Bound states and power counting in effective field theories

Michael Luke
Department of Physics, University of Toronto, 60 St. George St., Toronto, Ontario, Canada M5S 1A7

Aneesh V. Manohar
Department of Physics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0319

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The problem of bound states in effective field theories is studied. A rescaled version of nonrelativistic effective field theory is formulated which makes the velocity power counting of operators manifest. Results obtained using the rescaled theory are compared with known results from NRQCD. The same ideas are then applied to study Yukawa bound states in $1+1$ and $3+1$ dimensions, and to analyze when the Yukawa potential can be replaced by a δ-function potential. The implications of these results for the study of nucleon-nucleon scattering in chiral perturbation theory are discussed. [S0556-2821(97)03407-3]

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I. INTRODUCTION

Effective field theories are an extremely useful tool for studying the dynamics of particles at low energies. An effective Lagrangian typically has an expansion in inverse powers of some mass scale M, and describes dynamics at momentum scales which are much smaller than M. For example, heavy quark effective theory (HQET) [1–3] describes the dynamics of hadrons containing a single heavy quark of mass m_Q at momentum transfers much smaller than m_Q. The HQET Lagrangian has an expansion in inverse powers of m_Q, and is used to compute hadronic properties as an expansion in $\Lambda_{\rm QCD}/m_Q$, where $\Lambda_{\rm QCD}\sim 300$ MeV is a typical strong interaction scale. The scale at which the HQET Lagrangian ceases to be useful is the mass m_Q of the heavy quark. HQET can be used to study the interaction of a single heavy quark with light degrees of freedom, provided the momentum transfer is small compared with m_Q.

Systems containing a heavy quark and antiquark (or two heavy quarks) cannot be described by HQET. As $m_Q\rightarrow\infty$, the quark and antiquark form a Coulomb bound state of size $1/m_Q\alpha_s$ and typical momentum transfer $p\sim\alpha_s m_Q$. Perturbation theory is infrared divergent, with terms of the form $(\alpha/v)^n$, where v is the relative velocity of the Q and $\bar{Q}$ in the center-of-mass frame. These infrared singular terms cause a breakdown of perturbation theory, and must be resummed. Resummation of the most singular $(\alpha/v)^n$ terms is equivalent to solving the Schrödinger equation in a Coulomb potential, and the resummed $Q\bar{Q}$ scattering amplitude contains the corresponding bound state poles. One can construct a different effective field theory, nonrelativistic QCD (NRQCD), which is appropriate for the study of $Q\bar{Q}$ bound states in QCD (or its QED analogue, NRQED for the study of positronium) [4,5]. The terms in the NRQCD Lagrangian are of the same form as those for HQET, but NRQCD has a different power-counting scheme than HQET. In HQET, the power counting of operators is manifest. An operator with coefficient $1/m_Q^r$ has a matrix element of order $(\Lambda_{\rm QCD}/m_Q)^r$. Thus, the quark kinetic energy operator $Q^\dagger(\mathbf{D}^2/2m_Q)Q$ is of order $\Lambda_{\rm QCD}/m_Q$, and is subleading. The power counting is more subtle in NRQCD; $Q^\dagger(\mathbf{D}^2/2m_Q)Q$ is treated as a leading order operator, and higher dimensional operators are suppressed not by powers of $\Lambda_{\rm QCD}/m_Q$ but by powers of the relative three-velocity of the heavy quarks. We will refer to theories such as HQET, in which $Q^\dagger(iD^0)Q$ is of leading order but $Q^\dagger(\mathbf{D}^2/2m_Q)Q$ is small, as static theories, and will refer to theories such as NRQCD, in which $Q^\dagger(iD^0)Q$ and $Q^\dagger(\mathbf{D}^2/2m_Q)Q$ are both of leading order, as nonrelativistic theories.

