Decay constants and mixing parameters in a relativistic model for a $q\overline{Q}$ system

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We extend our recent work, in which the Dirac equation with a ''(asymptotically free) Coulomb + (Lorentz scalar $\gamma_0\sigma r$) linear'' potential is used to obtain the light quark wave function for $q\overline{Q}$ mesons in the limit $m_Q\rightarrow\infty$, to estimate the decay constant f_P and the mixing parameter B of the pseudoscalar mesons. We compare our results for the evolution of f_P and B with the meson mass M_P to the nonrelativistic formulas for these quantities and show that there is a significant correction in the subasymptotic region. For $\sigma=0.14\ \mathrm{GeV}^{-2}$ and $\Lambda_{\overline{\mathrm{MS}}}=0.240$ GeV we obtain $f_D=0.371$, $f_{D_s}=0.442$, $f_B=0.301$, $f_{B_s}=0.368$ GeV, and $B_D=0.88$, $B_{D_s}=0.89$, $B_B=0.95$, $B_{B_s}=0.96$, and $B_K=0.60$. [S0556-2821(97)00101-X]

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Recently, we presented our results for the form factor (Isgur-Wise function) which parametrizes the transition matrix elements of mesons containing a heavy quark [1]. In our relativistic treatment of a heavy-light system, we assumed that, with the heavy quark fixed at the origin, the light quark wave function obeys a Dirac equation with a spherically symmetric potential. This potential, an asymptotically free Coulomb term plus a linear confining term, reflects the QCD interactions both at short and long distances. In this way, we obtained the shape and the slope at zero recoil of the form factor in the leading $1/m_Q$ order by fitting our model parameters to the experimental results on the semileptonic B decays.

In this paper, we extend our results by calculating the decay constants f_P and mixing parameters B of the pseudoscalar $q\overline{Q}$ mesons in our model. f_P parametrizes the matrix element for the decay of a pseudoscalar meson through the axial-vector current A_μ^5:

$$\langle 0|A_\mu^5(x)|P(k)\rangle = ik_\mu f_P e^{-ik\cdot x}, \tag{1}$$

where k is the four-momentum of the meson. On the other hand, B is related to the matrix element for neutral-meson mixing which is conventionally written as

$$\langle\overline{P}|(V_\mu - A_\mu^5)^2|P\rangle = \frac{4}{3}f_P^2 M_P B, \tag{2}$$

where V_μ is the vector current and M_P is the meson mass. From a phenomenological point of view, knowledge of these matrix elements is necessary for extracting important quantities such as the quark mixing matrix and CP violation within the standard model. Experimentally $f_\pi=132$ MeV and $f_K=167$ MeV are well known. However, for heavy mesons only a few data with large uncertainties are available. Mark III sets an upper bound $f_D<290$ MeV [2] while WA75 [3], CLEO II [4] and BES [5] find $225\pm45\pm20\pm41$, $344\pm37\pm52$, and $430^{+150}_{-130}\pm40$ MeV for f_{D_s}, respectively.

There are various quark-model, QCD sum rules and lattice calculations of these parameters [6–9]. A frequently used point of comparison is a scaling law for f_P that is derived from a nonrelativistic (NR) quark model:

$$f_P^{\mathrm{NR}}(m_Q\rightarrow\infty)\propto\sqrt{\frac{1}{M_P}}, \tag{3}$$

where m_Q is the mass of the heavy quark. The mixing parameter is identically 1 in any NR quark model:

$$B^{\mathrm{NR}}(m_Q)=1\ . \tag{4}$$

In Refs. [8,9], it is shown that the combination of a relativistic dynamics and an asymptotically free Coulomb interaction for $q\overline{Q}$ system results in a significant deviation from the NR scaling law. Our approach here is complementary to Refs. [8,9] in the sense that we use a potential which is not only asymptotically free but also includes a linear confining term which determines the global behavior of the Dirac wave function. At the same time, we use a saturation value for the strong coupling $\alpha_s^\infty=\alpha_s(r\rightarrow\infty)$ therefore avoiding unphysical pair creation effects.

