

Avoidance of collapse by circular current-carrying cosmic string loops

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Earlier attempts to calculate the nonlinear dynamical evolution of Witten-type superconducting vacuum vortex defects relied on the use of approximate conducting string models that were too simple to take proper account of the effect of current saturation. This effect is however allowed for adequately in a newly developed class of rather more complicated, though still conveniently analytic, conducting string models. These more realistic models have recently been employed by Larsen and Axenides for investigating the collapse of circular string loops in the case for which angular momentum is absent. The present work extends this investigation to the generic case of circular string loops for which angular momentum is present, so that there will be a centrifugal potential barrier. This barrier will prevent collapse unless the initial conditions are such that the relevant current saturation limit is attained, in which case the string description of the vortex defect will break down, so that its subsequent fate is hard to foresee. On the other hand if saturation is avoided one would expect that the loop will eventually radiate away its excess energy and settle down into a vorton-type equilibrium state. [S0556-2821(97)00508-0]

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I. INTRODUCTION

Among the conceivable varieties of topological defects of the vacuum that might have been generated at early phase transitions, the *vortex-type* defects describable on a macroscopic scale as *cosmic strings* are the kind that is usually considered most likely to exist. This is because even if they were formed at the grand unified theory (GUT) scale, their density would be too low to induce a cosmological catastrophe, contrary to what happens in the cases of domain walls and monopoles [1,2]. However, this consideration applies only to the case of ordinary Goto-Nambu-type strings, which ultimately radiate away their energy and disappear. As was first pointed out by Davis and Shellard [3], the situation is drastically different for “superconducting” current-carrying strings of the kind originally introduced by Witten [4]. Indeed, it is becoming clearer and clearer [5,6] that the occurrence of stable currents in strings can lead to a real problem because loops can then be stabilized: The current, whether timelike or spacelike, breaks the Lorentz invariance along the string world sheet [7–10], thereby leading to the possibility of rotation [5]. The centrifugal effect of this rotation may then compensate the tension in such a way as to produce an equilibrium configuration, which, if it is stable, is what is known as a *vorton*. Whereas the energy density of nonconducting string distribution decays like that of a radiation gas, in contrast a distribution of relic vortons would scale like matter. Thus, depending [6] on when and how efficiently it

was formed, such a vorton distribution might eventually come to dominate the universe.

In view of this, it is very important to decide which rotating equilibrium configurations really would be stable over cosmologically significant time scales and what fraction of the original population of cosmic string loops would actually end up in such states. Dynamic stability with respect to small perturbations has been established [11–13] for most though not all of the relevant equilibrium states within the framework of the classical string description, but the question of stability against quantum tunneling processes remains entirely open, being presumably dependent on the postulated details of the underlying field theory. If the currents were rigorously conserved, the requirement that the corresponding quantum numbers should lie in the range consistent with stability would from the outset characterize the loops destined to survive as vortons, but in practice things will be more complicated: A lot of future work is needed to estimate the fraction of losses that can be expected from mechanisms such as collisions, longitudinal shocks, cusp formation, and occasional local violations of the permissible current magnitude limits, which may occur before a protovorton loop has finished radiating away its excess energy and settled down as an actual vorton.

A loss mechanism of a rather extreme kind, suggested originally for nonconducting strings by Hawking [14] and considered more recently in the context of conducting models by Larsen and Axenides [15], is that whereby a sufficiently large string loop ends up by undergoing “runaway collapse” to form a black hole. Events of this exotic kind are of intrinsic theoretical interest in their own right, even though it is evident that they must be far too rare to be of

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cosmological importance, since they can only occur for very exceptional cases of initial dynamical configurations with an extremely high degree of symmetry, meaning that they must be almost exactly circular.

The investigation by Larsen and Axenides was restricted to the reflection-symmetric case characterized by the absence of angular momentum, for which they showed [15], subject to the neglect of gravitational and electromagnetic self-interactions, that the presence of a current will not prevent an exactly circular loop from collapsing to what in the framework of the string description would be just a point, corresponding at a microscopic level to a configuration compressed within the radius characterizing the vacuum vortex core of the string, which will typically be of the order of the Compton wavelength associated with the relevant Higgs boson mass, m , say, so that gravitational collapse would follow if the total mass energy M were sufficiently large, $M \gtrsim m^{-1}$ in Planck units. In such a case the neglect [15] of an electromagnetic Coulomb barrier will automatically be justifiable, not because it is entirely absent, but simply because it will be dominated by gravitational attraction, provided the charge Q (if any) on the loop is sufficiently small, $Q \lesssim m^{-1}$. This condition will usually be satisfied because we shall have $Q = Ze$, where e is the relevant particle charge coupling constant, which must either be zero if the current is of electrically neutral type or else must be equal to the electron charge $e \approx 1/\sqrt{137} \approx 10^{-1}$, while Z is the integral charge quantum number: Since the latter arises just from random fluctuations, it will seldom exceed the relevant limit $e^{-1}m^{-1}$, which is of order 10^4 in the GUT case $m \approx 10^{-3}$ and even higher for lighter strings.

The present work extends the analysis of Larsen and Axenides [15] by treating generic circular states, for which the outcome is very different. Unlike the reflection-symmetric zero angular momentum case (which is the only possibility that can occur for a circular string loop of the simple nonconducting kind) a generic circular state for a conducting string loop will be subject to the centrifugal effect. Whereas the Coulomb barrier will usually be negligible, on the other hand, the centrifugal barrier will usually be of dominant importance. It is the centrifugal effect that makes possible the existence of vorton-type equilibrium states, and as will be seen below the associated centrifugal barrier will generically prevent the kind of collapse to a point that was envisaged by Larsen and Axenides. This means that while such a collapse must be very rare even in the nonconducting string case previously envisaged by Hawking, it will be much more extremely rare in the conducting string case envisaged here.

The motivation of the present work is not just to provide an explicit quantitative demonstration of the qualitatively obvious phenomenon of the existence of an infinite centrifugal barrier preventing the collapse of a generic circular configuration of a conducting vortex defect of the vacuum within the framework of the cosmic string description. A less trivial purpose is to explore the limits of validity of this *thin string* description by investigating the conditions under which the current may build up to the saturation limit beyond which the thin string approximation breaks down due to local (transverse or longitudinal) instabilities, so that a nonsingular description of the subsequent evolution would require the use of a more elaborate treatment beyond the scope of the

present work. In order to provide a physically complete analysis of such current saturation phenomena, the present study needs to be generalized to include noncircular configurations, whose treatment will presumably require the use of numerical as opposed to analytic methods. This consideration leads to a secondary motivation for the analytic investigation provided here, which is to provide some firm results that can be used for checking the reliability of the numerical programs that are already being developed for the purpose of treating conducting string loop dynamics in the general case.

There have been many previous studies of circular conducting string dynamics that, unlike the recent analysis of Larsen and Axenides [15], but like that of the present work, already included due allowance for the centrifugal effect. However, these earlier investigations were based on the use of conducting string models that were too highly simplified to provide a realistic description of Witten-type vortex defects. The most obvious example of such a highly simplified conducting string model is the linear type, which has recently been applied to the case of circular loops by Boisseau, Giacomini, and Polarski [16]. A more elegant model originally obtained from a Kaluza-Klein-type projection mechanism by Nielsen [17] has been used for studies of circular loops in various contexts by several authors [18], the application that is most closely related to the present work being that of Larsen [19]. The Nielsen model has the mathematically convenient property of being transonic (meaning that transverse and longitudinal perturbations travel at the same speed) and has been shown to provide an accurate macroscopic description of the effect of microscopic wiggles in an underlying Goto-Nambu-type (nonconducting) string [20]. However, this transonic model cannot describe the physically important saturation effect that arises for large currents in the more elaborate kind of string model [21] that is needed for a realistic description of the essential physical properties [9] of a naturally occurring vacuum vortex such as would result from the Witten mechanism [4].

II. CURRENT AND THE EQUATION OF STATE

To describe a vacuum vortex defect by a cosmic string model, meaning an approximation in terms of a structure confined to a two-dimensional world sheet, it is necessary to know enough about the relevant underlying field theoretical model to be able to obtain the corresponding cylindrical (Nielsen-Olesen-type) vortex configurations. The quantities such as the tension T and the energy per unit length U that are needed for the macroscopic description in terms of the appropriate thin string model are obtained from the relevant underlying vortex configuration by integration over a transverse section [9,10]. In simple nonconducting cases, the cosmic string models obtained in this way will be of the Goto-Nambu type, for which T and U are constant and equal to each other. For more general vortex-forming field theoretical models, the corresponding cosmic string models will be characterized by variable tension and energy which in a generic state will be related by an inequality of the form $T < U$. In many such cases and, in particular, in the category envisaged by Witten [4], to which the present analysis like that of Larsen and Axenides [15] is restricted, the only independent internal structure on the string world sheet will consist of a simple surface current c^μ , say (which may or may not be electrically charged), which implies that the dynamical

cal behavior of the string model will be governed by an equation of state specifying T and U as functions of the current magnitude,

$$\chi = c^\mu c_\mu, \quad (1)$$

and hence as functions of each other [7,8]. In view of the large number of different fields involved in realistic (GUT and electroweak) field theoretical models it is not unlikely that an accurate description of any vortex defects that may occur would require allowance for several independent currents, but even if that is the case, one might expect that in typical situations one particular current would dominate the others so that as a good approximation the effects of the others could be neglected.

