1/2}$.

It ought to be noted that the term $d\chi^2$ in Eq. (1) could alternatively has been written as $dr^2/(1-r^2)$, where r is the conventional nondimensional radius. Since $r=\sin\chi$, we see that the relationship between r and χ is unique when, as assumed here, $\chi\leqslant\pi/2$.

We shall numerate the coordinates according to $(x^0,x^1,x^2,x^3)=(t,\chi,\theta,\phi)$. It is often useful, when dealing with the Maxwell equations in curvilinear space, to write down the expressions for the fundamental electromagnetic tensors. There are two of them. First, there is the proper field tensor $F_{\mu\nu}$:

$$F_{\mu\nu}=\begin{bmatrix} 0 & -E_\chi & -E_\theta & -E_\phi \\ E_\chi & 0 & \sqrt{\gamma}B^\phi & -\sqrt{\gamma}B^\theta \\ E_\theta & -\sqrt{\gamma}B^\phi & 0 & \sqrt{\gamma}B^\chi \\ E_\phi & \sqrt{\gamma}B^\theta & -\sqrt{\gamma}B^\chi & 0 \end{bmatrix}, \qquad (2)$$

where $\sqrt{\gamma}=\sqrt{g_{\chi\chi}g_{\theta\theta}g_{\phi\phi}}=a_0^3\sin^2\chi\sin\theta$. Second, there is the tensor density $\mathcal{F}^{\mu\nu}=\sqrt{-g}F^{\mu\nu}=\sqrt{\gamma}F^{\mu\nu}$, containing the displacement $\mathbf{D}$ and the magnetic field $\mathbf{H}$:

$$\mathcal{F}^{\mu\nu}=\begin{bmatrix} 0 & \sqrt{\gamma}D^\chi & \sqrt{\gamma}D^\theta & \sqrt{\gamma}D^\phi \\ -\sqrt{\gamma}D^\chi & 0 & H_\phi & -H_\theta \\ -\sqrt{\gamma}D^\theta & -H_\phi & 0 & H_\chi \\ -\sqrt{\gamma}D^\phi & H_\theta & -H_\chi & 0 \end{bmatrix}. \qquad (3)$$

The Maxwell equations in empty space can be written as $F_{[\mu\nu,\rho]}=0$, $\mathcal{F}^{\mu\nu}_{\ \ ,\nu}=0$. We shall express them in three-dimensional form: taking the time factor to be $\exp(-i\omega t)$ in a complex classical representation for the fields, we have

$$\mathrm{div}\mathbf{E}=0, \quad \mathrm{div}\mathbf{B}=0,$$

$$\mathrm{curl}\mathbf{E}=i\omega\mathbf{B}, \quad \mathrm{curl}\mathbf{H}=-i\omega\mathbf{D}. \qquad (4)$$

Here the definitions are

$$(\mathrm{curl}\mathbf{E})^i=\epsilon^{ijk}\partial_j E_k, \quad \mathrm{div}\mathbf{B}=\gamma^{-1/2}\partial_i(\sqrt{\gamma}\mathbf{B}^i), \qquad (5)$$

where $\epsilon^{ijk}=\gamma^{-1/2}\delta_{ijk}$ is the antisymmetric pseudotensor, δ_{ijk} being the Levi-Civita symbol with $\delta_{123}=1$. The gravitational field in general acts like a medium with permittivity equal to permeability: $\epsilon=\mu=(-g_{00})^{-1/2}$. In the present case, therefore, the situation becomes simple since we can identify $\mathbf{D}$ with $\mathbf{E}$ and $\mathbf{B}$ with $\mathbf{H}$, and thus only work with one electric and one magnetic vectorial quantity.

In the following we shall work in an orthonormal basis, designated by carets:

$$\boldsymbol{\omega}^{\hat{t}}=\mathbf{d}t,$$

$$\boldsymbol{\omega}^{\hat{\chi}}=a_0\mathbf{d}\chi,$$

$$\boldsymbol{\omega}^{\hat{\theta}}=a_0\sin\chi\mathbf{d}\theta,$$

$$\boldsymbol{\omega}^{\hat{\phi}}=a_0\sin\chi\sin\theta\mathbf{d}\phi. \qquad (6)$$

The components of $\mathbf{E}$ in this basis are written as $(E_{\hat{\chi}},E_{\hat{\theta}},E_{\hat{\phi}})$, with a similar notation for $\mathbf{H}$. We write out Maxwell's equations in component form:

$$\frac{\sin\theta}{\sin\chi}\,\partial_\chi\left[\sin^2\chi\begin{pmatrix}E_{\hat{\chi}}\\ H_{\hat{\chi}}\end{pmatrix}\right]+\partial_\theta\left[\sin\theta\begin{pmatrix}E_{\hat{\theta}}\\ H_{\hat{\theta}}\end{pmatrix}\right]+\partial_\phi\begin{pmatrix}E_{\hat{\phi}}\\ H_{\hat{\phi}}\end{pmatrix}=0, \quad (7)$$

