Relevance of nucleon spin in an amplitude analysis of reactions $\pi^- p \to \pi^0\pi^0 n$ and $\pi^- p \to \eta\eta n$

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The measurements of reactions $\pi^- p \to \pi^-\pi^+ n$ and $\pi^+ n \to \pi^+\pi^- p$ on polarized targets at CERN found a strong dependence of pion production amplitudes on nucleon spin. Analyses of recent measurements of the $\pi^- p \to \pi^0\pi^0 n$ reaction on unpolarized targets by the GAMS Collaboration at 38 GeV/c and the BNL E852 Collaboration at 18 GeV/c use the assumption that pion production amplitudes do not depend on nucleon spin, in conflict with the CERN results on polarized targets. We show that measurements of $\pi^- p \to \pi^0\pi^0 n$ and $\pi^- p \to \eta\eta n$ on unpolarized targets can be analyzed in a model-independent way in terms of four partial-wave intensities and three independent interference phases in the mass region where S and D waves dominate. We also describe model-independent amplitude analysis of the $\pi^- p \to \pi^0\pi^0 n$ reaction measured on a polarized target, both in the absence and in the presence of G-wave amplitudes. We suggest that high statistics measurements of reactions $\pi^- p \to \pi^0\pi^0 n$ and $\pi^- p \to \eta\eta n$ be made on polarized targets at Protvino IHEP and at BNL, and that model-independent amplitude analyses of this polarized data be performed to advance hadron spectroscopy on the level of spin-dependent production amplitudes. [S0556-2821(97)01207-1]

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I. INTRODUCTION

The dependence of hadronic reactions on nucleon spin was discovered by Chamberlain and his group at Berkeley in 1957 in measurements of polarization in pp and np elastic scattering at 320 MeV [1]. The prevalent belief in the 1950s and 1960s was that in hadronic reactions spin is irrelevant and the spin effects observed by Chamberlain were expected to vanish at very high energies, such as 6 GeV/c. Instead, measurements of polarization in two-body reactions found significant dependence on spin up to 300 GeV/c at CERN [2] and Fermilab [3]. Measurements at BNL found large spin effects at very large momentum transfers [4,5]. Inclusively produced hyperons show large polarizations up to the equivalent of 2000 GeV/c [6]. Large spin effects in inclusive reactions were observed at the Fermilab Spin Facility with polarized proton and antiproton beams at 200 GeV/c [7,8]. Today, work is in progress to study the dependence of hadronic reactions on spin and nucleon spin structure with polarized colliding proton beams at the Relativistic Heavy Ion Collider (RHIC) collider at BNL [9].

The most remarkable feature of hadronic reactions is the conversion of kinetic energy of colliding hadrons into the matter of produced particles. This conversion process is characterized by conservation of total four-momentum and quantum numbers such as electric charge, baryon number, and strangeness. The conversion process depends also on the flavor content and spin of colliding hadrons.

The simplest production processes are single-pion production reactions such as $\pi N \to \pi^+\pi^- N$ and $KN \to K\pi N$. In 1978, Lutz and Rybicki showed [10] that measurements of these reactions on polarized targets yield enough observables that model-independent amplitude analysis is possible, determining the spin-dependent production amplitudes. The measurements of these reactions on polarized targets are thus of special interest because they permit one to study the spin dependence of pion creation directly on the level of production amplitudes. Several such measurements were actually done at the CERN Proton Synchrotron (PS).

The high statistics measurement of $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c on unpolarized targets [11] was later repeated with a transversely polarized proton target at the same energy [12–17]. Model-independent amplitude analyses were performed for various intervals of dimeson mass at small momentum transfers $-t = 0.005 - 0.2$ (GeV/c)^2 [12–15] and over a large interval of momentum transfer, $-t = 0.2 - 1.0$ (GeV/c)^2 [16,17].

Additional information was provided by the first measurement of $\pi^+ n \to \pi^+\pi^- p$ and $K^+ n \to K^+\pi^- p$ reactions on polarized deuteron targets at 5.98 and 11.85 GeV/c [18,19]. The data allowed us to study the t evolution of the mass dependence of moduli of amplitudes [20]. Detailed amplitude analyses [21,22] determined the mass dependence of amplitudes at larger momentum transfers of $-t = 0.2 - 0.4$ (GeV/c)^2.

The crucial finding of all these measurements was the strong dependence of production amplitudes on nucleon spin. The process of pion production is very closely related to nucleon transversity or the nucleon spin component in a direction perpendicular to the production plane. For instance, in $\pi^- p \to \pi^-\pi^+ n$ at small t and dipion masses below 1000 MeV, all amplitudes with recoil nucleon transversity down are smaller than the transversity up amplitudes, irrespective of dimeson spin and helicity. All recoil nucleon transversity down amplitudes also show suppression of resonance production in the ρ meson region.

The measurements of $\pi N \to \pi^+\pi^- N$ reactions on polarized targets also enabled a model-independent separation of S- and P-wave amplitudes. The S-wave amplitude with recoil nucleon transversity up is found to resonate at 750 MeV in both solutions [23–25] irrespective of the method of amplitude analysis [25]. The resonance is narrow and the most recent fits [25] determined its width to be 108 ± 53 MeV.

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Recently high statistics measurements of the $\pi^- p \to \pi^0 \pi^0 n$ reaction were made at 38 GeV/c by the GAMS Collaboration at IHEP Protvino [26–28] and at 18 GeV/c by the E852 Collaboration at BNL [29]. In principle one expects these experiments to confirm the existence of the $\sigma(750)$ state and to search for new states in higher partial waves. However, the situation is not so simple. The reason is that both groups analyze their well-acquired data using a strong simplifying assumption that the production amplitudes are independent of nucleon spin [30–34]. The purpose of this assumption is to reduce the number of unknown amplitudes by one-half and to enable one to proceed with amplitude analysis using such spin-independent ''amplitudes.''

At this point it is important to realize that one does not really make an assumption that production amplitudes are independent of nucleon spin. It is a well-known fact that nucleon helicity nonflip and flip amplitudes have an entirely different t dependence due to conservation of angular momentum. The helicity flip amplitudes vanish as $t \to 0$ while helicity nonflip amplitudes do not. The model-independent amplitude analyses of two-body reactions also found that the zero structures of flip and nonflip amplitudes are dramatically different. Moreover, pion production at small t proceeds mostly via pion exchange which contributes to helicity flip amplitudes. Thus the assumption that is really being made is that all nonflip amplitudes vanish.

The assumption that production amplitudes in $\pi^- p \to \pi^0 \pi^0 n$ do not depend on nucleon spin is in conflict with the general consensus that hadronic reactions depend on nucleon spin up to the highest energies and contradicts all that we have learned from measurements of $\pi N \to \pi^- \pi^+ N$ on polarized targets at CERN. Applied to the reactions $\pi^- p \to \pi^+ \pi^- n$ and $\pi^+ n \to \pi^+ \pi^- p$, the assumption has observable consequences that can be tested directly in measurements with polarized targets.

The first consequence is that all polarized moments p^L_M vanish identically. All experiments on polarized targets, however, found large nonzero polarized moments. An example is given in Fig. 1 which shows the polarized target asymmetry A related to the moment p^0_0. The polarized target asymmetry has large nonzero (negative) values in both reactions. Measurements of $K^+ n \to K^+ \pi^- p$ show similarly large values of A [19].

The experiments on polarized targets are best analyzed using nucleon transversity amplitudes rather than nucleon helicity amplitudes. The second consequence of the assumption of the independence of production amplitudes on nucleon spin is that all transversity amplitudes $|A|$ with recoil nucleon transversity ''up'' are equal in magnitude to transversity amplitudes $|\underline{A}|$ with recoil nucleon transversity ''down'' relative to the scattering plane $\pi^- N \to (\pi^- \pi^+) N$. In Fig. 2 we show the ratios of transversity amplitudes for S, P, D, and F waves for dimeson helicity $\lambda = 0$. The ratios are far from unity, indicating that production amplitudes depend strongly on nucleon spin.

If the assumption that the production amplitudes are independent of nucleon spin does not work in the reactions $\pi^- p \to \pi^- \pi^+ n$, $\pi^+ n \to \pi^+ \pi^- p$, and $K^+ n \to K^+ \pi^- p$, then there is no reason to assume that it will work in the $\pi^- p \to \pi^0 \pi^0 n$ reaction. We must conclude that some of the results of the analyses of $\pi^- p \to \pi^0 \pi^0 n$ by GAMS and E852 Collaborations are not reliable.

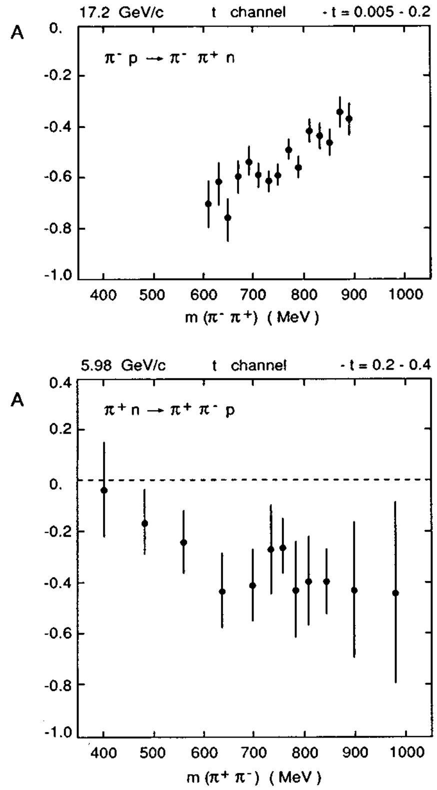

FIG. 1. Polarized target asymmetry in reactions $\pi^- p \to \pi^- \pi^+ n$ and $\pi^+ n \to \pi^+ \pi^- p$. The assumption that the pion production amplitudes do not depend on nucleon spin predicts that polarized target asymmetry will be zero.

The question of the reliability of amplitude analyses based on the assumption of the independence of production amplitudes on nucleon spin is of special importance to confirmation and further study of the narrow $\sigma(750)$ state in the $\pi^- p \to \pi^0 \pi^0 n$ reaction. The evidence for narrow $\sigma(750)$ is closely connected to the spin dependence of production amplitudes. In Fig. 3 we show the two S-wave production amplitudes for $\pi^- p \to \underline{\pi^-} \pi^+ n$. We see that while the transversity up amplitude $|S|^2\Sigma$ resonates in both solutions around 750 MeV, the transversity down amplitude $|\underline{S}|^2\Sigma$ is large and nonresonating. This results in a partial wave intensity $I_S = (|S|^2 + |\underline{S}|^2)\Sigma$ that does not necessarily show a narrow resonant behavior. As seen in Fig. 4, such is the case of solution $I_S(2,2)$.