The following work, like that of Larsen and Axenides, is based on the kind of string model [21] that is derivable on the basis of Witten's pioneering approach [4] to the treatment of currents in vacuum vortex defects. This approach is based on the plausible supposition [9,10] that the essential large scale features of such a phenomenon can be understood on the basis of an appropriately simplified field theoretical model governed by an effective action involving, in the simplest case just the gauged Higgs-type scalar field responsible for the local symmetry breaking on which the very existence of strings depends, together with a complex scalar "carrier" field Σ , which is subject to a local or global U(1) phase invariance group and which is confined to the vortex core of the string with a phase that may vary along the string world sheet, thereby determining a corresponding surface current [22]. Such a Witten-type scalar field model is not only applicable to cases where the underlying field responsible for the current is actually of this simple scalar type: It can also provide a useful approximation for fermionic fields [4] as well as for vector fields [23,24]. The carrier field will be expressible in the form

$$\Sigma = |\Sigma| \exp[i\varphi\{\sigma, \tau\}], \quad (2)$$

with σ and τ , respectively, the spacelike and timelike parameters describing the string's world sheet, where φ is a real phase variable, whose gradient will contain all the information needed to characterize a particular cylindrical equilibrium configuration of the vortex and, hence, to characterize the local state of the string in the cool limit for which short wavelength excitations are neglected. In conceivable cases for which short wavelength excitations contribute significantly to the energy a more elaborate "warm" string description would be needed [25], but on the basis of the assumption (which is commonly taken for granted in most applications) that the cool limit description is adequate, it follows that there will only be a single independent state parameter, w , say, which can conveniently be taken [7,8,21] to be proportional to the squared magnitude of the gauge-covariant derivative of the phase with components $\varphi_{|a}$, using Latin indices for the world sheet coordinates $\sigma^1 = \sigma$, $\sigma^2 = \tau$. We thus take the state parameter to be

$$w = \kappa_0 \gamma^{ab} \varphi_{|a} \varphi_{|b}, \quad (3)$$

where κ_0 is an adjustable positive dimensionless normalization constant, using the notation γ^{ab} for the inverse of the

induced metric γ_{ab} on the world sheet. The latter will be given, in terms of the background spacetime metric $g_{\mu\nu}$ with respect to four-dimensional background coordinates x^μ , by

$$\gamma_{ab} = g_{\mu\nu} x_{,a}^\mu x_{,b}^\nu, \quad (4)$$

using a comma to denote simple partial differentiation with respect to the world sheet coordinates σ^a . The gauge-covariant derivative $\varphi_{|a}$ would be expressible in the presence of a background electromagnetic field with Maxwellian gauge covector A_μ by $\varphi_{|a} = \varphi_{,a} - e A_{\mu} x_{,a}^\mu$. However, in the application developed below it will be assumed (as was done by Larsen and Axenides [15]) that the gauge term can be omitted, either because the carrier field is uncoupled, meaning $e=0$, or else because the electromagnetic background field is too weak to be important which (as discussed in the Introduction) will be a sufficiently good approximation for most relevant applications, so that it will be sufficient just to take $\varphi_{|a}$ to be the simple partial derivative operation, $\varphi_{|a} = \varphi_{,a}$. With even stronger justification it will also be assumed in the application to be developed below that the local background gravitational field is negligible, so that $g_{\mu\nu}$ can be taken to be flat.

Whether or not background electromagnetic and gravitational fields are present, the dynamics of such a system will be governed [7,8] by a Lagrangian scalar, \mathcal{L} , say, which is a function only of the state parameter w and which determines the corresponding conserved particle current vector, z^a , say, in the world sheet, according to the Noetherian prescription

$$z^a = - \frac{\partial \mathcal{L}}{\partial \varphi_{|a}}, \quad (5)$$

which implies

$$\mathcal{K} z^a = \kappa_0 \varphi^{|a} \quad (6)$$

(using the induced metric for internal index raising), where \mathcal{K} is given as a function of w by setting

$$2 \frac{d\mathcal{L}}{dw} = - \frac{1}{\mathcal{K}}. \quad (7)$$

This current z^a in the world sheet can be represented by the corresponding tangential current vector z^μ on the world sheet, where the latter is defined with respect to the background coordinates x^μ by

$$z^\mu = z^a x_{,a}^\mu. \quad (8)$$

The purpose of introducing the dimensionless scale constant κ_0 is to simplify macroscopic dynamical calculations by arranging for the variable coefficient \mathcal{K} to tend to unity when w tends to zero, i.e., in the limit for which the current is null. To obtain the desired simplification it is convenient not to work directly with the fundamental current vector z^μ that (in units such that the Dirac Planck constant \hbar is set to unity) will represent the quantized particle flux, but to work instead with a corresponding rescaled particle current c^μ , which is obtained by setting

$$z^\mu = \sqrt{\kappa_0} c^\mu. \quad (9)$$

In terms of the squared magnitude χ of this rescaled current c^μ , as given by Eq. (1) the primary state variable w will be given simply by

$$w = \mathcal{K}^2 \chi. \quad (10)$$

It is to be remarked that in the gauge-coupled case, i.e., if e is nonzero, there will be a corresponding electromagnetic current vector obtained by a prescription of the usual form $j^\mu = \partial \mathcal{L} / \partial A_\mu$ which simply gives $j^\mu = e z^\mu = e \sqrt{\kappa_0} c^\mu$.

An important role is played in the theory by the dual Lagrangian Λ , which is obtainable [7] from the original Lagrangian function \mathcal{L} by a Legendre-type transformation that gives

$$\Lambda = \mathcal{L} + \mathcal{K} \chi. \quad (11)$$

In the timelike current range where w is negative the tension and energy density will be, respectively, given by $T = -\mathcal{L}$, $U = -\Lambda$, whereas in the spacelike current range where w is positive they will be given by $T = -\Lambda$, $U = -\mathcal{L}$. Local stability requires the positivity of the squared speeds $c_E^2 = T/U$ and $c_L^2 = -dT/dU$ of extrinsic (wiggle) and longitudinal (sound-type) perturbations, and so the admissible range of variation of the state parameter w or, equivalently, of the squared current magnitude χ will be characterized by

$$\frac{\mathcal{L}}{\Lambda} > 0 > \frac{d\mathcal{L}}{d\Lambda}. \quad (12)$$

The appropriate function $\mathcal{L}\{w\}$ for such a string model is obtainable in principle by integrating the corresponding Lagrangian scalar for the underlying field theoretical model over a two-dimensional section through the relevant cylindrical vortex configuration. In practice this procedure can only be carried out with high precision by using a numerical treatment [9,10]. Progress was delayed for several years by the difficulty of using the output of such a numerical treatment for explicit dynamical applications. This problem has recently been solved by the discovery of very simple empirical formulas [21] (originally expressed using a systematic notation scheme employing a tilde for duality, so that $\tilde{\Lambda}$ and $\tilde{\chi}$ represent what are, respectively, expressed here as \mathcal{L} and $-w$) that provide a convenient analytic description, with sufficient accuracy for realism, within the limited range (12) of w for which the string description is actually valid.

The parameter w can take both positive and negative values depending on whether the current is spacelike or timelike, but for the Witten vortex model that we consider here, it turns out that the corresponding string description is valid only as long as it remains within a bounded range [7–10], outside of which vortex equilibrium states can still exist, but can no longer be stable. What transpires [21] is that the effective Lagrangian for the thin string description can be represented with reasonably good accuracy throughout the allowed range (and with very high accuracy in the timelike part for which $w < 0$) by a function \mathcal{L} that, for a suitably adjusted (typically order of unity) value of the normalization constant κ_0 , is expressible (even in the presence of electromagnetic and gravitational background fields) in terms of just two independent parameters m and m_* in the form

$$\mathcal{L}\{w\} = -m^2 - \frac{m_*^2}{2} \ln \left\{ 1 + \frac{w}{m_*^2} \right\}, \quad (13)$$

which leads to the very simple formula

$$\mathcal{K} = 1 + \frac{w}{m_*^2} \quad (14)$$

for the function introduced above. The allowed parameter range (12) is specifiable by the condition that this function should satisfy

$$e^{-2\alpha} < \mathcal{K} < 2, \quad (15)$$

where

$$\alpha = \left(\frac{m}{m_*} \right)^2. \quad (16)$$

(The lower limit is where the tension T and, hence, also the extrinsic ‘‘wiggle’’ speed tend to zero, while the upper limit is where the longitudinal perturbation speed tends to zero.) The fixed parameters m and m_* have the dimensions of mass and can be interpreted as expressing the respective orders of magnitude of the relevant Higgs field and the (presumably rather smaller) carrier mass scales. It is to be noted that the work of Larsen and Axenides [15] was based on a previously proposed alternative Lagrangian [21] that (in terms of the same parameters m and m_*) provides a somewhat more accurate treatment of the spacelike current range $w > 0$, whereas the newer version (13) provides a treatment that is considerably more accurate for large timelike currents. For our present purpose the slight difference between these alternative string models for Witten vortices is not of qualitative physical importance: Our main reason for preferring to use the newer version (13) has nothing to do with considerations of very high precision, but is just that it turns out to provide more conveniently explicit analytic expressions for the quantities that we shall need.

III. CONSERVATION LAWS

The dynamical equations for such a string model are obtained from the Lagrangian \mathcal{L} in the usual way, by applying the variation principle to a surface action integral of the form

$$S = \int d\sigma d\tau \sqrt{-\gamma} \mathcal{L}\{w\} \quad (17)$$

(using the notation $\gamma \equiv \det\{\gamma_{ab}\}$), in which the independent variables are the phase field φ on the world sheet and the position of the world sheet itself, as specified by the functions $x^\mu\{\sigma, \tau\}$.