We start with the time-independent Dirac equation

$$[\vec{\alpha}\cdot(-i\vec{\nabla})+V_c(r)+c_0+\gamma^0(\sigma r+m_q)]\Psi(\vec{r})=\epsilon_q\Psi(\vec{r}), \tag{5}$$

where m_q is the light quark mass. $V_c(r)$ is the asymptotically free Coulomb potential (tranforming as the zeroth component of a Lorentz four-vector):

$$V_c(r)=-\frac{4}{3}\frac{\alpha_s(r)}{r}, \tag{6}$$

where $\alpha_s(r)$ is obtained in the leading log approximation and is parametrized as

$$\alpha_s(r)=\frac{2\pi}{(11-2N_F/3)\ln[\delta+\beta/r]}. \tag{7}$$

*Deceased.

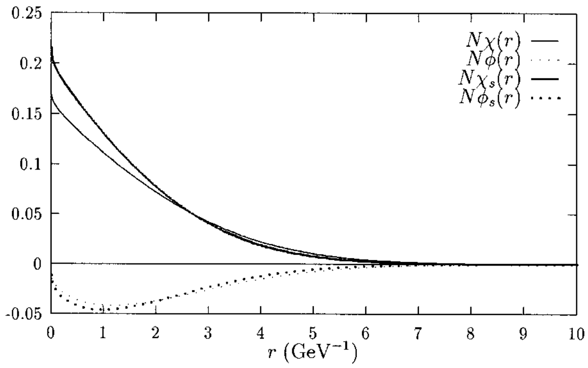

FIG. 1. The normalized large $\chi^n = N\chi(r)$ and small $\phi^n = N\phi(r)$ component of the wave function for $q\equiv u,d$ $(m_q=0)$ and $q\equiv s$ $(m_q = 0.175$ GeV).

The parameter δ defines the ''long distance'' saturation value for α_s. We use $\delta = 2.0$ which corresponds to $\alpha_s(r=\infty)\approx 1.0$. The parameter β is related to the QCD scale $\Lambda_{\overline{\rm MS}}$ by $\beta = (2.23\Lambda_{\overline{\rm MS}})^{-1}$ for $N_F=3$ (we use $N_F=3$ throughout this paper), where $\overline{\rm MS}$ denotes the modified minimal subtraction scheme. We take $\Lambda_{\overline{\rm MS}}\approx 0.240$ GeV, which corresponds to $\beta = 1.87$ GeV^{-1} in most of the calculations.

To describe the long distance behavior, a linear term $(\gamma_0\sigma r)$ is introduced in the potential which transforms as a Lorentz scalar. This form is favored based on many theoretical and phenomenological arguments [10–12]. For the parameter σ, we mainly use a value $\sigma=0.14$ which is favored by a recent lattice estimate [13] and can also be extracted from the Regge slope using a two-body generalization of the Klein-Gordon equation [11,12]. However, we also present results for $\sigma = 0.25$ and 0.18 GeV^2, as the former value is compatible with the experimentally available 0.45 GeV $2P$-$1S$ splitting for D and D_s system and the latter is extracted from Regge slope data using a string model [14]. As mentioned in the discussion of our results in Ref. [1], the best fit of our model to the recent CLEO 1994 data analysis favors a smaller value for the parameter σ.

The potential in the Dirac equation includes an additive constant c_0, which is clearly subleading both in the short and long distance regimes. The only role of this constant is to define the absolute scale of the light quark energy ϵ_q which is identified with the ''inertia'' parameter often introduced in heavy quark effective theory:

$$\epsilon_q \equiv \overline{\Lambda}_q = \lim_{m_Q\to\infty}(M_P - m_Q). \tag{8}$$

Therefore, only the difference $\overline{\Lambda}_q - c_0 \approx \epsilon_q - c_0$ can be extracted from Eq. (5). As we indicate later, in the heavy limit $m_Q\to\infty$, f_P and B are not very sensitive to $\overline{\Lambda}_q$ (and in turn to c_0).

