# Exact sum rule for transversely polarized DIS

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We have shown how, working in a field-theoretic framework, one can derive expressions for the even moments of the valence parts of $g_{1,2}(x)$. These expressions cannot be written as matrix elements of local operators and do not coincide with the analytic continuation to $n=$ even integer of the OPE results. Just as for the OPE one can in some cases argue that the hadronic matrix elements should be small, leading to approximate sum rules for the moments of the valence parts of $g_{1,2}(x)$. But, most importantly, for the case $n=2$ we have proved rigorously that the hadronic matrix element vanishes, yielding an exact sum rule. We have argued that the convergence properties of this sum rule are good and are a further test of QCD. [S0556-2821(97)04607-9]

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## I. INTRODUCTION

The inelastic form factors $G_1$ and $G_2$, and their scaling versions $g_1$ and $g_2$, describing spin-dependent or polarized deep inelastic scattering are attracting much attention at present with major experimental programs in progress at CERN and SLAC and planned for the DESY *ep* collider HERA. For a comprehensive account see the review article [1] by Anselmino, Efremov, and Leader (AEL). The theoretical and experimental status of $g_1$ and $g_2$ is rather different. There exists a simple partonic interpretation [2] of the scaling function $g_1(x)$ which is the only one of the two which survives in the strict Bjorken limit and, in that limit, completely controls the longitudinal polarization asymmetry. Longitudinal polarization dominates kinematically in this limit and is described in QCD as a leading (twist two) effect. The function $g_1(x)$ is also the easier one to measure experimentally [3,4]. The main theoretical issue is the subtle effect whereby the triangle anomaly induces an anomalous gluon contribution in $g_1(x)$, in particular in its first moment [5,6].

$G_2$ and the corresponding dimensionless scaling function $g_2$ are more complicated. They describe the difference between the properties of a longitudinally and a transversely polarized hadron, and QCD twist three effects, for which there is no probabilistic interpretation, contribute significantly [7]. The transverse polarization effects are suppressed as $M/Q$ ($M$ is the hadron mass; recall that a massless particle is always longitudinally polarized). This makes the experimental studies more complicated as well. The first results from the Spin Muon Collaboration (SMC) and SLAC have just appeared [8] and it is hoped that the high intensity lepton beam and jet target will make possible the measurement of $g_2$ with high accuracy by the HERMES Collaboration at HERA.

In this situation sum rules for $g_2$ are especially important. The Burkhardt-Cottingham superconvergence sum rule [9] is well known:

$$\int_0^1 g_2(x)dx=0, \tag{1.1}$$

though it is not always realized that it does not follow from the operator product expansion (OPE) and that it may contradict the expected small-$x$ Regge behavior [1,10].

The other sum rules that are often quoted are the approximate Wandzura-Wilczek sum rules [11]

$$\int_0^1 x^{n-1}\left[\frac{n-1}{n}g_1(x)+g_2(x)\right]dx=0, \quad n=1,3,5,\ \ldots, \tag{1.2}$$

which, as is discussed below, involve the neglect of twist three contributions and which assumes the validity of Eq. (1.1).[1] If the sum rules in (1.2) are assumed to hold also for even values of $n$, one obtains the remarkable result

$$g_1(x)+g_2(x)=\int_x^1\frac{g_1(x)}{x}dx. \tag{1.3}$$

The function $g_2(x)$ defined by (1.3) is often called $g_2^{WW}(x)$.

In [11] it is argued, on the basis of a model, that the twist three terms in $g_2$ can be neglected. However, this argument is generally unreliable, since the self-same model gives unacceptable results for $F_{1,2}(x)$ and $g_1(x)$.

In the following, $g_2(x)$ refers to pure electromagnetic deep inelastic scattering (DIS), unless explicitly indicated otherwise. We shall discuss the derivation of sum rules in-

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[1] Similar sum rules, for weak boson-mediated deep inelastic scattering (DIS), based on the neglect of twist three contributions, have recently appeared [20].

 

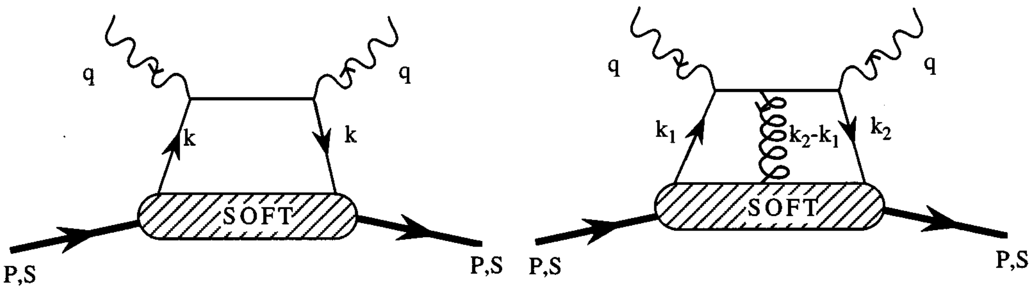


FIG. 1. Simplest Feynman diagrams contributing to DIS at twist two and twist three level. (Crossed diagrams are not shown.)

volving $g_2(x)$ from two different points of view. One is based upon the imposition of gauge invariance in a specific lepton-hadron reaction, namely, polarized DIS, the second upon a study of the properties of the hadronic matrix elements involved in $g_{1,2}(x)$, without reference to any specific reaction.