Independently of the detailed form of the complete system, one knows in advance, as a consequence of the local or global U(1) phase invariance group, that the corresponding Noether current will be conserved, a condition which is expressible as

$$(\sqrt{-\gamma} z^a)_{,a} = 0. \quad (18)$$

For a closed string loop, this implies (by Green's theorem) the conservation of the corresponding flux integral

$$Z = \oint d\sigma^a \epsilon_{ab} z^b, \quad (19)$$

where ϵ is the antisymmetric surface measure tensor (whose square is the induced metric $\epsilon_{ab}\epsilon^b{}_c = \gamma_{ac}$), meaning that for any circuit around the loop one will obtain the same value for the quantum number Z , which is interpretable as the integral value of the number of carrier particles in the loop. The loop will also be characterized by a second independent quantum number whose conservation is trivially obvious, namely, the topologically conserved phase winding number N that is defined by

$$2\pi N = \oint d\varphi = \oint d\sigma^a \varphi_{,a}. \quad (20)$$

As usual, the stress momentum energy density distribution $\hat{T}^{\mu\nu}$ on the background spacetime is derivable from this action by varying the background metric, according to the specification

$$\hat{T}^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} \equiv \frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}. \quad (21)$$

This leads to an expression of the standard form

$$\sqrt{-g}\hat{T}^{\mu\nu} = \int d\sigma d\tau \sqrt{-\gamma} \delta^{(4)}[x^\rho - x^\rho\{\sigma, \tau\}] \bar{T}^{\mu\nu}, \quad (22)$$

in which the *surface* stress energy momentum tensor on the world sheet (from which the surface energy density U and the string tension T are obtainable as the negatives of its eigenvalues) can be seen to be given [7,8] by

$$\bar{T}^{\mu\nu} = \mathcal{L} \eta^{\mu\nu} + \mathcal{K} c^\mu c^\nu, \quad (23)$$

using the notation

$$\eta^{\mu\nu} = \gamma^{ab} x_{,a}^\mu x_{,b}^\nu \quad (24)$$

for what is interpretable as the (first) fundamental tensor of the world sheet.

Independently of the particular form of the Lagrangian, the equations of motion obtained from the action (17) will be expressible in the standard form [7,8]

$$\bar{\nabla}_\mu \bar{T}^\mu{}_\nu = \bar{f}_\nu, \quad (25)$$

in which $\bar{\nabla}_\mu$ denotes the operator of surface-projected covariant differentiation and where \bar{f}_μ is the external force density acting on the world sheet. When the effect of electromagnetic coupling is significant this will be given in terms of the field $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ by $\bar{f}_\mu = e F_{\mu\nu} z^\nu$. Even if this force density is nonzero, its contraction with the current vector z^μ or with the corresponding rescaled current vector c^μ will vanish, and hence it can be seen from the preceding formulas that the equations of motions automatically imply the surface current conservation law

$$\bar{\nabla}_\mu c^\mu = 0, \quad (26)$$

which is the equivalent, in background tensorial notation, of the condition expressed above in terms of z^a using what was expressed above in world sheet coordinate notation. The background tensorial operator $\bar{\nabla}$ in the foregoing equations is definable formally by

$$\bar{\nabla}^\mu \equiv \eta^{\mu\nu} \nabla_\nu \equiv x_{,a}^\mu \gamma^{ab} \nabla_b, \quad (27)$$

where ∇ is the usual operator of covariant differentiation with respect to the Riemannian background connection. Thus for any closed loop there will be a corresponding conserved circuit integral C given by

$$C = \oint dx^\mu \varepsilon_{\mu\nu} c^\nu, \quad (28)$$

where $\varepsilon_{\mu\nu}$ is the background spacetime version of the surface measure tensor ϵ_{ab} , which means that its contravariant version is the antisymmetric tangential tensor that is given by

$$\varepsilon^{\mu\nu} = \epsilon^{ab} x_{,a}^\mu x_{,b}^\nu. \quad (29)$$

This constant C is of course just a rescaled version of the integer particle quantum number Z , which will be given in terms of it by

$$Z = \sqrt{\kappa_0} C. \quad (30)$$

In the following work (as in the preceding work of Larsen and Axenides [15]) it will be assumed either that the current is uncoupled or else (as will more commonly be the case) that $F_{\mu\nu}$ is negligible, so that we can simply take

$$\bar{f}_\mu = 0. \quad (31)$$

As well as neglecting electromagnetic correction effects, we shall now also restrict our attention to cases in which the background is both axisymmetric and stationary, as is the case for the flat space in which we are in the end most particularly interested. This means that there will be corresponding vectors, ℓ^μ and k^μ , say, which satisfy the Killing equations

$$\nabla_\mu \ell_\nu + \nabla_\nu \ell_\mu = 0, \quad \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0, \quad (32)$$

and which will be, respectively, interpretable as generators of rotations and time translations, so that when suitably normalized, their effect can be expressed in the form

$$k^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}, \quad \ell^\mu \frac{\partial}{\partial x^\mu} = 2\pi \frac{\partial}{\partial \phi}, \quad (33)$$

where t is an ignorable time coordinate and ϕ is an ignorable angle coordinate. This normalization is such that the total circumferential length of the circular trajectory of the angle Killing vector will simply be given by ℓ where

$$\ell^2 = \ell^\mu \ell_\mu. \quad (34)$$

These Killing vectors can be employed in the usual way to define the corresponding angular momentum surface current vector \mathcal{J}^μ and the corresponding energy current vector \mathcal{E}^μ by setting

$$\bar{T}^\mu{}_\nu \ell^\nu = 2\pi \mathcal{J}^\mu, \quad \bar{T}^\mu{}_\nu k^\nu = -\mathcal{E}^\mu. \quad (35)$$

These currents will then satisfy surface conservation laws

$$\bar{\nabla}_\mu \mathcal{J}^\mu = 0, \quad \bar{\nabla}_\mu \mathcal{E}^\mu = 0, \quad (36)$$

which have the same form as that satisfied by the current c^μ . This means that for a closed loop there will be corresponding conserved angular momentum and mass-energy integrals, J and M , say, which will be given by

$$J = \oint dx^\mu \varepsilon_{\mu\nu} \mathcal{J}^\nu, \quad M = \oint dx^\mu \varepsilon_{\mu\nu} \mathcal{E}^\nu. \quad (37)$$

IV. CONSTANTS OF CIRCULAR MOTION

We now restrict ourselves to cases for which the string configuration itself shares the background spacetime property of being symmetric with the action generated by the Killing vector ℓ^μ . This entails that ℓ^μ should be tangential to the world sheet, i.e.,

$$\ell^\mu = \lambda^a x^\mu_{,a}, \quad (38)$$

where λ^a is a corresponding Killing vector with respect to the intrinsic geometry of the world sheet, which, on the understanding that ℓ^μ is interpretable, in the manner described above, as the generator of angular rotation about a symmetry axis, means that the string configuration is *circular*, its circumference at any instant being given by the local value of ℓ .

In such a case, this Killing vector ℓ^μ can be used to generate the, in that case circular, circuit used for evaluating these integrals, i.e., the infinitesimal displacement in the integrand can be taken to be given by $2\pi d\sigma^a = \lambda^a d\phi$, so that we obtain

$$2\pi dx^\mu = x^\mu_{,a} \lambda^a d\phi = \ell^\mu d\phi, \quad (39)$$

where ϕ is an ignorable angle coordinate of the usual kind with period 2π as introduced above. (Such an angle coordinate can be conveniently used to specify the first world sheet coordinate σ^1 by setting $\sigma = \phi$, so that by taking the second world sheet coordinate $\sigma^2 = \tau$ to be constant on the circular symmetry trajectories the components of the intrinsic Killing vector are obtained in the form $\{\lambda^1, \lambda^2\} = \{1, 0\}$.) Substituting this ansatz for dx^μ in the corresponding integral formulas, it can be seen that the global integrals (19) and (20) for the winding number N and the particle number Z will be given directly in the circular case by corresponding *locally defined* Bernoulli-type constants of motion, B and C , say, according to the relations

$$B = 2\pi \sqrt{\kappa_0} N, \quad C = \frac{Z}{\sqrt{\kappa_0}}, \quad (40)$$

where these quantities—of which the latter C is directly identifiable with the global flux of the current c^μ so that it is justifiable to designate it by the same symbol—are now to be thought of as being constructed according to the prescriptions of the purely local form [26]

$$B = \sqrt{\kappa_0} \lambda^a \varphi_{|a}, \quad C = \ell^\mu \varepsilon_{\mu\nu} c^\nu. \quad (41)$$

The reason why the single symmetry generator ℓ^μ gives rise to not just one but two independent Bernoulli-type constants in this way is attributable to the the string duality property [7,8]. (It is to be noted that instead of using the rationalized constants B and C , the work of Larsen [15,19] uses corresponding unrationalized constants n and Ω , which are expressible in terms of our present notation by $\Omega = -C/2\pi$ and $n = \sqrt{\kappa_0} N = B/2\pi$.)

In a similar manner the mass (when it is defined) and angular momentum integrals introduced in the previous section will also be expressible in terms of purely locally defined constants of the motion in the circular case. To do this it is convenient [26] to start by introducing an effective momentum tangent vector Π^μ given in terms of the relevant Killing vector, namely, ℓ^μ in the case with which we are concerned, by the ansatz

$$\Pi^\mu = \ell^\nu \varepsilon_{\nu\rho} \bar{T}^{\rho\mu}. \quad (42)$$

Since the surface stress momentum energy tensor $\bar{T}^{\mu\nu}$ only has components in directions tangential to the world sheet, it is evident that this formula provides a vector Π^μ that automatically has the required property of being tangent to the string world sheet.