$$\frac{1}{a_0\sin\chi\,\sin\theta}\left[\partial_\theta\left(\sin\theta\begin{pmatrix}E_{\hat{\phi}}\\ H_{\hat{\phi}}\end{pmatrix}\right)-\partial_\phi\begin{pmatrix}E_{\hat{\theta}}\\ H_{\hat{\theta}}\end{pmatrix}\right]=i\omega\begin{pmatrix}H_{\hat{\chi}}\\ -E_{\hat{\chi}}\end{pmatrix}, \quad (8)$$

$$\frac{1}{a_0\sin\chi\,\sin\theta}\left[\partial_\phi\begin{pmatrix}E_{\hat{\chi}}\\ H_{\hat{\chi}}\end{pmatrix}-\sin\theta\partial_\chi\left(\sin\chi\begin{pmatrix}E_{\hat{\phi}}\\ H_{\hat{\phi}}\end{pmatrix}\right)\right]$$

$$=i\omega\begin{pmatrix}H_{\hat{\theta}}\\ -E_{\hat{\theta}}\end{pmatrix}, \quad (9)$$

$$\frac{1}{a_0\sin\chi}\left[\partial_\chi\left(\sin\chi\begin{pmatrix}E_{\hat{\theta}}\\ H_{\hat{\theta}}\end{pmatrix}\right)-\partial_\theta\begin{pmatrix}E_{\hat{\chi}}\\ H_{\hat{\chi}}\end{pmatrix}\right]=i\omega\begin{pmatrix}H_{\hat{\phi}}\\ -E_{\hat{\phi}}\end{pmatrix}. \quad (10)$$

Manipulating these equations we can construct a second-order differential equation for $E_{\hat{\chi}}$ (or $H_{\hat{\chi}}$). We separate variables, for the electric field meaning $E_{\hat{\chi}}(\chi,\theta,\phi)=E_{\hat{\chi}}(\chi)Y_{\ell m}(\theta,\phi)$, where $Y_{\ell m}$ are the usual spherical harmonics, defined in accordance with [17]. We obtain

$$\frac{d^2}{d\chi^2}(\sin^2\chi X)+(\omega a_0)^2\sin^2\chi X-\ell(\ell+1)X=0, \quad (11)$$

where X stands for $E_{\hat{\chi}}$ or $H_{\hat{\chi}}$. The solution of Eq. (11) is a known function:

$$X\propto\sin^{\ell-1}\chi C_{n-\ell}^{(\ell+1)}(\cos\chi). \qquad (12)$$

Here n is an integer, and $C_n^{(\alpha)}(x)$ is the Gegenbauer polynomial, satisfying the basic equation

$$(1-x^2)\,\frac{d^2}{dx^2}\,C_n^{(\alpha)}(x)-(2\alpha+1)x\,\frac{d}{dx}\,C_n^{(\alpha)}(x)$$

$$+n(n+2\alpha)C_n^{(\alpha)}(x)=0. \qquad (13)$$

The eigenfrequencies can be found by inserting solution (12) into Eq. (11). They are

$$\omega_n=\frac{n+1}{a_0}, \quad n\geqslant\ell. \qquad (14)$$

Several general properties of the Gegenbauer polynomials are listed in [18]. We write down the following relations, which are useful in our context $(\alpha\neq0)$:

$$C_n^{(\alpha)}(1)=\binom{n+2\alpha-1}{n}, \tag{15}$$

$$C_n^{(\alpha)}(0)=\begin{cases}0, & n=2m+1,\\ (-1)^{n/2}\dfrac{\Gamma(\alpha+n/2)}{\Gamma(\alpha)(n/2)!}, & n=2m,\end{cases} \tag{16}$$

$$C_0^{(\alpha)}(x)=1, \quad C_1^{(\alpha)}(x)=2\alpha x. \tag{17}$$

Also we shall need the derivative formula

$$(1-x^2)\frac{d}{dx}C_n^{(\alpha)}(x)=-nxC_n^{(\alpha)}(x)+(n+2\alpha-1)C_{n-1}^{(\alpha)}(x). \tag{18}$$

We consider henceforth, as mentioned above, a three-dimensional spherical volume, spanned by the ''radius'' χ and the angular coordinates θ and ϕ, closed by the perfectly conducting two-dimensional surface at $\chi=\pi/2$. We shall work out the specific solutions, starting from Eq. (12), for the electric and magnetic modes separately. To facilitate the writing, we omit the carets on the field components from now on.

