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# Relativistic radiation hydrodynamics: Shock and deflagration waves

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In the equilibrium diffusion limit, transport effects vanish and the radiation pressure tensor is diagonal and isotropic. The radiation pressure $P_r$ and energy density $\rho_r$ are then related by $P_r=\frac{1}{3}\rho_r$. This assumption will fail near the boundary surface from which radiation escapes freely. This paper deals with a relativistic hydrodynamic approach of radiation phenomena taking into account the effects of the radiation pressure, energy density, and energy flux. From physical and geometrical considerations we derive an energy-momentum tensor for radiation. This tensor, which generalizes the isotropic case, may describe a certain model of radiation field which is of some interest for astrophysics and cosmology. Next, we examine the propagation of shock waves in the radiating fluid considered. The Rankine-Hugoniot jump conditions are deduced. The case of a radiation-dominated gas is considered in detail. The study of Rankine-Hugoniot curves with Eddington's factor as a parameter allows us to point out the key role that this factor plays in radiation phenomena. In particular, the Eddington factor is used as a convenient parameter to study the deflagration waves. [S0556-2821(97)02908-1]

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## I. INTRODUCTION

Radiation phenomena play a crucial role in diverse astrophysical and cosmological situations such as stellar stability, supernovae explosions, and early stages of the universe. Many works are devoted to radiation, radiation transport, and the dynamics of radiating fluids. Thomas [1], Simon [2], and Mihalas and Mihalas [3] have calculated the energy-momentum tensor for a fluid consisting of some material plus radiation quanta (photons, neutrinos, or gravitons). They have solved the relativistic transport equation for the radiation. Lindquist [4] and Anderson [5] have used the kinetic description of a gas and radiation field together. From the usual conservation laws Weinberg [6] has derived the equations of radiative hydrodynamics and obtained expressions for radiative contributions to various transport coefficients associated with the system.

In a fundamental work [7] Levermore and Pomraning posed a theoretical foundation of the flux-limited diffusion theory. In recent works Anile and Sammartino [8], Anile and Romano [9], and Bonanno and Romano [10], have presented a covariant flux-limited diffusion theory for radiative phenomena from the microscopic viewpoint.

In the present paper from the hydrodynamic viewpoint we propose a model of a radiating fluid (material plus radiation) taking into account the effects of the radiation energy density, pressure, and energy flux. Our study generalizes the case where the radiation field is assumed to be isotropic. The radiating shock being one of the most interesting phenomena in radiation hydrodynamics, we study the propagation of shock waves in the considered model. This study allows us to point out the key role that Eddington's factor plays in radiation phenomena.

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## II. THE RADIATING FLUID

### A. The stress-energy tensor of the material medium

The material medium under consideration is assumed to be a perfect fluid which is described by its local thermodynamic variables such as the baryon number density $n$, the energy density $\rho_m$, and the pressure $P_m$ (the subscript $m$ stands for matter):

$$T_m^{\mu\nu}=\rho_m u^\mu u^\nu - P_m \gamma^{\mu\nu}, \tag{1}$$

where $u^\mu$ is the fluid unit four-vector $u^\mu u_\mu=1$, $\gamma^{\mu\nu}=g^{\mu\nu}-u^\mu u^\nu$ is the projection tensor onto the three-space orthogonal to $u^\mu$, and $g$ is the Lorentzian metric with the signature $+---$. Following Taub [11] and Lichnerowicz [12] we set

$$\rho_m=n(c^2 m_o+\epsilon_m)=n e_m, \tag{2}$$

where $n$ is the baryon number density, $m_0$ the average baryonic rest mass, and $\epsilon_m$ is the ''classical'' specific (per baryon) internal energy.

We introduce the relativistic specific enthalpy

$$f_m=c^2 m_o+h_m, \tag{3}$$

where $h_m=\epsilon_m+P_m/n$ is the ''classical'' specific enthalpy. From the preceding relation it follows that

$$\rho_m+P_m=n f_m. \tag{4}$$

The material under consideration is assumed to satisfy the state equation

$$P_m=(\gamma-1)\rho_m \quad \text{with} \quad 1\leqslant\gamma\leqslant 2, \tag{5}$$

where $\gamma$ is the adiabatic index assumed to be a constant. Equation (4) becomes

$$n f_m=\rho_m+P_m=\gamma\rho_m.$$

 

The stress-energy tensor can then be written as

$$T_m^{\mu\nu}=nf_m u^\mu u^\nu - P_m \gamma^{\mu\nu}. \tag{6}$$

The conservation of matter is expressed by the equation

$$\nabla_\mu(nu^\mu)=0, \tag{7}$$

where $\nabla_\mu$ is the covariant derivative operator.

### B. The stress-energy tensor of radiation

In the equilibrium diffusion limit and in a local Galilean reference of frame the radiation energy tensor is of the form

$$T_{\mu'\nu'}=\begin{pmatrix} & \rho_r & \\ & P_r & \\ 0 & P_r & 0 \\ & P_r & \end{pmatrix}, \tag{8}$$

where $\rho_r$ and $P_r$ are the radiation energy density and pressure, respectively. In this approximation the transport effects vanish. The aim of this paper is to account for the effects of the radiation energy density, pressure, and energy flux.

#### *1. The pressure tensor of radiation*

In this section and the following, all thermodynamical quantities referring to radiation are the proper values as measured in the comoving coordinate systems defined as those in which the observer is at rest relative to the material. At the considered point let us introduce the principal frame of reference, i.e., the unit orthonormal tetrad $V_\alpha^{(\lambda')}$ such that $V_\alpha^{(0')}=u_\alpha$ defines the time-axis and the remaining three $V_\alpha^{(\lambda')}$ define the proper space associated with $V_\alpha^{(0')}$:

$$V^{(o')\alpha}V_{(o')\alpha}=u^\alpha u_\alpha=1,$$

$$V_\alpha^{(i')}V_{(i')}^\alpha=-1$$

(no summation with respect to the index $i'$),

$$V_\alpha^{(o')}V^{(i')\alpha}=u_\alpha V^{(i')\alpha}=0. \tag{9}$$

In a local Galilean frame of reference we take the radiation pressure tensor of the form (Levermore [13], Anile *et al.* [14], Kremer and Müller [15])

$$\Pi^{i'k'}=a_r\delta^{i'k'}+bJ^{i'k'}, \tag{10}$$

$J^{i'}$ being the radiation energy flux. The subscript $r$ stands for radiation. The variables $a_r$ and $b$ are functions of the radiation energy density $\rho_r$ (which we introduce below) and $J^2$. We now seek the spacetime generalization for Eq. (10).