Whenever the background spacetime is invariant under the action of another Killing vector k^μ , which in the application to be considered here will be interpreted as expressing stationarity, so that the corresponding integral formula (37) for M is well defined, it can be seen using Eq. (39) again that this globally defined quantity will now be obtainable as a purely local constant of the motion from the formula

$$k^\mu \Pi_\mu = -M. \quad (43)$$

It can also be seen that whether or not the background has a (stationary) symmetry generated by k^μ , the other (angular momentum) constant J provided according to Eq. (37) by the original (axisymmetry) Killing vector ℓ^μ itself will similarly be obtainable, in the circular case, as a purely local constant of the motion given by

$$\ell^\mu \Pi_\mu = 2\pi J. \quad (44)$$

It is, however, to be noted that this last constant is not independent of the ones presented above: It can be seen by substituting the formula (23) in Eq. (42) and using the defining relations (6) and (9) for the current that it will be expressible in terms of the two (mutually dual) Bernoulli constants (41) by the simple product formula

$$J = \frac{BC}{2\pi} = NZ. \quad (45)$$

This shows that the integral quantization of the winding and particle numbers N and Z automatically entails the integral quantization of the angular momentum J .

V. DYNAMICS OF CIRCULAR MOTION

Whenever the string motion shares the symmetry generated by a background spacetime Killing vector, the problem of solving the equations of motion for its two-dimensional world sheet can naturally be reduced to a problem of finding a one-dimensional trajectory tangential to the world sheet but not aligned with the symmetry generator, since when such a trajectory has been found it is trivial to extend it to the complete two-dimensional world sheet by the symmetry action. The general procedure for obtaining such a tangential trajectory for symmetric solutions of the equations of motion of conducting string models was originally developed by Carter, Frolov, and Heinrich [26], who applied this method to study stationary solutions in a Kerr black hole background. This method was adapted to the kind of situation with which the present application is concerned, namely, circular instead of stationary symmetry, by Larsen [19]. The original derivation [26] involved the use of quotient space with respect to the relevant symmetry action, but a more recent and general treatment [8] (including allowance for the possibility of axionic as well as electromagnetic coupling) has provided a more direct route that does not need such an auxiliary construction.

What this procedure provides is a particular kind of world-sheet-generating trajectory that is characterized by having a tangent vector Π^μ given in terms of the relevant Killing vector, namely, ℓ^μ in the case with which we are concerned, by the ansatz (42). The procedure makes use of the fact [8,26] that, for given values of the Bernoulli constants B and C , the state function w and, hence, also the squared magnitude of the tangent vector Π^μ , namely, the quantity

$$\Psi = \Pi^\mu \Pi_\mu \quad (46)$$

(which will play the role of a potential), can be specified in advance as a *scalar field* over the entire background space (not just a single string world sheet), in such a way as to agree with the respective physical values of Ψ on the particular world sheet under consideration.

In the generic case this is done by expressing Ψ as a function of one of the independent state variables, w , say, which will itself be expressible, as a function of the squared Killing vector magnitude ℓ^2 and, hence, as a scalar field over the entire background space by solving the equation

$$\ell^2 = \frac{B^2}{w} - \frac{C^2}{\chi}, \quad (47)$$

where χ is the squared magnitude (1) of the current vector, which is obtainable from the Lagrangian \mathcal{L} as a function of w in the manner described above. It is to be remarked that in the recent derivation [8] the corresponding formula (expressed using the notation β for C , $\tilde{\beta}$ for B , and $-\tilde{\chi}$ for w) contains a transcription error that has the effect of replacing w and χ by their respective squares. Except in the ‘‘chiral,’’ i.e., *null current*, limit for which w and χ vanish so that

a special treatment is needed, the self-dual formula (47) is immediately obtainable by using the Bernoulli formulas (41) to evaluate the respective components of ℓ^μ parallel to and orthogonal to the current.

Having used this procedure to obtain w as a field over the background spacetime, one can then use the result to obtain the corresponding value of the squared magnitude Ψ , which, in a recent derivation [8], was written as X^2 , though it need not be positive, since the vector Π^μ may be timelike or null as well as spacelike. This quantity will be obtainable using Eq. (47) as a function of the squared Killing vector magnitude ℓ^2 (which in the case of symmetry must necessarily be positive) by the manifestly self-dual formula

$$\Psi = \frac{C^2 \Lambda^2}{\chi} - \frac{B^2 \mathcal{L}^2}{w}, \quad (48)$$

where Λ is the dual Lagrangian function, as given by Eq. (11).

In order to obtain a treatment that remains valid in the ‘‘chiral’’ limit, for which w and χ vanish, so that formulas (47) and (48) become indeterminate, it is convenient to rewrite Ψ in a manner that sacrifices manifest self-duality by expressing it in the form

$$\Psi = \frac{B^2 C^2}{\ell^2} - Y^2, \quad (49)$$

where (unlike the quantity $X = \sqrt{\Psi}$ which may be imaginary) the quantity Y defined in this way will always be real in the admissible range (12), as can be seen by expressing it in either of the equivalent, mutually dual, alternative forms

$$Y = \frac{B^2}{\mathcal{K}\ell} - \Lambda\ell = \frac{C^2 \mathcal{K}}{\ell} - \mathcal{L}\ell, \quad (50)$$

of which the latter is the most convenient for practical calculations starting from a given form of the Lagrangian \mathcal{L} .

In the generic case, $B^2 \neq C^2$, the required quantities \mathcal{K} and \mathcal{L} are obtained indirectly as functions of ℓ by solving Eq. (47) which will give a result that is always non-null; i.e., w and χ will never pass through zero, and so the current will preserve a character that is permanently timelike or permanently spacelike as the case may be. The exception is the ‘‘chiral’’ case, which is characterized by the equality $B^2 = C^2$ and for which the only possible states are of the null kind characterized by $w = \chi = 0$, so that the required quantities \mathcal{K} and \mathcal{L} will be given directly, independently of ℓ , as the constants $\mathcal{K} = 1$ and $\mathcal{L} = -m^2$.

It is of particular interest for the dynamical applications that follow to obtain the derivative of the field Y with respect to the cylindrical radial coordinate ℓ with respect to which it is implicitly or (in the chiral case) explicitly defined: It is obvious in the chiral case since in this case $\Lambda = \mathcal{L}$, and it can also be verified [using the relation

$$\frac{d\ell}{dw} = -\frac{\ell}{2w} + \frac{C^2(d\mathcal{K}/dw)}{2\ell w(d\mathcal{L}/dw)} \quad (51)$$

for the variations of ℓ in the generic case for which \mathcal{L} and \mathcal{K} are variable that this derivative will be expressible in either of the very simple, mutually dual, equivalent forms

$$\frac{dY}{d\ell} = -\frac{C^2\mathcal{K}}{\ell^2} - \Lambda = -\frac{B^2}{\mathcal{K}\ell^2} - \mathcal{L}, \quad (52)$$

of which again it is the latter that is most convenient for practical calculations starting from a given form of the Lagrangian \mathcal{L} .

After having obtained the field Ψ in this way, the final step in the procedure for obtaining the string tangent vector Π^μ is to integrate its equations of motion, which can easily be shown [8,26,27] to have the form

$$2\Pi^\nu\nabla_\nu\Pi_\mu = \nabla_\mu\Psi, \quad (53)$$

subject to the constraint

$$\ell^\mu\Pi_\mu = BC, \quad (54)$$

whose conservation is an automatic consequence of the symmetry. This constant is interpretable according to Eq. (45) as being proportional to the angular momentum J .

The foregoing formulation depends only on the existence of the symmetry generated by ℓ^μ , which is postulated to apply not only to the background but also to the string solution itself, so that the further supposition that this symmetry is that of rotation about an axis means that the string configuration is circular. It does not depend on the additional postulate that the background spacetime is also subject to another symmetry generated by the independent Killing vector k^μ that is responsible for the existence of the mass-energy constant M given by Eq. (43).

In order to solve the equation of motion (53), which can be seen to be interpretable as that of a geodesic with respect to a conformal modified background metric given by $\Psi g_{\mu\nu}$, it is useful to employ a Hamiltonian formulation of the standard form

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial \Pi_\mu}, \quad \frac{d\Pi_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}, \quad (55)$$

where τ is parameter along the trajectory, which can conveniently be used to specify the choice of the second internal coordinate σ^2 (the first one σ^1 , having already been chosen to be the angle coordinate ϕ). Such a formulation of the conformal geodesic equation is readily obtainable by taking the Hamiltonian to be given by

$$2H = g^{\mu\nu}\Pi_\mu\Pi_\nu - \Psi, \quad (56)$$

with the understanding that the system is to be solved subject to the constraint

$$H = 0, \quad (57)$$

which ensures the correct normalization of the tangent vector, which by the first Hamiltonian equation will be given directly by

$$\Pi^\mu = \frac{dx^\mu}{d\tau}. \quad (58)$$

VI. SOLUTION FOR THE STATE FUNCTION

In order to carry out the procedure summarized in the preceding section, we first have to solve Eq. (47) for the state variable w . In terms of the magnitude ℓ of the axisymmetry generator ℓ^μ , which will give the local value of the circumference of the circular string loop, this equation will be expressible as

$$\ell^2 w = B^2 - C^2 \mathcal{K}^2. \quad (59)$$

For a general conducting string model this equation would be hard to solve explicitly, but there are special cases for which a convenient analytic solution is available, the first known example being that of the transonic string model, for which the equation for the field w is found to be simply *linear*, so that it can immediately be solved to provide a system that turns out to be completely integrable by separation of variables in a Kerr black hole background when the symmetry under consideration is stationarity [26], though unfortunately not when it is axisymmetry [19], in which case such complete integrability is available only for purely equatorial configurations, including in the limit, the kind of flat space ring configurations with which the present study is ultimately concerned. Another case whose application to circular configurations has recently been considered [16] and in which Eq. (47) is also simply linear, is that of the even cruder fixed trace model (for which \mathcal{K} is just a constant) that was originally suggested by Witten himself [4] to describe the effect of his mechanism for currents that are very small compared with saturation, but which turns out [9] to be misleading (because subsonic) even in that limit.