In this paper, we study the problem of nonrelativistic bound states in effective field theories. In Sec. II, we define a rescaled version of NRQCD (RNRQCD) in which the power counting of operators is manifest. Most of the results of RNRQCD follow trivially from well-known results in NRQCD, but there are a few advantages, which are discussed in Sec. II. In Sec. III we analyze nonrelativistic Yukawa bound states due to the exchange of a massive scalar, in both $1+1$ and $3+1$ dimensions, and compare the results to those in an effective theory in which the scalar is integrated out and replaced by a four-fermion interaction. Naively, the momentum expansion of the four-fermion effective theory has a convergence radius equal to the mass of the scalar, but this is not necessarily the case when there are weakly bound states. We study whether the effective theory can ''produce its own bound state,'' that is, whether one can obtain a composite weakly bound state in an effective theory where the higher dimensional operators have small coefficients. We show that in $1+1$ dimensions, the effective four-fermion theory has a radius of convergence of order the scalar mass, and produces a weakly bound state with perturbative coefficients for higher dimensional operators when the parameters of the Yukawa theory are such that there is one weakly bound state in the spectrum. If the Yukawa theory has two or more bound states, of which one is weakly bound, the radius of convergence of the four-fermion effective theory vanishes as the weakly bound state approaches threshold, and the higher dimension operators have large coefficients. In $3+1$ dimensions, the four-fermion effective theory has a finite radius of convergence and higher dimensional operators with small coefficients only when there is no bound state near threshold in the Yukawa theory.

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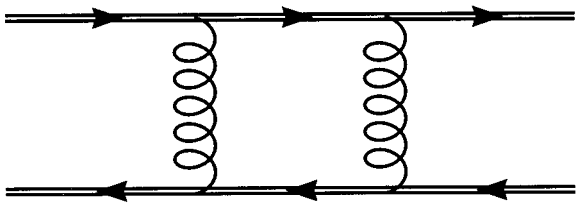

FIG. 1. The box graph in the $Q\bar{Q}$ sector. The double lines represent nearly on-shell nonrelativistic quarks and antiquarks, and the curly lines represent gluons.

We discuss the implications of our analysis for recent attempts to describe nonperturbative aspects of nucleon-nucleon scattering by solving the Schrödinger equation for a chiral Lagrangian in which the nucleon-nucleon interaction arises from both pions exchange and contact terms [6–9]. We present our conclusions in Sec. IV.

II. POWER COUNTING IN NONRELATIVISTIC QCD

The HQET Lagrangian at leading order in the $1/m_Q$ expansion is

$$\mathcal{L}=Q^\dagger(iD^0)Q, \tag{2.1}$$

where Q is the annihilation field for a nonrelativistic quark. It is well known that the Lagrangian, Eq. (2.1), cannot be used for systems containing two heavy quarks. The basic problem arises from the Feynman graph in Fig. 1, when both the intermediate fermions are simultaneously almost on shell. The box graph evaluated in QCD [with propagators $1/(\not{p}+m)$] has terms which are of order $m_Q/|\mathbf{k}|=1/|\mathbf{v}|$, where $\mathbf{k}$ and $\mathbf{v}$ are the three-momentum and velocity of the external quarks, respectively. The box diagrams in HQET cannot reproduce this behavior since the Feynman rules are independent of m_Q. As a result, QCD cannot be matched onto HQET in the $Q\bar{Q}$ sector. The box graph in HQET has a loop integral of the form

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^0+i\epsilon}\frac{1}{-k^0+i\epsilon}. \tag{2.2}$$

The k^0 integral has a pinch singularity, and is divergent. This singularity is a signal that higher dimensional operators in HQET are important in the $Q\bar{Q}$ sector. One constructs a different effective field theory, NRQCD, with the leading order quark Lagrangian

$$\mathcal{L}=Q^\dagger(iD^0)Q+Q^\dagger\left(\frac{\mathbf{D}^2}{2m_Q}\right)Q+\cdots, \tag{2.3}$$

in which $\mathbf{D}^2/2m_Q$ is considered to be of the same order as D^0. The box graph in NRQCD has a loop integral of the form

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^0-|\mathbf{k}|^2/2m_Q+i\epsilon}\frac{1}{-k^0-|\mathbf{k}|^2/2m_Q+i\epsilon}\cdots, \tag{2.4}$$

instead of Eq. (2.2). The k^0 integral is now finite, and gives

$$-i\int\frac{d^3k}{(2\pi)^3}\frac{m_Q}{|\mathbf{k}|^2}\cdots, \tag{2.5}$$

which, when the gluon propagators are included, reproduces the $m_Q/|\mathbf{k}|$ enhancement of the box graph in QCD.