In an arbitrary frame of reference we will have

$$\Pi^{i'k'}=a_r\delta^{i'k'}+bJ^{i'}J^{k'}. \tag{11}$$

It is easy to show that

$$A_\alpha^{0'}=V_\alpha^{(0')}=u_\alpha, \quad A_{0'}^\alpha=V_{(0')}^\alpha=u^\alpha,$$

$$A_\alpha^{i'}=-V_\alpha^{(i')}, \quad A_{i'}^{\dot\alpha}=V_{(i')}^\alpha. \tag{12}$$

From these relations we obtain

$$g_{\alpha\beta}=A_\alpha^{\lambda'}A_\beta^{\mu'}\Pi_{\lambda'\mu'}=u_\alpha u_\beta - V_\alpha^{(i')}V_\beta^{(k')}\delta_{i'k'}. \tag{13}$$

Formula (13) gives the projection tensor onto the three-space orthogonal to $V^{(0')\alpha}=u^\alpha$:

$$\gamma_{\alpha\beta}=g_{\alpha\beta}-u_\alpha u_\beta=-V_\alpha^{(i')}V_\beta^{(k')}\delta_{i'k'}. \tag{14}$$

Making use of Eqs. (12) and (14), the relation (11) becomes

$$\Pi_{\alpha\beta}=-a_r\gamma^{\alpha\beta}+bV_{(i')}^\alpha V_{(k')}^\beta J^{i'}J^{k'}. \tag{15}$$

In Eqs. (14) and (15) the summation convention is assumed. We set

$$J^\alpha=V_{(i')}^\alpha J^{i'}$$

and

$$J^\beta=V_{(k')}^\beta J^{k'} \tag{16}$$

and obtain the expression of the pressure tensor of radiation in an arbitrary frame of reference:

$$\Pi^{\alpha\beta}=-a_r\gamma^{\alpha\beta}+bJ^\alpha J^\beta. \tag{17}$$

The pressure tensor thus obtained satisfies $\Pi^{\mu\nu}u_\nu=0$.

#### *2. The radiation stress-energy tensor*

We define $\rho_r$ as the energy density of the radiation as measured locally by an observer with four-velocity $u^\mu$. Let $T_r^{\mu\nu}$ be the stress-energy tensor of the radiation. Our goal is to account for the effects of the radiation energy, pressure, and energy flux. The irreversible processes which may occur in gas is the radiation flow. We then ignore the viscosity and heat conduction and take the radiation stress-energy tensor to be of the form

$$T_r^{\mu\nu}=\rho_r u^\mu u^\nu+\Pi^{\mu\nu}+u^\mu J^\nu+u^\nu J^\mu. \tag{18}$$

In what follows we will assume that

$$\rho_r=-\Pi_\alpha^\alpha. \tag{19}$$

This assumption is equivalent to

$$T_{r\mu}^\mu=0. \tag{20}$$

Equation (19) or (20) yields

$$b=(\rho_r-3a_r)/J^2. \tag{21}$$

Substituting Eq. (21) into Eq. (17) we find

$$\Pi^{\mu\nu}=-a_r\gamma^{\mu\nu}+Eh^\mu h^\nu, \tag{22}$$

where we set

$$h^{\mu}=\frac{J^{\mu}}{|J|}. \tag{23}$$

$h^{\mu}$ is the unit vector in the direction of the flux.

The pressure tensor of radiation defined by Eq. (22) constitutes the generalization of the isotropic case. In Eq. (22) the last term describes the anisotropy due to the radiation field. Therefore, the quantity $a_r$ does not reduce to a simple hydrostatic pressure.

In a local Galilean frame of reference the pressure tensor $\Pi_{\mu\nu}$ defined by Eq. (22) has the form

$$\Pi_{i'k'}=a_r\delta_{i'k'}+Eh_{i'}h_{k'} \tag{24}$$

or

$$\Pi_{i'k'}=\begin{pmatrix} & a_r & \\ 0 & a_r & 0 \\ & E+a_r & \end{pmatrix}=\begin{pmatrix} & a_r & \\ 0 & a_r & 0 \\ & a_r & \end{pmatrix}+\begin{pmatrix} & 0 & \\ 0 & 0 & 0 \\ & E & \end{pmatrix}, \tag{25}$$

where $h_{i'}$ has been chosen as $h_{i'}=V^{(3)}_{i'}$.

If we introduce the projection tensor onto the two-space orthogonal to $u^{\mu}$ and $h^{\mu}$,

$$\Gamma^{\mu\nu}=\gamma^{\mu\nu}+h^{\mu}h^{\nu} \quad \text{with} \quad \Gamma^{\mu\nu}u_{\nu}=\Gamma^{\mu\nu}h_{\nu}=0, \tag{26}$$

then we obtain

$$\Pi^{\mu\nu}=-a_r\Gamma^{\mu\nu}+(E+a_r)h^{\mu}h^{\nu}. \tag{27}$$

It follows that

$$-\Pi^{\alpha\lambda}h_{\lambda}=(E+a_r)h^{\alpha},$$

$$-\Pi^{\alpha\lambda}Y_{\lambda}=a_rY^{\alpha}, \tag{28}$$

where $Y$ is any vector belonging to the two-plane orthogonal to the two-plane defined by $(u,h)$:

$$Y^{\alpha}=\Gamma^{\alpha\lambda}Y_{\lambda}.$$

From Eq. (28) it follows that $E+a_r$ and $a_r$ are the eigenvalues of $(-\Pi^{\mu\nu})$ corresponding to the eigenvectors $h^{\mu}$ and $Y^{\mu}$, respectively. $(a_r+E)$ and $a_r$ may then be interpreted as the components of the radiation pressure along the direction of the flux $h^{\mu}$ and in the two-space orthogonal to this direction and to $u^{\mu}$.

Let us mention that $\Gamma^{\mu\nu}$ cannot vanish as it has been claimed in [18] (p. 2889). $\Gamma^{\mu\nu}$ is an operator:

$$\Gamma^{\mu\nu}\Gamma_{\mu\rho}=\Gamma^{\nu}_{\rho}.$$

The dimensionless quantity

$$\chi=(E+a_r)/\rho_r \tag{29}$$

is called the Eddington factor [13,14,15]. But the pressure component along $h^{\mu}$ is

$$P^{*}_r=a_r+E=\chi\rho_r \quad \text{with} \quad \frac{1}{3}\leqslant\chi\leqslant 1. \tag{30}$$

Equation (28) yields

$$a_r=(\rho_r/2)(1-\chi), \quad E=(\rho_r/2)(3\chi-1). \tag{31}$$

Then the pressure tensor defined by Eq. (27) is expressed in terms of $\chi$ as

$$\Pi^{\mu\nu}=\rho_rE^{\mu\nu},$$

where

$$E^{\mu\nu}=-\tfrac{1}{2}(1-\chi)\Gamma^{\mu\nu}+\chi h^{\mu}h^{\nu}. \tag{32}$$