It is therefore necessary to establish what quantities can be determined from the measurements of $\pi^- p \to \pi^0 \pi^0 n$ on unpolarized targets without the assumption of the independence of production amplitudes on nucleon spin. Furthermore, it is necessary to find out if a model-independent amplitude analysis of $\pi^- p \to \pi^0 \pi^0 n$ in measurements on polarized targets is possible. The purpose of this work is to provide answers to these questions. We shall show that in measurements of $\pi^- p \to \pi^0 \pi^0 n$ on unpolarized targets in the region where S and D waves dominate, one can measure four spin-averaged partial wave intensities and three unrelated phases connected with the spin-averaged interference terms. We will also show that model-independent amplitude

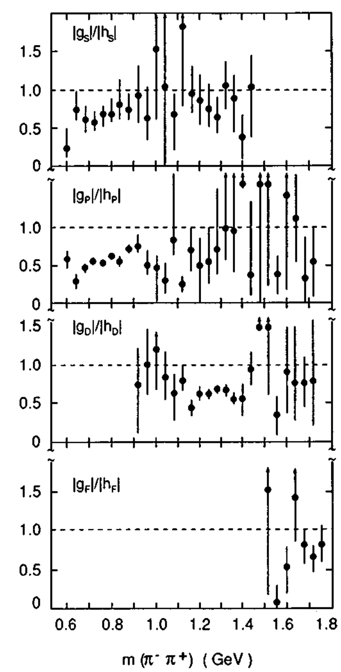

FIG. 2. The ratio of amplitudes with recoil nucleon transversity ''down'' and ''up'' with dimeson helicity $\lambda=0$ in $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c and $-t=0.005-0.2$ (GeV/c)^2. The assumption that the pion production amplitudes do not depend on nucleon spin predicts that all ratios will be equal to 1. The deviation from unity shows the strength of the dependence of production amplitudes on nucleon spin. Based on Fig. 6 of Ref. [14].

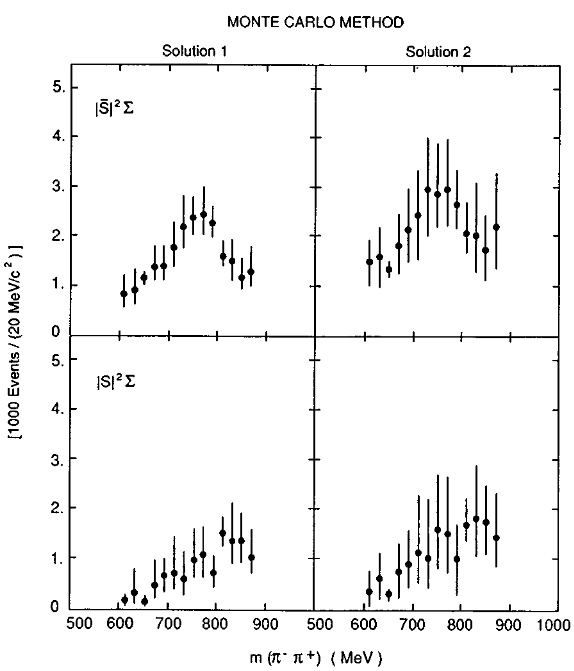

FIG. 3. Mass dependence of unnormalized amplitudes $|\bar{S}|^2\Sigma$ and $|S|^2\Sigma$ measured in $\pi^- p_\uparrow \to \pi^- \pi^+ n$ at 17.2 GeV/c at $-t=0.005-0.20$ (GeV/c)^2 using the Monte Carlo method for amplitude analysis [24]. Both solutions for the transversity ''up'' amplitude $|\bar{S}|^2\Sigma$ resonate while the transversity ''down'' amplitude $|S|^2\Sigma$ is nonresonating in both solutions.

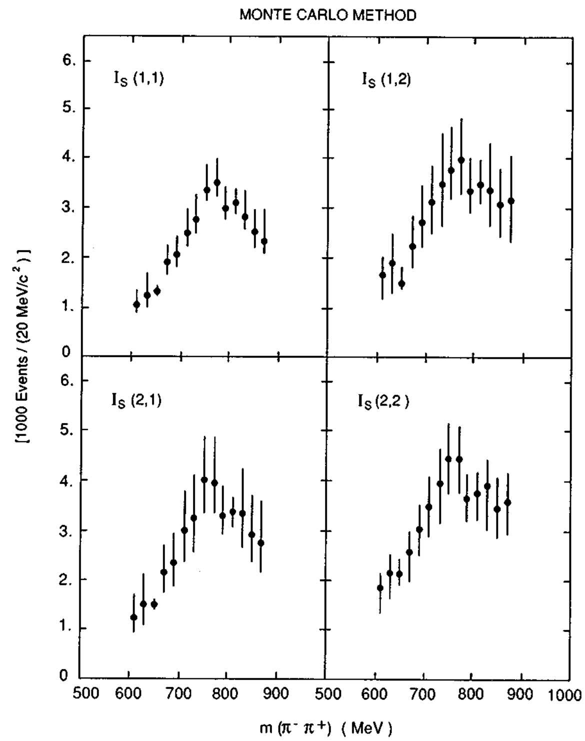

FIG. 4. Four solutions for the S-wave intensity I_S measured in the reaction $\pi^- p_\uparrow \to \pi^- \pi^+ n$ at 17.2 GeV/c and $-t=0.005-0.20$ GeV/c using the Monte Carlo method for amplitude analysis [24]. Although both solutions for the amplitude $|\bar{S}|^2\Sigma$ resonate, the intensity $I_S(2,2)$ appears nonresonating.

analysis is possible when measurements of $\pi^- p \to \pi^0 \pi^0 n$ are made on polarized targets, both in the region where S and D waves dominate as well as in the region where G waves also contribute. We shall propose that such measurements are a natural extension of measurements on unpolarized targets and should be performed at both IHEP in Protvino and at BNL using Brookhaven Multi Particle Spectrometer.

The paper is organized in seven sections. The kinematics, observables, and pion production amplitude are introduced in Sec. II. The method of the model-independent analysis of data on unpolarized targets is described in Sec. III. In Sec. IV we compare this method with model-dependent analyses of the GAMS and E852 Collaborations. In Sec. V we describe a model-independent amplitude analysis of $\pi^- p \to \pi^0 \pi^0 n$ on polarized targets in the absence of G waves. In Sec. VI we extend the model-independent amplitude analysis to include the G-wave amplitudes. The paper closes with a summary and proposals for measurements of $\pi^- p \to \pi^0 \pi^0 n$ and $\pi^- p \to \eta\eta n$ on polarized targets in Sec. VII.

II. KINEMATICS, OBSERVABLES, AND AMPLITUDES

A. Kinematics

Various aspects of phase space, kinematics, and amplitudes in pion production in $\pi N \to \pi\pi N$ reactions are described in several books [35–37]. The kinematical variables used to describe the dimeson production on a polarized target at rest are $(s,t,m,\theta,\phi,\psi,\delta)$ where s is the center-of-mass system (c.m.s.) energy squared, t is the four-momentum

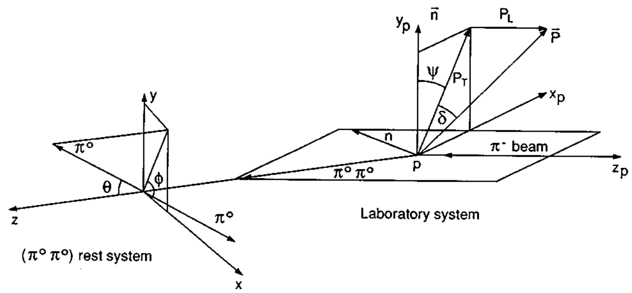

FIG. 5. Definition of the coordinate systems used to describe the target polarization $\vec{P}$ and the decay of the dimeson $\pi^0\pi^0$ system.

transfer to the nucleon squared, and m is the dimeson invariant mass. The angles (θ,ϕ) describe the direction of π^0 in the $\pi^0\pi^0$ rest frame. The angle ψ is the angle between the direction of the target transverse polarization and the normal to the scattering plane (Fig. 5). The angle δ is the angle between the direction of the target polarization vector and its transverse component (projection of the polarization vector into the x,y plane). The analysis is usually carried out in the t-channel helicity frame for the $\pi^0\pi^0$ dimeson system. The helicities of the initial and final nucleons are always defined in the s-channel helicity frame.

B. Observables

In our discussion of observables measured in $\pi^- p \to \pi^0\pi^0 n$ with polarized targets we follow the notation of Lutz and Rybicki [10]. When the polarization of the recoil nucleon is not measured, the unnormalized angular distribution $I(\theta,\phi,\psi,\delta)$ of $\pi^0\pi^0$ (or $\eta\eta$) production on polarized nucleons at rest of fixed s, m, and t can be written as

$$I(\Omega,\psi,\delta)=I_U(\Omega)+P_T\cos\psi I_C(\Omega)+P_T\sin\psi I_S(\Omega)+P_L I_L(\Omega), \tag{2.1}$$

where $P_T=P\cos\delta$ and $P_L=P\sin\delta$ are the transverse and longitudinal components of target polarization $\vec{P}$ with respect to the incident momentum (Fig. 5). The simple $\cos\psi$ and $\sin\psi$ dependence is due to a spin $\frac{1}{2}$ of the target nucleon [10,38]. Parity conservation requires I_U and I_C to be symmetric and I_S and I_L to be antisymmetric in ϕ. In the data analysis of the angular distribution of the dimeson system, it is convenient to use expansions of the angular distributions into spherical harmonics. In the notation of Lutz and Rybicki we have

$$I_U(\Omega)=\sum_{L,M} t_M^L \mathrm{Re}Y_M^L(\Omega),$$

$$I_C(\Omega)=\sum_{L,M} p_M^L \mathrm{Re}Y_M^L(\Omega),$$

$$I_S(\Omega)=\sum_{L,M} r_M^L \mathrm{Im}Y_M^L(\Omega),$$

$$I_L(\Omega)=\sum_{L,M} q_M^L \mathrm{Im}Y_M^L(\Omega). \tag{2.2}$$

The expansion coefficients t, p, r, and q are called multipole moments. The moments t_M^L are unpolarized. The moments p_M^L, r_M^L, and q_M^L are polarized moments. Experiments with transversely polarized targets measure only transverse moments p_M^L and r_M^L but not the longitudinal moments q_M^L.