The ground-state solution to Eq. (5) has the form

$$\psi(\vec{r}) = N\begin{pmatrix}\chi(r)\\ -i\sigma\cdot\hat{r}\phi(r)\end{pmatrix}, \tag{9}$$

where N is a normalization constant. Figure 1 illustrates the functional behavior of the normalized large component $\chi^n[=N\chi(r)]$ and small component $\phi^n[=N\phi(r)]$ of the wave function $\psi(\vec{r})$ for the cases where $m_q=0$ (appropriate for $q\equiv u,d$) and $m_q=0.175$ GeV (for $q\equiv s$). We observe that $\chi(r)$ is finite and $\phi(r)\to 0$ as $r\to 0$. This, as noted in Refs. [8,9], is due to using a running $\alpha_s(r)$ in the Coulomb potential [Eq. (6)] rather than a fixed α which would result in a singular wave function at the origin. On the other hand, since our model incorporates the long distance behavior of QCD interactions (through linear confining potential) and at the same time avoids a singularity in the Coulomb potential [by introducing a saturation value for $\alpha_s(r)$], the resulting wave function is physically meaningful for the whole range of r. This is our main improvement over previous works where a Dirac equation along with the leading-log Coulomb potential has been applied to heavy-light mesons [8,9]. The functions $\chi(r)$, $\phi(r)$ and the normalization constant N are independent of m_Q in our leading order (in $1/m_Q$) approach where we assume that the heavy quark is fixed at the origin. However, as it is illustrated in Fig. 1, the SU(3)$_F$ symmetry breaking strange quark mass $m_s=0.175$ GeV results in about 20% larger value for N [$\chi(r)$ is independent of m_q as $r\to 0$]. The normalization constant N depends on the global behavior of the potential, i.e., β and σ [see Eqs. (5), (6), and (7)]. We mainly use $\beta=1.87$ (corresponding to $\Lambda_{\overline{\rm MS}}=0.240$ GeV) and $\sigma = 0.14$ GeV^{-1} [see the discussion following Eq. (7)], however, we do vary these parameters to investigate the sensitivity of our results.

We now proceed to compute the decay constant f_P and mixing parameter B. Within our model, the zero component of Eq. (1) for a meson localized around the origin becomes

$$\sqrt{6}\int d^3\vec{r}e^{i\vec{k}\cdot\vec{r}}\Psi_Q(\vec{r})^*\chi(\vec{r}) = M_P f_P F(\vec{k}). \tag{10}$$

It is assumed that the heavy antiquark wave function is nonrelativistic and therefore has a nonzero axial-vector matrix element only with the large component χ of the light quark wave function. The factor $\sqrt{6}$ is ''geometric,'' arising from the spin and color matrix elements. If the meson is localized at the origin, the form factor $F(k)$ is constant. The weak decay constant is determined by assuming that the meson is localized in a region of dimension M_P^{-1} and that the form factor $F(\vec{k})$ is normalized

$$\int d^3\vec{k}|F(k)|^2 = (2\pi)^3(2M_P)^{-1}. \tag{11}$$

This normalization of the current, which is conventional, interprets the current as that associated with a meson of mass M_P [15,16]. Substituting Eq. (10) into Eq. (11) gives

$$f_P^2 = \frac{12}{M_P}N^2\int d^3\vec{r}|\Psi_Q(\vec{r})|^2\chi^2(r). \tag{12}$$

Considering that $\Psi_Q(\vec{r})$ is ''spread'' over a distance $r_Q = O(1/m_Q)$, one can reduce Eq. (12) to the expression [9]

$$f_P(m_Q) = \sqrt{\frac{12}{M_P}}N\chi(r_Q). \tag{13}$$

At this point, we need to make a definite connection between r_Q and the heavy quark mass m_Q. It is assumed that $r_Q = \kappa/m_Q$, where $\kappa \geqslant 1$ is expected on physical grounds. We obtain a value for κ by extrapolating the decay constant

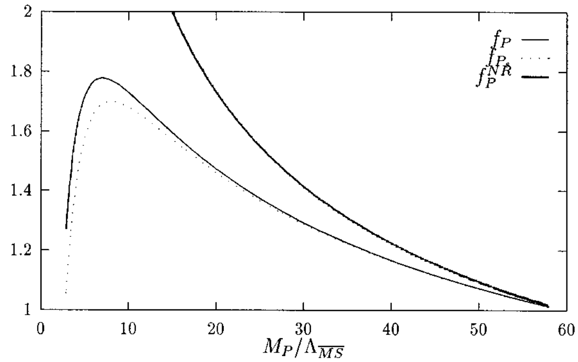

FIG. 2. The evolution of the decay constant with the meson mass ($m_q=0$ and $m_q=0.175$ GeV) compared with the NR scaling law.