Then relations (28) become

$$-E^{\mu\nu}h_{\nu}=\chi h^{\mu}, \quad -E^{\mu\nu}Y_{\nu}=\tfrac{1}{2}(1-\chi)Y^{\mu}. \tag{33}$$

Thus $\chi$ and $(1-\chi)/2$ are the eigenvalues of $(-E^{\mu\nu})$ corresponding to the eigenvectors $h^{\mu}$ and $Y^{\mu}$, respectively. Inserting Eq. (31) into Eq. (27) we obtain the same expression for $\Pi^{\mu\nu}$ as in [16]. Making use of Eq. (22), Eq. (18) becomes

$$T^{\mu\nu}_r=\rho_ru^{\mu}u^{\nu}-a_r\gamma^{\mu\nu}+u^{\mu}J^{\nu}+u^{\nu}J^{\mu}+Eh^{\mu}h^{\nu}. \tag{34}$$

One may compare this expression with that obtained from the microscopic viewpoint [8,9,10,17,18]. In the most extreme cases, the flux out of an optically thin zone predicted by diffusion may exceed the energy density times the velocity of light. Therefore, in the following section we will define $J=|J|$ as [13,14]

$$J=\rho_rf, \tag{35}$$

$f=|f|$ being the normalized flux such that $0\leqslant f\leqslant 1$:

$$f=\begin{cases} 0 & \text{in the isotropic case,} \\ 1 & \text{in the free-streaming case.} \end{cases} \tag{36}$$

$\chi$ and $f$ are related by [14,15]

$$\chi=\frac{5}{3}-\frac{2}{3}\sqrt{4-3f^2}. \tag{37}$$

It follows from Eq. (37) that

$$\chi\in[\tfrac{1}{3},1]\Leftrightarrow f\in[0,1].$$

In what follows we shall use the expression (34) of the stress-energy tensor of the radiation.

### C. The stress-energy tensor of radiating fluid

Our goal here is to account for the effects of the radiation energy, pressure, and energy flux, so we shall ignore viscosity and thermal conduction and as in [3] we shall adopt the following assumptions.

(i) The material medium is a perfect gas which remains in local thermodynamical equilibrium with all species of particles as the same kinetic temperature.

(ii) The radiation field can be treated in the nonequilibrium diffusion.

The radiating fluid (matter plus radiation) may then be described by the total stress-energy tensor

$$T^{\mu\nu}=T^{\mu}_m+T^{\mu\nu}_r=\rho u^{\mu}u^{\nu}+\Pi^{\mu\nu}+u^{\mu}J^{\nu}+u^{\nu}J^{\mu}, \tag{38}$$

where

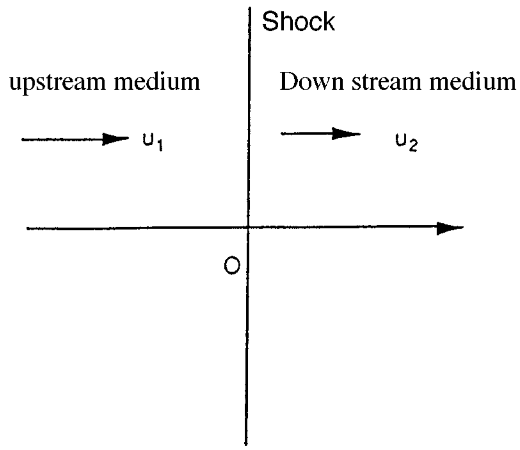


FIG. 1. A relativistic deflagration wave (shock) in the local frame of the shock front.

$$\Pi^{\mu\nu} = -P\gamma^{\mu\nu} + Eh^\mu h^\nu, \quad P = P_m + a_r, \quad \rho = \rho_m + \rho_r. \tag{39}$$

The tensor given by Eq. (38) can also be written as

$$T^{\mu\nu} = Wu^\mu u^\nu - Pg^{\mu\nu} + u^\mu J^\nu + u^\nu J^\mu + Eh^\mu h^\nu, \tag{40}$$

where

$$W = \rho + P = nF. \tag{41}$$

## III. SHOCK WAVES IN RADIATING FLUID

### A. Shock invariants

The shock wave is here a hypersurface of discontinuity denoted as $\Sigma$ for the fluid and radiation variables. The total stress-energy tensor $T^{\mu\nu}$ satisfies the local energy and momentum conservation law:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{42}$$

The conservation of baryon flux density is written as

$$\nabla_\alpha (nu^\alpha) = 0. \tag{43}$$

From the main system (42) and (43) we obtain by a classical argument the general shock equations

$$1_\alpha [nu^\alpha] = 0, \quad 1_\alpha [T^{\alpha\beta}] = 0, \tag{44}$$

where the square brackets correspond to the discontinuity of a quantity $Q$ across the shock $\Sigma$:$[Q] = Q_1 - Q_0$. The subscripts ''0'' and ''1'' denote upstream and downstream variables, respectively. The thermodynamical quantities are measured in the shock system. The quantities $u_0$ and $u_1$ are thus the flow four velocities of the upstream and downstream fluids, respectively, with respect to the rest frame of the shock front (Fig. 1).

Equations (44) express the invariance across $\Sigma$ of the scalar

$$a = nl_\alpha u^\alpha \tag{45}$$

and of the vector

$$W^\beta = l_\alpha T^{\alpha\beta},$$

where

$$T^{\alpha\beta} = nFu^\alpha u^\beta - Pg^{\alpha\beta} + u^\alpha J^\beta + u^\beta J^\alpha + Eh^\alpha h^\beta. \tag{46}$$

We introduce the quantity

$$h_n^2 = \frac{(l_\alpha h^\alpha)^2}{(-\gamma^{\mu\nu} l_\mu l_\nu)}. \tag{47}$$

$h_n$ is the component of $h$ in the spatial direction of propagation of the waves. Here $l^\alpha$ is a unit four-vector normal to $\Sigma$. It follows from the timelike character of $\Sigma$ that $l^\alpha l_\alpha = -1$.

It is easy to show that the equality $h_n^2 = 1$ holds only if the normal to the wave front $l^\alpha$ belongs to the two-plane defined by the vectors $u^\alpha$ and $h^\alpha$: that is,

$$l^\alpha = gu^\alpha + kh^\alpha,$$

where

$$g = l^\alpha u_\alpha$$

and

$$k = -l^\alpha h_\alpha.$$

It follows from these considerations that

$$\Gamma^{\mu\nu} l_\mu = 0$$

and

$$\Gamma^{\mu\nu} l_\mu l_\nu = l^\mu l_\mu - (l^\mu u_\mu)^2 + (l_\mu h^\mu)^2 = 0,$$

$$(l_\mu h^\mu)^2 = 1 + (l^\mu u_\mu)^2. \tag{48}$$

In all the following sections we shall consider the last relation of Eq. (48).