The multipole moments are obtained from the experimentally observed distributions in each (m,t) bin by means of the optimization of the maximum likelihood function which takes into account the acceptance of the apparatus [11,39]. In these fits it is usually assumed that moments with $M>2$ vanish. However, it was pointed out by Sakrejda [16] that moments up to $M=4$ may have to be taken into account at larger momentum transfers extending to 1.0 $(\mathrm{GeV}/c)^2$.

The expansion coefficients t, p, r, and q are simply connected to moments of angular distributions [10]:

$$t_M^L=\epsilon_M\langle \mathrm{Re}_M^L\rangle=\frac{\epsilon_M}{2\pi}\int I(\Omega,\psi,\delta)\mathrm{Re}Y_M^L(\Omega)d\Omega',$$

$$p_M^L=2\epsilon_M\langle\cos\psi \mathrm{Re}_M^L\rangle =\frac{2\epsilon_M}{2\pi}\int I(\Omega,\psi,\delta)\mathrm{Re}Y_M^L(\Omega)\cos\psi\cos\delta d\Omega',$$

$$r_M^L=4\langle\sin\psi \mathrm{Im}Y_M^L\rangle =\frac{4}{2\pi}\int I(\Omega,\psi,\delta)\mathrm{Im}Y_M^L\sin\psi\cos\delta d\Omega',$$

$$q_M^L=4\langle \mathrm{Im}Y_M^L\rangle=\frac{4}{2\pi}\int I(\Omega,\psi,\delta)\mathrm{Im}Y_M^L\sin\delta d\Omega', \tag{2.3}$$

where $d\Omega'=d\Omega d\psi\, d(-\sin\delta)$. In Eqs. (2.3), $\epsilon_M=1$ for $M=0$ and $\epsilon_M=2$ for $M\neq 0$. Integrated over the solid angles (θ,ϕ), the distribution (2.1) becomes

$$I(\psi,\delta)=(1+AP_T\cos\psi)\frac{d^2\sigma}{dm\ dt}, \tag{2.4}$$

where $A=A(s,t,m)=\sqrt{4\pi}p_0^0$ is the polarized target asymmetry analogous to the polarization parameter measured in two-body reactions. In Eq. (2.4), $d^2\sigma/dm\ dt$ is the integrated reaction cross section:

$$\frac{d^2\sigma(s,t,m)}{dm\ dt}=\int I(\Omega,\psi,\delta)d\Omega'. \tag{2.5}$$

Finally we note the relation of moments t_M^L to moments $H(LM)$ introduced by Chung [31,32]:

$$t_M^L=\epsilon_M\langle \mathrm{Re}Y_M^L\rangle=\epsilon_M\sqrt{\frac{2L+1}{4\pi}}H(LM). \tag{2.6}$$

C. Amplitudes

The reaction $\pi^- p\to\pi^0\pi^0 n$ is described by the production amplitudes $H_{\lambda_n,0\lambda_p}(s,t,m,\theta,\phi)$ where λ_p and λ_n are the helicities of the proton and neutron, respectively. The production amplitudes can be expressed in terms of production amplitudes corresponding to definite dimeson spin J using an angular expansion

$$H_{\lambda_n,0\lambda_p}=\sum_{J=0}^{\infty}\sum_{\lambda=-J}^{+J}(2J+1)^{1/2}H^J_{\lambda\lambda_n,0\lambda_p}(s,t,m)d^J_{\lambda 0}(\theta)e^{i\lambda\phi}, \tag{2.7}$$

where J is the spin and λ the helicity of the $(\pi^0\pi^0)$ dimeson system. Because of the identity of the two final-state mesons, the "partial waves" with odd J are absent so that $J=0,2,4,\ldots$.

In the following we will consider only S-wave $(J=0)$, D-wave $(J=2)$, and G-wave $(J=4)$ amplitudes. Furthermore, we will restrict the dimeson helicity λ to values $\lambda=0$ or ± 1 only in accordance with the assumption that moments with $M>2$ vanish. This assumption is supported by experiments.

The "partial wave" amplitudes $H^J_{\lambda\lambda_n,0\lambda_p}$ can be expressed in terms of nucleon helicity amplitudes with definite t-channel exchange naturality. The nucleon s-channel helicity amplitudes describing the production of the $(\pi^0\pi^0)$ [or $(\eta\eta)$] system in the S-, D-, and G-wave states are

$$0^-\frac{1}{2}^+\rightarrow 0^+\frac{1}{2}^+: H^0_{0+,0+}=S_0,\quad H^0_{0+,0-}=S_1,$$

$$0^-\frac{1}{2}^+\rightarrow 2^+\frac{1}{2}^+: H^2_{0+,0+}=D^0_0,\quad H^2_{0+,0-}=D^0_1,$$

$$H^2_{\pm 1+,0+}=\frac{D_0^+\pm D_0^-}{\sqrt{2}},\quad H^2_{\pm 1+,0-}=\frac{D_1^+\pm D_1^-}{\sqrt{2}},$$

$$0^-\frac{1}{2}^+\rightarrow 4^+\frac{1}{2}^+: H^4_{0+,0+}=G^0_0,\quad H^4_{0+,0-}=G^0_1,$$

$$H^4_{\pm 1+,0+}=\frac{G_0^+\pm G_0^-}{\sqrt{2}},\quad H^4_{\pm 1+,0-}=\frac{G_1^+\pm G_1^-}{\sqrt{2}}. \tag{2.8}$$

At large s, the amplitudes S_n, D^0_n, D^-_n, G^0_n, and G^-_n, $n=0,1$, are dominated by unnatural exchanges while the amplitudes D^+_n and G^+_n, $n=0,1$, are dominated by natural exchanges. The index $n=|\lambda_p-\lambda_n|$ is nucleon helicity flip.

The observables obtained in experiments on transversely polarized targets in which recoil nucleon polarization is not observed are most simply related to the nucleon transversity amplitudes of definite naturality [10,19,40]. For S, D, and G waves they are defined as

$$S=k(S_0+iS_1),\quad \overline{S}=k(S_0-iS_1), \tag{2.9}$$

$$D^0=k(D^0_0+iD^0_1),\quad \overline{D^0}=k(D^0_0-iD^0_1),$$

$$D^-=k(D^-_0+iD^-_1),\quad \overline{D^-}=k(D^-_0-iD^-_1),$$

$$D^+=k(D^+_0-iD^+_1),\quad \overline{D^+}=k(D^+_0+iD^+_1),$$

$$G^0=k(G^0_0+iG^0_1),\quad \overline{G^0}=k(G^0_0-iG^0_1),$$

$$G^-=k(G^-_0+iG^-_1),\quad \overline{G^-}=k(G^-_0-iG^-_1),$$

$$G^+=k(G^+_0-iG^+_1),\quad \overline{G^+}=k(G^+_0+iG^+_1),$$

where $k=1/\sqrt{2}$. The formal proof that the amplitudes defined in Eqs. (2.9) are actually transversity amplitudes is given from the definition in the Appendix in Ref. [19].

The nucleon helicity and nucleon transversity amplitudes differ in the quantization axis for the nucleon spin. The transversity amplitudes S, D^0, D^-, D^+, G^0, G^-, and G^+ ($\overline{S}$, $\overline{D^0}$, $\overline{D^-}$, $\overline{D^+}$, $\overline{G^0}$, $\overline{G^-}$, and $\overline{G^+}$) describe the production of the dimeson state with the recoil nucleon spin antiparallel or "down" (parallel or "up") relative to the normal $\vec{n}$ to the production plane. The direction of normal $\vec{n}$ is defined according to the Basel convention by $\vec{p}_\pi\times\vec{p}_{\pi\pi}$ where $\vec{p}_\pi$ and $\vec{p}_{\pi\pi}$ are the incident pion and dimeson momenta in the target proton rest frame.

Using the symbols ↑ and ↓ for the nucleon transversities up and down, respectively, the following table shows the spin states of target protons and recoil neutrons and the dimeson helicities corresponding to the transversity amplitudes (2.9):

	p	n	$(\pi^0\pi^0)$
S,D^0,G^0	↑	↓	0
$\overline{S},\overline{D^0},\overline{G^0}$	↓	↑	0
D^-,G^-	↑	↓	+1 or −1
$\overline{D^-},\overline{G^-}$	↓	↑	+1 or −1
D^+,G^+	↓	↓	+1 or −1
$\overline{D^+},\overline{G^+}$	↑	↑	+1 or −1

Parity conservation requires that in the transversity frame dimeson production with helicities ± 1 depends only on the transversities of the initial and final nucleons. The amplitudes D^-, $\overline{D^-}$, $\ldots$, G^+, $\overline{G^+}$ do not distinguish between dimeson helicity states with $\lambda=+1$ or -1. Also, dimeson production with helicity $\lambda=0$ is forbidden by parity conservation when the initial and final nucleons have the same transversities.

D. Observables in terms of amplitudes

It is possible to express the moments t^L_M and p^L_M in terms of quantities that do not depend explicitly on whether we use nucleon helicity or nucleon transversity amplitudes. However, eventually we are going to work with transversity amplitudes. The quantities we shall need are the spin-averaged partial wave intensity

$$I_A=|A|^2+|\overline{A}|^2=|A_0|^2+|A_1|^2 \tag{2.10}$$

and partial wave polarization

$$P_A=|A|^2-|\overline{A}|^2=2\epsilon_A \mathrm{Im}(A_0A_1^*), \tag{2.11}$$

where $\epsilon_A=+1$ for $A=S,D^0,D^-,G^0,G^-$ and $\epsilon_A=-1$ for $A=D^+,G^+$. We also introduce the spin-averaged interference terms

$$R(AB)=\mathrm{Re}(AB^*+\overline{AB^*})=\mathrm{Re}(A_0B_0+\epsilon_A\epsilon_B A_1B_1^*), \tag{2.12}$$

$$Q(AB)=\mathrm{Re}(AB^*-\overline{AB^*})=\mathrm{Re}(\epsilon_B A_0 B_1^* - \epsilon_A A_1 B_0^*). \tag{2.13}$$

Then moments t_M^L can be expressed in terms of spin-averaged intensities I_A and spin-averaged interference terms $R(AB)$. The moments p_M^L are then expressed in terms of polarizations P_A and interference terms $Q(AB)$. The formulas for p_M^L are obtained from those for t_M^L using a replacement $I_A \rightarrow \epsilon_A P_A$ and $R(AB)\rightarrow Q(AB)$ for $\epsilon_A=\epsilon_B=1$ and $R(AB)\rightarrow -Q(AB)$ for $\epsilon_A=\epsilon_B=-1$. There is no mixing of natural and unnatural exchange amplitudes in the moments t_M^L and p_M^L.