HQET has a simple power-counting scheme in which D^μ is of order $\Lambda_{\rm QCD}$. The importance of various operators is then manifest from the Lagrangian. An operator in the Lagrangian of dimension $4+r$ has a coefficient of order $1/m_Q^r$ and is of relative order $(\Lambda_{\rm QCD}/m_Q)^r$. It is also trivial to count powers of $1/m_Q$ in loop graphs. The quark and gluon propagators are independent of m_Q, and so any Feynman graph with V vertices of order $1/m_Q^{r_1}\cdots 1/m_Q^{r_V}$ has an overall factor of $1/m_Q^r$ where $r=\Sigma_{i=1}^V r_i$. NRQCD has a more complicated power-counting scheme, which is discussed in detail in Ref. [5]. In NRQCD, both D^0 and $\mathbf{D}^2/2m_Q$ are of the same order, and so the quark propagator is $i/(k^0-\mathbf{k}^2/2m_Q+i\epsilon)$. The quark propagator depends on m_Q, so that one can get factors of m_Q from loop graphs, and the power-counting rules for loop graphs in NRQCD are not as straightforward as in HQET. Furthermore, there are several relevant scales in NRQCD: m_Q, the three-moment of the heavy quarks m_Qv, the kinetic energy of the heavy quarks $m_Qv^2/2$, and $\Lambda_{\rm QCD}$. The matrix elements of higher dimensional operators in NRQCD are suppressed by powers of v, but the v counting is not manifest in the Lagrangian.

It is advantageous to have an effective field theory with manifest power-counting rules. One can achieve this for NRQCD by rescaling the fields and coordinates of the usual NRQCD Lagrangian. In a nonrelativistic system, E and p are of order m_Qv^2 and m_Qv, respectively. Therefore, it is useful to rescale the coordinates so that these are the natural sizes of the energy and momentum. Define new coordinates $\mathbf{X}$ and T, and new fields Ψ, $\mathcal{A}^0$ and $\mathcal{A}$ by

$$\mathbf{x}=\lambda_x\mathbf{X},\quad t=\lambda_tT,\quad Q=\lambda_Q\Psi,\quad A^0=\lambda_{A^0}\mathcal{A}^0,\quad \mathbf{A}=\lambda_{\mathbf{A}}\mathcal{A}, \tag{2.6}$$

where $\lambda_x=1/m_Qv$. Requiring ∂^0 and $\nabla^2/2m_Q$ to be both of the same order determines $\lambda_t=m_Q\lambda_x^2=1/m_Qv^2$. This gives the relation between the rescaled energy and momentum K^0 and $\mathbf{K}$ and the original variables k^0 and $\mathbf{k}$:

$$K^0=k^0/m_Qv^2,\quad \mathbf{K}=\mathbf{k}/m_Qv. \tag{2.7}$$

In a nonrelativistic system the rescaled energy and momentum are both of order unity.

Upon rescaling, the Lagrangian density picks up an overall factor of $\lambda_x^3\lambda_t$ from the change of integration variables $d^3x\,dt\to\lambda_x^3\lambda_t d^3X\,dT$, so the $\Psi^\dagger(iD^0)\Psi$ term is canonically normalized if $\lambda_Q=\lambda_x^{-3/2}$. The gauge field quadratic terms get rescaled to

$$(\nabla\times\mathbf{A})^2\rightarrow m_Q\lambda_x^3\lambda_A^2(\nabla\times\mathcal{A})^2,$$

$$(\nabla A^0)^2\rightarrow m_Q\lambda_x^3\lambda_{A^0}^2(\nabla\mathcal{A}^0)^2,$$

$$\left(\frac{\partial\mathbf{A}}{\partial t}\right)^2\rightarrow\frac{\lambda_x\lambda_A^2}{m_Q}\left(\frac{\partial\mathcal{A}}{\partial t}\right)^2,$$

$$\frac{\partial\mathbf{A}}{\partial t}\cdot\boldsymbol{\nabla}A^0\rightarrow\lambda_x^2\lambda_{A^0}\lambda_A\frac{\partial\mathbf{A}}{\partial t}\cdot\boldsymbol{\nabla}A^0. \tag{2.8}$$

In the infrared, $\lambda_x\rightarrow\infty$, and the dominant gauge kinetic terms are $(\nabla\times\mathbf{A})^2$ and $(\nabla A^0)^2$. These terms are properly normalized if