Before closing this section it is worthwhile to mention that the corresponding elementary wave solution in the case of a scalar field Φ is [2]

$$\Phi\propto\sin^{\ell}\chi C_{n-\ell}^{(\ell+1)}(\cos\chi).$$

III. CLASSICAL MODES. ELECTROMAGNETIC ENERGY

A. The magnetic (TE) modes

This is the simplest case. The magnetic radial component H_χ is, in general, different from zero. The boundary condition is

$$H_\chi=0, \quad \chi=\pi/2. \tag{19}$$

For the electromagnetic field, we know that $\ell\geqslant 1$ always. In Eq. (12), we must have $n-\ell\geqslant 0$, in order to avoid infinities at the origin $\chi=0$. This property can be seen, for instance, from the relationship between $C_n^{(\alpha)}$ and the hypergeometric function F:

$$C_n^{(\alpha)}(x)=\frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}F\left(-n,n+2\alpha;\alpha+\frac{1}{2};\frac{1-x}{2}\right). \tag{20}$$

Moreover condition (19), together with the first line of Eq. (16), show that the subscript $(n-\ell)$ in Eq. (12) must be an odd number. It is convenient now to change the meaning of n, and write the basic magnetic mode solution as

$$H_\chi=H_0\ell(\ell+1)\sin^{\ell-1}\chi C_{2n+1}^{(\ell+1)}(\cos\chi)Y_{\ell m}(\theta,\phi), \tag{21}$$

where $n=0,1,2,\ldots$. H_0 is a normalization constant.

The eigenfrequencies in this case become, according to the general Eq. (14),

$$\omega_n=\frac{2n+\ell+2}{a_0}, \tag{22}$$

showing that the lowest-order magnetic mode is $(\ell+2)/a_0$, i.e., $\omega=3/a_0$ when $\ell=1$.

It is worth noticing that the spatial variation of the $n=0$ mode is quite simple: using Eq. (17) we see that

$$H_\chi(n=0)=2H_0\ell(\ell+1)^2\sin^{\ell-1}\chi\cos\chi Y_{\ell m}. \tag{23}$$

When $\ell=1$, H_χ is different from zero at the origin; it decreases as a cosine function towards the value $H_\chi=0$ at the boundary. When $\ell>1$, $H_\chi=0$ at the origin.

From Maxwell's equations we calculate the remaining field components:

$$H_\theta=\frac{H_0}{\sin\chi}\frac{d}{d\chi}[\sin^{\ell+1}\chi C_{2n+1}^{(\ell+1)}(\cos\chi)]\partial_\theta Y_{\ell m}, \tag{24}$$

$$H_\phi=\frac{\mathrm{im}H_0}{\sin\chi}\frac{d}{d\chi}[\sin^{\ell+1}\chi C_{2n+1}^{(\ell+1)}(\cos\chi)]\frac{Y_{\ell m}}{\sin\theta}, \tag{25}$$

$$E_\theta=-m(2n+\ell+2)H_0\sin^{\ell}\chi C_{2n+1}^{(\ell+1)}(\cos\chi)\frac{Y_{\ell m}}{\sin\theta}, \tag{26}$$

$$E_\phi=-i(2n+\ell+2)H_0\sin^{\ell}\chi C_{2n+1}^{(\ell+1)}(\cos\chi)\partial_\theta Y_{\ell m}, \tag{27}$$

where in the two last expressions we took Eq. (22) into account. The time factor $\exp(-i\omega t)$ is understood everywhere.

B. The electric (TM) modes

The governing equation for the electric modes is again Eq. (11), where now $X=E_\chi$. The boundary conditions at the perfectly conducting surface $\chi=\pi/2$ follow from the general electromagnetic theory in orthogonal coordinates, when account is taken of the metric (1):

$$\partial_\chi(\sin\chi H_\theta)=0, \quad \partial_\chi(\sin\chi H_\phi)=0, \quad \chi=\pi/2, \tag{28}$$

cf., for instance, p. 484 in [19].

When E_χ contains the factor $\sin^{\ell-1}\chi$, as Eq. (12) shows, then the components H_θ and H_ϕ will in view of Maxwell's equations have to contain the factor $\sin^{l}\chi$, and so we obtain from Eq. (28) the condition

$$\partial_\chi[\sin^{\ell+1}\chi C_{n-\ell}^{(\ell+1)}(\cos\chi)]=0, \quad \chi=\pi/2. \tag{29}$$

Use of formula (18), with $x=\cos\chi=0$, then tells us that $(n-\ell)$ must be an even integer. Again changing the meaning of the integer n, we write the electric mode in the form

$$E_\chi=E_0\ell(\ell+1)\sin^{\ell-1}\chi C_{2n}^{(\ell+1)}(\cos\chi)Y_{\ell m}(\theta,\phi), \tag{30}$$

with $n=0,1,2,\ldots$. E_0 is a normalization constant.

Using Eq. (14) we see that the eigenfrequencies now become

$$\omega_n=\frac{2n+\ell+1}{a_0}, \tag{31}$$

so that the lowest order electric mode is $(\ell+1)/a_0$. When $\ell=1$, $\omega=\omega_{\min}=2/a_0$. This is the lowest of all possible modes, both electric and magnetic.