The more recent work of Larsen and Axenides [15] was more advanced in that it used the newer kind of string model [21] that (unlike the simple transonic model and its cruder fixed trace predecessor) can provide a realistic account of the current saturation effect that is a salient feature of the Witten mechanism; this work was, however, much more specialized than the preceding investigations cited above, as it only considered the nonrotating case of vanishing winding number N , which in our present notation means $B=0$. As well as allowing for a nonzero winding number, the present work involves a physically unimportant but technically valuable improvement in that we use more recently proposed [21] string model characterized by Eq. (13), which turns out to be particularly convenient for the present stage in our analysis since it leads to an equation for w that although not actually linear, as was the case for the transonic model, can be seen to be the next best thing, meaning that it is just *quadratic* (whereas the version used by Larsen and Axenides gives an equation for w that has a much more awkward quartic form). The result of using Eq. (13) is expressible in the form

$$C^2 \mathcal{K}^2 + m_*^2 \ell^2 (\mathcal{K} - 1) - B^2 = 0, \quad (60)$$

which can immediately be solved to give

$$\mathcal{K} = \frac{-m_*^2 \ell^2 + \sqrt{4C^2(B^2 + m_*^2 \ell^2) + m_*^4 \ell^4}}{2C^2}, \quad (61)$$

choosing the positive sign for the square root because \mathcal{K} is positive throughout the admissible range (12) for the state parameter w .

In terms of this explicit formula for \mathcal{K} the state function w itself is immediately obtainable using the expression

$$w = m_*^2 (\mathcal{K} - 1), \quad (62)$$

which applies for this model. Since our Lagrangian (13) can be expressed directly in terms of \mathcal{K} as

$$\mathcal{L} = -m_*^2 (\alpha + \ln \sqrt{\mathcal{K}}), \quad (63)$$

our explicit formula for \mathcal{K} can also be directly applied to obtain the required potential Ψ , for which we obtain the formula

$$\Psi = \frac{B^2 C^2}{\ell^2} - \ell^2 \left(\frac{C^2 \mathcal{K}}{\ell^2} + m_*^2 (\alpha + \ln \sqrt{\mathcal{K}}) \right)^2. \quad (64)$$

VII. MOTION IN A FLAT BACKGROUND

Up to this point we have been using a formulation that is valid for an arbitrary stationary axisymmetric background, including, for example, that of a Kerr black hole. In order to obtain a result that is completely integral in explicit form and because it is the case of greatest physical importance, we shall now restrict our attention to the case of a flat space background, for which there will be no loss of generality in supposing the circular string loop to be confined to an equatorial hyperplane with three-dimensional spacetime metric given in terms of circular coordinates $\{r, \phi, t\}$ by

$$ds^2 = dr^2 + r^2 d\phi^2 - dt^2, \quad (65)$$

so that the Killing vectors used in the discussion above will be identifiable as $\{k^1, k^2, k^3\} = \{0, 0, 1\}$ and $\{\ell^1, \ell^2, \ell^3\} = \{0, 2\pi, 0\}$.

In these circumstances the circumferential length field ℓ that played a fundamental role in the preceding discussion will be given simply by

$$\ell = 2\pi r, \quad (66)$$

and the evolution of the circular string world sheet will be given simply by specifying the radius r as a function of the background time t . We shall use a dot to denote differentiation with respect to this time t , which will vary proportionally to the Hamiltonian time τ , with a coefficient given by the energy constant, so that we shall have

$$\frac{dt}{d\tau} = M. \quad (67)$$

For a complete physical description of the solution, it would also be necessary to specify the distribution over the world sheet of the phase field φ , which must evidently have the form

$$\varphi = q + N\phi, \quad (68)$$

where N is the conserved winding number as defined above and q is a function only of t .

The fact that the fourth (azimuthal) direction can be ignored in this particularly simple case means that the complete set of equations of motion is provided directly in first integrated form by the constants of the motion. Using the formulas (6) and (9) to work out the expression (41) for the Bernoulli constant C , it can be seen that the time derivative of this function q will be given in terms of that of the radius r by

$$\dot{q} = C\mathcal{K} \frac{\sqrt{1 - \dot{r}^2}}{2\pi r \sqrt{\kappa_0}}. \quad (69)$$

By similarly using the formula (23) to work out the expression (43) for the mass-energy constant M the evolution equation for r can be obtained in the first integrated form

$$M \sqrt{1 - \dot{r}^2} = Y, \quad (70)$$

where Y is the quantity that is given by the formula (50), whose evaluation as a function of the circumference, $\ell = 2\pi r$ is discussed in Sec. V.

Instead of going through the detailed evaluation of the expression (43) using Eq. (23), a more elegant albeit less direct way of obtaining the same equation of motion for r is to apply the Hamiltonian formalism described in Sec. V. It is evident from Eq. (67) that in the flat background (65) the radial momentum component Π_1 will be given by

$$\Pi_1 = \frac{dr}{d\tau} = M\dot{r}, \quad (71)$$

and under these conditions the Hamiltonian (56) will reduce to the simple form

$$H = \frac{1}{2} \left(\Pi_1^2 + \frac{J^2}{r^2} - M^2 - \Psi \right). \quad (72)$$

It is to be remarked that the term J^2/r^2 in this formula has the form of the centrifugal barrier potential that is familiar in the context of the analogous problem for a point particle. By what is a rather remarkable cancellation, it can be seen that the effect of the extra potential Ψ , taking account of the elastic internal structure of the string, is merely to replace the familiar centrifugal barrier contribution J^2/r^2 by a modified barrier contribution given simply by Y^2 where Y is the scalar field (50) introduced in the previous section, since it can be seen that the relevant combination of terms turns out to be expressible simply as

$$\frac{J^2}{r^2} - \Psi = Y^2. \quad (73)$$

The normalization expressed by the constraint that the Hamiltonian should vanish can thus be seen to give the equation of motion for r in the convenient first integrated form

$$M^2 \dot{r}^2 = M^2 - Y^2, \quad (74)$$

which is evidently equivalent to the radial evolution equation (70) given above.

VIII. STATIONARY “VORTON” STATES

As an immediate particular consequence of this equation of motion, it can be seen that there will be *vorton-type* equilibrium solutions, with mass energy given by

$$M = Y, \quad (75)$$

wherever the relevant effective energy function Y satisfies the stationarity condition

$$\frac{dY}{dr} = 0. \quad (76)$$

The formula (52) given above for the derivative of Y can be used to write this stationarity condition in the form

$$C^2 \mathcal{L}_w = B^2 \Lambda \chi. \quad (77)$$

This is recognizable as the equilibrium requirement that is well known from previous more specialized studies of circular equilibrium states [28], according to which the propagation speed c_E of extrinsic (wobble) perturbations determines the effective rotation velocity v , namely, that of the current in the timelike case, for which one obtains $v^2 = T/U = \mathcal{L}/\Lambda$, and that of the orthonormal tangent direction if the current is spacelike, in which case one obtains $v^2 = T/U = \Lambda/\mathcal{L}$. Finally, in the “chiral” case for which the current is null, both formulas are valid simultaneously: One will have $\Lambda = \mathcal{L}$ and $v = 1$.

In all of these cases the vorton circumference will be given by

$$\ell_v = \frac{|B|}{\sqrt{-\mathcal{K}\mathcal{L}}}, \quad (78)$$

and the equilibrium condition to be solved for the state function of the vorton will be expressible in the more directly utilizable form

$$\mathcal{K}^2 \frac{\mathcal{L}}{\Lambda} = b^2, \quad (79)$$

using the abbreviation b for the *Bernoulli ratio*, as defined by

$$b = \frac{B}{C} = 2\pi\kappa_0 \frac{N}{Z}, \quad (80)$$

where N and Z are the corresponding integer-valued winding number and particle quantum number (of which, if the current were characterized by a nonzero electric coupling constant e , the latter would determine the vorton’s total ionic charge, namely, $Q = Ze$, as in ordinary atomic physics).

From the well-known theorem [11] that (although there may be instabilities with respect to nonaxisymmetric perturbations in certain cases) the circular equilibrium states are *always* stable with respect to perturbations that preserve their circular symmetry, it follows that within the admissible range (12) the effective energy function Y can be extreme only at a minimum but never at a maximum. This evidently implies that, within a continuously connected segment of the admissible range, there can be at most a single such extre-

um: In other words for a given value of the conserved ratio b^2 , the vorton equation can have *at most* one solution for the state variable w and, hence, for any function thereof, such as the derived variable \mathcal{K} and the corresponding vorton circumference ℓ , which will thus be *uniquely* determined. It will be seen in the next section that in some cases there will be no solution at all; i.e., there are values of the ratio b^2 for which Y is monotonic throughout the allowed range, so that a corresponding vorton state does not even exist.