The speed $V^\Sigma$ of the wave front $\Sigma$ with respect to the fluid, that is, with respect to the time direction $u$, is given by [12]

$$(V^\Sigma / c)^2 = \beta^2 = \frac{(l_\alpha u^\alpha)^2}{(-\gamma^{\mu\nu} l_\mu l_\nu)}. \tag{49}$$

We set $u^2 = (l^\alpha u_\alpha)^2$, and Eq. (49) gives

$$u^2 = \frac{\beta^2}{1 - \beta^2}. \tag{50}$$

Thus, $u$ is the four-velocity of the fluid.

The identity

$$(l_\alpha u^\alpha)^2 \left( \frac{1}{\beta^2} - 1 \right) = 1 \tag{51}$$

follows from Eq. (50). Equations (48) and (49) yield

$$l_\alpha h^\alpha = -\frac{l_\alpha u^\alpha}{\beta}. \tag{52}$$

We have chosen the sign $-$ assuming that $l^\alpha u_\alpha$ and $l^\alpha h_\alpha$ have opposite sign.

### B. Basic shock equations

We deduce from the invariance of the vector $W^\beta$ that the scalar

$$D=1_\rho W^\rho=W(l_\alpha u^\alpha)^2+P+2l_\alpha J^\alpha l_\rho u^\rho+E(l_\alpha h^\alpha)^2 \tag{53}$$

is invariant across $\Sigma$. Making use of Eqs. (51) and (52), Eq. (53) can be written as

$$D=\omega^*(l_\alpha u^\alpha)^2+P^*. \tag{54}$$

In Eq. (54) we have set

$$P^*=P+E=P_m+P_r^*,$$
$$\omega^*=W+E-\frac{2J}{\beta}=\rho+P^*-\frac{2J}{\beta}, \tag{55}$$

where $P_r^*$ is given by Eq. (30).

In the following sections we shall set

$$\tau^*=\frac{\omega^*}{n^2}. \tag{56}$$

Using Eq. (54) we obtain

$$D=a^2\tau^*+P^*. \tag{57}$$

We decompose $u$ and $h$ according to tangential components and normal components with respect to $\Sigma$:

$$u^\beta=v^\beta+\frac{l_\alpha u^\alpha}{l^\rho l_\rho}\,l^\beta=v^\beta-l_\alpha u^\alpha l^\beta,$$
$$h^\beta=t^\beta+\frac{l_\alpha h^\alpha}{l^\rho l_\rho}\,l^\beta=t^\beta-l_\alpha h^\alpha l^\beta. \tag{58}$$

$W$ is then written as

$$W^\beta=Z^\beta-(P^*+a^2\tau^*)l^\beta, \tag{59}$$

where $Z^\beta$ is defined by

$$Z^\beta=Av^\beta+Bt^\beta \tag{60}$$

with

$$A=W(l_\alpha u^\alpha)+Jl_\alpha h^\alpha,$$
$$B=Jl_\alpha u^\alpha+E(l_\alpha h^\alpha). \tag{61}$$

The vector $Z^\beta$ is tangent to $\Sigma$. The invariance of $W^\beta$ is equivalent to the invariance of $Z^\beta$ and of that of tangential component. The scalar $Z^\rho Z_\rho$ is then invariant across $\Sigma$.

Using Eq. (59) we obtain the following scalar invariant across $\Sigma$:

$$Z^\rho Z_\rho=A^2v^\rho v_\rho+B^2t^\rho t_\rho+2ABt^\rho v_\rho. \tag{62}$$

It is easy to obtain the relations

$$v^\rho v_\rho=1+(l_\alpha u^\alpha)^2=(l_\alpha h^\alpha)^2,$$
$$t^\rho t_\rho=-1+(l_\alpha h^\alpha)^2=(l_\alpha u^\alpha)^2, \tag{63}$$
$$t^\rho v_\rho=l^\alpha u_\alpha l^\beta h_\beta.$$

Making use of Eq. (63), Eq. (62) gives

$$Z^\rho Z_\rho=A^2(l_\alpha h^\alpha)^2+B^2(l_\alpha u^\alpha)^2+2ABl_\alpha u^\alpha l_\rho h^\rho$$
$$=(Al_\alpha h^\alpha+Bl_\alpha u^\alpha)^2. \tag{64}$$

Thus the invariance of $Z_\rho Z^\rho$ across $\Sigma$ is equivalent to that of the scalar

$$L=Al_\alpha h^\alpha+Bl_\alpha u^\alpha. \tag{65}$$

Making use of Eqs. (52) and (55), Eq. (65) can be expressed as

$$L=Al_\alpha h^\alpha+Bl_\alpha u^\alpha=-\left[\frac{\omega^*}{\beta}\,(l_\alpha u^\alpha)^2+J\right].$$

Thus the quantity

$$g=\frac{\omega^*}{\beta}\,(l_\alpha u^\alpha)^2+J=\frac{\tau^*a^2}{\beta}+J \tag{66}$$

is invariant across $\Sigma$.

Using Eq. (61) in (64) we obtain

$$Z^\rho Z_\rho=L^2=\frac{\omega^{*2}}{\beta^2}\,(l_\alpha u^\alpha)^4+2J\,\frac{\omega^*}{\beta}\,(l_\alpha u^\alpha)^2+J^2. \tag{67}$$

By using the identity (51), Eq. (67) can be written as

$$Z^\rho Z_\rho=\omega^{*2}(l_\alpha u^\alpha)^4+\omega^*(W+E)(l_\alpha u^\alpha)^2+J^2. \tag{68}$$

Thus the following scalar is invariant across $\Sigma$:

$$K=\frac{Z^\rho Z_\rho}{a^2}=a^2\tau^{*2}+\tau^*(W+E)+\frac{J^2}{a^2}. \tag{69}$$

Thus we have obtained the following radiation Rankine-Jump conditions:

$$n_0l_\alpha u_0^\alpha=n_1l_\alpha u_1^\alpha=a,$$
$$a^2\tau_0^*+P_0^*=a^2\tau_1^*+P_1^*=D,$$
$$a^2\tau_0^{*2}+\tau_0^*(W_0+E_0)+\frac{J_0^2}{a^2}=a^2\tau_1^{*2}+\tau_1^*(W_1+E_1)+\frac{J_1^2}{a^2}$$
$$=K. \tag{70}$$

The set of equations (70) is supplemented by an equation of state of the fluid and of the radiation after the shock. These state equations will be taken to be of the form given by Eq. (5) for the matter and by Eqs. (30) and (35) for the radiation. The relations (5), (30), (35), and (70) then completely determine downstream quantities in terms of the known upstream parameters.