From Maxwell's equations we calculate the remaining field components:

$$E_\theta=\frac{E_0}{\sin\chi}\frac{d}{d\chi}[\sin^{\ell+1}\chi C_{2n}^{(\ell+1)}(\cos\chi)]\partial_\theta Y_{\ell m}, \tag{32}$$

$$E_\phi=\frac{\mathrm{i}mE_0}{\sin\chi}\frac{d}{d\chi}[\sin^{\ell+1}\chi C_{2n}^{(\ell+1)}(\cos\chi)]\frac{Y_{\ell m}}{\sin\theta}, \tag{33}$$

$$H_\theta=m(2n+\ell+1)E_0\sin^{\ell}\chi C_{2n+1}^{(\ell+1)}(\cos\chi)\frac{Y_{\ell m}}{\sin\theta}, \tag{34}$$

$$H_\phi=i(2n+\ell+1)E_0\sin^{\ell}\chi C_{2n+1}^{(\ell+1)}(\cos\chi)\partial_\theta Y_{\ell m}, \tag{35}$$

again with the factor $\exp(-i\omega t)$ understood.

C. Electromagnetic energy

The total electromagnetic energy is

$$W=\int w\, dV, \tag{36}$$

where w is the energy density

$$w=\frac{1}{4}(|E_\chi|^2+|E_\theta|^2+|E_\phi|^2+|H_\chi|^2+|H_\theta|^2+|H_\phi|^2). \tag{37}$$

We consider again one of the magnetic modes first, corresponding to $E_\chi=0$. It is convenient to make use of the vector spherical harmonics

$$\mathbf{X}_{\ell m}=\frac{1}{\sqrt{\ell(\ell+1)}}\mathbf{L}Y_{\ell m}, \tag{38}$$

whose components are

$$(X_{\ell m})_\chi=0,$$

$$(X_{\ell m})_\theta=\frac{-m}{\sqrt{\ell(\ell+1)}}\frac{Y_{\ell m}}{\sin\theta},$$

$$(X_{\ell m})_\phi=\frac{-i}{\sqrt{\ell(\ell+1)}}\partial_\theta Y_{\ell m}. \tag{39}$$

Because of the orthogonality of $\mathbf{X}_{\ell m}$, one has

$$\int |\mathbf{X}_{\ell m}|^2 d\Omega=1, \tag{40}$$

where the integration is taken over all angles. For ease in writing we also introduce the symbol Q_n, defined as

$$Q_n=\sin^{\ell+1}\chi C_n^{(\ell+1)}(\cos\chi). \tag{41}$$

Inserting Eq. (21) and Eqs. (24)–(27) into Eq. (37) we get, after integrating over all angles and observing that $dV=a_0^3\sin^2\chi\, d\chi\, d\Omega$, the mode energy

$$W_{n\ell m}=\frac{1}{4}a_0^3|H_0|^2\ell(\ell+1)\int_0^{\pi/2}\left\{\left[\frac{\ell(\ell+1)}{\sin^2\chi}+(2n+\ell+2)^2\right]Q_{2n+1}^2+(\partial_\chi Q_{2n+1})^2\right\}d\chi. \tag{42}$$

In the last term we perform a partial integration, observing that the contribution from the surface term is zero due to the boundary condition (19). Moreover taking into account the governing equation for the Gegenbauer polynomials, which can conveniently be written in the form

$$\partial_\chi^2 Q_n=-\left[(n+\ell+1)^2-\frac{\ell(\ell+1)}{\sin^2\chi}\right]Q_n, \tag{43}$$

we get

$$W_{n\ell m}=\frac{1}{2}a_0^3|H_0|^2\ell(\ell+1)(2n+\ell+2)^2\int_0^{\pi/2}Q_{2n+1}^2 d\chi. \tag{44}$$

The integration over χ can be performed, if we invoke the orthogonality relation for the Gegenbauer polynomials [18]:

$$\int_{-1}^{1}(1-x^2)^{\alpha-1/2}[C_n^{(\alpha)}(x)]^2dx=\frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}, \tag{45}$$

and observe that $C_n^{(\alpha)}(-x)=(-1)^nC_n^{(\alpha)}(x)$. The final result then becomes, for the magnetic modes,

$$W_{n\ell m}=\frac{\pi}{8}a_0^3|H_0|^2\ell(\ell+1)(2n+\ell+2)\times\frac{2^{-2\ell}\Gamma(2n+2\ell+3)}{(2n+1)![\Gamma(\ell+1)]^2}. \tag{46}$$

The electric modes can be handled similarly. The energy, when expressed in a form analogous to Eq. (42), is then

$$W_{n\ell m}=\frac{1}{4}a_0^3|E_0|^2\ell(\ell+1)\int_0^{\pi/2}\left\{\left[\frac{\ell(\ell+1)}{\sin^2\chi}+(2n+\ell+1)^2\right]Q_{2n}^2+(\partial_\chi Q_{2n})^2\right\}d\chi. \tag{47}$$

This expression is further processed by means of a partial integration, taking the boundary condition (29) into account. We obtain

$$W_{n\ell m}=\frac{1}{2}a_0^3|E_0|^2\ell(\ell+1)(2n+\ell+1)^2\int_0^{\pi/2}Q_{2n}^2 d\chi \tag{48}$$

which, again in view of the orthogonality condition (45), can be expressed in the form

$$W_{n\ell m}=\frac{\pi}{8}\,a_0^3|E_0|^2\ell(\ell+1)(2n+\ell+1)\times\frac{2^{-2\ell}\Gamma(2n+2\ell+2)}{(2n)![\Gamma(\ell+1)]^2}. \tag{49}$$

This is the final classical result for the electric mode energy.