In both these approaches we are able to produce new sum rules that do not follow from the operator product expansion. The OPE only makes statements about the odd moments of $g_{1,2}(x)$, corresponding to the fact that one is essentially dealing with forward virtual Compton scattering, which, viewed in the $t$ channel, involves $\overline{p}p \to \gamma\gamma$, and thus only involves positive parity states.

In our more general field-theoretic approach we obtain results also for the even moments of $g_{1,2}(x)$. But what is fascinating, and at first sight surprising, is that they involve only the valence contributions to the structure functions.

Among these the most interesting is the case $n=2$, the so-called Efremov-Leader-Teryaev (ELT) sum rule, since it is exact and does not rely on any neglect of twist three contributions:

$$\int_0^1 dx\, x[g_1{}^V(x)+2g_2{}^V(x)]=0 \tag{1.4}$$

and which, as we shall discuss, can be tested experimentally.

Note that in charged current DIS the OPE gives expressions for the odd moments for the sum of $\nu$ and $\overline{\nu}$ reactions (as in the electromagnetic case) but for the even moments for the difference of $\nu$ and $\overline{\nu}$ reactions. In the latter case we can also derive our sum rule for the charged current version of $g_2^{W^+}-g_2^{W^-}$ by starting from the OPE [12].

In Secs. II and III we show how to derive these generalized sum rules first by appealing to gauge invariance in polarized DIS, second by a detailed study of the properties of various hadronic matrix elements. Section IV discusses some aspects of the ELT sum rules and their generalization and in Sec. V we consider how the new sum rules might be tested experimentally.

## II. SUM RULE FROM GAUGE INDEPENDENCE IN DIS

Consider first the field-theoretic calculation of the antisymmetric part of the hadronic tensor $W_{\mu\nu}^{(A)}$ which controls polarized deep inelastic scattering, via the Feynman diagrams of Fig. 1.

Because we are dealing with twist three effects it is necessary to consider the quark-quark-gluon correlators

$$b_A(x_1,x_2)=\int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}e^{i\lambda_1(x_1-x_2)+i\lambda_2 x_2}\widetilde{b}_A(\lambda_1,\lambda_2), \tag{2.1}$$

where

$$\widetilde{b}_A(\lambda_1,\lambda_2)=\frac{1}{2M}\langle P,S|\overline{\psi}(0)\not{n}\gamma^5 S\cdot D(\lambda_1 n)\psi(\lambda_2 n)|P,S\rangle, \tag{2.2}$$

and

$$b_V(x_1,x_2)=\int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}e^{i\lambda_1(x_1-x_2)+i\lambda_2 x_2}\widetilde{b}_V(\lambda_1,\lambda_2), \tag{2.3}$$

where

$$\widetilde{b}_V(\lambda_1,\lambda_2)=-\frac{i}{2M}\epsilon^{\mu\nu\rho\sigma}S_\nu P_\rho n_\sigma \times\langle P,S|\overline{\psi}(0)\not{n}D_\mu(\lambda_1 n)\psi(\lambda_2 n)|P,S\rangle, \tag{2.4}$$

where $D^\mu$ is the covariant derivative and $n^\mu$ is a lightlike gauge-fixing vector: $n^2=0$, $n\cdot A=0$, $n\cdot P=1$. It also defines the transverse direction; for example, for the covariant spin vector $S^\mu$,

$$S_T^\mu=S^\mu-(S\cdot n)P^\mu. \tag{2.5}$$

The fractions $x_1$ and $x_2$ correspond to the fractions of the hadron momentum carried by the quarks. In these definitions the correlators $b_A(x_1,x_2)$ and $b_V(x_1,x_2)$ are real and dimensionless. They are related to the correlators used in AEL by $b_V=iB^V/2$ and $b_A=-B^A/2$. They have the symmetry properties

$$b_V(x_1,x_2)=-b_V(x_2,x_1), \quad b_A(x_1,x_2)=b_A(x_2,x_1). \tag{2.6}$$

In Eqs. (2.1)–(2.4) we have suppressed the flavor label $f$ on the quark fields.

Use of the equation of motion for the quark field of a given flavor leads to a very general relation between $b_V, b_A$, and the quark-quark correlator function $f_T(x)$ which directly gives the field-theoretic expression for the transverse combination of $g_1$ and $g_2$: namely,

$$g_T(x)\equiv g_1(x)+g_2(x)=\frac{1}{2}\sum_f Q_f^2[f_T(x)+f_T(-x)], \tag{2.7}$$

where

$$f_T(x)=\int\frac{d\lambda}{2\pi}e^{i\lambda x}\tilde{f}_T(\lambda) \tag{2.8}$$

and

$$\tilde{f}_T(\lambda)=\frac{1}{2M}\langle P,S|\overline{\psi}(0)\gamma_5 \not{S}\psi(\lambda n)|P,S\rangle, \tag{2.9}$$

where, again, we suppress the flavor label.