Using the results of Lutz and Rybicki [10] and of Chung [32], we obtain the following expressions for moments in terms of quantities (2.10)–(2.13) and a constant $c=\sqrt{4\pi}$:

Unpolarized moments:

$$ct_0^0=I_S+I_{D^0}+I_{D^-}+I_{D^+}+I_{G^0}+I_{G^-}+I_{G^+}, \tag{2.14}$$

$$ct_0^2=\sqrt{5}\left\{\frac{2}{\sqrt{5}}R(SD^0)+\frac{2}{7}I_{D^0}+\frac{1}{7}(I_{D^-}+I_{D^+}) + \frac{12}{7\sqrt{5}}R(D^0G^0)+2\frac{\sqrt{6}}{7}[R(D^-G^-)+R(D^+G^+)] + \frac{20}{77}I_{G^0}+\frac{17}{77}(I_{G^-}+I_{G^+})\right\},$$

$$ct_1^2=2\sqrt{5}\left\{\frac{2}{\sqrt{10}}R(SD^-)+\frac{\sqrt{2}}{7}R(D^0D^-)+\frac{2\sqrt{3}}{7}R(D^0G^-) - \frac{4}{5}\sqrt{\frac{2}{5}}R(D^-G^0)+\frac{2\sqrt{15}}{77}R(G^0G^-)\right\},$$

$$ct_2^2=2\sqrt{5}\left\{\frac{1}{7}\sqrt{\frac{3}{2}}(I_{D^-}-I_{D^+})-\frac{1}{7}[R(D^-G^-)-R(D^+G^+)] + \frac{5\sqrt{6}}{77}(I_{G^-}-I_{G^+})\right\},$$

$$ct_0^4=\sqrt{9}\left\{\frac{2}{7}I_{D^0}-\frac{4}{21}(I_{D^-}+I_{D^+})+\frac{2}{3}R(SG^0) + \frac{40\sqrt{5}}{231}R(D^0G^0)+\frac{162}{1001}I_{G^0}+\frac{10}{77}\sqrt{\frac{2}{3}}[R(D^-G^-) + R(D^+G^+)]+\frac{81}{1001}(I_{G^-}+I_{G^+})\right\},$$

$$ct_1^4=2\sqrt{9}\left\{\frac{2}{7}\sqrt{\frac{5}{3}}R(D^0D^-)+\frac{\sqrt{2}}{3}R(SG^-) + \frac{17\sqrt{10}}{231}R(D^0G^-) + \frac{10}{77\sqrt{3}}R(D^-G^0)+\frac{81\sqrt{2}}{1001}R(G^0G^-)\right\},$$

$$ct_2^4=2\sqrt{9}\left\{\frac{\sqrt{10}}{21}(I_{D^-}-I_{D^+})+\frac{6\sqrt{15}}{154}[R(D^-G^-) - R(D^+G^+)]+\frac{27\sqrt{10}}{1001}(I_{G^-}-I_{G^+})\right\},$$

$$ct_0^6=\sqrt{13}\left\{\frac{30\sqrt{5}}{143}R(D^0G^0)-\frac{20\sqrt{6}}{143}[R(D^-G^-) + R(D^+G^+)]+\frac{20}{143}I_{G^0}-\frac{1}{143}(I_{G^-}+I_{G^+})\right\},$$

$$ct_1^6=2\sqrt{13}\left\{\frac{10\sqrt{21}}{143}R(D^0G^-)+\frac{10\sqrt{35}}{143\sqrt{2}}R(D^-G^0) + \frac{2\sqrt{105}}{143}R(D^0G^-)\right\},$$

$$ct_2^6=2\sqrt{13}\left\{\frac{4\sqrt{70}}{143}[R(D^-G^-)-R(D^+G^+)] + \frac{\sqrt{105}}{143}(I_{G^-}-I_{G^+})\right\},$$

$$ct_0^8=\sqrt{17}\left\{\frac{490}{2431}I_{G^0}-\frac{392}{2431}(I_{G^-}+I_{G^+})\right\},$$

$$ct_1^8=2\sqrt{17}\left\{\frac{294\sqrt{5}}{2431}R(G^0G^-)\right\},$$

$$ct_2^8=2\sqrt{17}\left\{\frac{42\sqrt{35}}{2431}(I_{G^-}-I_{G^+})\right\}.$$

Polarized moments p_M^L:

$$cp_0^0=P_S+P_{D^0}+P_{D^-}-P_{D^+}+P_{G^0}+P_{G^-}-P_{G^+}, \tag{2.15}$$

$$cp_0^2=\sqrt{5}\left\{\frac{2}{\sqrt{5}}Q(SD^0)+\frac{2}{7}P_{D^0}+\frac{1}{7}(P_{D^-}-P_{D^+}) + \frac{12}{7\sqrt{5}}Q(D^0G^0)+\frac{2\sqrt{6}}{7}[Q(D^-G^-)-Q(D^+G^+)] + \frac{20}{77}P_{G^0}+\frac{17}{77}(P_{G^-}-P_{G^+})\right\},$$

$$cp_1^2=2\sqrt{5}\left\{\frac{2}{\sqrt{10}}Q(SD^-)+\frac{\sqrt{2}}{7}Q(D^0D^-)+2\frac{\sqrt{3}}{7}Q(D^0G^-)-\frac{4}{5}\sqrt{\frac{2}{5}}Q(D^-G^0)+\frac{2\sqrt{15}}{77}Q(G^0G^-)\right\},$$

$$cp_2^2=2\sqrt{5}\left\{\frac{1}{7}\sqrt{\frac{3}{2}}(P_{D^-}+P_{D^+})-\frac{1}{7}[Q(D^-G^-)+Q(D^+G^+)]+\frac{5\sqrt{6}}{77}(P_{G^-}+P_{G^+})\right\},$$

$$cp_0^4=\sqrt{9}\left\{\frac{2}{7}P_{D^0}-\frac{4}{21}(P_{D^-}-P_{D^+})+\frac{2}{3}Q(SG^0)+\frac{40\sqrt{5}}{231}Q(D^0G^0)+\frac{162}{1001}P_{G^0}+\frac{10}{77}\sqrt{\frac{2}{3}}[Q(D^-G^-)-Q(D^+G^+)]+\frac{81}{1001}(P_{G^-}-P_{G^+})\right\},$$

$$cp_1^4=2\sqrt{9}\left\{\frac{2}{7}\sqrt{\frac{5}{3}}Q(D^0D^-)+\frac{\sqrt{2}}{3}Q(SG^-)+\frac{17\sqrt{10}}{231}Q(D^0G^-)+\frac{10}{77\sqrt{3}}Q(D^-G^0)+\frac{81\sqrt{2}}{1001}Q(G^0G^-)\right\},$$

$$cp_2^4=2\sqrt{9}\left\{\frac{\sqrt{10}}{21}(P_{D^-}+P_{D^+})+\frac{6\sqrt{15}}{154}[Q(D^-G^-)+Q(D^+G^+)]+\frac{27\sqrt{10}}{1001}(P_{G^-}+P_{G^+})\right\},$$

$$cp_0^6=\sqrt{13}\left\{\frac{30\sqrt{5}}{143}Q(D^0G^0)-\frac{20\sqrt{6}}{143}[Q(D^-G^-)-Q(D^+G^+)]+\frac{20}{143}P_{G^0}-\frac{1}{143}(P_{G^-}-P_{G^+})\right\},$$

$$cp_1^6=2\sqrt{13}\left\{\frac{10\sqrt{21}}{143}Q(D^0G^-)+\frac{10\sqrt{35}}{143\sqrt{2}}Q(D^-G^0)+\frac{2\sqrt{105}}{143}Q(G^0G^-)\right\},$$

$$cp_2^6=2\sqrt{13}\left\{\frac{4\sqrt{70}}{143}[Q(D^-G^-)+Q(D^+G^+)]+\frac{\sqrt{105}}{143}(P_{G^-}+P_{G^+})\right\},$$

$$cp_0^8=\sqrt{17}\left\{\frac{490}{2431}P_{G^0}-\frac{392}{2431}(P_{G^-}-P_{G^+})\right\},$$

$$cp_1^8=2\sqrt{17}\left\{\frac{294\sqrt{5}}{2431}Q(G^0G^-)\right\},$$

$$cp_2^8=2\sqrt{17}\left\{\frac{42\sqrt{35}}{2431}(P_{G^-}+P_{G^+})\right\}.$$

Polarized moments r_M^L:

$$cr_1^2=-2\sqrt{2}\mathrm{Re}(SD^{+*}-\overline{SD^+}{}^*)-\frac{2\sqrt{10}}{7}\mathrm{Re}(D^0D^{+*}-\overline{D^0}\,\overline{D^+}{}^*),$$

$$cr_2^2=-\frac{2\sqrt{30}}{7}\mathrm{Re}(D^-D^{+*}-\overline{D^-}\,\overline{D^+}{}^*),$$

$$cr_1^4=-\frac{4\sqrt{15}}{7}\mathrm{Re}(D^0D^{+*}-\overline{D^0}\,\overline{D^+}{}^*),$$

$$cr_2^4=-\frac{4\sqrt{10}}{7}\mathrm{Re}(D^-D^{+*}-\overline{D^-}\,\overline{D^+}{}^*). \qquad (2.16)$$

We do not include G-wave contributions in the polarized moments r_M^L. In general, these moments are not well determined in measurements on transversely polarized targets and, as can be seen in the Appendix, the calculation of relative phases between the natural exchange amplitude D^+ and the unnatural exchange amplitudes S, D^0, and D^- already involves a high degree of ambiguity. The inclusion of G waves would make the situation even less tractable.

III. MODEL-INDEPENDENT ANALYSIS OF MEASUREMENTS ON UNPOLARIZED TARGETS

We will now show that in the mass region where only S and D waves dominate, i.e., up to about 1500 MeV, it is possible to perform an analysis of the measurements of $\pi^-p\to\pi^0\pi^0n$ and $\pi^-p\to\eta\eta n$ on unpolarized targets without the simplifying assumption that production amplitudes do not depend on nucleon spin. However, we will find that data on unpolarized targets measure in a model-independent way only the partial wave intensities and three unrelated interference phases, and not the production amplitudes which remain undetermined.

When only S and D waves contribute there are six independent observables to determine seven unknowns: four partial wave intensities and three spin-averaged interference terms. Since there are more unknowns than observables, it is necessary to express the maximum likelihood function $\mathcal{L}$ in terms of the partial wave intensities and the interference terms and fit $\mathcal{L}$ to the observed data to find a solution.