IV. QUANTUM THEORY: THE BOGOLIUBOV TRANSFORMATION

Consider now the quantum electrodynamic theory of the half Einstein universe. We shall analyze the following problem: what is the amount of produced radiation energy associated with a ''sudden'' creation of this particular universe from an initial Minkowski universe having the same proper radius? Information about the energy production can give us some physical insight about the importance of quantum mechanics for cosmology in general, in spite of the fact that the quantum system that we study here is after all quite simple.

Our assumption about a ''sudden'' transformation from one kind of universe to another kind is of course an idealization. It is however an attractive model since it has both mathematical and physical merits: mathematically, the formalism becomes easily tractable; physically, the suddenness is known to be quite appropriate as regards the formation of *local* systems, such as cosmic strings, in the early universe. Cosmic strings were handled in this way by Parker, in the context of the scalar field [15] (cf. also [16]). One may object here, of course, that a universe is after all not a local system. However, our calculation as such does not require that the scale factor a_0 is a very large quantity. The calculation can equally well be looked upon as a treatment of a ''miniuniverse'' of the Einstein type, in which a_0 can be arbitrarily small. In the following we shall consider the electric modes only. As noted above, the lowest eigenfrequency ($\omega_{\min}=2/a_0$) is found for just the $\ell=1$ electric mode.

Let us begin by writing the classical expression (48) for the mode energy as

$$W_{n\ell m}=\frac{(2n+\ell+1)^2}{\ell(\ell+1)}\int E_\chi^2\sin^2\chi\,dV, \tag{50}$$

where now E_χ is the *real* field in the radial direction. The classical energy can accordingly be expressed in terms of one single component only. In quantum theory it is therefore natural to regard E_χ, now considered a Hermitian operator, as the basic field quantity. We expand it as follows, summing over all modes:

$$E_\chi=\frac{1}{a_0\sin\chi}\sum_{n=0}^{\infty}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\ell(\ell+1)}\times(a_{n\ell m}^E u_{n\ell m}^E+a_{n\ell m}^{E\dagger}u_{n\ell m}^{E*}). \tag{51}$$

Here $a_{n\ell m}^E$ and $a_{n\ell m}^{E\dagger}$ are annihilation and creation operators, satisfying the commutation rules

$$[a_{n\ell m}^E,a_{n'\ell' m'}^{E\dagger}]=\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}. \tag{52}$$

Subscripts E refer to the Einstein metric. The $u_{n\ell m}^E$ are mode functions. They will be normalized in accordance with the general expression

$$(u_1,u_2)=-i\int u_1^*\overleftrightarrow{\partial^\mu}u_2 d^3\Sigma_\mu, \tag{53}$$

where $d^3\Sigma_\mu=(1/3!)\epsilon_{\mu\nu\rho\sigma}\mathbf{d}x^\mu\wedge\mathbf{d}x^\nu\wedge\mathbf{d}x^\sigma$, $\epsilon_{\mu\nu\rho\sigma}=\sqrt{-g}\,\delta_{\mu\nu\rho\sigma}$, with $\delta_{0123}=1$. This means that the scalar product becomes

$$(u_1^E,u_2^E)=ia_0^3\int(u_1^{E*}\overleftrightarrow{\partial_0}u_2^E)\,\sin^2\chi d\chi d\Omega. \tag{54}$$

As mode functions we take the form

$$u_{n\ell m}^E=\frac{1}{a_0}\sqrt{\frac{2(2n)!}{\pi}}\frac{2^\ell\Gamma(\ell+1)}{[\Gamma(2n+2\ell+2)]^{1/2}}\times\sin^\ell\chi C_{2n}^{(\ell+1)}(\cos\chi)Y_{\ell m}\exp(-i\omega_n t) \tag{55}$$

[recall that $\omega_n=(2n+\ell+1)/a_0$ in the exponential factor]. Insertion of Eq. (55) into Eq. (54) shows that the scalar product thereby becomes orthonormal:

$$(u_{n\ell m}^E,u_{n'\ell' m'}^E)=-(u_{n\ell m}^{E*},u_{n'\ell' m'}^{E*})=\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}, \tag{56}$$

the other scalar products being zero. Moreover, Eqs. (50) and (51) show that the mode energy $W_{n\ell m}$ can be written as the frequency ω_n times the photon occupation number, plus a zero-point term:

$$W_{n\ell m}=\omega_n\left(\langle a_{n\ell m}^{E\dagger}a_{n\ell m}^E\rangle+\frac{1}{2}\right). \tag{57}$$