For an arbitrary test function $\sigma(x)$ one finds [13]

$$\int dxdy\{[\sigma(x)+\sigma(y)]b_A(x,y)+[\sigma(x)-\sigma(y)]b_V(x,y)\}$$
$$=-2\int dx\sigma(x)xf_T(x). \tag{2.10}$$

Further, demanding that the results for $W_{\mu\nu}^{(A)}$ be independent of the gauge-fixing vector $n^\mu$ leads to a relation between $b_A$ and the quark-quark correlator function $h_L(x)$ which gives the field-theoretic formula for $g_1(x)$: namely,

$$g_1(x)=\frac{1}{2}\sum_f Q_f^2[h_L(x)+h_L(-x)], \tag{2.11}$$

where

$$h_L(x)=\int\frac{d\lambda}{2\pi}e^{i\lambda x}\tilde{h}_L(\lambda) \tag{2.12}$$

and

$$\tilde{h}_L(\lambda)=\frac{1}{2Mn\cdot S}\langle P,S|\overline{\psi}(0)\not{n}\gamma_5\psi(\lambda n)|P,S\rangle. \tag{2.13}$$

One finds [13]

$$\int dxdy\left[\frac{\sigma(x)-\sigma(y)}{x-y}\right]b_A(x,y)$$
$$=\int dx\sigma(x)[f_T(x)-h_L(x)]. \tag{2.14}$$

Note that in Eqs. (2.10) and (2.14) the range of integration is $|x|\leqslant 1$, $|y|\leqslant 1$ and $|x-y|\leqslant 1$.

For the longitudinal case it is possible to associate $h_L$, for each flavor, with a polarized quark or antiquark number density,

$$\Delta q(x)=h_L(x),\quad \Delta\overline{q}(x)=h_L(-x), \tag{2.15}$$

but such a connection is not possible for the transverse spin case.

In Eq. (2.14) let us now choose $\sigma(x)=x^{n-1}$, with $n$ odd. The integral $-1\leqslant x\leqslant 1$ on the right-hand side (RHS) of Eq. (2.14) can then be converted into an integral $0\leqslant x\leqslant 1$, leading via Eqs. (2.7) and (2.8) to [13]

$$\int_0^1 dxx^{n-1}g_2(x)=\frac{1}{2}\sum_f Q_f^2\int dxdy\left[\frac{n-1}{2}(x^{n-2}+y^{n-2})\right.$$
$$\left.+\phi_{n-1}(x,y)\right]b_A(x,y)\quad (n\text{ odd}), \tag{2.16}$$

where

$$\phi_n(x,y)\equiv\frac{x^n-y^n}{x-y}-\frac{n}{2}(x^{n-1}+y^{n-1}). \tag{2.17}$$

Note that

$$\phi_n(x,y)=0\quad\text{if}\quad x=y. \tag{2.18}$$

Now, let us choose $\sigma(x)=x^{n-2}$ in Eq. (2.10) with $n$ odd. By analogous arguments, Eq. (2.10) becomes

$$\int_0^1 dx\, x^{n-1}\left[\frac{n-1}{n}g_1(x)+g_2(x)\right]$$
$$=-\frac{1}{4}\sum_f Q_f^2\int dxdy\{(x^{n-2}+y^{n-2})b_A(x,y)$$
$$+(x^{n-2}-y^{n-2})b_V(x,y)\}\ \ (n\text{ odd}). \tag{2.19}$$

It follows that [13],

$$\int_0^1 dx\, x^{n-1}\left[\frac{n-1}{n}g_1(x)+g_2(x)\right]$$
$$=\frac{1}{4(n+1)}\sum_f Q_f^2\int dxdy\left\{\phi_{n-1}(x,y)b_A(x,y)\right.$$
$$\left.-\frac{n-1}{2}(x^{n-2}-y^{n-2})b_V(x,y)\right\}\quad n=1,3,5,\ldots. \tag{2.20}$$

The set of relations (2.20) is perfectly equivalent to what one obtains from the operator product expansion for $n=3,5,7,\ldots$. The OPE, however, says nothing about the case $n=1$. Indeed, Eq. (2.20) may not be valid for $n=1$ because the integrals could diverge.

We see that the left-hand side (LHS) of Eq. (2.20) is just the LHS of the Wandzura-Wilczek (WW) sum rule (1.2). The WW sum rule was originally derived from the operator product expansion by neglecting twist three operators on the RHS and by assuming that the operator product result can be continued smoothly to $n=1$, where, of course, the WW sum rule just reduces to the Burkhardt-Cottingham (BC) sum rule (1.1).

There are good reasons to believe that BC sum rule will fail because the expected Regge behavior for $g_2(x)$ as $x\to 0$ might make the integral over $g_2(x)$ diverge [10].