$$\lambda_A=\lambda_{A^0}=(m_Q\lambda_x^3)^{-1/2}. \tag{2.9}$$

With this rescaling the NRQCD Lagrangian becomes

$$\begin{aligned}\mathcal{L}^R=&\Psi^\dagger\left(i\partial_0-\frac{g}{\sqrt{v}}\mathcal{A}_0^aT^a\right)\Psi-\frac{1}{2}\Psi^\dagger(i\nabla-g\sqrt{v}\mathcal{A}^aT^a)^2\Psi\\&-\frac{1}{4}(\partial_i\mathcal{A}_j^a-\partial_j\mathcal{A}_i^a-g\sqrt{v}f_{abc}\mathcal{A}_i^b\mathcal{A}_j^c)^2\\&+\frac{1}{2}(\partial_i\mathcal{A}_0^a-v\partial_0\mathcal{A}_i^a-g\sqrt{v}f_{abc}\mathcal{A}_i^b\mathcal{A}_0^c)^2\\=&\Psi^\dagger\left(i\partial_0+\frac{\nabla^2}{2}-\frac{g}{\sqrt{v}}\mathcal{A}_0^aT^a\right)\Psi-\frac{1}{4}(\partial_i\mathcal{A}_j^a-\partial_j\mathcal{A}_i^a)^2\\&+\frac{1}{2}(\partial_i\mathcal{A}_0^a)^2+O(v,g\sqrt{v}),\end{aligned} \tag{2.10}$$

which will be referred to as the RNRQCD (rescaled NRQCD) Lagrangian. (The effective Lagrangian also contains the corresponding terms for the heavy antiquark field.)

It is clear from the form of the RNRQCD Lagrangian that the effective coupling constant in the $\Psi\Psi$ sector for Coulomb gluons (i.e., $\mathcal{A}^0$) is α/v, not α. At low velocities, the Coulomb gluon interaction must be summed to all orders. Transverse gluons have a coupling constant of order $g\sqrt{v}$, and decouple as $v\rightarrow0$. Loop integrals with Coulomb gluons are independent of g, v, and m_Q, and so the v dependence of a graph may be easily read off from the vertex factors. There are no hidden enhancements factors from loop graphs.[1] This is in contrast with the usual formulation of NRQCD, where $1/v$ enhancements arise due to factors of $m_Q/|\mathbf{k}|$ in the loop graphs. For nonrelativistic bound states, v is of order α, and the Coulomb interaction in Eq. (2.10) becomes strongly coupled and must be summed to all orders. Other interactions, such as those due to transverse gluons, are suppressed by powers of v and may be treated using perturbation theory. These well-known results follow simply from the scaling of the various terms in the rescaled Lagrangian, Eq. (2.10). The derivation using the original NRQCD Lagrangian is more involved [5,10].

The effective Lagrangian, Eq. (2.10), contains additional higher dimensional operators which are also suppressed by factors of v. Terms in the effective Lagrangian are relevant, irrelevant, or marginal, depending on whether the power of v in the coefficient is negative, positive or zero.[2] Note that in RNRQCD the fields and derivatives are all dimensionless, and so terms with additional fields or derivatives are not suppressed unless they appear with additional powers of v. For example, the operator

$$\frac{1}{m_Q^3}\psi^\dagger\nabla^4\psi \tag{2.11}$$

in NRQCD becomes, in the rescaled theory,

$$\frac{1}{m_Q^3}\frac{m_Q\lambda_x^5\lambda_Q^2}{\lambda_x^4}\Psi^\dagger\nabla^4\Psi=v^2\overline{\Psi}\nabla^4\Psi, \tag{2.12}$$

and is of order v^2, which agrees with the power counting in Ref. [5]. The relation between our power counting and that of Ref. [5] is slightly more subtle for operators containing $\vec{E}$ and $\vec{B}$ fields. For example, the chromomagnetic moment operator

$$\frac{g}{m_Q}\psi^\dagger\sigma^{\mu\nu}G_{\mu\nu}\psi \tag{2.13}$$

becomes, in the rescaled theory,

$$\frac{g}{m_Q}m_Q\lambda^4\lambda_Q^2\lambda_A\Psi^\dagger\sigma^{\mu\nu}\mathcal{G}_{\mu\nu}\Psi=g\sqrt{v}\Psi^\dagger\sigma^{\mu\nu}\mathcal{G}_{\mu\nu}\Psi, \tag{2.14}$$

whereas in the power counting of Ref. [5] this operator is of order v^2. However, the NRQCD power counting refers to the size of the matrix element of the operator in a quark-antiquark state. The chromomagnetic gluon must therefore be attached to one of the external quark lines, which costs an additional power of $g\sqrt{v}$; hence, the matrix element of the operator Eq. (2.14) is of order $g^2v\sim v^2$ in a quarkonium state, as expected.