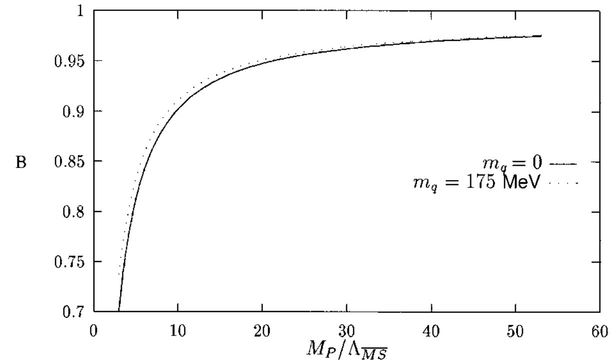

FIG. 3. The scaling of the mixing parameter B with the meson mass.

formula (15) to the K-meson system where $f_K=167$ MeV is known experimentally. This extrapolation, even though of uncertain reliability due to finite m_Q effects, has been frequently made in purely NR models. For $m_s^{\rm constituent}\approx M_K=0.495$ GeV, we obtain $\kappa=1.67$ for $\Lambda_{\overline{\rm MS}}=0.240$ and $\kappa=1.73$ for $\Lambda_{\overline{\rm MS}}=0.200$. In comparing our results for the decay constant with the NR scaling law, Eq. (3), we also include a finite renormalization factor suggested by Voloshin and Shifman [17] and Politzer and Wise [18]. In Fig. 2, comparison is made between the ''improved'' NR scaling law

$$f_P^{\rm NR}(M_P)\propto\sqrt{\frac{1}{M_P}}\left(\frac{1}{\tilde{\alpha}_s(M_P)}\right)^{\gamma}, \qquad (14)$$

and our relativistic results

$$f_P(M_P)\propto\sqrt{\frac{1}{M_P}}\chi\left(\frac{\kappa}{M_P}\right)\left(\frac{1}{\tilde{\alpha}_s(M_P)}\right)^{\gamma}, \qquad (15)$$

where in the latter we have made the approximation $M_P\approx m_Q$ in the expression for r_Q. One can justify this approximation in view of the uncertainty in the determination of r_Q from the extrapolation to the K-meson system. Because of this assumption, r_Q does not depend on the inertia parameter Λ_q and therefore is also independent of the constant c_0. Also in Eq. (14), $\gamma=2/9$ for $N_F=3$ and $\tilde{\alpha}_s(M_P)=2\pi/9\ln(M_P/\Lambda_{\overline{\rm MS}})$ in the renormalization correction factor. In the figure, the above functions are normalized to their value at a large mass scale ($\sim 60\Lambda_{\overline{\rm MS}}$) where the physics (of $m_Q\to\infty$) is well understood. Our results are presented for $m_q=0$ and $m_q=0.175$ GeV. The main feature that distinguishes the relativistic graph from the NR scaling law is the maximum at around the D meson mass, as noted in Refs. [8,9]. We would like to emphasize again that our improved model is free of unphysical behavior for the whole range of r. Therefore, here we rigorously confirm that the maximum in the scaling law is *physical* and is due entirely to the relativistic dynamics of the light quark at the short distances. The position of this maximum is around $7.0\Lambda_{\overline{\rm MS}}$ for $m_q=0$ (for $\Lambda_{\overline{\rm MS}}=0.240$ GeV, $M_P^{\rm peak}\approx1.68$ GeV which is somewhat below $M_D\approx1.87$ GeV) and around $8.1\Lambda_{\overline{\rm MS}}$ for $m_q=0.175$ GeV (resulting $M_{P_s}^{\rm peak}\approx1.94$ GeV which is roughly the same as $M_{D_s}\approx1.96$ GeV). From Fig. 2, we also notice that, roughly speaking, from the B meson mass scale onward, $SU(3)_F$ is a good symmetry as far as the scaling of the decay constant is concerned.