Contrary to the operator product approach, one can certainly choose $\sigma(x)=x^{n-1}$ with $n$ even in Eq. (2.14) and $\sigma(x)=x^{n-2}$ with $n$ even in Eq. (2.10), to obtain a totally new set of relations, which, however, neither involve $g_1(x)$ or $g_2(x)$ as such, but a part of them, $g_1^V(x)$ and $g_2^V(x)$, which can be regarded as the valence contribution to them. For $g_1(x)$, which has a simple partonic interpretation, this is straightforward. For $g_2(x)$, which does not have a partonic interpretation it is not clear what $g_2^V(x)$ means physically. However, it is a well-defined object, which can be measured, and thus sum rules involving it are of physical importance.

The difference between $n$ odd and $n$ even appears in the following way. The LHS's of Eqs. (2.16) and (2.19) originally involve integrals of the form, for example,

$$\frac{1}{2}\sum_f Q_f^2\int_{-1}^{1}dx\; x^{n-1}h_L(x).$$

Because $n$ was odd, this could be written as

$$\frac{1}{2}\sum_f Q_f^2\int_0^1 dx\; x^{n-1}[h_L(x)+h_L(-x)]=\int_0^1 dx\; x^{n-1}g_1(x).$$

For $n$ even, the last step will lead to expressions of the form

$$\frac{1}{2}\sum_f Q_f^2\int_0^1 dx\; x^{n-1}[h_L(x)-h_L(-x)]$$

$$=\frac{1}{2}\sum_f Q_f^2\int_0^1 dx\; x^{n-1}[\Delta q_f(x)-\Delta\overline{q_f}(x)] \quad (2.21)$$

$$=\int_0^1 dx\; x^{n-1}g_1^V(x). \quad (2.22)$$

We shall define $g_2^V(x)$ by [see Eq. (2.7)]

$$g_2^V(x)=-g_1^V(x)+\frac{1}{2}\sum_f Q_f^2[f_T(x)-f_T(-x)]. \quad (2.23)$$

Then, the relations (2.16), (2.19), and (2.20) hold also for even $n$ with $g_1(x)\to g_1^V(x)$ and $g_2(x)\to g_2^V(x)$.

Of particular interest is the case $n=2$, because the contribution of the twist three correlators on the RHS of Eq. (2.20) vanishes when $n=2$. Thus, one has

$$\int_0^1 dx\; x[g_1^V(x)+2g_2^V(x)]=0. \quad (2.24)$$

This so-called Efremov-Leader-Teryaev (ELT) sum rule was incorrectly stated in AEL [1] where the label ''$V$'' was not indicated.

We shall return to discuss certain aspects of the ELT sum rule, the possibility of testing it physically, its convergence properties, and whether or not it can be generalized, after first discussing a quite different approach to the sum rule.

## III. SUM RULE FROM PROPERTIES OF HADRONIC MATRIX ELEMENTS

The derivation of the sum rule in Sec. II is a little unsatisfactory in that it appeals to a particular lepton-hadron reaction to derive properties inherent to the nucleon. The following derivation deals only with nucleon matrix elements. The sum rules can be derived from a careful study of the structure and gauge properties of the matrix elements and use of the equation of motion of the quark field. In this approach [1], one sees very clearly why sum rules such as the Burkhardt-Cottingham one may fail because of the noninvertability of certain Fourier transforms.

Consider first the forward matrix element of the bilocal operator

$$\overline{\psi(0)}\gamma^\mu\gamma_5\psi(x)$$

on the light cone $x^2=0$. Its most general form is

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma^\mu\gamma_5\psi(x)\rangle_{P,S}=A_1S^\mu+(x\cdot S)A_2P^\mu+(x\cdot S)A_3x^\mu, \quad (3.1)$$

where $\langle\cdots\rangle$ is short for $\langle P,S|\cdots|P,S\rangle$. The scalar functions $A_{1,2,3}$ are functions only of $x\cdot P$.

From Eq. (3.1) we deduce

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma^\mu\gamma_5\partial^\nu\psi(x)\rangle_{P,S}=A_1'S^\mu P^\nu+A_2P^\mu S^\nu+A_3x^\mu S^\nu$$
$$+(x\cdot S)[A_2'P^\mu P^\nu+A_3'x^\mu P^\nu$$
$$+A_3g^{\mu\nu}], \quad (3.2)$$

where

$$A'\equiv\frac{dA(x\cdot P)}{d(x\cdot P)}. \quad (3.3)$$

We now put $x^\mu=\lambda n^\mu$. Then,

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma^\mu\gamma_5\psi(\lambda n)\rangle_{P,S}=A_1S^\mu+\lambda(n\cdot S)[A_2P^\mu+\lambda A_3n^\mu], \quad (3.4)$$

where now $A_i=A_i(\lambda)$, and

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma^\mu\gamma_5\partial^\nu\psi(\lambda n)\rangle_{P,S}=A_1'S^\mu P^\nu+A_2P^\mu S^\nu$$
$$+\lambda\{A_3n^\mu S^\nu+(n\cdot S)[A_2'P^\mu P^\nu$$
$$+\lambda A_3'n^\mu P^\nu+A_3g^{\mu\nu}]\}. \quad (3.5)$$

We assume that all scalar functions are such that $\lambda A(\lambda)\to 0$ as $\lambda\to 0$ for all terms occurring in Eqs. (3.4) and (3.5). This is in accord with expectation from OPE, since the limit $\lambda\to 0$ in our light-cone operators corresponds to them becoming local operators. Then, at $\lambda=0$ we have the simple structures