### C. Rankine-Hugoniot adiabats

The invariance across $\Sigma$ of the scalars $D$ and $K$ yields

$$\tau_1^*(W_1+E_1)-\tau_0^*(W_0+E_0)-(P_1^*-P_0^*)(\tau_1^*+\tau_0^*)+\frac{1}{a^2}(J_1^2-J_0^2)=0. \quad (71)$$

Taking into account the expressions of $W$ and $P^*$ this equation can be transformed as

$$\tau_1^*(\rho_1+P_0^*)-\tau_0^*\rho_0-P_1^*\tau_0^*+\frac{1}{a^2}(J_1^2-J_0^2)=0. \quad (72)$$

For a simplified notation we introduce the dimensionless quantities

$$X=\frac{\tau^*}{\tau_0^*},\quad \Pi=\frac{P^*}{P_0^*},\quad \lambda=\frac{\rho}{P^*}. \quad (73)$$

$\lambda$ is known if the equations of state of the fluid and the radiation are given. Making use of Eq. (73), Eq. (72) may be written as

$$X_1(\lambda_1\Pi_1+1)-\lambda_0-\Pi_1+\frac{1}{a^2\tau_0^*P_0^*}(J_1^2-J_0^2)=0. \quad (74)$$

Equations (35) and (73) yield

$$\frac{J_1^2}{a^2\tau_0^*P_0^*}=\frac{1}{(1+\sigma_1)^2}\alpha_0 f_n^2\lambda_1^2\Pi_1^2,$$
$$\frac{J_0^2}{a^2\tau_0^*P_0^*}=\frac{1}{(1+\sigma_0)^2}\alpha_0 f_0^2\lambda_0^2, \quad (75)$$

where we have set

$$\alpha_0=\frac{P_0^*}{\tau_0^*a^2}=\frac{P_0^*}{\omega_0^*u_0^2}=\frac{(\gamma_0-1)\sigma_0+\chi_0}{u_0^2(\gamma_0\sigma_0+1+\chi_0-2f_0/\beta_0)}$$
$$\sigma=\rho_m/\rho_r. \quad (76)$$

Then (74) becomes

$$X_1(\lambda_1\Pi_1+1)-\lambda_0-\Pi_1+\frac{\alpha_0 f_1^2\lambda_1^2\Pi_1^2}{(1+\sigma_1)^2}-\frac{\alpha_0 f_0^2\lambda_0^2}{(1+\sigma_0)^2}=0. \quad (77)$$

We introduce the parameter

$$\beta^*=\frac{P_m}{P^*}=\frac{P_m}{P_m+\chi\rho_r}, \quad (78)$$

which is in the range $0\leqslant\beta^*\leqslant 1$.

Equation (78) gives

$$\beta^*=\frac{(\gamma-1)\sigma}{(\gamma-1)\sigma+\chi},$$

which yields

$$\sigma=\frac{\beta^*\chi}{(\gamma-1)(1-\beta^*)}. \quad (79)$$

Then $\lambda$ given by (73) is written as

$$\lambda=\frac{\rho}{P^*}=\frac{\beta^*}{\gamma-1}\left(1+\frac{1}{\sigma}\right)=\frac{1}{\chi}+\beta^*\left(\frac{1}{\gamma-1}-\frac{1}{\chi}\right). \quad (80)$$

Focusing mainly on the radiation effects we shall consider the case corresponding to $\beta^*\ll 1$, that is, to the radiation-dominated gas. In this case Eqs. (55), (73), and (80) yield

$$P^*=P_r^*=\chi\rho_r;\omega^*=\rho_r(1+\chi-2f/\beta),$$

$$\Pi=\frac{P_r^*}{P_{0r}^*};\lambda=\frac{1}{\chi},$$

$$\alpha_0=\frac{\chi_0}{u_0^2(1+\chi_0-2f_0/\beta_0)}, \quad (81)$$

and Eq. (77) becomes

$$\alpha_0 f^2\lambda^2\Pi^2+(\lambda X-1)\Pi-(\phi_0-X)=0, \quad (82)$$

where $(X,\Pi)$ is any post shock state and

$$\phi_0=\phi(\lambda_0)=\lambda_0+\lambda_0^2 f_0^2\alpha_0. \quad (83)$$

Making use of Eq. (37) and the expression of $\lambda$ from Eq. (81) we obtain

$$4f^2\lambda^2=(3\lambda-1)(3-\lambda). \quad (84)$$

Equation (82) is a quadratic equation with $X$ as an unknown quantity and $\lambda$ as a parameter. In the $(X,\Pi)$ plane, the equation of the straight line $D$ is written as

$$\Pi(X)=-\frac{X}{\alpha_0}+1+\frac{1}{\alpha_0}. \quad D$$

Now we shall discuss of some of the possible applications of the shock equations (70) through particular cases of physical interest.

*Case I:* Both the upstream and the downstream medium is assumed to be extremely opaque and in thermal equilibrium. The shock is then in an optically thick medium and all radiation emanating from the front is reabsorbed in the upstream medium.

This case corresponds to

$$f_0=f=0\Leftrightarrow\chi_0=\chi=\tfrac{1}{3}\Rightarrow\lambda_0=\lambda=3. \quad (85)$$

This case is appropriate for very high-temperature flows (e.g., in stellar, envelope), where the contributions of the radiation pressure and energy density are significant.

Making use of Eq. (85) in Eq. (82) we obtain

$$\Pi(X)=\frac{3-X}{3X-1}. \quad (86)$$

The corresponding curve will be denoted as $C_0$. $C_0$ is the Taub adiabat for a shock wave. By using Eq. (85), the expression of $\alpha_0$ given by Eq. (81) becomes

$$\alpha_0 = \frac{1}{4u_0^2}. \tag{87}$$

The intersections of the straight line $D$ with the curve $C_0$ are determined by the equation

$$\frac{3-X}{3X-1} = \frac{X}{\alpha_0} + 1 + \frac{1}{\alpha_0},$$

which yields

$$3X^2 - 4(1+\alpha_0)X + 4\alpha_0 + 1 = 0. \tag{88}$$

The discriminant of this quadratic equation is

$$\Delta^t = 4(\alpha_0 - \tfrac{1}{2})^2. \tag{89}$$

Then the roots of Eq. (88) are

$$X = \tfrac{2}{3}[(\alpha_0+1) \pm (\alpha_0 - \tfrac{1}{2})].$$

Thus we obtain the following intersection points of $C_0$ and $D$:

$$X_1 = \frac{4\alpha_0 + 1}{3}$$

and

$$\Pi_1 = \frac{2-\alpha_0}{3\alpha_0} \tag{90}$$

and

$$X_2 = \Pi_2 = 1. \tag{91}$$

For $\alpha_0 = \frac{1}{2}$(or $u_0^2 = \frac{1}{2} \Leftrightarrow \beta_0^2 = 1/\sqrt{3}$) the discriminant $\Delta'$ given by Eq. (89) vanishes and the equation of the straight line $D$ becomes

$$\Pi(X) = 3 - 2X. \qquad D_0$$

$D_0$ is tangent to the curve $C_0$ at the point (1,1).