A. Tranverse gluons

The quark propagator in RNRQCD is

$$\frac{i}{K^0-|\mathbf{K}|^2/2+i\epsilon}, \tag{2.15}$$

and the propagator for A^0 is

$$\frac{i}{|\mathbf{K}|^2}, \tag{2.16}$$

[1] v counting for transverse gluons is more complicated, and is discussed in Sec. II A.

[2] This is in the renormalization group sense. Irrelevant operators inserted in loop graphs which are sufficiently divergent can produce effects that do not vanish as $v\rightarrow0$.

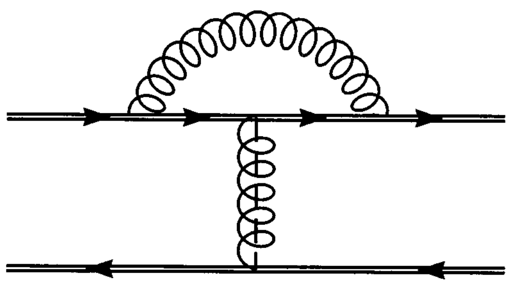

FIG. 2. Radiative correction to Coulomb scattering. The gluon with a dashed line is a Coulomb gluon.

which are both independent of v. The transverse gluon propagator is

$$-i\left(\delta_{ij}-\frac{K_iK_j}{|\mathbf{K}|^2}\right)\frac{1}{|\mathbf{K}|^2-v^2(K^0)^2}, \tag{2.17}$$

which depends on v, and so there are potential $1/v$ enhancements from graphs with internal transverse gluons. A generic loop integral can be evaluated by first doing the K^0 integration using residues. The residue of the transverse gluon propagator at the K^0 pole $|\mathbf{K}|/v$ is $-1/v|\mathbf{K}|$, which is enhanced by $1/v$. The transverse gluon propagator contribution from other poles (such as fermion poles) is typically of order unity, since these poles are at values of K^0 of order unity. Thus transverse gluon loops can have $1/v$ enhancements from regions in the momentum integral where the transverse gluon is on shell. Consider, for example, the graph in Fig. 2, where the gluon in the loop is a transverse gluon. The diagram has a $1/v$ enhancement from the region in loop integral where the fermion and virtual gluon are on shell, i.e., from physical tranverse gluon radiation.

The transverse gluon coupling constant is $g\sqrt{v}$, which is a factor of v smaller than that for Coulomb gluons. Thus transverse gluon loops obey naive v counting, and are v^2 suppressed, unless the cut part of the graph contributes to transverse gluon radiation. In the latter case, the transverse gluon graph has a $1/v$ enhancement over the naive v counting, and is only suppressed by one power of v.

It is not possible to choose a rescaling scheme which has a manifest v-counting scheme for real and virtual transverse gluons. The reason is that quarks behave like nonrelativistic particles when transverse gluons are exchanged between them, but like static particles when one of them radiates an on-shell transverse gluon.

B. Coulomb scattering

At leading order in v, the only diagrams which contribute to $Q\bar{Q}$ scattering in the effective theory are the Coulomb ladder graphs of Fig. 3, where the $\mathcal{A}^0$ propagators are denoted by gluons with dashed lines. Crossed ladder graphs such as Fig. 4 vanish when both gluons are Coulomb gluons. The one-loop box graph has an integral of the form

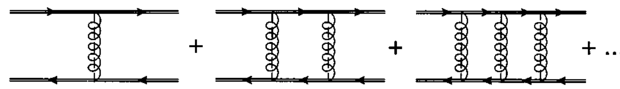

FIG. 3. The leading order contribution to $Q\bar{Q}$ scattering. The gluons with dashed lines represent Coulomb gluons.

$$\int\frac{dK^0}{(2\pi)}\frac{d^3\mathbf{K}}{(2\pi)^3}\frac{1}{|\mathbf{K}|^2}\frac{1}{|\mathbf{K}+\mathbf{P}_1'-\mathbf{P}_1|^2}\times\frac{1}{(P_1-K)^0-|\mathbf{P}_1-\mathbf{K}|^2/2+i\epsilon}\times\frac{1}{(P_2+K)^0-|\mathbf{P}_2+\mathbf{K}|^2/2+i\epsilon}. \tag{2.18}$$