IX. SOLUTION OF THE EQUATIONS OF MOTION

The results in the immediately preceding section are independent of the particular form of the Lagrangian \mathcal{L} . If we now restrict ourselves to the specific case of the model (13), we can use the results of the earlier sections to rewrite the effective barrier energy function Y in the form

$$Y = m_*^2 \ell \left[\alpha + \ln \sqrt{\mathcal{K}} + \left(\frac{C}{m_* \ell} \right)^2 \mathcal{K} \right], \quad (81)$$

with \mathcal{K} given explicitly as a function of ℓ by Eq. (61). A convenient way of applying this formula is to think of \mathcal{K} as the independent variable, with the circumference ℓ (and hence the radius $r = \ell/2\pi$) given by

$$m_*^2 \ell^2 = \frac{B^2 - C^2 \mathcal{K}^2}{\mathcal{K} - 1}. \quad (82)$$

In the case $b^2 < 1$, which means $B^2 < C^2$, this determines ℓ as a monotonically increasing function of \mathcal{K} in the *timelike* current range, $e^{-2\alpha} < \mathcal{K} < 1$. In the case $b^2 > 1$, which means $B^2 > C^2$, this determines ℓ as a monotonically decreasing function of \mathcal{K} in the *spacelike* current range, $1 < \mathcal{K} < 2$. In either case, we finally obtain the effective barrier energy function in the form

$$Y = m_* |C| \sqrt{\frac{b^2 - \mathcal{K}^2}{\mathcal{K} - 1}} \left(\alpha + \ln \sqrt{\mathcal{K}} + \frac{\mathcal{K}(\mathcal{K} - 1)}{b^2 - \mathcal{K}^2} \right), \quad (83)$$

as a fully explicit function just of \mathcal{K} . The formula (52) for the derivative of this function gives

$$\frac{dY}{d\ell} = m_*^2 \left(\alpha + \ln \sqrt{\mathcal{K}} - \frac{b^2(\mathcal{K} - 1)}{\mathcal{K}(b^2 - \mathcal{K}^2)} \right). \quad (84)$$

It can thus be seen that the vorton equilibrium requirement (79), expressing the condition (76) that this derivative should vanish, will be given for this particular string model by

$$\mathcal{K} = \mathcal{K}_v, \quad (85)$$

where \mathcal{K}_v is obtained by solving the equation

$$b^2 = \frac{\mathcal{K}_v^2 (\alpha + \ln \sqrt{\mathcal{K}_v})}{\alpha - 1 + \ln \sqrt{\mathcal{K}_v} + \mathcal{K}_v^{-1}}. \quad (86)$$

Whenever an admissible solution exists, it can be seen that the corresponding value

$$M = M_v \quad (87)$$

of the mass of the vorton state will be given by

$$\frac{M_v}{m_*} = |C| \sqrt{\frac{\mathcal{K}_v - 1}{b^2 - \mathcal{K}_v^2} \left(\frac{b^2}{\mathcal{K}_v} + \mathcal{K}_v \right)}. \quad (88)$$

X. CONFINEMENT EFFECT AND CLASSIFICATION OF SOLUTIONS

Since \mathcal{K} tends to unity for large values of ℓ , it can be seen from Eq. (81) that the effective potential Y grows linearly with radius at large distances. This means that no matter how large its energy may be, the loop can never expand to infinity: It is subject to a confinement effect (not unlike that which motivated early attempts to use string models to account for the phenomenon of quark confinement in hadron theory [29]).

The fact that it admits no possibility of unbound trajectories distinguishes the loop problem considered here from cases such as the familiar example a point particle, of mass m , say, moving in the Newtonian gravitational field of a central mass, M_* , say. In that case, the orbits can be classified as type 0, type 1, and type 2, where type 0 means the special case of constant radius (circular) orbits, type 1 means the generic case of varying radius but nevertheless bound orbits, and type 2 denotes unbound orbits. These types can be subclassified into categories A and B, where A stands for “always regular” or “avoiding trouble” and B stands for “badly terminating.” For type 0 orbits, the “good” subcategory A is clearly the only possibility. However, while type 1 orbits are generically of type 1A, which for an inverse square law means the elliptic case, there is also the possibility of type 1B orbits, meaning bound trajectories of purely radially moving type, which end by plunging into the central singularity. Similarly type 2 orbits are generically of type 2A, which for an inverse square law means the parabolic and hyperbolic cases, but there is also the possibility of type 2B orbits, meaning unbound trajectories of purely radially moving type which begin or end at the central singularity. In the simple point particle case the only relevant parameters are the orbital binding energy E , say, and the angular momentum J , say. Subcategory B corresponds to the special case $J=0$. In the inverse square law case the classification is simplified by the property of self-symmetry with respect to the transformations $E \mapsto E/s$, $J \mapsto J\sqrt{s}$ where s is a scale factor: Thus, for the generic subcategory A, the classification depends just on the invariant dimensionless combination $EJ^2/m^3M_*^2$, being type 0 for its absolute minimum value, which is $-1/2$, type 1 for a higher but still negative value, and type 2 otherwise.

The same principles can be applied to the classification of solutions of the circular string loop problem, for which one only needs the type 1, with “good” and “bad” subcategories 1A and 1B, and the type 0, which in this case means a vorton state, which can only be “good.” There is no analogue of type 2 for the string loop problem because the possibility of an unbound orbit does not exist. This is because the relevant effective potential function Y does not only diverge to infinity (due to the centrifugal effect) as the radius r becomes small, i.e., as $\ell \rightarrow 0$, which corresponds to $\mathcal{K} \rightarrow |b|$: It is evident that Y must also diverge (due to the energy needed for stretching the string) in the large r limit,

i.e., as $\ell \rightarrow \infty$, which corresponds to $\mathcal{K} \rightarrow 1$.

Despite the fact that instead of the five possibilities (namely, 0, 1A, 1B, 2A, and 2B) needed for the point particle problem there are only three (namely, 0, 1A, and 1B) in the circular string loop problem, the state of affairs for this latter problem is considerably more complicated because the orbits are not fully characterized just by the mass energy parameter M and the angular momentum parameter J : They also depend on the Bernoulli constants B and C [which, by Eq. (40), are, respectively, proportional to the microscopic winding number N and the particle number Z]. According to Eq. (45) these constants are related by the condition $BC = 2\pi J$, but that still leaves three independent parameters which may conveniently be taken to be M , B , C , say, instead of the two that were sufficient for the point particle case. As in the inverse square law case for a point particle, the flat space string loop problem is self-similar with respect to scale transformations, which are expressible in this case by $B \mapsto Bs$, $C \mapsto Cs$, and $M \mapsto Ms$ (so that $J \mapsto Js^2$). Thus, whereas all that mattered qualitatively in the inverse square law was a *single* dimensionless ratio (namely, that between J^2 and E^{-1}), in a corresponding manner the not so simple behavior of the circular string loop is qualitatively dependent on the *two* independent dimensionless ratios relating B^2 , C^2 , and M^2 . A further complication is that the nature of this dependence depends on the dimensionless parameter α characterizing the underlying string model.

Unlike the mass-energy parameter M , whose conservation depends on the stationary character of the space time background, and would no longer hold exactly when allowance is made for losses from gravitational radiation, the winding number and particle number satisfy conservation laws of a less conditional nature, and so (although their local conservation is also symmetry dependent) the corresponding Bernoulli parameters B and C provide more fundamental information about the string loop. It is therefore appropriate to use their ratio b as the primary variable in a classification of the solution (with the understanding that $b = \infty$ means $C=0$).

Proceeding on this basis, the relevant parameter space can be described in terms of five consecutive zones for the parameter b^2 . The reason why there are so many possibilities is that the range of ℓ , from 0 to ∞ corresponds, according to Eq. (82), to a range of \mathcal{K} from 1 to $|b|$, which may extend beyond the range (15) that is physically admissible according to the criterion (12).

Between the limits where it diverges, $Y \rightarrow +\infty$, as $\mathcal{K} \rightarrow 1$ and $\mathcal{K} \rightarrow |b|$, the effective potential energy function Y will vary smoothly with at least one local minimum. However, according to the theorem recalled at the end of the previous section, Y can have at most one local minimum and no local maximum within the admissible range (15). Moreover, since α is strictly positive by its construction (16), it is evident that the large radius limit $\mathcal{K} \rightarrow 1$ will always lie safely within the physically admissible range (15). This leaves only two alternative possibilities, which are either that Y should be monotonic, with $dY/d\ell > 0$, throughout the physically admissible range (15), or else that this admissible range should include a turning point at a critical value of ℓ within which the derivative $dY/d\ell$ will become negative, in which case it will have to remain negative all the way to the inner limit of the ad-

missible range. It is directly apparent from the expression (84) for $dY/d\ell$ that there is no possibility for it to remain positive near the limit of the admissible range in the timelike current case, i.e., as $\ln\sqrt{\mathcal{K}} \rightarrow -\alpha$, and so for $b^2 < 1$ the vorton equilibrium equation (86) will always have a physically admissible solution. However, in the spacelike current case, $b^2 > 1$, for which the relevant limit of the admissible range is given (independently of α) by $\mathcal{K} \rightarrow 2$, it can be seen that it is indeed possible for the gradient (84) to remain positive, the condition for this being the criterion for the first of the qualitatively different zones listed as follows.

Zone I. This is the ‘‘fatal’’ spacelike zone characterized by

$$b^2 \left(1 - \frac{1}{2\alpha + \ln 2} \right) \geq 4, \tag{89}$$

for which no admissible vorton solution exists. For such a scenario it can be seen from Eq. (83) that the mass energy must necessarily satisfy the condition

$$M \geq M_s, \tag{90}$$

where the mass limit M_s is given by

$$\frac{M_s}{m_*} = \sqrt{B^2 - 4C^2} \left(\alpha + \frac{1}{2} \ln 2 + \frac{2}{b^2 - 4} \right). \tag{91}$$

In this case, after a possible phase of expansion to a maximum radius obtained by solving $Y = M$, the loop will inevitably contract until it reaches the current saturation limit at $\mathcal{K} = 2$, at which stage our classical string description will break down. This means that in terms of the terminology introduced above, all zone I trajectories are of type 1B. In Fig. 1 is displayed the potential Y against (top) the value of \mathcal{K} and (bottom) that of the circumference ℓ , all quantities being rescaled with the Kibble mass m . It should be noted that for $\alpha \geq 1$, the potential is roughly (i.e., up to negligible logarithmic corrections) independent of α when seen as function of ℓ but not as a function of \mathcal{K} . This shows that the most relevant parameter for cosmological applications is m and not m_* even though the latter is essential for the very existence of vorton states. The starting point of this zone marks the end of the curves in Fig. 6.

It is to be remarked that in order for this zone to be of finite extent, the carrier mass scale m_* must not be too large compared with the Kibble mass scale m , the precise condition being that the value of α given by Eq. (16) should satisfy the inequality

$$\alpha > (1 - \ln 2)/2. \tag{92}$$

If this condition were not satisfied, which would be unlikely in a realistic model, since the Witten mechanism cannot be expected to work if the carrier mass is too large [4,9,10], then zone I would consist only of the extreme limit $b^2 = \infty$, i.e., the case $C = 0$, for which the string falls radially inwards with a spacelike current but zero angular momentum.