The inequality $X_1 < 1$ (or $\Pi_1 > 1$) implies $\alpha_0 < \frac{1}{2} \Leftrightarrow u_0^2 > \frac{1}{2}$ (or $\beta_0^2 < \frac{1}{3}$).

Thus the curve $C_0$ and the straight line $D$ through (1,1) will have another intersection with the upper branch if $\alpha_0 < \frac{1}{2}$ and with the lower branch if $\alpha_0 > \frac{1}{2}$.

For $\alpha_0 = \frac{1}{5}(<\frac{1}{2})$ and for $\alpha_0 = \frac{3}{2}(>\frac{1}{2})$ we obtain the following equations of the straight line $D$:

$$\Pi(X) = 6 - 5X \qquad d_1$$

and

$$\Pi(X) = -\frac{2X}{3} + \frac{5}{3}. \qquad d_2$$

The curve $C_0$ and the straight lines $d_1$ and $d_2$ are plotted in Fig. 2. They pass through the point (1,1).

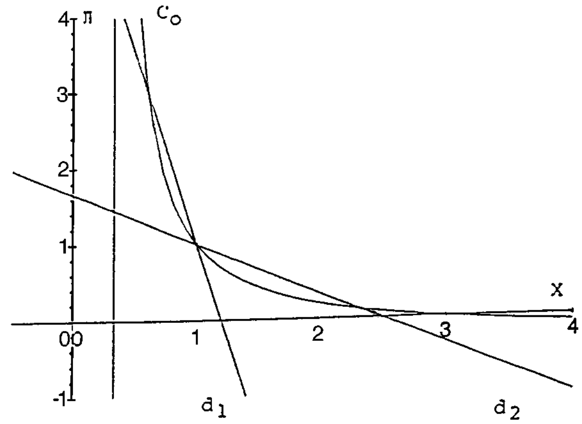


FIG. 2. Plot of Taub adiabat $C_0$, $d_1(\alpha_0 = \frac{1}{5})$, and $d_2(\alpha_0 = \frac{3}{2})$ in the $(X,\Pi)$ plane. The curve $C_0$ and the straight lines $d_1$ and $d_2$ pass through the point (1,1). The intersection of $d_1$ and $d_2$ with $C_0$ gives the values of the downstream quantities $(\Pi,X)$ if we assume the upstream states are known.

*Case II:* We suppose now that we have a radiating shock propagating into the upstream medium assumed to be transparent. The downstream medium is assumed to be opaque.

In this case we have

$$f_0 \in ]0,1] \Leftrightarrow \chi_0 \in ]\tfrac{1}{3},1] \Leftrightarrow \lambda_0 \in [1,3[ \; f=0 \Leftrightarrow \chi = \tfrac{1}{3} \Rightarrow \lambda = 3. \tag{92}$$

The radiation originating in the post shock medium may then flow freely across the front and escape to infinity. An example is a strong shock emerging from the photosphere or chromosphere of a star (see [3], p. 562).

The discontinuity of the Eddington factor is $[\chi] = \chi - \chi_0 < 0$.

By using Eq. (92), Eq. (82) yields

$$\Pi(X) = \frac{\Phi_0 - X}{3X-1}, \tag{93}$$

where

$$\phi_0 = \lambda_0 + \lambda_0^2 f_0^2 \alpha_0 = \lambda_0 + \frac{\alpha_0}{4}(3\lambda_0 - 1)(3 - \lambda_0) \tag{94}$$

with

$$\alpha_0 = \frac{\chi_0}{u_0^2(1 + \chi_0 - 2f_0/\beta_0)}. \tag{95}$$

The curve corresponding to Eq. (93) will be denoted as $C$. $C$ is called the Hugoniot curve.

The equation of state of the unburnt gas is taken in the form

$$P_{0r}^* = \chi_0 \rho_{0r}.$$

The equation of state of the burnt fluid is that of a gas of radiation:

$$P_r = \frac{\rho_r}{3}.$$

$C$ then represents the states of the burnt fluid. We assume that it lies above the Taub adiabat $C_0$ which represents the states of the unburnt fluid. This implies that $\Phi_0>3$ and that the curve $C$ does not pass through the initial point (1,1).

The intersections of the straight line $D$ with the curve $C$ are determined by the equation

$$\frac{\Phi_0-X}{3X-1}=-\frac{X}{\alpha_0}+1+\frac{1}{\alpha_0} \tag{96}$$

or

$$3X^2-4X(1+\alpha_0)+(1+\alpha_0+\alpha_0\Phi_0)=0. \tag{97}$$

The discriminant of this quadratic equation is

$$\Delta=4\alpha_0^2-\alpha_0(3\Phi_0-5)+1. \tag{98}$$

Equation (98) is also a quadratic polynomial, its discriminant may be written as

$$\delta_0=9\Phi_0^2-30\Phi_0+9=3(\Phi_0-3)(3\Phi_0-1)\,. \tag{99}$$

It follows that $\Phi_0>3\Rightarrow\delta_0>0$.

Equation (98) then has two positive roots given by

$$\alpha_{01}=\frac{3\Phi_0-5-\sqrt{\delta_0}}{8},$$

$$\alpha_{02}=\frac{3\Phi_0-5+\sqrt{\delta_0}}{8}. \tag{100}$$

Thus $\Delta>0$, if $\alpha_0<\alpha_{01}$, or $\alpha_0>\alpha_{02}$. For each value of $\Phi_0>3$, we can determine $\alpha_{01}$ and $\alpha_{02}$ and may choose $\alpha_0$ such that $\alpha_0<\alpha_{01}$ or $\alpha_0>\alpha_{02}$.

The roots of the quadratic equation (97) are

$$X=\frac{2(1+\alpha_0)\pm\sqrt{\Delta}}{3}. \tag{101}$$

These solutions represent the two intersections which the ray $D$ through the point (1,1) makes with the curve $C$.

We deduce from Eq. (94) that

$$\lambda_0=\frac{2+5\alpha_0\pm2\sqrt{\Delta}}{3\alpha_0}, \tag{102}$$

where $\alpha_0$ is given by (100). In order to satisfy the inequalities $1\leqslant\lambda_0<3$ we have to consider in Eq. (102) a value of $\alpha_0$ which is greater than $\alpha_{02}$.