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma^\mu\gamma_5\psi(0)\rangle_{P,S}=A_1(0)S^\mu \quad (3.6)$$

and

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma^{\mu}\gamma_5\partial^{\nu}\psi(0)\rangle_{P,S}=A_1'(0)S^{\mu}P^{\nu}+A_2(0)P^{\mu}S^{\nu}. \tag{3.7}$$

We shall also require, from Eq. (3.5),

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma_5\not\partial\psi(\lambda n)\rangle_{P,S}=-\lambda(n\cdot S)[M^2A_2'+5A_3+\lambda A_3'] \tag{3.8}$$

so that, at $\lambda=0$,

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma_5\not\partial\psi(0)\rangle_{P,S}=0. \tag{3.9}$$

Finally note, from Eq. (3.5), that

$$\begin{aligned}\frac{1}{M}\langle\overline{\psi(0)}\gamma^{\mu}\gamma_5 n\cdot\partial\psi(\lambda n)\rangle_{P,S}&=A_1'S^{\mu}+(n\cdot S)[(A_2+\lambda A_2')P^{\mu}\\&\quad+\lambda(2A_3+\lambda A_3')n^{\mu}]\\&=\frac{1}{M}\frac{d}{d\lambda}\langle\overline{\psi(0)}\gamma^{\mu}\gamma_5\psi(\lambda n)\rangle_{P,S}.\end{aligned} \tag{3.10}$$

Consider now the gluonic matrix element

$$\frac{1}{M}\langle\overline{\psi(0)}\gamma^{\mu}\gamma_5 gA^{\nu}(x)\psi(x)\rangle_{P,S}$$

with $x=\lambda n$. Its most general form is

$$\begin{aligned}&\lambda(S\cdot n)[B_1P^{\mu}P^{\nu}+\lambda B_2P^{\mu}n^{\nu}+\lambda B_3n^{\mu}P^{\nu}+\lambda^2B_4n^{\mu}n^{\nu}]\\&\quad+B_5S^{\mu}P^{\nu}+B_6P^{\mu}S^{\nu}+\lambda B_7S^{\mu}n^{\nu}+\lambda B_8n^{\mu}S^{\nu}.\end{aligned} \tag{3.11}$$

The gauge condition $n_{\mu}A^{\mu}=0$ implies that

$$B_5=0,\quad \lambda B_1=-B_6,\quad \lambda B_3=-B_8, \tag{3.12}$$

so that

$$\begin{aligned}&\frac{1}{M}\langle\overline{\psi(0)}\gamma^{\mu}\gamma_5 gA^{\nu}(\lambda n)\psi(\lambda n)\rangle_{P,S}\\&\quad=\lambda B_1[(S\cdot n)P^{\mu}P^{\nu}-P^{\mu}S^{\nu}]+\lambda(S\cdot n)[B_2P^{\mu}n^{\nu}\\&\qquad+\lambda B_4n^{\mu}n^{\nu}]+\lambda^2B_3[(S\cdot n)n^{\mu}P^{\nu}-n^{\mu}S^{\nu}]+\lambda B_7S^{\mu}n^{\nu}.\end{aligned} \tag{3.13}$$

Notice the crucial feature that the imposition of the gauge condition, together with the assumptions about the vanishing of products such as $\lambda B(\lambda)$ as $\lambda\to 0$, leads to the vanishing of Eq. (3.13) at $\lambda=0$, i.e.,

$$\langle\overline{\psi(0)}\gamma^{\mu}\gamma_5 gA^{\nu}(0)\psi(0)\rangle_{P,S}=0. \tag{3.14}$$

This result will be crucial for deriving the ELT sum rule.

Let us now relate some of the above coefficients to the functions occurring in the discussion of $g_1$ and $g_2$. From Eqs. (2.13) and (3.4), we have

$$\tilde{h}_L(\lambda)=\frac{1}{2}[A_1(\lambda)+\lambda A_2(\lambda)]. \tag{3.15}$$

From Eqs. (2.9) and (3.4),

$$\tilde{f}_T(\lambda)=\frac{1}{2}A_1(\lambda). \tag{3.16}$$

Then, from Eqs. (2.11) and (2.12), if the Fourier transforms can be inverted,

$$\int_0^1 dx g_1(x)=\frac{Q_f^2}{2}\tilde{h}_L(0)=\frac{Q_f^2}{4}A_1(0)\quad\text{by Eq. (3.15)}. \tag{3.17}$$

Similarly, from Eqs. (2.7) and (2.8)

$$\begin{aligned}\int_0^1 dx[g_1(x)+g_2(x)]&=\frac{Q_f^2}{2}\tilde{f}_T(0)\\&=\frac{Q_f^2}{4}A_1(0)\quad\text{by Eq. (3.16)}.\end{aligned} \tag{3.18}$$

Equations (3.17) and (3.18) imply the Burkhardt-Cottingham sum rule

$$\int_0^1 dx g_2(x)=0. \tag{3.19}$$

As is discussed in Ref. [10] the above derivation may fail because of the noninvertability of the Fourier transforms. We turn now to the ELT sum rule.