The K^0 integral can be done by contour integration to give

$$\int\frac{dK^0}{(2\pi)}\frac{1}{(P_1-K)^0-|\mathbf{P}_1-\mathbf{K}|^2/2+i\epsilon}\times\frac{1}{(P_2+K)^0-|\mathbf{P}_2+\mathbf{K}|^2/2+i\epsilon}=-\frac{i}{(P_1+P_2)^0-|\mathbf{P}_1-\mathbf{K}|^2/2-|\mathbf{P}_2+\mathbf{K}|^2/2}, \tag{2.19}$$

which is the Schrödinger Green function of nonrelativistic quantum mechanics. The ladder graphs of Fig. 3 can be evaluated by first doing the K^0 loop integrals. It is easy to see that each loop gives the Schrödinger Green function for the intermediate two-fermion state. The sum of all the ladder graphs gives the nonrelativistic Schrödinger equation in momentum space for a Coulomb potential. This result is, of course, well known. What is different here is that in the rescaled theory, the sum of the leading order graphs is identical to the Schrödinger equation. While this is also the case for NRQCD in the Coulomb gauge, the rescaling allows similar results to be obtained in nongauge theories, such as Yukawa theory (which we will discuss at length in the next section), where there is no gauge freedom in choosing the propagator.

C. Heavy-light systems

One can see explicitly why $\mathbf{D}^2/2m_Q$ is a relevant operator for heavy-heavy bound states, but not for heavy-light bound states. Consider a nonrelativistic bound state of two particles with different masses m_H and m_L, with $m_H \gg m_L$. Applying the rescaling, Eq. (2.9), with $m_Q\to m_L$, one finds that the fermion kinetic terms are

$$\psi_L^\dagger i\partial^0\psi_L+\frac{1}{2}\psi_L^\dagger\nabla^2\psi_L \tag{2.20}$$

for the light fermion and

$$\psi_H^\dagger i\partial^0\psi_H+\frac{m_L}{2m_H}\psi_H^\dagger\nabla^2\psi_H \tag{2.21}$$

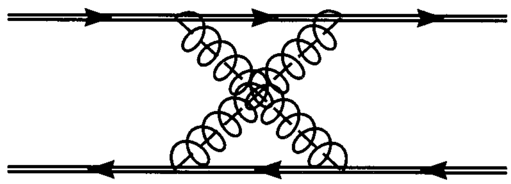

FIG. 4. The crossed box graph.

for the heavy fermion. The $\psi^\dagger\nabla^2\psi$ operator is comparable to $\psi^\dagger\partial^0\psi$ for the light particle, but is smaller by m_L/m_H for the heavier particle. The heavier particle can be treated as a static source (as in HQET), but the lighter particle must be treated as nonrelativistic (as in NRQCD).

D. Scaling dimensions

The velocity scaling rules have a renormalization group interpretation. In the nonrelativistic limit, the scaling dimensions of space and time should be chosen to be

$$[\mathbf{x}]=-1,\quad [t]=-2, \tag{2.22}$$

instead of the usual choice in relativistic theories,

$$[\mathbf{x}]=-1,\quad [t]=-1. \tag{2.23}$$

With the choice Eq. (2.22), one finds that

$$[\mathcal{L}]=5, \tag{2.24}$$

$$[\psi]=3/2,\quad [\mathcal{A}^0]=3/2,\quad [\mathcal{A}]=3/2,\quad [k^0]=2,\quad [\mathbf{k}]=1. \tag{2.25}$$

Operators are relevant, irrelevant, or marginal depending on whether their dimensions are less than, greater than, or equal to $[\mathcal{L}]=5$. For example, the Coulomb interaction term $\Psi^\dagger A^0\Psi$ has dimension $9/2=5-1/2$, and so the Coulomb interaction is relevant, and of order $v^{-1/2}$. The transverse gluon interaction term is $\Psi^\dagger\mathbf{p}\cdot\mathbf{A}\Psi$ and has dimension $11/2=5+1/2$, and so the term is irrelevant, and of order $v^{1/2}$. This agrees with the powers of v in the Lagrangian, Eq. (2.10).

III. POWER COUNTING IN CHIRAL PERTURBATION THEORY

There has been much recent interest in applying the techniques of nonrelativistic effective theories to nucleon-nucleon scattering [6–9]. The goal is to describe low-energy nucleon-nucleon scattering using a chiral Lagrangian where massive excitations (such as ρ's and ω's) have been integrated out and replaced by an effective four point interaction. Nonperturbative effects, such as the large scattering length in the $^1S_0$ channel or the deuteron in the $^3S_1$ channel, could be described by solving the Schrödinger equation in the effective theory.