The straight line $D$ is tangent to $C$ only if the discriminant $\Delta$ given by Eq. (98) is null: the two points of intersection coalesce at the Jouguet point $J$ and $\alpha_0=\alpha_{20}$ (the root $\alpha_0=\alpha_{10}$ being excluded). It follows from (101) that at the Jouguet point we have

$$X_J=\frac{2}{3}\,(1+\alpha_{02}). \tag{103}$$

The equation of the tangent to $C$ at this point is

$$\Pi_J(X)=-\frac{X}{\alpha_{02}}+1+\frac{1}{\alpha_{02}}. \qquad T$$

Since the tangent to either curve at a point $(\Pi,X)$ is related to the speed of sound in the medium corresponding to that point one observes that the hydrodynamical processes are devided into weak $(\beta<\beta_s)$, Jouguet $(\beta=\beta_s)$, and strong $(\beta>\beta_s)$ deflagrations. $\beta_s$ designs the sonic speed and $\beta$ is the speed of the deflagration wave in the burnt fluid.

The existence of deflagration as a possible mode for a phase transition in the early universe has recently been examined (see, for example, [19], [20], and [21]). From Landau and Lifchitz [22] one knows that

$$\beta_1^2=\frac{(P_2-P_1)(e_2+P_1)}{(e_2-e_1)(e_1+P_2)},$$
$$\beta_2^2=\frac{(P_2-P_1)(e_1+P_2)}{(e_2-e_1)(e_2+P_1)}, \tag{104}$$

where $e$, $P$ have their usual meanings. Using our notations, the relations (104) may be written as

$$\beta_0^2=\frac{(\Pi-1)(\lambda\Pi+1)}{(\lambda\Pi-\lambda_0)(\lambda_0+\Pi)}\ ,$$
$$\beta^2=\frac{(\Pi-1)(\lambda_0+\Pi)}{(\lambda\Pi+1)(\lambda\Pi-\lambda_0)}. \tag{105}$$

It follows from Eq. (105) that

$$\beta_0\to1,\quad \beta\to\frac{1}{\lambda}=\chi,\quad \text{for}\ \Pi\gg1.$$

Any straight line $D$ such that $\alpha_0>\alpha_{02}$ intersects the deflagration branch in two points: along it from the point (1,1) the pressure $\Pi$ decreases and the volume $X$ increases.

Thus the relation

$$\Pi(X)=\frac{\Phi_0-X}{3X-1}$$

gives various deflagration adiabat with $\lambda_0$ as a parameter. In order to ensure $1\leqslant\lambda_0<3$ it is necessary that $\Phi_0>3$. It is interesting to notice the similarity of our relation with that obtained from the equation (3.8) of [23]

$$\Pi(x)=\frac{K-x}{3x-1}\quad \text{with}\ \ \Pi=P_2/P_1\quad \text{and}\ \ x=x_2/x_1,$$

where

$$K=\frac{4\varepsilon}{P_1}+3 \tag{106}$$

for various detonation adiabats as a function of the vacuum energy $e$.

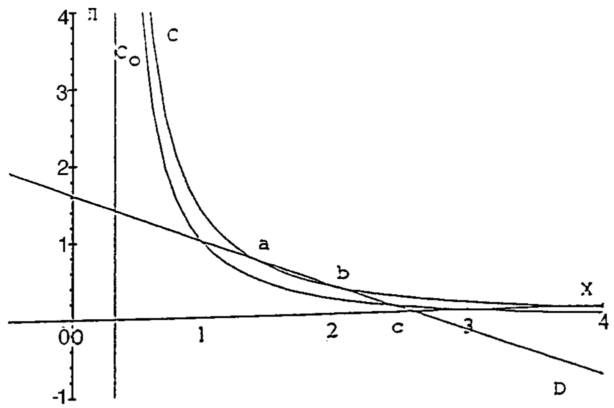

FIG. 3. Plot of $C_0$, $C(\Phi_0=3.8)$, and $D(\alpha_0=8/5)$ in the $(X,\Pi)$ plane.

For $\Phi_0=3.8$ (100) gives $\alpha_{01}=0.1755$ and $\alpha_{02}=1.4245$. We may choose $\alpha_0=0.16<\alpha_{01}$ or $\alpha_0=8/5>\alpha_{02}$ to have $\Delta>0$. Equation (102) then yields

$$\lambda_0\begin{cases}\frac{14}{3}\\ 7\end{cases} \text{if } \alpha_0=0.16$$

and

$$\lambda_0=\begin{cases}\frac{5}{3}\\ \frac{5}{2}\end{cases} \text{if } \alpha_0=\tfrac{8}{5}. \tag{107}$$

We shall exclude the value $\alpha_0=0.16$ as $\lambda_0$ must be in the range $1\leqslant\lambda_0<3$. In this example Eq. (93) becomes

$$\Pi(X)=\frac{3.8-X}{3X-1} \tag{108}$$

with the equation of the straight line $D$ being

$$\Pi(X)=\frac{5X}{8}+\frac{13}{8}. \tag{109}$$

The curves $C_0$ and Eq. (108) and the straight line Eq. (109) are plotted in Fig. 3. The equation of the tangent $T$ becomes

$$\Pi_J(X)=-0.702X+1.702. \tag{110}$$

The curves $C_0$ and Eq. (108), and the straight line (110) are plotted in Fig. 4. Thus for $\Phi_0>3$ we obtain various Hugoniot adiabats which lie above the Taub adiabat $C_0$.

*Case III:* The upstream material is an optically thin medium while the downstream medium is an optically thick one. This case may correspond to

$$\begin{aligned} &f_0=1\Leftrightarrow\chi_0=1\Rightarrow\lambda_0=1,\\ &f=0\Leftrightarrow\chi=\tfrac{1}{3}\Rightarrow\lambda=3. \end{aligned} \tag{111}$$

The relations (111) are the extreme case of Eq. (92).