Zone II. This is the ‘‘dangerous’’ spacelike zone characterized by

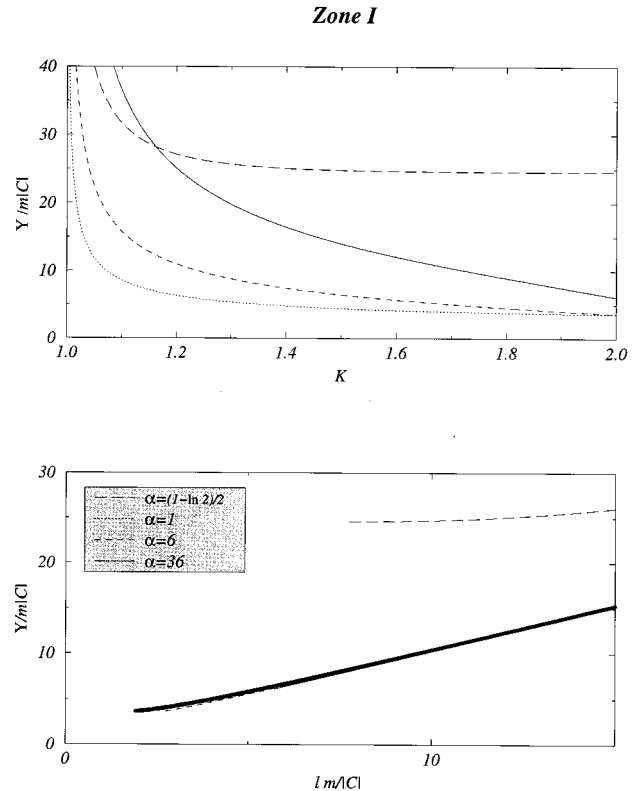


FIG. 1. The potential $Y/(m|C|)$ as a function of \mathcal{K} (top) and $m\ell/|C|$ (bottom) for zone I. Here and on the following figures, it is found that for $\alpha \geq 1$, the curves as functions of $m\ell/|C|$ all coincide (up to negligible logarithmic corrections) so they can be shown for different values of α ranging from 1 to 100 by the same thick curve. (Note that this simplification depends on normalizing with respect to the Higgs boson mass m rather than the carrier mass scale m_* .) It is clear, however, that the variations with \mathcal{K} are strongly dependent on the ratio α .

$$b^2 \geq 4 > b^2 \left(1 - \frac{1}{2\alpha + \ln 2} \right) \tag{93}$$

[which would consist of the entire range $b^2 \geq 4$ if Eq. (92) were not satisfied], for which the trajectory may be of (stationary) type 0, (well-behaved oscillatory) type 1A, or (badly behaved) type 1B, depending on its energy. The type 1B case is that for which M satisfies an inequality of the form (90), in which case the loop will evolve in the same way as in the previous scenario, and thus will again end up by contracting to a state of current saturation. The ‘‘good’’ type 1A possibility is characterized by the condition that the mass should lie in the range

$$M_s > M > M_v, \tag{94}$$

where the maximum beyond which the current will ultimately saturate is given by the preceding formula (91) for M_s , and the minimum value M_v is the mass of the relevant vorton state as characterized by Eq. (88): When this latter condition is satisfied the loop will oscillate in a well-behaved manner between a minimum and a maximum radius that are obtained by solving $Y = M$. Finally the type 0 possibility is that of the vorton state itself, as given by the minimum value $M = M_v$. Similarly to Fig. 1, Fig. 2 shows the potential in

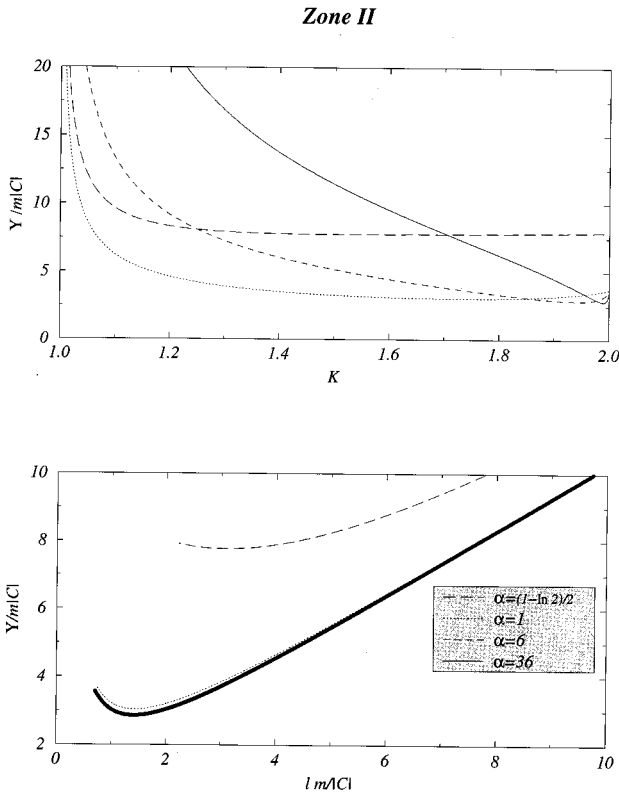


FIG. 2. Same as Fig. 1 for zone II.

this zone II, against either \mathcal{K} (top) and $\ell m/|C|$ (bottom), with the same remark as before when $\alpha \geq 1$.

Zone III. This is the “safe” zone characterized by

$$4 > b^2 > e^{-4\alpha}, \tag{95}$$

for which there is no danger of bad behavior; i.e., the only possibilities are the well-behaved type 1A, which applies to the entire range

$$M > M_v, \tag{96}$$

and the vortonic type 0, as given by $M = M_v$.

It is to be remarked that this “safe” zone consists of three qualitatively distinct parts, namely, a subrange of spacelike current solutions, zone III{+}, say (Fig. 3), given by

$$2 > |b| > 1, \tag{97}$$

a subrange of timelike current solutions, zone III{-}, say (Fig. 4), given by

$$1 > |b| > e^{-2\alpha}, \tag{98}$$

and in between the special “chiral” case of null current solutions, zone III{0}, say, which is given just by $|b| = 1$.

Zone IV. This is the “dangerous” timelike zone characterized by

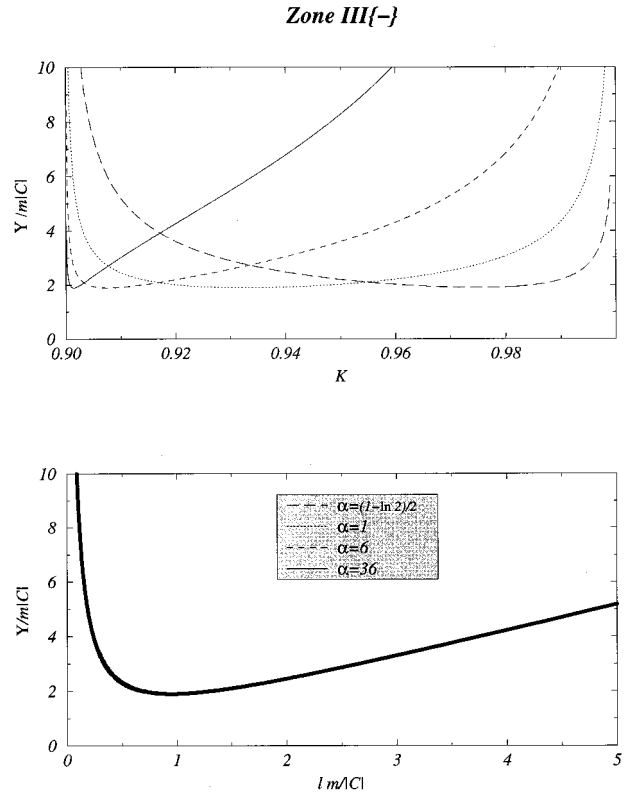


FIG. 3. Same as Fig. 1 for zone III{-}.

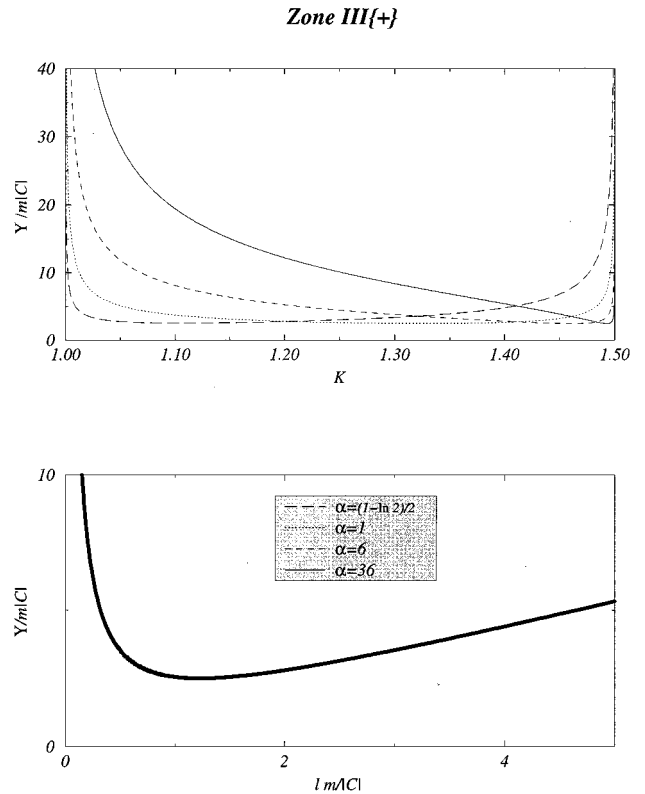


FIG. 4. Same as Fig. 1 for zone III{+}.