For this purpose we will now show that the interference terms $R(AB)$ in Eqs. (2.14) have a general form

$$R(AB)=\sqrt{I_A}\sqrt{I_B}\cos(\delta_{AB}). \qquad (3.1)$$

From the definition (2.12) we have

$$R(AB)=\sum_{n=0}^{1}\mathrm{Re}(A_nB_n^*)=\sum_{n=0}^{1}|A_n||B_n|\cos(\phi_n^A-\phi_n^B).$$

We can write

$$R(AB)=\sqrt{I_A}\sqrt{I_B}Z_{AB}. \tag{3.2}$$

With definitions for $n=0,1$,

$$\xi_n^{AB}=\frac{|A_n|}{\sqrt{I_A}}\frac{|B_n|}{\sqrt{I_B}},\quad \varphi_n^{AB}=\phi_n^A-\phi_n^B, \tag{3.3}$$

we have

$$Z_{AB}=\xi_0^{AB}\cos\varphi_0^{AB}+\xi_1^{AB}\cos\varphi_1^{AB}. \tag{3.4}$$

We now recall a theorem from wave theory [41],

$$A_1\sin(\omega t+\varphi_1)+A_2\sin(\omega t+\varphi_2)=A\sin(\omega t+\varphi), \tag{3.5}$$

where

$$A^2=A_1^2+A_2^2+2A_1A_2\cos(\varphi_2-\varphi_1),$$

$$\tan\varphi=\frac{A_1\sin\varphi_1+A_2\sin\varphi_2}{A_1\cos\varphi_1+A_2\cos\varphi_2}. \tag{3.6}$$

For $\omega t=\pi/2$ we get

$$A_1\cos\varphi_1+A_2\cos\varphi_2=A\cos\varphi, \tag{3.7}$$

with A and φ given above. We can apply Eq. (3.7) to Eq. (3.4) and get

$$Z_{AB}=\xi_{AB}\cos\varphi_{AB},$$

where ξ_{AB} and φ_{AB} are given by Eqs. (3.6) with the appropriate substitutions from Eq. (3.4). After some algebra it is possible to show that

$$0\leqslant\xi_{AB}\leqslant+1, \tag{3.8}$$

so that $-1\leqslant Z_{AB}\leqslant+1$. Thus we can actually write $Z_{AB}\equiv\cos\delta_{AB}$ which proves the statement (3.1). The phase δ_{AB} is not simply related to the two relative phases $\phi_0^A-\phi_0^B$ and $\phi_1^A-\phi_1^B$ of the helicity amplitudes A_n,B_n, $n=0,1$. Moreover, $\cos\delta_{AB}$ is a measurable parameter along with the intensities I_A and I_B.

We will refer to δ_{SD^0}, δ_{SD^-}, and $\delta_{D^0D^-}$ in Eq. (3.1) as interference phases. Notice again that interference phases are not relative phases between amplitudes and are thus independent. Whereas relative phases satisfy for $n=0,1$,

$$(\phi_n^S-\phi_n^{D^0})+(\phi_n^{D^-}-\phi_n^S)+(\phi_n^{D^0}-\phi_n^{D^-})=0, \tag{3.9}$$

there is no such relation for the interference phases.

We can use Eq. (3.1) to express the maximum likelihood function $\mathcal{L}$ in terms of the four intensities I_S, I_{D^0}, I_{D^-}, and I_{D^+} and three interference phases δ_{SD^0}, δ_{SD^-}, and $\delta_{D^0D^-}$ and fit $\mathcal{L}$ to the observed angular distributions to find a solution for these quantities in each (m,t) bin. We can conclude that an analysis of the data on $\pi^-p\to\pi^0\pi^0n$ unpolarized targets is possible without the assumption that production amplitudes are independent of nucleon spin. However, the data on unpolarized targets cannot determine the eight moduli and six cosines of dependent relative phases of production amplitudes. As we show below, for that determination measurements on polarized targets are necessary. The measurements on unpolarized targets determine only four partial wave intensities and three interference phases in a model-independent way.

In a mass region where G waves contribute, measurements on unpolarized targets measure 12 independent unpolarized moments t_M^L. There are 7 intensities and 11 spin-averaged interference terms in Eqs. (2.14) for a total of 18 unknowns. In this case a model-independent amplitude analysis is not possible. However, we shall see below that a model-independent analysis including G waves is possible for measurements on polarized targets.

IV. COMPARISON WITH MODEL-DEPENDENT ANALYSES OF $\pi^-p\to\pi^0\pi^0n$ ON UNPOLARIZED TARGETS

Both the GAMS Collaboration and BNL E852 Collaboration use the assumption of the independence of production amplitudes on nucleon spin [31,32] but employ different strategies in actual fits to the observed angular distributions [33,34]. We will confine our discussion to the mass region where S and D waves dominate.

The assumption of the independence of production amplitudes on nucleon spin means that formally there is one S-wave amplitude S and three D-wave amplitudes D^0, D^-, and D^+. The amplitudes have no nucleon spin index. However, as we have argued above, these amplitudes are essentially the single flip helicity amplitudes $(n=1)$ while all helicity nonflip amplitudes $(n=0)$ are assumed to vanish.

In the GAMS approach [33] the unpolarized moments are then written as (with $c=\sqrt{4\pi}$)

$$ct_0^0=|S|^2+|D^0|^2+|D^-|^2+|D^+|^2,$$

$$ct_0^2=2\mathrm{Re}(SD^{0*})+\frac{2\sqrt{5}}{7}|D^0|^2+\frac{\sqrt{5}}{7}(|D^-|^2+|D^+|^2),$$

$$ct_1^2=2\sqrt{2}\mathrm{Re}(SD^{-*})+\frac{2\sqrt{10}}{7}\mathrm{Re}(D^0D^{-*}),$$

$$ct_2^2=\frac{\sqrt{30}}{7}(|D^-|^2-|D^+|^2),$$

$$ct_0^4=\frac{6}{7}|D^0|^2-\frac{4}{7}(|D^-|^2+|D^+|^2),$$

$$ct_1^4=\frac{4}{7}\sqrt{15}\mathrm{Re}(D^0D^{-*}),$$

$$ct_2^4=\frac{2}{7}\sqrt{10}(|D^-|^2-|D^+|^2). \tag{4.1}$$

There are six independent equations for seven unknowns: four moduli and three cosines of relative phases. The GAMS Collaboration determines these quantities by expressing the maximum likelihood function $\mathcal{L}$ in terms of the amplitudes (moduli and cosines) and fitting $\mathcal{L}$ to the observed angular distribution to find solutions for the moduli and relative phases [27,28,33]. Formally this approach is equivalent to our approach (described in the previous section) with an additional assumption that the interference phases are not independent but satisfy a constraint

$$\delta_{SD^0}+\delta_{D^-S}+\delta_{D^0D^-}=0. \tag{4.2}$$

What the GAMS Collaboration is actually doing is determining partial wave intensities I_A, $A=S$, D^0, D^-, D^+, and interference phases subject to the constraint (4.2). When the constraint (4.2) is removed, their approach becomes a fully model-independent determination, not of amplitudes, but of partial wave intensities.

The BNL E852 employs a different approach [34]. They express the moduli squared and interference terms in Eqs. (4.1) in terms of real and imaginary parts for amplitudes S, D^0, and D^-. Since there is no interference with D^+, only $|D^+|^2$ is retained. Moreover, it is possible to set the overall phase such that one of the amplitudes (e.g., S) is purely real. Thus there are six unknown quantities. The maximum likelihood function is then expressed in terms of these unknown real and imaginary parts of S, D^0, D^-, and $|D^+|^2$ and fitted to the observed angular distributions to find the solution for the amplitudes [34]. Formally this approach is different from our model-independent method and relies more explicitly on the assumption that the nonflip helicity amplitudes all vanish.

V. MODEL-INDEPENDENT AMPLITUDE ANALYSIS OF $\pi^- p_\uparrow \to \pi^0\pi^0 n$ MEASURED ON POLARIZED TARGETS WITH *G* WAVES ABSENT

In the following we will assume that unpolarized and polarized moments t^L_M and p^L_M (and r^L_M) have been determined using the maximum likelihood method in data analysis of measurements of $\pi^- p \to \pi^0\pi^0 n$ and $\pi p \to \eta\eta n$ on polarized targets in a manner previously used in the reactions $\pi N_\uparrow \to \pi^-\pi^+ N$ [11–19]. In this section we show that an analytical solution exists for S and D waves in the mass region where these waves dominate. In the next section we extend the solution to include the G-wave amplitudes. In both cases we will find it useful to work with nucleon transversity amplitudes (2.9).

In the mass region where S and D waves dominate and the G wave is absent, there are seven unpolarized moments t^L_M, seven polarized moments p^L_M, and four polarized moments r^L_M measured in each (m,t) bin. Looking at Eqs. (2.14) and (2.15) and recalling definitions (2.10)–(2.13), we see that it is advantageous to introduce new observables which are the sum and difference of the corresponding moments t^L_M and p^L_M. We thus define (with $c=\sqrt{4\pi}$) the first set of equations

$$a_1=\frac{c}{2}(t^0_0+p^0_0)=|S|^2+|D^0|^2+|D^-|^2+|\overline{D^+}|^2, \tag{5.1}$$

$$a_2=\frac{c}{2}(t^2_0+p^2_0)=2\,\mathrm{Re}(SD^0{*})+\frac{2\sqrt{5}}{7}|D^0|^2+\frac{\sqrt{5}}{7}(|D^-|^2+|D^+|),$$

$$a_3=\frac{c}{2}(t^2_1+p^2_1)=2\sqrt{2}\,\mathrm{Re}(SD^-{*})+\frac{2\sqrt{10}}{7}\,\mathrm{Re}(D^0D^-{*}),$$

$$a_4=\frac{c}{2}(t^2_2+p^2_2)=\frac{\sqrt{30}}{7}(|D^-|^2-|\overline{D^+}|^2),$$

$$a_5=\frac{c}{2}(t^4_0+p^4_0)=\frac{6}{7}|D^0|^2-\frac{4}{7}(|D^-|^2+|D^+|^2),$$

$$a_6=\frac{c}{2}(t^4_1+p^4_1)=\frac{4}{7}\sqrt{15}\,\mathrm{Re}(D^0D^-{*}),$$

$$a_7=\frac{c}{2}(t^4_2+p^4_2)=\frac{2}{7}\sqrt{10}(|D^-|^2-|\overline{D^+}|^2).$$