The rescaling of the previous section can be extended to chiral perturbation theory for heavy nucleons [11,6,9]. The leading terms in the nucleon-pion chiral Lagrangian have the form

$$\mathcal{L}=\psi^\dagger\left(i\partial^0+\frac{\nabla^2}{2M}\right)\psi-\frac{g}{f_\pi}\nabla\phi\cdot\psi^\dagger\sigma\psi+\frac{1}{2}(\partial^0\phi)^2 -\frac{1}{2}(\nabla\phi)^2-\frac{c_1}{f_\pi^2}(\psi^\dagger\psi)^2-\frac{c_2}{f_\pi^2}(\psi^\dagger\sigma\psi)^2, \tag{3.1}$$

describing the interaction of a pseudoscalar Goldstone boson ϕ with decay constant f_π and coupling g, where we have suppressed flavor indices. The leading effects of massive excitations are contained in the dimension-6 operators $(\psi^\dagger\psi)^2$ and $(\psi^\dagger\sigma\psi)^2$ with dimensionless coefficients c_1 and c_2. In $3+1$ dimensions the appropriate rescaling is

$$\mathbf{x}=\lambda_x\mathbf{X},\quad t=M\lambda_x^2T,\quad \psi=\lambda_x^{-3/2}\Psi,\quad \phi=(M\lambda_x^3)^{-1/2}\Phi, \tag{3.2}$$

giving the rescaled Lagrangian

$$\mathcal{L}=\Psi^\dagger\left(i\partial^0+\frac{\nabla^2}{2}\right)\Psi-\frac{gM\sqrt{v}}{f_\pi}\nabla\Phi\cdot\Psi^\dagger\sigma\Psi +\frac{v^2}{2}(\partial^0\Phi)^2-\frac{1}{2}(\nabla\Phi)^2-\frac{c_1M^2v}{f_\pi^2}(\Psi^\dagger\Psi)^2 -\frac{c_2M^2v}{f_\pi^2}(\Psi^\dagger\sigma\Psi)^2. \tag{3.3}$$

Both the derivative interaction and the dimension-6 operators are irrelevant in the infrared. Each pion exchange in a Feynman diagram contributes a factor of g^2M^2v/f_π^2 which has the same dependence of M and v as an insertion of one of the dimension-6 four-nucleon operators. This reproduces the power counting of Ref. [6]: A graph with n insertions of dimension-6 operators and m ladder pion exchanges scales as

$$\left(\frac{M^2v}{f_\pi^2}\right)^{n+m}=\left(\frac{MQ}{f_\pi^2}\right)^{n+m}, \tag{3.4}$$

where $Q=Mv$ is the three-momentum of the scattering nucleons (in the center-of-mass frame). Furthermore, with this rescaling the power counting of Ref. [9], in which poles in k^0 in the pion propagator were treated as higher order in the nonrelativistic expansion, is manifest. Since pion emission is kinematically forbidden as $v\to 0$, the temporal piece of the pion kinetic term may be treated as a higher-order insertion.

Since the pion-nucleon interaction terms are irrelevant in the infrared, unlike the interactions in QED and QCD, perturbation theory for nucleon-nucleon scattering does not break down at threshold for weak coupling. Multiloop bubble graphs are not enhanced by powers of $1/v$, but instead suppressed by powers of v. It was argued in Refs. [6–9] that the large scattering lengths in the $^1S_0$ channel and the deuteron in the $^3S_1$ channel in nucleon-nucleon scattering are signs of the breakdown of perturbation theory, due to the fact that for large M the factor in Eq. (3.4) is not small. It was proposed by these authors that the appropriate description of nucleon-nucleon scattering could be obtained by summing all terms of order $(QM)^n$; corrections to this would be suppressed by powers of Q/M, and could be calculated systematically.

However, as noted in Ref. [9], the validity of this approach rests on the assumption that coefficients of higher dimensional four-nucleon operators in the chiral Lagrangian with n spatial derivatives are smaller than M^n. Otherwise, the effective theory description would break down, since the contributions from higher dimensional operators with unknown coefficients would be as large as the graphs which are being summed in the effective theory. In [9] it was argued that this assumption was consistent in the ${}^1S_0$ (but not in the ${}^3S_1$) channel, since no divergences were found in the Feynman diagrams contributing to ${}^1S_0$ channel scattering which would require counterterms scaling like $(QM)^n$.