Let us now consider the Minkowski space. Similarly as above, we assume the region under investigation to be bounded by a perfectly conducting spherical surface. Scaling the ordinary Minkowski radius r according to $r=a_0\chi$, where a_0 has the same meaning as above, we can write the line element as

$$ds^2=-dt^2+a_0^2(d\chi^2+\chi^2d\Omega^2). \tag{58}$$

We shall fix the external radius R to be

$$R=a_0\pi/2. \tag{59}$$

Thus R, corresponding to $\chi=\pi/2$, is taken to be the same as the proper radial distance from the origin to the outer surface $\chi=\pi/2$ in the Einstein metric (1). In this sense the Einstein half universe and our Minkowski ''universe'' are analogous to each other.

The field components for the electric modes in Minkowski space can be written, when the time factors $\exp(-i\omega t)$ are omitted, in the form [7,20,21]

$$E_\chi=\frac{E_0\ell(\ell+1)}{\chi^2}\,\psi_\ell(\omega_s a_0\chi)Y_{\ell m}, \tag{60}$$

$$E_\theta=\frac{E_0}{\chi}\,\partial_\chi\psi_\ell(\omega_s a_0\chi)\partial_\theta Y_{\ell m}, \tag{61}$$

$$E_\phi=\frac{\mathrm{i}mE_0}{\chi}\,\partial_\chi\psi_\ell(\omega_s a_0\chi)\,\frac{Y_{\ell m}}{\sin\theta}, \tag{62}$$

$$H_r=0, \tag{63}$$

$$H_\theta=\frac{m\omega a_0E_0}{\chi}\,\psi_\ell(\omega_s a_0\chi)\,\frac{Y_{\ell m}}{\sin\theta}, \tag{64}$$

$$H_\phi=\frac{i\omega a_0E_0}{\chi}\,\psi_\ell(\omega_s a_0\chi)\partial_\theta Y_{\ell m}. \tag{65}$$

Here $\psi_\ell(x)=xj_\ell(x)$ is the (real) Riccati-Bessel function, j_ℓ being the spherical Bessel function. The eigenfrequencies ω_s in Minkowski space are determined from the boundary condition, for the electric mode, at the surface $r=R$:

$$\partial_\chi\psi_\ell(\omega_s a_0\chi)|_{\chi=\pi/2}=0. \tag{66}$$

Note that at this point the formulation of the theory in Minkowski space differs from that in Einstein space: in the latter case, the determination of the eigenfrequencies in Eq. (14) was made on the basis of the governing equation (11) alone; the boundary condition at the outer surface was not drawn into consideration explicitly. Near the origin $\chi=0$ it follows, in view of the relationship

$$\psi_\ell(x)\to\frac{x^{\ell+1}}{(2\ell+1)!!},\quad x\ll 1, \tag{67}$$

that Eq. (60) leads to $E_\chi\propto\chi^{\ell-1}$. This is seen to be in qualitative agreement with Eq. (30), when one notes, in accordance with Eq. (15), that $C_{2n}^{(\ell+1)}(1)$ is a constant.

As expected, for general values of χ the Einstein space modes (30) and (32)–(35) are analogous to the Minkowski space modes (60)–(65). The analogy can be regarded as reflecting the following correspondence between the basis functions:

$$\psi_\ell(\omega_s a_0\chi)\leftrightarrows\sin^{\ell+1}\chi C_{2n}^{(\ell+1)}(\cos\chi). \tag{68}$$

Note also that the order s of the root ω_s in Minkowski space corresponds to the order $2n$ of the Gegenbauer polynomial in Einstein space.

We now intend to construct the Minkowski mode functions u^M corresponding to the Einstein mode functions u^E above. First consider the electromagnetic energy: inserting Eqs. (60)–(65) into Eq. (37) we get, taking into account that the volume element now is $dV=a_0^3\chi^2d\chi\,d\omega$,

$$W_{s\ell m}=\frac{1}{4}\,a_0^3|E_0|^2\ell(\ell+1)\times\int_0^{\pi/2}\left\{\left[\frac{\ell(\ell+1)}{\chi^2}+\omega_s^2a_0^2\right]\psi_\ell^2+(\partial_\chi\psi_\ell)^2\right\}d\chi, \tag{69}$$

cf. the analogous Eq. (47). By means of a partial integration, observing condition (66), and also taking into account the governing equation

$$\psi_\ell''(x)=-\left[1-\frac{\ell(\ell+1)}{x^2}\right]\psi_\ell(x) \tag{70}$$

for the Riccati-Bessel functions, we obtain

$$W_{s\ell m}=\frac{1}{2}\,a_0^3|E_0|^2\ell(\ell+1)(\omega_s a_0)^2\int_0^{\pi/2}\psi_\ell^2 d\chi. \tag{71}$$