The second set of equations is obtained by defining observables $\overline{a_1},\overline{a_2},\ldots,\overline{a_7}$ which are the difference of corresponding moments. We obtain

$$\overline{a_1}=\frac{c}{2}(t^0_0-p^0_0)=|\overline{S}|^2+|\overline{D^0}|^2+|\overline{D^-}|^2+|D^+|^2,$$

$$\overline{a_2}=\frac{c}{2}(t^2_0-p^2_0)=2\,\mathrm{Re}(\overline{SD^0}{*})+\frac{2\sqrt{5}}{7}|\overline{D^0}|^2$$

$$+\frac{\sqrt{5}}{7}(|\overline{D^-}|^2+|D^{+|^2}),$$

$$\overline{a_3}=\frac{c}{2}(t^2_1-p^2_1)=2\sqrt{2}\,\mathrm{Re}(\overline{SD^-}{*})+\frac{2\sqrt{10}}{7}\,\mathrm{Re}(\overline{D^0}\,\overline{D^-}{*}),$$

$$\overline{a_4}=\frac{c}{2}(t^2_2-p^2_2)=\frac{\sqrt{30}}{7}(|\overline{D^-}|^2-|D^+|^2),$$

$$\overline{a_5}=\frac{c}{2}(t^4_0-p^4_0)=\frac{6}{7}|\overline{D^0}|^2-\frac{4}{7}(|\overline{D^-}|^2+|D^+|^2),$$

$$\overline{a_6}=\frac{c}{2}(t^4_1-p^4_1)=\frac{4\sqrt{15}}{7}\,\mathrm{Re}(\overline{D^0}\,\overline{D^-}{*}),$$

$$\overline{a_7}=\frac{c}{2}(t^4_2-p^4_2)=\frac{2}{7}\sqrt{10}(|\overline{D^-}|^2-|D^+|^2). \tag{5.2}$$

The first set of six independent equations involves four moduli $|S|$, $|D^0|$, $|D^-|$, and $|\overline{D^+}|$ and three cosines of relative phases $\cos(\gamma_{SD^0})$, $\cos(\gamma_{SD^-})$, and $\cos(\gamma_{D^0D^-})$. The second set of six independent equations involves the amplitudes of opposite transversity: four moduli $|\overline{S}|$, $|\overline{D^0}|$, $|\overline{D^-}|$, and $|D^+|$ and three cosines of their relative phases, $\cos(\overline{\gamma_{SD^0}})$, $\cos(\overline{\gamma_{SD^-}})$, and $\cos(\overline{\gamma_{D^0D^-}})$. The two sets are entirely independent and the relative phase between transversity amplitudes up and down is unknown in measurements on transversely polarized targets.

To proceed with the analytical solution, we first find, from Eq. (5.1),

$$|D^0|^2 = \frac{4}{10}(a_1 - |S|^2) + \frac{7}{10}a_5,$$

$$|D^-|^2 = \frac{3}{10}(a_1 - |S|^2) - \frac{7}{10}a_5 + \frac{7}{2\sqrt{30}}a_4,$$

$$|\overline{D^+}|^2 = \frac{3}{10}(a_1 - |S|^2) - \frac{7}{10}a_5 - \frac{7}{2\sqrt{30}}a_4, \tag{5.3}$$

$$\cos\gamma_{SD^0} = \frac{1}{|S||D^0|}\left(A + \frac{1}{2\sqrt{5}}|S|^2\right),$$

$$\cos\gamma_{SD^-} = \frac{1}{|S||D^-|}B,$$

$$\cos\gamma_{D^0D^-} = \frac{1}{|D^0||D^-|}C, \tag{5.4}$$

where

$$A = \frac{1}{2}\left\{a_2 - \frac{1}{\sqrt{5}}a_1 + \frac{1}{2\sqrt{5}}a_5\right\},$$

$$B = \frac{1}{2}\left\{\frac{1}{\sqrt{2}}a_3 - \frac{1}{2\sqrt{3}}a_6\right\},$$

$$C = \frac{1}{2}\left\{\frac{7}{4\sqrt{15}}a_6\right\}. \tag{5.5}$$

Notice that a_7 is not independent and does not enter in the above equations. Similar solutions can be derived from the second set (5.2) for amplitudes of opposite transversity. However, we need one more equation in each set: $\overline{\text{one}}$ equation for $|S|^2$ in the first set and another one for $|\overline{S}|^2$ in the second set.

The additional equations are provided by the relative phases which are not independent:

$$\gamma_{SD^0} - \gamma_{SD^-} + \gamma_{D^0D^-} = (\phi_S - \phi_{D^0}) - (\phi_S - \phi_{D^-}) + (\phi_{D^0} - \phi_{D^-}) = 0,$$

$$\overline{\gamma}_{SD^0} - \overline{\gamma}_{SD^-} + \overline{\gamma}_{D^0D^-} = (\overline{\phi}_S - \overline{\phi}_{D^0}) - (\overline{\phi}_S - \overline{\phi}_{D^-}) + (\overline{\phi}_{D^0} - \overline{\phi}_{D^-}) = 0. \tag{5.6}$$

These conditions lead to nonlinear relations between the cosines:

$$\cos^2(\gamma_{SD^0}) + \cos^2(\gamma_{SD^-}) + \cos^2(\gamma_{D^0D^-}),$$

$$-2\cos(\gamma_{SD^0})\cos(\gamma_{SD^-})\cos(\gamma_{D^0D^-}) = 1,$$

$$\cos^2(\overline{\gamma}_{SD^0}) + \cos^2(\overline{\gamma}_{SD^-}) + \cos^2(\overline{\gamma}_{D^0D^-}),$$

$$-2\cos(\overline{\gamma}_{SD^0})\cos(\overline{\gamma}_{SD^-})\cos(\overline{\gamma}_{D^0D^-}) = 1. \tag{5.7}$$

Similar relations also hold for the sines. Next we define combinations of observables

$$D = \frac{4}{10}a_1 - \frac{7}{10}a_5,$$

$$E = \frac{3}{10}a_1 - \frac{7}{20}a_5 + \frac{7}{2\sqrt{3}}a_4, \tag{5.8}$$

so that

$$|D^0|^2 = D - \frac{4}{10}|S|^2,$$

$$|D^-|^2 = E - \frac{3}{10}|S|^2. \tag{5.9}$$

Substituting into Eqs. (5.7) first from Eqs. (5.4) for the cosines and then from Eqs. (5.9) for $|D^0|^2$ and $|D^-|^2$, we obtain a cubic equation for $x \equiv |S|^2$,

$$ax^3 + bx^2 + cx + d = 0, \tag{5.10}$$

where

$$a = \frac{27}{200},$$

$$b = \frac{1}{10}\left(\frac{1}{\sqrt{5}}A - 3D - \frac{9}{2}E\right),$$

$$c = \frac{1}{10}(3A^2 + 4B^2 - 10C^2 + 2\sqrt{5}BC - 2\sqrt{5}AE + 10DE)$$

$$d = 2ABC - A^2E - B^2D. \tag{5.11}$$

A similar cubic equation can be derived for the amplitude $|\overline{S}|^2$.

Analytical expressions for the three roots of the cubic equation (5.10) are given in Table I of Ref. [21]. It is seen from the table that three real solutions exist; one of them is negative and it is rejected. There are in general two positive solutions for $|S|^2$ which lead to two solutions in set 1. Similarly there are two solutions in set 2 of opposite transversity. Since the two sets are independent, there are four solutions for partial wave intensities:

$$I_A(i,j) = |A(i)|^2 + |\overline{A(j)}|^2, i, \quad j = 1,2. \tag{5.12}$$

The error propagation in the cubic equation and the calculation of errors on the moduli, cosines, and partial wave intensities as well as the treatment of unphysical complex solutions are best handled using the Monte Carlo method described in detail in Ref. [24].

The determination of relative phases between the natural exchange amplitude D^+ and unnatural exchange amplitudes S, D^0, and D^- is described in the Appendix.

VI. MODEL-INDEPENDENT AMPLITUDE ANALYSIS OF $\pi^- p_\uparrow \to \pi^0\pi^0 n$ MEASURED ON POLARIZED TARGETS WITH G WAVES INCLUDED

In the mass region where S, D, and G waves all contribute (expected above 1500 MeV), the measurement of $\pi^- p_\uparrow \to \pi^0\pi^0 n$ on polarized targets will yield 13 unpolarized moments t^L_M, 13 polarized moments p^L_M, and 8 polarized moments r^L_M. Central to our discussion are again the moments t^L_M and p^L_M given by Eqs. (2.14) and (2.15). Using the definitions (2.10)–(2.13) we see again that it is useful to define two new sets of observables, one with the sums $t^L_M + p^L_M$ and another one with the differences $t^L_M - p^L_M$. With $c=\sqrt{4\pi}$ we obtain, for the first set (sums),

$$a_1=\frac{c}{2}(t^0_0+p^0_0)=|S|^2+|D^0|^2+|D^-|^2+|\overline{D^+}|^2+|G^0|^2+|G^-|^2+|\overline{G^+}|^2, \tag{6.1}$$

$$a_2=\frac{c}{2}(t^2_0+p^2_0)=\sqrt{5}\Bigg\{\frac{2}{\sqrt{5}}\mathrm{Re}(SD^0*)+\frac{2}{7}|D^0|^2+\frac{1}{7}(|D^-|^2+|\overline{D^+}|^2)$$
$$+\frac{12}{7\sqrt{5}}\mathrm{Re}(D^0G^0*)+\frac{2\sqrt{6}}{7}[\mathrm{Re}(D^-G^-*)+\mathrm{Re}(\overline{D^+}\,\overline{G^+}*)]+\frac{20}{77}|G^0|^2+\frac{17}{77}(|G^-|^2+|\overline{G^+}|^2)\Bigg\},$$

$$a_3=\frac{c}{2}(t^2_1+p^2_1)=2\sqrt{5}\Bigg\{\frac{2}{\sqrt{10}}\mathrm{Re}(SD^-*)+\frac{\sqrt{2}}{7}\mathrm{Re}(D^0D^-*)+\frac{2\sqrt{3}}{7}\mathrm{Re}(D^0G^-*)$$
$$-\frac{4\sqrt{2}}{5\sqrt{5}}\mathrm{Re}(D^-G^0*)+\frac{2\sqrt{15}}{77}\mathrm{Re}(G^0G^-*)\Bigg\},$$

$$a_4=\frac{c}{2}(t^2_2+p^2_2)=2\sqrt{5}\Bigg\{\frac{1}{7}\sqrt{\frac{3}{2}}(|D^-|^2-|\overline{D^+}|^2)-\frac{1}{7}[\mathrm{Re}(D^-G^-*)-\mathrm{Re}(\overline{D^+}\,\overline{G^+}*)]+\frac{5\sqrt{6}}{77}(|G^-|^2-|\overline{G^+}|^2)\Bigg\},$$