Consider first Eq. (2.10) which followed from the equations of motion. Choosing $\sigma(x)=\delta(x-z)$ and then integrating over $z$, using Eqs. (2.6), (2.1), and (2.8), there results:

$$\tilde{b}_A(0,0)=-i\left.\frac{d\tilde{f}_T}{d\lambda}\right|_{\lambda=0}, \tag{3.20}$$

where we have taken the quark mass to be zero for simplicity and where we have taken, on the basis of Eq. (2.8),

$$xf_T(x)=i\int\frac{d\lambda}{2\pi}e^{i\lambda x}\frac{d\tilde{f}_T}{d\lambda}(\lambda). \tag{3.21}$$

Now, because of Eq. (3.14), from Eq. (2.2),

$$\tilde{b}_A(0,0)=\frac{i}{2M}\langle\overline{\psi(0)}\not n\gamma_5(S_T\cdot\partial)\psi(0)\rangle_{P,S}$$

so that, via Eq. (3.7),

$$\tilde{b}_A(0,0)=-\frac{i}{2}A_2(0). \tag{3.22}$$

Use of this and Eq. (3.16) in Eq. (3.20) yields

$$A_2(0)=\frac{d}{d\lambda}A_1(\lambda)|_{\lambda=0}=A_1'(0). \tag{3.23}$$

Now, by arguments similar to those that lead to Eq. (3.21), we have

$$\int_{-1}^{1} dx\; x h_L(x) = i \frac{d\tilde{h}}{d\lambda}\bigg|_{\lambda=0}$$
$$= \frac{i}{2}[A_1'(0) + A_2(0)] \quad \text{by Eq. (3.15)}$$
$$= iA_1'(0) \quad \text{by Eq. (3.23)}. \tag{3.24}$$

Similarly, we have, using Eqs. (3.21) and (3.16),

$$2\int_{-1}^{1} dx\; x f_T(x) = 2i \frac{d\tilde{f}_T}{d\lambda}\bigg|_{\lambda=0} = iA_1'(0). \tag{3.25}$$

Subtracting Eq. (3.24) from Eq. (3.25) and repeating the kind of argument that led to Eq. (2.22), we obtain, once again, the ELT sum rule

$$\int_0^1 dx\; x[g_1^V(x) + 2g_2^V(x)] = 0. \tag{3.26}$$

## IV. DISCUSSION OF THE ELT SUM RULE AND A GENERALIZATION

We discuss here first the question of the convergence of the ELT sum rule (2.24), then consider an analogous sum rule involving the complete functions $g_{1,2}(x)$ and not just their valence parts and then comment upon an implication for the concept of handedness of jets.

As mentioned earlier, the Burkhardt-Cottingham sum rule (1.1) may well diverge because of a possible $1/x^2$ growth of $g_2(x)$ as $x \to 0$. It is important to note that such a singular behavior will not spoil the convergence of the ELT sum rule (2.24), since the singularity will cancel out in the subtraction in Eq. (2.13).

Consider now the question of the analogue of Eq. (2.24) for the complete functions $g_{1,2}(x)$. In contrast with the operator product expansion, the sum rule (2.10) holds for $\sigma(x) = x^{n-1}$ with $n$ odd or even and the sum rule (2.14) holds for $\sigma(x) = x^{n-2}$ with $n$ odd or even. For $n$ odd and $\geqslant 3$ they reproduce the OPE results for the moments of $g_{1,2}(x)$. For $n$ even they produce new sum rules for the moments of the valence parts of $g_{1,2}(x)$. However, it is possible to consider sum rules for $n$ even from a different point of view, namely, from the analytic continuation in $n$ of the results for $n$ odd. Hence, we wish to begin with Eqs. (2.16) and (2.19) and analytically continue in $n$. As written, the RHS's of Eqs. (2.16) and (2.19) do not have a unique analytic continuation since $x$ and $y$ can be negative so that terms of the form $x^n$ and $y^n$ effectively reproduce factors of $(-1)^n$ which grow exponentially in the imaginary $n$ direction and spoil the uniqueness of the analytic continuation. However, starting with $n$ odd we can rewrite all the integrals in Eqs. (2.16) and (2.19) in such a way that $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$ after which the analytic continuation is unique. We shall not give the detailed results for arbitrary $n$, but for $n=2$ we find

$$\int_0^1 dx\; x[g_1(x) + 2g_2(x)]$$
$$= \sum_f Q_f^2 \int_0^1 dy \int_0^{1-y} dx \left[\frac{x-y}{x+y} b_A(x,-y) - b_V(x,-y)\right]. \tag{4.1}$$

The matrix elements on the RHS of Eq. (4.1) are not zero and cannot be expressed as a finite series of matrix elements of local operators. However, they are of twist three and are proportional to the square root of the product of the probability to find a gluon and the probability to find a $q\bar{q}$ pair in the nucleon. The latter was estimated to be small from the study of QCD sum rules by Shuryak and Vainshtein [14]. So, it may be that the RHS of Eq. (4.1) is negligible, corresponding to the Wandzura-Wilczek sum rules (1.2) continued to $n=2$. Together with the Burkhardt-Cottingham sum rule (1.1), this means that $g_2^{WW}(x)$ should intersect the experimental $g_2(x)$ at least twice in the interval $0<x<1$ which seems compatible with the present SLAC data [8].