From the recursion relations for the spherical Bessel functions one can derive the integration formula $[\psi_\ell=\psi_\ell(x)]$

$$\int\psi_\ell^2(x)=\frac{1}{2}\,x\left\{(\psi_\ell')^2-\frac{1}{x}\,\psi_\ell\psi_\ell'+\left[1-\frac{\ell(\ell+1)}{x^2}\right]\psi_\ell^2\right\}, \tag{72}$$

the use of which in Eq. (71) leads to

$$W_{s\ell m}=\frac{\pi}{8}\,a_0^3|E_0|^2\ell(\ell+1)(\omega_s a_0)^2\times\left[1-\frac{\ell(\ell+1)}{(\omega_sR)^2}\right]\psi_\ell^2(\omega_sR). \tag{73}$$

This is the final expression for the energy of the electric Minkowski modes.

The Minkowski mode functions $u_{s\ell m}^M$ can now be introduced via the expansion

$$E_\chi=\frac{1}{a_0\chi}\sum_s\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\ell(\ell+1)}(a_{s\ell m}^Mu_{s\ell m}^M+a_{s\ell m}^{M\dagger}u_{s\ell m}^{M*}), \tag{74}$$

where the creation and annihilation operators satisfy

$$[a_{slm}^M,a_{s'\ell' m'}^{M\dagger}]=\delta_{ss'}\delta_{\ell\ell'}\delta_{mm'}. \tag{75}$$

The Minkowski scalar product

$$(u_1^M,u_2^M)=ia_0^3\int(u_1^{M*}\overleftrightarrow{\partial_0}u_2^M)\chi^2d\chi\,d\Omega \tag{76}$$

is required to be orthogonal, in analogy to the Einstein scalar product (54) above. Explicitly, the mode functions become

$$u_{s\ell m}^M=\frac{1}{a_0\sqrt{\omega_sR}}\,\frac{[1-\ell(\ell+1)/(\omega_sR)^2]}{\psi_\ell(\omega_sR)}\,\frac{\psi_\ell(\omega_s a_0\chi)}{\chi}\times Y_{\ell m}\exp(-i\omega_st). \tag{77}$$

The classical mode energy, which in accordance with Eq. (71) can be expressed as

$$W_{s\ell m}=\frac{(\omega_s a_0)^2}{\ell(\ell+1)}\int E_x^2\chi^2dV, \tag{78}$$

corresponds quantum mechanically to the form

$$W_{s\ell m}=\omega_s\left(\langle a_{s\ell m}^{M\dagger}a_{s\ell m}^M\rangle+\frac{1}{2}\right). \tag{79}$$

Armed with the above formalism, we are now able to analyze the following physical problem: assume that the Einstein universe was suddenly formed at the instant $t=0$. Prior to this event the Minkowski universe was present. We assume that E_χ was continuous at $t=0$. The two sets of mode functions, u^E and u^M, may be related via a Bogoliubov transformation:

$$u^E_{n\ell m}=\sum_{s\ell' m'}[\gamma(n\ell m|s\ell' m')u^M_{s\ell' m'}+\delta(n\ell m|s\ell' m')u^{M*}_{s\ell' m'}], \qquad (80)$$

γ and δ being the Bogoliubov coefficients. The corresponding relation for the operators is

$$a^M_{s\ell m}=\sum_{n\ell' m'}[\gamma(n\ell m|s\ell' m')a^E_{n\ell' m'}+\delta^*(n\ell m|s\ell' m')a^{E^\dagger}_{n\ell' m'}]. \qquad (81)$$

The average number of particles produced in the mode s, ℓ, m from the initial vacuum is

$$N_{s\ell m}=\sum_{n\ell' m'}|\delta(n\ell m|s\ell' m')|^2. \qquad (82)$$

From Eq. (80) we see that the Bogoliubov coefficient is

$$\delta(n\ell m|s\ell' m')=-(u^E_{n\ell m},u^{M*}_{s\ell' m'})_M, \qquad (83)$$

an extra subscript M is added to indicate that the scalar product is taken in accordance with the Minkowski normalization (76). Inserting Eq. (55) and Eq. (77) into Eq. (83) we see that the coefficient vanishes unless when $\ell'=\ell$ and $m'=-m$. From Eq. (82) we then obtain

$$N_{s\ell m}=\frac{2^{2\ell+1}}{\pi\omega_s R\psi^2_\ell(\omega_s R)}\frac{[\Gamma(\ell+1)]^2}{1-\ell(\ell+1)/(\omega_s R)^2}\times\sum_n\frac{a_0^2(\omega_s-\omega_n)^2(2n)!}{\Gamma(2n+2\ell+2)}I, \qquad (84)$$

where I is the integral

$$I=\int_0^{\pi/2}\chi\,\sin^\ell\chi\,C^{(\ell+1)}_{2n}(\cos\chi)\psi_\ell(\omega_s a_0\chi)d\chi \qquad (85)$$

(recall that $R=a_0\pi/2$). Note that expression (84) is degenerate with respect to m. Thus, if we were to sum $N_{s\ell m}$ over the Minkowski quantum numbers, we would have to take into account that there are $2\ell+1$ m values for each ℓ.