The mixing parameter B arises in the theory of meson-antimeson mixing and mass difference [19] and describes the product

$$\sum_n \langle\overline{P}|V-A|n\rangle\langle n|V-A|P\rangle. \qquad (16)$$

Within our model, if the vector part of the interaction is ignored, the state n has the quantum numbers of the vacuum and this is proportional to f_P^2 and by definition the mixing parameter B is 1. However, the sum can also include intermediate states of opposite parity and the vector part of the interaction can contribute. When this is included B can be written as the integral [20]

$$B(m_Q)=\frac{12}{f_P^2M_P}N^2\int d^3\vec{r}|\Psi_Q(\vec{r})|^2[\chi^2(r)-\phi^2(r)], \qquad (17)$$

which will be approximated by

$$B(m_Q)=1-\left[\frac{\phi(r_Q)}{\chi(r_Q)}\right]^2, \qquad (18)$$

by the same argument that followed Eq. (12). The nonrelativistic scaling law for mixing parameter [Eq. (4)] is recovered for $m_Q\to\infty$ (i.e., $r_Q\to0$) as $\phi(r\to0)=0$ (see Fig. 1).

Figure 3 illustrates the evolution of the mixing parameter B with the meson mass M_P (as in the case of the decay constant, $r_Q=\kappa/m_Q$ and we use the approximation $M_P\approx m_Q$) for $m_q=0$ and $m_q=0.175$ GeV. We observe that in the subasymptotic region, deviation from the NR scaling law is 5% and 12% for the B and D meson mass scale, respectively. On the other hand, the $SU(3)_F$ symmetry breaking effects in the scaling, even for $M_D\lesssim M_P\lesssim M_B$, is negligible.

In Table I, we present our numerical predictions for the decay constant and mixing parameter of D,D_s,B,B_s mesons plus an estimate of B_K, the mixing parameter for the K meson. To test the sensitivity of these predictions, we vary the model parameters β and σ. For example, changing $\Lambda_{\overline{\rm MS}}$ from 0.240 GeV ($\beta=1.87\ {\rm GeV}^{-1}$) to 0.200 GeV ($\beta=2.24\ {\rm GeV}^{-1}$), results in a very small change (ranging from 1% to 4%) in f_P and B. The ratios $f_{B_s}/f_B=1.22$ and $f_{D_s}/f_D=1.18$ (for $\sigma=0.14\ {\rm GeV}^{-1}$) are independent of

TABLE I. The estimated decay constants (in GeV) and mixing parameters for various choices of the model parameters.

	$\sigma=0.25$	$\sigma=0.18$	$\sigma=0.14$
$\beta=1.87$	$f_D=0.519, f_B=0.427$ $f_{D_s}=0.587, f_{B_s}=0.496$ $B_D=0.87, B_B=0.95$ $B_{D_s}=0.89, B_{B_s}=0.95$ $B_K=0.58$	$f_D=0.429, f_B=0.350$ $f_{D_s}=0.497, f_{B_s}=0.417$ $B_D=0.87, B_B=0.95$ $B_{D_s}=0.89, B_{B_s}=0.95$ $B_K=0.59$	$f_D=0.371, f_B=0.301$ $f_{D_s}=0.442, f_{B_s}=0.368$ $B_D=0.88, B_B=0.95$ $B_{D_s}=0.89, B_{B_s}=0.96$ $B_K=0.60$
$\beta=2.24$	$f_D=0.514, f_B=0.418$ $f_{D_s}=0.580, f_{B_s}=0.485$ $B_D=0.88, B_B=0.95$ $B_{D_s}=0.89, B_{B_s}=0.96$ $B_K=0.59$	$f_D=0.424, f_B=0.341$ $f_{D_s}=0.491, f_{B_s}=0.406$ $B_D=0.88, B_B=0.95$ $B_{D_s}=0.90, B_{B_s}=0.96$ $B_K=0.60$	$f_D=0.367, f_B=0.292$ $f_{D_s}=0.434, f_{B_s}=0.356$ $B_D=0.88, B_B=0.95$ $B_{D_s}=0.90, B_{B_s}=0.96$ $B_K=0.61$

$\Lambda_{\overline{MS}}$. On the other hand, the sensitivity to the linear potential coefficient is much more significant. In addition to $\sigma=0.14\ \mathrm{GeV}^{-1}$, the decay constants and mixing parameters are also estimated for $\sigma=0.18, 0.25\ \mathrm{GeV}^{-1}$ [see the paragraph on the linear potential following Eq. (7)]. An increase of 30% to 40% in the decay constants is observed for increasing σ from 0.14 to $0.25\ \mathrm{GeV}^{-1}$. However, the mixing parameters are far less sensitive to this model parameter.