The radiation originating in the postshock medium can then flow freely and escape to infinity. In this case, from Eq. (30) we have $E=0$ and $a_{0r}=P_{0r}=\rho_{0r}$, and it follows from Eqs. (50) and (95) that

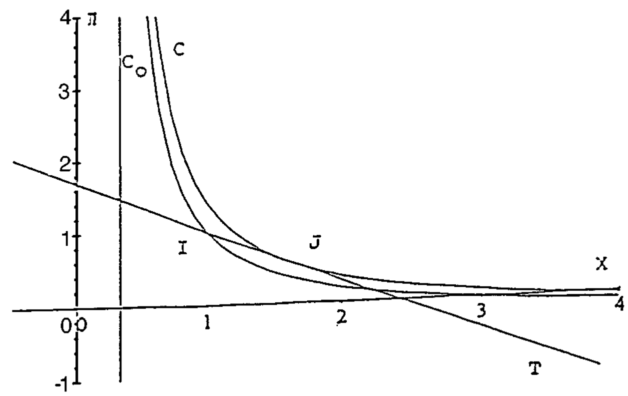

FIG. 4. Plot of $C_0$, $C(\Phi_0=3.8)$ and $T(\alpha_0=1.4245)$ in the $(X,\Pi)$ plane.

$$\alpha_0=-\frac{1+\beta_0}{2\beta_0}. \tag{112}$$

Equation (83) then gives $\Phi_0=1+\alpha_0$. It follows that $\Phi_0>3\Leftrightarrow\alpha_0>2$. The equation (96) yields

$$3X^2-4X(1+\alpha_0)+(1+\alpha_0)^2=0. \tag{113}$$

The discriminant of this quadratic equation is

$$\delta=(1+\alpha_0)^2. \tag{114}$$

The roots of Eq. (110) are

$$\begin{aligned} X_1&=\frac{2(1+\alpha_0)-\sqrt{\delta}}{3}=\frac{1+\alpha_0}{3},\\ X_2&=\frac{2(1+\alpha_0)-\sqrt{\delta}}{3}=1+\alpha_0. \end{aligned} \tag{115}$$

The straight line $D$ is tangent to the curve $C$ if $\delta=0\Leftrightarrow\alpha_0=-1$ which yields $\Phi_0=0$ and Eq. (115) gives $X_1=X_2=0$. Thus the case where $\chi_0=1$, $\chi=\frac{1}{3}$, and $\alpha_0=-1$ leads to results which are not consistent with shock waves.

It may be remarked that if $\chi_0=1$ and $\alpha_0=-1$ then we obtain from Eq. (95) that $\beta_0=1$. We obtain for $C$ the equation

$$\Pi(X)=\frac{\Phi_0-X}{3X-1}=\frac{(1+\alpha_0)-X}{3X-1}. \tag{116}$$

For $\alpha_0=\frac{57}{20}$, Eq. (116) becomes

$$\Pi(X)=\frac{3.85-X}{3X-1}, \tag{117}$$

the equation of the straight line $D$ being

$$\Pi(X)=-\frac{20X}{57}+\frac{77}{57}. \tag{118}$$

For $\alpha_0=\frac{57}{20}$, Eq. (115) gives the two intersections of the straight line $D$ with the curve $C$:

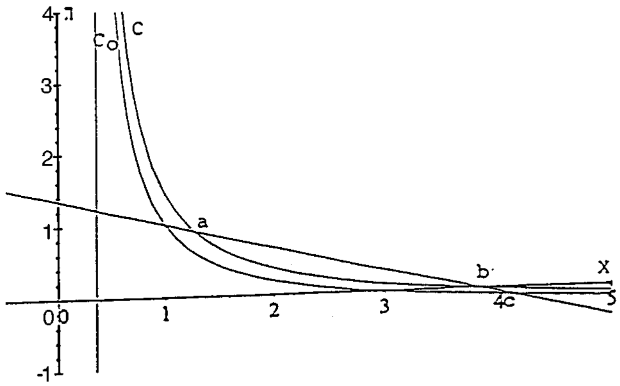


FIG. 5. Plot of $C_0$, $C(\Phi_0=3.85)$ and $D(\alpha_0=\frac{57}{20})$ in the $(X,\Pi)$ plane.

$$X_1=\frac{1}{3}(1+\alpha_0)=\frac{77}{60},\quad \Pi_1=-\frac{20X_1}{57}+\frac{77}{57}=\frac{154}{171},\tag{119}$$
$$X_2=1+\alpha_0=\phi_0=\frac{77}{20},\quad \Pi_2=0.$$

The straight line $D$ through (1,1) intersects the Taub adiabat $C_0$ at another point whose coordinates are given by Eq. (90):

$$X_1=\frac{1}{3}(1+4\alpha_0)=\frac{62}{15}\;\Pi_1=\frac{2-\alpha_0}{3\alpha_0}=-\frac{17}{171}.\tag{120}$$

Equation (120) represents a thermodynamic state with a negative pressure, the final state on the curve $C$ corresponding to the strong deflagration is $(X_2=\Phi_0,\Pi_2=0)$ given by the last relation of Eq. (119). The curves $C_0$ and Eqs. (117) and (118) are plotted in Fig. 5.

## IV. REMARKS

The above results are obtained considering a radiation-dominated gas, that is assumes that $\beta_1^*\leqslant 1$. The same results may be obtained considering that the matter and the radiation are not in thermal equilibrium, i.e., the energy tensors given by Eqs. (1) and (34) are not evaluated at equilibrium states [18]. The conservation law (42) then reads

$$\nabla_\alpha T_m^{\alpha\beta}=0,\quad \nabla_\alpha T_r^{\alpha\beta}=0.$$

## V. CONCLUSION

The main purpose of this paper was to derive from the macroscopic viewpoint the radiation energy momentum tensor taking into account the effects of the radiation pressure, energy density, and flux and to extend previously known results on the relativistic shock waves to the case of relativistic radiation hydrodynamics. It is shown that the Eddington factor is a convenient parameter to study the Rankine-Hugoniot adiabats. Our main conclusions are the following.

(i) For $\chi_0=\chi=\frac{1}{3}$ and $u_0^2>\frac{1}{2}$ (or $\beta_0>1/\sqrt{3}$) we obtain the Taub adiabat for shock waves (see Fig. 2).

(ii) For $\frac{1}{3}<\chi_0\leqslant 1$ and $\chi=\frac{1}{3}$ and for each value of the quantity $\Phi_0$ such that $\Phi_0>3$ the curve $C$ lies above the Taub adiabat. It is shown that in this case one has to consider only the intersections of the straight line $D$ with the deflagration branch. The slope of the straight line $T$ which is tangent to this branch at the Jouguet point is determined by the great root $\alpha_{02}$ of the equation $\Delta(\alpha_0)=0$ given by Eq. (98). Any ray with the slope greater than that of this tangent $(\alpha_0>\alpha_{02})$ intersects the deflagration branch in two points (Fig. 3).

(iii) If $\chi_0=1$ (free streaming) and $\chi=\frac{1}{3}$, then the straight line $D$ through the point (1,1) cannot be tangent to the deflagration curve and null pressure corresponds to its first intersection (strong deflagration) with the curve (Fig. 5). The speed of the downstream medium is then $\beta=1$. The radiation emetted across the front from the hot downstream medium escapes freely to infinity.

We hope that some applications of our results to the radiation dominated epoch of the Universe will be possible. A forthcoming paper will be devoted to the study of the denotation waves.

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