The electromagnetic energy in mode s, ℓ, m produced by the sudden formation of the Einstein universe is $W_{s\ell m}=\omega_s N_{s\ell m}$.

Inspection of Eq. (84) shows the remarkable fact that the produced number of particles is independent of the value of the scale factor a_0. The reason is that the scale factor occurs only in the nondimensional combinations $a_0\omega_s$ and $a_0\omega_n$; for the lowest-lying states these numbers are of order unity.

Expression (84) is complicated, and we consider henceforth only the contribution from the lowest mode. It corresponds to $\ell=1$, $s=1$, $n=0$. We have in this case

$$\psi_1(x)=\frac{\sin x}{x}-\cos x, \quad C^{(\ell+1)}_0(\cos\chi)=1, \qquad (86)$$

$$\omega_s R=2.75, \quad \omega_n a_0=2, \qquad (87)$$

cf. Eqs. (17) and (31) (for notational clarity we here keep the general subscripts s and n on ω to distinguish the frequencies). Some calculation leads to

$$I=-\frac{\omega_s a_0}{\omega_s^2 a_0^2-1}\left(\frac{2\omega_s a_0\cos\omega_s R}{\omega_s^2 a_0^2-1}+\frac{\pi}{2}\sin\omega_s R\right)=0.820, \qquad (88)$$

which again means that

$$N_{11m}=0.009\,47. \qquad (89)$$

The number of produced photons in this particular mode is thus very small. The produced energy, in dimensional units, becomes

$$W_{11m}=\frac{2.75\hbar c}{R}N_{11m}=\frac{1.75\hbar c}{a_0}N_{11m}. \qquad (90)$$

To get an appreciable amount of energy, we have to envisage a miniuniverse whose scale factor a_0 is very small. From a physical point of view, the dependence of W_{11m} upon some inverse power of a_0 is just what we would expect. The curvature effect, and hence also the quantum-mechanical effect, become more pronounced the smaller the value of a_0 is. The curvature effect is reflected, for instance, by the fact that the scalar curvature of the Einstein metric is equal to $6/a_0^2$.

[1] J. S. Dowker, Ann. Phys. (N.Y.) **62**, 361 (1971); J. S. Dowker and R. Critchley, Phys. Rev. D **15**, 1484 (1977); J. S. Dowker and B. M. Altaie, *ibid.* **17**, 417 (1978); T. Inagaki, K. Ishikawa, and T. Muta, Prog. Theor. Phys. **96**, 847 (1996). As for textbook references, see, for instance, C. Møller, *The Theory of Relativity*, 2nd ed. (Clarendon Press, Oxford, 1972), Sec. 12.6; H. Stephani, *General Relativity* (Cambridge University Press, Cambridge, England, 1982), Sec. 25.
[2] L. H. Ford, Phys. Rev. D **11**, 3370 (1975).
[3] L. H. Ford, Phys. Rev. D **14**, 3304 (1976).
[4] G. Kennedy and S. D. Unwin, J. Phys. A **13**, L253 (1980).
[5] S. S. Bayin and M. Ozcan, Phys. Rev. D **48**, 2806 (1993).
[6] S. S. Bayin and M. Ozcan, Phys. Rev. D **49**, 5313 (1994).
[7] M. Born and E. Wolf, *Principles of Optics*, 6th ed. (Pergamon, Oxford, 1991), Sec. 13.5.
[8] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), Sec. 90.
[9] J. Van Bladel, *Relativity and Engineering*, Springer Series in

Electrophysics Vol. 15 (Springer-Verlag, Berlin, 1984), Sec. 9.
[10] I. Brevik, J. Math. Phys. (N.Y.) **28**, 2241 (1987); I. Brevik and S. Straumsnes, Gen. Relativ. Gravit. (to be published).
[11] B. Mashhoon, Phys. Rev. **8**, 4297 (1973).
[12] G. V. Skrotskii, Dok. Akad. Nauk SSSR **114**, 73 (1957) [Sov. Phys. Dokl. **2**, 226 (1957)].
[13] J. Plebanski, Phys. Rev. **118**, 1396 (1960).
[14] A. M. Volkov and V. A. Kiselev, Zh. Eksp. Teor. Fiz. **57**, 1353 (1969) [Sov. Phys. JETP **30**, 733 (1970)].
[15] L. Parker, Phys. Rev. Lett. **59**, 1369 (1987).
[16] I. Brevik and T. Toverud, Phys. Rev. D **51**, 691 (1995).
[17] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
[18] *Handbook of Mathematical Functions*, Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, DC, 1964).
[19] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[20] M. Kerker, *The Scattering of Light* (Academic Press, New York, 1969).
[21] J. P. Barton, D. R. Alexander, and S. A. Schaub, J. Appl. Phys. **66**, 4594 (1989).