The method used in Sec. III to derive sum rules for the first and second moments of $g_{1,2}(x)$ highlights an interesting aspect of the Burkhardt-Cottingham sum rule. The assumption that all the scalar functions $A(\lambda)$ are well behaved as $\lambda \to 0$, as implied by the assumed behavior $\lambda A(\lambda) \to 0$ as $\lambda \to 0$ means, as can be seen from Eq. (3.4), that the first moments of the longitudinal $g_L(x) \equiv g_1(x)$ and the transverse $g_T(x) \equiv g_1(x) + g_2(x)$ depend on the matrix element of the axial-vector current which is proportional to just the single vectorial structure $S^\mu$. There is no reference to any direction which could differentiate longitudinal from transverse, so the first moments of $g_L(x)$ and $g_T(x)$ coincide. This seems very similar to the ''naive'' derivation of the BC sum rule from rotational invariance [2] as well as to the early QCD derivation [15]. [It would be interesting to understand analogously the *physical* meaning of Eq. (2.24) written as $\int_0^1 dx\; x g_L^V(x) = 2\int_0^1 dx\; x g_T^V(x)$.]

An analogous situation arises for the new spin-dependent variable handedness (H) introduced in [16], which allows the study of the polarization of a quark or gluon which has fragmented into a jet. H is given as a product of the quark polarization times the analyzing power A of the fragmentation reaction. The analyzing power is described by light-cone functions analogous to $h_L(x)$ and $f_T(x)$. As discussed in [16] longitudinal and transverse analyzing powers coincide in the case of particle decay as a consequence of rotational invariance, but in the ''decay'' of the jet the light-cone vector $n^\mu$ ''remembers'' the jet direction resulting in a difference between longitudinal and transverse analyzing powers. But by the same reasoning as above, the first moment of the longitudinal and transverse analyzing powers should coincide. The integration variable in this case is $z$, the fraction of the parton's momentum carried by a pair (or triple) of particles used to define the jet.

Let us now consider how the new sum rules can be used to learn about $g_2(x)$ and to test QCD.

## V. PHENOMENOLOGICAL TESTS OF THE ELT SUM RULE

The general field-theoretic expression for $g_2(x)$ in terms of hadronic matrix elements of operators is given in Eqs.

(2.7)–(2.9). As mentioned earlier, despite appearances to the contrary, $g_2(x)$ does not have any simple probabilistic parton model interpretation even though only quark operators appear in the matrix element (2.9). Nonetheless, it is given by a sum over contributions coming from quark operators of definite flavor $f$ (the flavor label was suppressed in Sec. II), so that the contribution of a given flavor of quark or antiquark to $g_2(x)$ is meaningful.

Moreover, since the flavor label is clearly irrelevant in the derivation, it must be true that Eq. (2.24) holds for the contribution to $g_2(x)$ of each flavor. Hence one has, for each flavor $f$,

$$\int_0^1 dx\; x[g_{1,f}^V(x)+2g_{2,f}^V(x)]=0. \tag{5.1}$$

Information about the contributions of a given flavor to $g_2(x)$ can be obtained by studying reactions with different targets and by studying nonpurely electromagnetic DIS, for example, charge-changing DIS involving $W^{\pm}$ exchange or, at large $Q^2$, interference between $\gamma$ and $Z^0$ exchange. There is also the possibility of focusing on specific flavors by looking at semi-inclusive DIS.

There thus appear to be several possibilities to learn about $g_{2,f}^V(x)$.

(1) Assuming, as usual, that the contributions from sea quarks are the same in protons and neutrons, we can derive a kind of analogue of the Bjorken sum rule. For, then, from Eq. (2.24) or (5.1),

$$\int_0^1 dx\; x\{[g_1(x)+2g_2(x)]_p-[g_1(x)+2g_2(x)]_n\}=0. \tag{5.2}$$

Hence, we have the interesting new sum rule

$$\int_0^1 dx\; x[g_2^p(x)-g_2^n(x)]=\frac{1}{2}\int_0^1 dx\; x[g_1^n(x)-g_1^p(x)]. \tag{5.3}$$

(2) In unpolarized semi-inclusive DIS it is claimed that the study of meson production

$$\ell+N\rightarrow\ell'+M+X,$$

where $M=\pi^{\pm},\pi^0,K^{\pm},K^0,\overline{K^0}$, etc. allows one to identify the contribution of a given $q_f$ or $\overline{q_f}$ to the unpolarized structure functions and it is proposed to use the same approach, but with a longitudinally polarized target at CERN [17] to identify the individual $\Delta q_f(x)$ and $\Delta\overline{q_f}(x)$ contributions to $g_1(x)$.

We suggest that the same method, but using a transversely polarized target, will allow the identification of the contributions $g_{2,f}(x)$ to $g_2(x)$ coming from a given flavor quark or antiquark.