$$a_5=\frac{c}{2}(t^4_0+p^4_0)=\sqrt{9}\Bigg\{\frac{2}{7}|D^0|^2-\frac{4}{21}(|D^-|^2+|\overline{D^+}|^2)+\frac{2}{3}\mathrm{Re}(SG^0*)+\frac{40\sqrt{5}}{231}\mathrm{Re}(D^0G^0*)+\frac{162}{1001}|G^0|^2$$
$$+\frac{81}{1001}(|G^-|^2+|\overline{G^+}|^2)+\frac{10\sqrt{2}}{77\sqrt{3}}[\mathrm{Re}(D^-G^-*)+\mathrm{Re}(\overline{D^+}\,\overline{G^+}*)]\Bigg\},$$

$$a_6=\frac{c}{2}(t^4_1+p^4_1)=2\sqrt{9}\Bigg\{\frac{2}{7}\sqrt{\frac{5}{3}}\mathrm{Re}(D^0D^-*)+\frac{\sqrt{2}}{3}\mathrm{Re}(SG^-*)+\frac{17\sqrt{10}}{231}\mathrm{Re}(D^0G^-*)$$
$$+\frac{10}{77\sqrt{3}}\mathrm{Re}(D^-G^0*)+\frac{81\sqrt{2}}{1001}\mathrm{Re}(G^0G^-*)\Bigg\},$$

$$a_7=\frac{c}{2}(t^4_2+p^4_2)=2\sqrt{9}\Bigg\{\frac{\sqrt{10}}{21}(|D^-|^2-|\overline{D^+}|^2)+\frac{6\sqrt{15}}{154}[\mathrm{Re}(D^-G^-*)-\mathrm{Re}(\overline{D^+}\,\overline{G^+}*)]+\frac{27\sqrt{10}}{1001}(|G^-|^2-|\overline{G^+}|^2)\Bigg\},$$

$$a_8=\frac{c}{2}(t^6_0+p^6_0)=\frac{\sqrt{13}}{143}\{30\sqrt{5}\mathrm{Re}(D^0G^0*)-20\sqrt{6}[\mathrm{Re}(D^-G^-*)+\mathrm{Re}(\overline{D^+}\,\overline{G^+}*)]+20|G^0|^2-(|G^-|^2+|\overline{G^+}|^2)\},$$

$$a_9=\frac{c}{2}(t^6_1+p^6_1)=\frac{2\sqrt{13}}{143}\Bigg\{10\sqrt{21}\mathrm{Re}(D^0G^-*)+\frac{10\sqrt{35}}{\sqrt{2}}\mathrm{Re}(D^-G^0*)+2\sqrt{105}\mathrm{Re}(G^0G^-*)\Bigg\},$$

$$a_{10}=\frac{c}{2}(t^6_2+p^6_2)=\frac{2\sqrt{13}}{143}\{4\sqrt{70}[\mathrm{Re}(D^-G^-*)-\mathrm{Re}(\overline{D^+}\,\overline{G^+}*)]+\sqrt{105}(|G^-|^2-|\overline{G^+}|^2)\},$$

$$a_{11}=\frac{c}{2}(t^8_0+p^8_0)=\frac{\sqrt{17}}{2431}\{490|G^0|^2-392(|G^-|^2+|\overline{G^+}|^2)\},$$

$$a_{12}=\frac{c}{2}(t_1^8+p_1^8)=\frac{2\sqrt{17}}{2431}\{294\sqrt{5}\mathrm{Re}(G^0G^{-*})\},$$

$$a_{13}=\frac{c}{2}(t_2^8+p_2^8)=\frac{2\sqrt{17}}{2431}\{42\sqrt{35}(|G^-|^2-|\overline{G^+}|^2)\}.$$

The second set of observables $\overline{a_i}$, $i=1,2,\ldots,13$, is formed similarly by the differences $t_M^L-p_M^L$. It has the same form as set 1 but involves the amplitudes of opposite transversity.

The first set a_i, $i=1,\ldots,13$, involves seven moduli

$$|S|,|D^0|,|D^-|,|\overline{D^+}|,|G^0|,|G^-|,|\overline{G^+}|, \tag{6.2}$$

ten cosines of relative phases between unnatural amplitudes,

$$\cos(\gamma_{SD^0}),\cos(\gamma_{SD^-}),\cos(\gamma_{SG^0}),\cos(\gamma_{SG^-}), \tag{6.3}$$

$$\cos(\gamma_{D^0D^-}),\cos(\gamma_{D^0G^0}),\cos(\gamma_{D^0G^-}), \tag{6.4}$$

$$\cos(\gamma_{D^-G^0}),\cos(\gamma_{D^-G^-}),\cos(\gamma_{G^0G^-}), \tag{6.5}$$

and one cosine of relative phase between the two natural amplitudes,

$$\cos(\overline{\gamma_{D^+G^+}}). \tag{6.6}$$

The second set $\overline{a_i}$, $i=1,\ldots,13$, involves the same amplitudes but of opposite transversity. We will now show that the cosines (6.4) and (6.5) can be expressed in terms of cosines (6.3). For instance, we can write

$$\gamma_{D^0D^-}=\phi_{D^0}-\phi_{D^-}=(\phi_S-\phi_{D^-})-(\phi_S-\phi_{D^0})$$
$$=\gamma_{SD^-}-\gamma_{SD^0}. \tag{6.7}$$

Hence

$$\cos\gamma_{D^0D^-}=\cos\gamma_{SD^0}\cos\gamma_{SD^-}+\sin\gamma_{SD^0}\sin\gamma_{SD^-}.$$

Since the signs of the sines $\sin\gamma_{SD^0}$ and $\sin\gamma_{SD^-}$ are not known, we write

$$\sin\gamma_{SD^0}=\epsilon_{SD^0}|\sin\gamma_{SD^0}|,\quad \sin\gamma_{SD^-}=\epsilon_{SD^-}|\sin\gamma_{SD^-}|. \tag{6.8}$$

Then

$$\cos\gamma_{D^0D^-}=\cos\gamma_{SD^0}\cos\gamma_{SD^-}+\epsilon_{D^0D^-}\sqrt{(1-\cos^2\gamma_{SD^0})(1-\cos^2\gamma_{SD^-})}, \tag{6.9}$$

where $\epsilon_{D^0D^-}=\pm 1$ is the sign ambiguity. The remaining cosines in Eqs. (6.4) and (6.5) can be written in the form similar to Eq. (6.9) with their own sign ambiguities. The sign ambiguities of all cosines (6.4) and (6.5) can be written in terms of sign ambiguities ϵ_{SD^0}, ϵ_{SD^-}, ϵ_{SG^0}, and ϵ_{SG^-} corresponding to the sines $\sin\gamma_{SD^0}$, $\sin\gamma_{SD^-}$, $\sin\gamma_{SG^0}$, and $\sin\gamma_{SG^-}$. We can write

$$\epsilon_{D^0D^-}=\epsilon_{SD^0}\epsilon_{SD^-}, \tag{6.10}$$

$$\epsilon_{D^0G^0}=\epsilon_{SD^0}\epsilon_{SG^0},$$
$$\epsilon_{D^0G^-}=\epsilon_{SD^0}\epsilon_{SG^-},$$
$$\epsilon_{D^-G^0}=\epsilon_{SD^-}\epsilon_{SG^0},$$
$$\epsilon_{D^-G^-}=\epsilon_{SD^-}\epsilon_{SG^-},$$
$$\epsilon_{G^0G^-}=\epsilon_{SG^0}\epsilon_{SG^-}. \tag{6.11}$$

First we notice that reversal of all signs ϵ_{SD^0}, ϵ_{SD^-}, ϵ_{SG^0}, and ϵ_{SG^-} yields the same sign ambiguities (6.10) and (6.11). Next we notice that the sign ambiguities (6.11) of cosines (6.5) are uniquely determined by the sign ambiguities (6.10) for cosines (6.4). Only sign ambiguities (6.10) are independent and there are eight sign combinations (6.10). The following table lists all eight allowed sets of sign ambiguities of cosines (6.4) and (6.5):

	1	2	3	4	5	5	7	8
$\epsilon_{D^0D^-}$	+	−	+	+	−	−	+	−
$\epsilon_{D^0G^0}$	+	+	−	+	−	+	−	−
$\epsilon_{D^0G^-}$	+	+	+	−	+	−	−	−
$\epsilon_{D^-G^0}$	+	−	−	+	+	−	−	+
$\epsilon_{D^-G^-}$	+	−	+	−	−	+	−	+
$\epsilon_{G^0G^-}$	+	+	−	−	−	−	+	+

Using expressions like Eq. (6.9) for cosines (6.4) and (6.5), we have 12 unknowns in each nonlinear set of 13 equations a_i, $i=1,2,\ldots,13$, with one choice of sign ambiguities for cosines (6.4) and (6.5) from the above table. The nonlinear set can be solved numerically or by the χ^2 method [25]. In each (m,t) bin we thus have 8 solutions for moduli (6.2) and cosines (6.3)–(6.5), and 8 solutions for amplitudes of opposite transversity from the set $\overline{a_i}$, $i=1,2,\ldots,13$. Since each solution is uniquely labeled by the choice of sign ambiguities, there is no problem linking solutions in neighboring (m,t) bins.

Since the 8 solutions from the first set a_i, $i=1,2,\ldots,13$, are independent from the 8 solutions from the second set $\overline{a_i}$, $i=1,2,\ldots,13$, there will be a 64-fold ambiguity in the partial wave intensities. For $A=$ S, D^0, D^-, D^+, G^0, G^-, and G^+ we can write

$$I_A(i,j)=|A(i)|^2+|\overline{A(j)}|^2,\quad i,j=1,2,\ldots,8. \tag{6.12}$$

We now will discuss constraints on the moments that should be taken into account at the time of fitting maximum likelihood function $\mathcal{L}$ to the observed angular distribution in the process of constrained optimization.