The results obtained for f_D all exceed the upper bound of 0.290 of Ref. [2]. However, the finite renormalization factor is uncertain at the lower mass of the D meson; if this factor, which is 1.27, is not included, f_D is 0.292 for $\sigma=0.14$. The results may therefore not be incompatible with Ref. [2] at the smaller value of σ.

A comparison between our predictions (Table I) and other theoretical results [6] also favor a smaller value for σ. Even though our results are larger than other theoretical predictions, for $\sigma=0.14\ \mathrm{GeV}^{-1}$ and taking into account the uncertainty factor 1.27 (from the finite renormalization factor below the D-meson mass scale) there are agreements with some lattice and potential model estimates. However, QCD sum rules predictions are generally smaller than ours. Our estimated B_B and B_{B_s} is on the low side of a recent lattice prediction by UKQCD [7].

In conclusion, we used a relativistic model with a phenomenological potential that accounts for QCD interactions at all length scales, to estimate the decay constant f_P and mixing parameter B of heavy-light mesons. The evolution of f_P and B with the meson mass significantly deviates from the NR scaling law in the phenomenologically interesting subasymptotic region $M_D \lesssim M_P \lesssim M_B$.

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[1] M. R. Ahmady, R. R. Mendel, and J. D. Talman, Phys. Rev. D **52**, 254 (1995).
[2] MARK III Collaboration, J. Adler *et al.*, Phys. Rev. Lett. **60**, 1375 (1988).
[3] WA75 Collaboration, S. Aoki *et al.*, Prog. Theor. Phys. **89**, 131 (1993).
[4] CLEO II Collaboration, D. Acosta *et al.*, Phys. Rev. D **49**, 5690 (1994).
[5] BES Collaboration, J. Z. Bai *et al.*, Phys. Rev. Lett. **74**, 4599 (1995).
[6] See Table IV of the review by J. D. Richman and P. R. Burchat, Rev. Mod. Phys. **67**, 893 (1995).
[7] UKQCD Collaboration, A. K. Ewing *et al.*, Phys. Rev. D **54**, 3526 (1996).
[8] R. R. Mendel and H. D. Trottier, Phys. Lett. B **231**, 312 (1989).
[9] R. R. Mendel and H. D. Trottier, Phys. Rev. D **46**, 2068 (1992).
[10] V. D. Mur, V. S. Popov, Yu. A. Simonov, and V. P. Yurov, Sov. Phys. JETP **78**, 1 (1994), and references therein.
[11] R. R. Mendel and H. D. Trottier, Phys. Rev. D **42**, 911 (1990), and references therein.
[12] W. Lucha, F. F. Schoberl, and D. Gromes, Phys. Rep. **200**, 127 (1991).
[13] K. D. Born, E. Laermann, R. Sommer, P. M. Zerwas, and T. F. Walsh, Phys. Lett. B **329**, 325 (1994).
[14] See Sec. 8.10 in D. H. Perkins, *Introduction to High Energy Physics*, 3rd ed. (Addison-Wesley, Reading, MA, 1987).
[15] H. D. Trottier, Ph.D. thesis, McGill University, 1987.
[16] J. F. Donoghue and K. Johnson, Phys. Rev. D **21**, 1975 (1980).
[17] M. B. Voloshin and M. A. Shifman, Yad. Fiz. **45**, 463 (1987) [Sov. J. Nucl. Phys. **45**, 333 (1987)].
[18] H. D. Politzer and M. B. Wise, Phys. Lett. B **206**, 682 (1988).
[19] M. K. Gaillard and B. W. Lee, Phys. Rev. D **10**, 897 (1974).
[20] R. E. Schrock and S. B. Treiman, Phys. Rev. D **19**, 2148 (1979); P. Colic *et al.*, Nucl. Phys. **B221**, 141 (1983).