Hence, in principle, the valence contribution to $g_2(x)$ of a given flavor, $g_{2,f}^V(x)$, can be measured.

(3) A simpler method is to assume dominance of the $u$ and $d$ contributions and to study

$$\ell+N\rightarrow\ell'+\text{jet}+X,$$

using a transversely polarized target and with identification of the charge of the jet ($\pm$). If the differences of cross sections when the transverse spin is reversed, $\Delta_T d\sigma^{\text{jet}+}$ and $\Delta_T d\sigma^{\text{jet}-}$, are measured then $[\Delta_T d\sigma^{\text{jet}+}-\Delta_T d\sigma^{\text{jet}-}]$ will involve the combinations [18]

$$(g_{2,u}+g_{2,\bar{d}})-(g_{2,d}+g_{2,\bar{u}})=g_{2,u}^V-g_{2,d}^V. \tag{5.4}$$

It would seem possible to carry out such a measurement in the upgraded SMC experiment HMC with a forward magnetic spectrometer or in the HERMES experiment at HERA which uses a polarized gas jet target.

(4) In charge-changing DIS mediated by $W^{\pm}$ bosons, the coupling to quarks and antiquarks is of opposite sign. If the cross-section differences under reversal of the transverse nucleon polarization can be measured for

$$\mu^+N\rightarrow\overline{\nu_\mu}+X,$$

and for

$$\mu^-N\rightarrow\nu_\mu+X$$

then, for the difference of these, one has [12]

$$\Delta_T d\sigma^{\mu^+\rightarrow\overline{\nu_\mu}}-\Delta_T d\sigma^{\mu^-\rightarrow\nu_\mu}\propto g_2^{W^+}-g_2^{W^-}. \tag{5.5}$$

The precise relation between cross sections and scaling functions is given in Ref. [1]. However, the expression for $g_2^W(x)$ given there, which was taken from Ref. [19], is incorrect. In fact, $g_2^W(x)$ is given in terms of the function $f_T(x)$ as it occurs in Eq. (2.7). The only difference is in the coupling constants involved. Hence, the combination occurring in Eq. (5.5) can be expressed in terms of the purely electromagnetic $g_2(x)$ valence parts discussed above:

$$g_2^{W^+}(x)-g_2^{W^-}(x)=18g_{2,d}^V(x)-\frac{9}{2}g_{2,u}^V(x). \tag{5.6}$$

For an isoscalar target $A_0$ one then has

$$[g_2^{W^+}(x)-g_2^{W^-}(x)]^{A^0}_{\substack{\text{per}\\ \text{nucleon}}}=\frac{25}{4}[g_{2,u}^V(x)+g_{2,d}^V(x)]. \tag{5.7}$$

In principle, one could combine Eqs. (5.7) and (5.4) to study the individual $u$ and $d$ valence contributions to $g_2(x)$.

(5) If an asymmetry measurement with transversely polarized target can be done at sufficiently large $Q^2$, so that $\gamma$–$Z^0$ interference is important, then

$$g_2^{\gamma Z}(x)=2\sum_f\left(\frac{g_V^f}{Q_f}\right)g_{2,f}(x), \tag{5.8}$$

where $g_V^u=1/2-(4/3)\sin^2\theta_W$, $g_V^d=-1/2+(2/3)\sin^2\theta_W$, $Q_f$ is the charge, and $g_{2,f}(x)$ is the flavor $f$ contribution to the pure electromagnetic $g_2(x)$. Measurement of $g_2^{\gamma Z}(x)$ thus provides further information about the flavor $f$ contributions to $g_2(x)$.

## VI. CONCLUSIONS

It is known that, for electromagnetic DIS, the operator product expansion provides expressions for the $n$th moments of $g_1(x)$ and $g_2(x)$ in terms of hadronic matrix elements of local operators for $n=$ odd integer. In some cases these matrix elements are expected to be small, leading to approximate sum rules for the odd moments of $g_{1,2}(x)$. We have shown how, working in a field-theoretic framework, one can derive expressions for the even moments of the valence parts of $g_{1,2}(x)$. These expressions cannot be written as matrix elements of local operators and do not coincide with the analytic continuation to $n=$ even integer of the OPE, results.

Just as for the OPE, one can in some cases argue that the hadronic matrix elements should be small, leading to approximate sum rules for the moments of the valence parts of $g_{1,2}(x)$. But, most importantly, for the case $n=2$ we have proved rigorously that the hadronic matrix element vanishes, yielding the exact ELT sum rule

$$\int_0^1 dx\ x[g_1^V(x)+2g_2^V(x)]=0.$$

In the case of the charged current DIS the OPE provides expressions for the even moments of $g_2^{W^+}-g_2^{W^-}$ and can be used as a starting point for the derivation of our sum rule.

We have argued that the convergence properties of these sum rules are good and have discussed how they can be used to get information about $g_2(x)$ from inclusive and semiinclusive electromagnetic DIS, from charged current DIS and eventually from $\gamma$-$Z$ interference in neutral current DIS. It is important to test the sum rule since it is a direct consequence of QCD.

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