The observables a_i and $\overline{a_i}$, $i=1,2,\ldots,13$, are not all linearly independent. In fact one finds two relations [32]

$$8\sqrt{14}a_4-4\sqrt{42}a_7+\frac{91}{\sqrt{13}}a_{10}-\frac{119}{2}\sqrt{\frac{3}{17}}\;a_{13}=0,$$

$$8\sqrt{14}\overline{a_4}-4\sqrt{42}\overline{a_7}+\frac{91}{\sqrt{13}}\overline{a_{10}}-\frac{119}{2}\sqrt{\frac{3}{17}}\overline{a_{13}}=0. \tag{6.13}$$

By adding and subtracting the last two equations we get the same relationship for corresponding moments t_M^L and p_M^L:

$$8\sqrt{14}t_2^2-4\sqrt{42}t_2^4+\frac{91}{\sqrt{13}}t_2^6-\frac{119}{2}\sqrt{\frac{3}{17}}t_2^8=0,$$

$$8\sqrt{14}p_2^2-4\sqrt{42}p_2^4+\frac{91}{\sqrt{13}}p_2^6-\frac{119}{2}\sqrt{\frac{3}{17}}p_2^8=0. \tag{6.14}$$

Additional constraints can be obtained by solving for $|G^-|^2+|\overline{G^+}|^2$ from a_{11} and substituting into a_1. Proceeding in the same way also for $|\overline{G^-}|^2+|G^+|^2$ from $\overline{a_{11}}$ and substituting into $\overline{a_1}$, we get

$$a_1+\frac{2431}{392\sqrt{17}}a_{11}>0,\quad \overline{a_1}+\frac{2431}{392\sqrt{17}}\overline{a_{11}}>0. \tag{6.15}$$

By adding the two inequalities we get

$$t_0^0+\frac{2431}{392\sqrt{17}}t_0^8>0. \tag{6.16}$$

The constraints (6.13) and (6.14), or (6.15) and (6.16), are self-consistency constraints which follow from the assumption that only S, D, and G waves contribute. These constraints should be imposed on the maximum likelihood function during the fit to the observed angular distribution. We then deal with constrained optimization [42–44]. A program MINOS 5.0 has been developed at Stanford University for constrained optimization with equality and inequality constraints [45].

VII. SUMMARY

The dependence of hadronic reactions on nucleon spin is now a well-established experimental fact. Measurements of the reactions $\pi^-p\rightarrow\pi^-\pi^+n$ and $\pi^+n\rightarrow\pi^+\pi^-p$ on polarized targets at CERN found a strong dependence of pion production amplitudes on nucleon spin. The assumption that pion production amplitudes are independent of nucleon spin is in direct conflict with these experimental findings. The analyses of the $\pi^-p\rightarrow\pi^0\pi^0n$ data based on this assumption thus are not sufficient and may not be fully reliable.

We have shown in Sec. III that unpolarized data provide model-independent information only on the spin-averaged partial wave intensities and cosines of three interference phases. To obtain information about the production amplitudes, measurements of $\pi^-p\rightarrow\pi^0\pi^0n$ on polarized targets are necessary. We have shown in Secs. V and VI how to perform model-independent amplitude analyses of $\pi^-p\rightarrow\pi^0\pi^0n$ measured on polarized targets. A model-independent analysis is possible in the mass region where only S- and D-wave amplitudes contribute, as well as in the mass region where also G-wave amplitudes contribute. Our only assumption was that amplitudes with dimeson helicity $\lambda\geqslant2$ do not significantly contribute to the $\pi^0\pi^0$ production. This assumption is supported by the available data.

On this basis we propose that high statistics measurements of $\pi^-p\rightarrow\pi^0\pi^0n$ and $\pi^-p\rightarrow\eta\eta n$ be made at the BNL Multiparticle Spectrometer and at IHEP in Protvino and that model-independent amplitude analyses of these reactions be performed. We note that this amplitude analysis will require the unpolarized moments t_M^L which should be determined from the data on unpolarized targets in the same (m,t) bins.

We suggest that the extensions of the GAMS and BNL E852 programs to measurements on polarized targets will significantly contribute to new developments of hadron spectroscopy on the level of spin-dependent production amplitudes and to our understanding of hadron dynamics.

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APPENDIX: CALCULATION OF THE PHASES γ_{D^+S} AND $\overline{\gamma_{D^+S}}$

In this appendix we solve Eqs. (2.16) for the helicity frame invariant phases $\gamma_{D^+S}=\phi_{D^+}-\phi_S$ and $\overline{\gamma_{D^+S}}=\overline{\phi_{D^+}}-\overline{\phi_S}$. Other phases in Eq. (2.16) are then expressed in terms of these phases and the phases (5.4):

$$\gamma_{D^+D^0}=\phi_{D^+}-\phi_{D^0}=(\phi_{D^+}-\phi_S)+(\phi_S-\phi_{D^0})$$
$$=\gamma_{D^+S}-\gamma_{D^0S},$$

$$\gamma_{D^+D^-}=\phi_{D^+}-\phi_{D^-}=(\phi_{D^+}-\phi_S)+(\phi_S-\phi_{D^-})$$
$$=\gamma_{D^+S}-\gamma_{D^-S}, \tag{A1}$$

with similar relations for $\overline{\gamma_{D^+D^0}}$ and $\overline{\gamma_{D^+D^-}}$. The system of equations (2.16) can then be written as

$$b_1=-\frac{7\sqrt{4\pi}}{2\sqrt{30}}r_2^2=|D^+||D^-|\cos\gamma_{D^+D^-} \tag{A2}$$
$$-|\overline{D^+}||\overline{D^-}|\cos\overline{\gamma_{D^+D^-}},$$

$$b_2=-\frac{7\sqrt{4\pi}}{4\sqrt{15}}r_1^4=|D^+||D^0|\cos\gamma_{D^+D^0}$$
$$-|\overline{D^+}||\overline{D^0}|\cos\overline{\gamma_{D^+D^0}},$$

$$b_3 = -\frac{\sqrt{4\pi}}{2\sqrt{2}} r_1^2 - \sqrt{\frac{5}{7}}\, b_2 = |D^+||S|\cos\gamma_{D^+S} - |\overline{D^+}||\overline{S}|\cos\overline{\gamma_{D^+S}}\,.$$

From b_3 we obtain

$$\cos\overline{\gamma_{D^+S}} = \frac{|D^+||S|\cos\gamma_{D^+S} - b_3}{|\overline{D^+}||\overline{S}|}. \tag{A3}$$

Using Eqs. (A1) we obtain, from b_2,

$$\sin\overline{\gamma_{D^+S}} = -\cos\overline{\gamma_{D^+S}}(\cos\overline{\gamma_{D^0S}}/\sin\overline{\gamma_{D^0S}}) + \frac{b_2 - |D^+||D^0|(\cos\gamma_{D^+S}\cos\gamma_{D^0S} + \sin\gamma_{D^+S}\sin\gamma_{D^0S})}{|\overline{D^+}||\overline{D^0}|\sin\overline{\gamma_{D^0S}}}. \tag{A4}$$

We now define

$$c_1 \equiv |D^0||S|\sin\gamma_{D^0S} = \epsilon_1 \sqrt{|D^0|^2|S|^2 - \left(A + \frac{1}{2\sqrt{5}}|S|^2\right)^2},$$

$$c_2 \equiv |D^-||S|\sin\gamma_{D^-S} = \epsilon_2 \sqrt{|D^-|^2|S|^2 - B^2},$$

$$c_3 = |D^-||D^0|\sin\gamma_{D^-D^0} = \epsilon_3 \sqrt{|D^-|^2|D^0|^2 - C^2}, \tag{A5}$$

where $\epsilon_k = \pm 1, k = 1,2,3$ is the ambiguity sign of the sines. The c_3 and the sign ϵ_3 are not independent of c_1 and c_2:

$$|S|^2 c_3 = \left(A + \frac{1}{2\sqrt{5}}|S|^2\right) c_2 - B c_1. \tag{A6}$$

Similarly we define $\overline{c_1}$, $\overline{c_2}$, and $\overline{c_3}$ for amplitudes of opposite transversity. Substituting for $\cos\overline{\gamma_{D^+S}}$ and $\sin\overline{\gamma_{D^+S}}$ from Eqs. (A3) and (A4) in the equation for b_1 and using the above definitions for c_k, $\overline{c_k}$, $k = 1,2,3$, we obtain

$$\begin{aligned}(b_1\overline{c_1} + b_2\overline{c_2} + b_3\overline{c_3})|\overline{S}|^2|S| &= \sin\gamma_{D^+S}|D^+||\overline{S}|^2(c_1\overline{c_2} + \overline{c_1}c_2) \\ &+ \cos\gamma_{D^+S}|D^+|\Bigg\{\overline{c_1}(B|\overline{S}|^2 - \overline{B}|S|^2) + \overline{c_2} \\ &\times\left[\left(A + \frac{1}{2\sqrt{5}}|S|^2\right)|\overline{S}|^2 + \left(\overline{A} + \frac{1}{2\sqrt{5}}|\overline{S}|^2\right)|S|^2\right]\Bigg\}.\end{aligned} \tag{A7}$$

Define

$$d = \frac{b_1\overline{c_1} + b_2\overline{c_2} + b_3\overline{c_3}}{c_1\overline{c_2} + \overline{c_1}c_2}\left(\frac{|S|}{|D^+|}\right),$$ (A8)

$$\tan\alpha = \left\{\overline{c_1}(B|\overline{S}|^2 - \overline{B}|S|^2) + \overline{c_2}\left[\left(A + \frac{1}{2\sqrt{5}}|S|^2\right)|\overline{S}|^2 + \left(\overline{A} + \frac{1}{2\sqrt{5}}|\overline{S}|^2\right)|S|^2\right]\right\} \Big/ (c_1\overline{c_2} + \overline{c_1}c_2)|\overline{S}|^2.$$

With this notation Eq. (A6) takes the form

$$\sin\gamma_{D^+S} + \cos\gamma_{D^+S}\tan\alpha = d. \tag{A9}$$

Its solution is

$$\cos\gamma_{D^+S} = \frac{1}{1+\tan^2\alpha}\{d\tan\alpha \pm \sqrt{1+\tan^2\alpha - d^2}\},$$

$$\sin\gamma_{D^+S} = \frac{1}{1+\tan^2\alpha}\{d \mp \tan\alpha\sqrt{1+\tan^2\alpha - d^2}\}. \tag{A10}$$

Using Eqs. (A10) we obtain $\cos\overline{\gamma_{D^+S}}$ and $\sin\overline{\gamma_{D^+S}}$ from Eqs. (A3) and (A4).

There are four combinations of solutions for moduli $|A|^2$ and $|\overline{A}|^2$, $A = S$, D^0, D^-, and D+, entering the calculation of d and $\tan\alpha$. In addition each such combination is accompanied by the fourfold sign ambiguity from the undetermined signs ϵ_k and $\overline{\epsilon_k}$, $k = 1,2$. This 16-fold ambiguity increases to 32-fold ambiguity due to sign ambiguity in Eqs. (A10).

The solvability of Eq. (A9) imposes a nonlinear constraint on the data and on the solution for moduli squared:

$$d^2 - 1 \leqslant \tan^2\alpha. \tag{A11}$$

Additional constraints follow from the requirement that cosines and sines of γ_{D^+S} and $\overline{\gamma_{D^+S}}$ have physical values. In principle, these constraints could reduce the overall ambiguity of solution (A10).

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