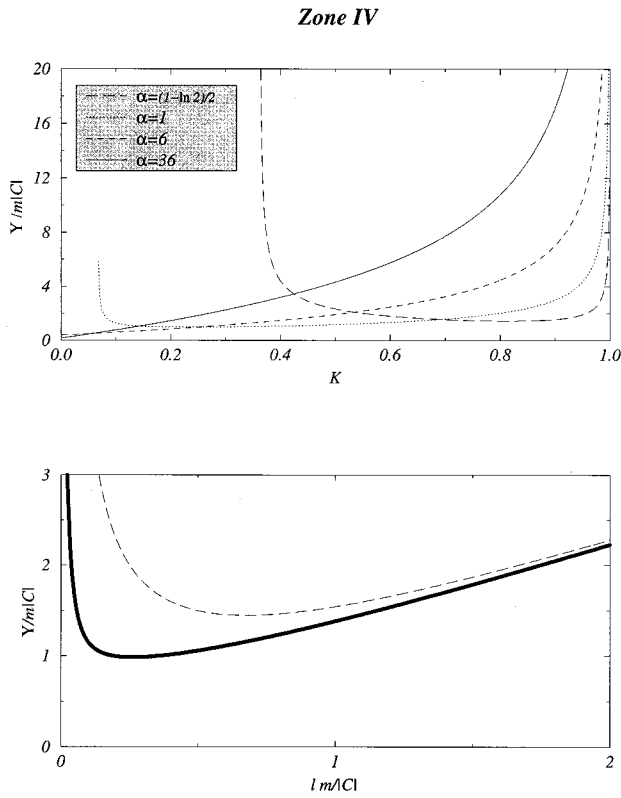


FIG. 5. Same as Fig. 1 for zone IV. In this zone, for large values of α , the minimum value of the potential is attained only for very small values of \mathcal{K} and hence are not visible on the figure.

$$e^{-4\alpha} \geq b^2 > 0, \quad (99)$$

for which (as in zone II) the trajectory may be of (stationary) type 0, (well-behaved oscillatory) type 1A, or (badly behaved) type 1B, depending on its energy (see Fig. 5). The latter will occur whenever

$$M \geq M_r, \quad (100)$$

where the relevant minimum mass, above which the loop will contract to a state of complete relaxation, i.e., zero tension, is given by

$$\frac{M_r}{m_*} = |C| e^{-2\alpha} \sqrt{\frac{1 - e^{-2\alpha}}{e^{-4\alpha} - b^2}}. \quad (101)$$

The ‘good’ type 1A possibility is characterized by the condition that the mass should lie in the range

$$M_r > M > M_v, \quad (102)$$

where as before M_v is the vorton mass value given by Eq. (88), while finally the type 0 possibility occurs when $M = M_v$. It is to be remarked that the zone (99) includes a subzone characterized by the strict condition $c_E < c_L$, where $c_L^2 = -dT/dU$ is the wobble velocity, a condition that is expressible as [21] $\mathcal{K} < e^{2(1-2\alpha)}$, and which has been conjectured to be sufficient to ensure classical stability of the corresponding vorton state.

Zone V. This is the ‘fatal’ timelike zone characterized by

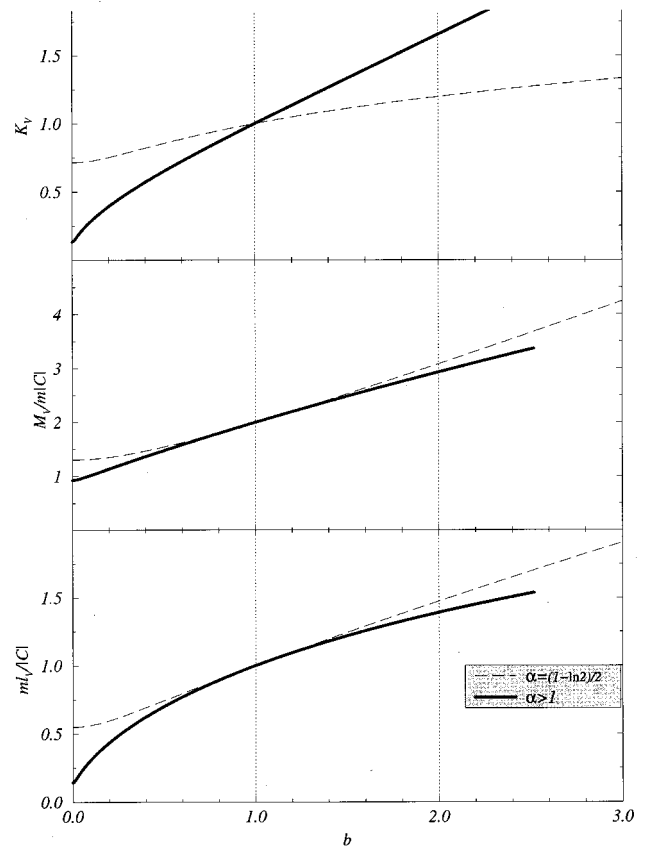


FIG. 6. The vorton state function \mathcal{K}_v , mass $M_v/m|C|$, and length $m/l|C|$ against the Bernoulli ratio $|b|$. From $\alpha = (1 - \ln 2)/2$ to $\alpha = 1$, the curve is smoothly deformed from the long-dashed one to the thick one which includes many values of α , showing explicitly the independence in α .

$$b^2 = 0, \quad (103)$$

i.e., the case for which $B = 0$ (to which the investigation of Larsen and Axenides [15] was devoted) for which it can be seen that the mass must satisfy the inequality

$$\frac{M}{m_*} > |C| \sqrt{1 - e^{-2\alpha}} \quad (104)$$

(in which the lower bound is the common limit to which M_v and M_r converge as B tends to zero). In this case (as in the more extensive range covered by zone I) the trajectory must be of type 1B, its ultimate fate being to reach a state of relaxation, $T \rightarrow 0$, as in zone IV when (100) is satisfied.

All these zones are shown in Fig. 6 where, as functions of the parameter $|b|$ are plotted the value of the function \mathcal{K} that minimizes the potential in all but zone I (top), the corresponding value of the vorton mass M_v (middle) and length l (bottom), all in units of the Kibble mass m . It should be clear in this figure that in most cases the latter two are almost independent of α , the largest dependence occurring in zones II and IV.

XI. CONCLUSIONS

In view of the potential cosmological interest of vorton formation, it is of interest to distinguish the range of conditions under which a cosmic string loop can survive in an ‘‘A-type’’ oscillatory state, which will ultimately damp down towards a stationary vorton configuration, from the alternative range of conditions under which the loop will undergo a ‘‘B-type’’ evolution, whereby it reaches a configuration for which the classical string description breaks down, in which case the investigation of its subsequent fate and, in particular, of the question of whether the underlying vacuum vortex defect will ultimately survive at all will need more sophisticated methods of analysis than are presently available.

The present investigation is restricted to the case of exactly circular loops for which it is shown, on the basis of the best available classical string model [21], that there is an extensive range of parameter space, including the whole of zone III in the above classification, for which the ‘‘A-type’’ solutions (which are propitious for ultimate vorton formation) will indeed be obtained. On the other hand, it is also shown that (unlike what occurs in the classical point particle problem) badly behaved ‘‘B-type’’ solutions are not limited to the special zero angular momentum case, zone V, to which a preceding study [15] of this problem was restricted, but are of generic occurrence, occupying the whole of zone I and extensive parts of zones II and IV. It remains an open question whether these results are representative of what will happen in the more general case of initially noncircular loops.

The foregoing results are based on an analysis that is purely classical in the sense that it neglects both quantum effects and also the general relativistic effects of the gravitational field. In realistic cases of cosmological interest, involving cosmic strings produced at or below the GUT transition level, it is to be expected that the neglect of gravitational effects will be a very good approximation: As remarked in the Introduction, the relevant Schwarzschild radius will usually be so small that the question of black hole formation will be utterly academic, while the effect of gravitational radiation, although it may become cumulatively important, will be allowable for in the short run in terms of a very slow ‘‘secular’’ variation of the mass parameter M , whereby an oscillatory (type 1A) trajectory will gradually settle down towards a stationary (type 0) vorton state.

Unlike the usually small corrections that will arise from

gravitation, the effects of quantum limitations may be of dominant importance for realistic cosmological applications. The preceding analysis should be valid for loops characterized by sufficiently large values of the winding number N and particle quantum number Z , and thus for correspondingly large values of the Bernoulli constants B and C and hence of M . However, it can be expected to break down whenever the loop length ℓ becomes small enough to be comparable with the Compton wavelength

$$\ell_* = m_*^{-1} \quad (105)$$

associated with the carrier mass scale m_* . It can be seen from Eq. (82) that the current saturation limit $\mathcal{K} \rightarrow 2$ cannot be attained without violating the classicality condition

$$\ell \geq \ell_* \quad (106)$$

unless the corresponding dimensionless Bernoulli constants B and C [which by Eq. (40) will have the same order of magnitude provided κ_0 is of order unity] are such as to satisfy the condition

$$B^2 - 4C^2 \geq 1. \quad (107)$$

This differs from the corresponding purely classical condition $B^2 > 4C^2$ (characterizing zones I and II) by having 1 instead of 0 on the right-hand side. It can similarly be seen that the relaxation ($T \rightarrow 0$) limit $\mathcal{K} \rightarrow e^{-2\alpha}$ cannot be obtained without violating the classicality condition (106) unless the Bernoulli constants satisfies the condition

$$C^2 e^{-4\alpha} - B^2 \geq 1 - e^{-2\alpha}, \quad (108)$$

which is similarly stronger than the corresponding purely classical condition $C^2 e^{-4\alpha} > B^2$ (characterizing zones IV and V).

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