In this section we will investigate the validity of this power counting in a simplified model. We will neglect the Goldstone bosons, and simply consider a theory of a nonrelativistic fermion of mass M coupled to a scalar of mass $m \ll M$,

$$\mathcal{L}_Y = \psi^\dagger\left(i\partial^0 + \frac{\nabla^2}{2M}\right)\psi - g\phi\psi^\dagger\psi + \frac{1}{2}(\partial^0\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{3.5}$$

The scalar plays the role of the ρ, ω, and other excitations. At low momenta, $p \ll m$, the scalar may be integrated out, resulting in an effective four-fermion theory

$$\mathcal{L}_\delta = \psi^\dagger\left(i\partial^0 + \frac{\nabla^2}{2M}\right)\psi + h(\psi^\dagger\psi)^2 + \cdots, \tag{3.6}$$

where the ellipsis represents higher dimensional operators, suppressed by powers of p/m. The Yukawa potential arising from scalar exchange in Eq. (3.5) is replaced by a series of δ functions and their derivatives in the effective theory. We will refer to these two theories as NRY and NRδ for nonrelativistic Yukawa and nonrelativistic δ function, respectively.

We now wish to ask whether NRδ correctly describes the $\psi\psi$ scattering amplitude of NRY at low momentum $p \leq p_{\max}$, in a parameter regime where the Born approximation fails. By low momentum we mean $p \leq p_{\max} \ll m$, where $p_{\max}$ is held fixed as one varies the parameters in the Yukawa theory. This is analogous to the question of whether the dimension-6 operators in Eq. (3.3) correctly describe nucleon-nucleon scattering at momenta much smaller than m_ρ, in a regime where the Born approximation fails. We consider the problem in both $1+1$ and $3+1$ dimensions. In $1+1$ dimensions, we will show that NRδ correctly describes the scattering for weak coupling; however, for a coupling large enough that more than one bound state exists the higher dimensional operators in NRδ become important, and the effective theory breaks down below $p_{\max}$. In this case, the full theory is required to correctly describe the low-energy physics. In $3+1$ dimensions, there are no bound states for sufficiently weak coupling. In this case NRδ correctly describes the scattering and no resummation of ladder graphs is necessary. However, when the coupling is strong enough to form bound states, the situation is analogous to the $(1+1)$-dimensional case with excited states. Once again the higher dimensional operators become important, and again the effective theory description breaks down below $p_{\max}$. We find by explicit calculation that there are, in fact, higher dimensional operators in the low-energy theory which scale like $(QM)^n$, and spoil the power counting of Refs. [6–9]. We will comment on the relation between our results and the effective range expansion, as well as the case where $p_{\max}$ is allowed to vary, in Sec. III C.

A. 1+1 dimensions

In $1+1$ dimensions, the mass dimensions of the fields and couplings in Eq. (3.5) are

$$[\psi]=1/2, \quad [\phi]=0, \quad [M]=1, \quad [m]=1, \quad [g]=1. \tag{3.7}$$

The Yukawa potential due to scalar exchange is

$$V(x) = -g^2\int\frac{dq}{(2\pi)}\frac{e^{iqx}}{q^2+m^2} = -\frac{g^2}{2m}e^{-m|x|}. \tag{3.8}$$

An attractive potential in $1+1$ dimensions always has a bound state. The Schrödinger equation for a Yukawa potential is not analytically soluble. However, when g is small, the state is weakly bound and spread out over a large region. In this limit the Yukawa potential can be approximated by the δ function,

$$-\frac{g^2}{2m}e^{-m|x|} \to -\frac{g^2}{m^2}\delta(x). \tag{3.9}$$

The δ-function potential has a bound state with energy $E = -g^4M/4m^4$, and wave function

$$\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}, \quad \kappa = \frac{g^2M}{m^2}. \tag{3.10}$$

Corrections to the δ-function result will be suppressed by powers of κ/m, the ratio of the size of the state to the range of the potential. For $\kappa/m \sim 1$, excited bound states appear, and the δ-function approximation completely breaks down.

In effective field theory language, this indicates that the nonrelativistic δ (NRδ) potential will only correctly describe bound states for weak coupling, $\kappa/m \ll 1$, and will break down at larger values of the coupling. Since an excited state may be arbitrarily close to threshold, the effective theory may therefore break down for $\psi\psi$ scattering at arbitrarily low energies, which is not the usual behavior one expects from a low-energy effective theory, where the importance of higher dimensional operators is set by powers of p/m. In this case, the effects of higher dimensional operators must be suppressed instead by powers of κ/m. Therefore, it is instructive to analyze this problem by matching the nonrelativistic Yukawa (NRY) onto the NRδ potential and seeing how the scale κ enters the problem.

To make the power counting manifest we use a rescaled Lagrangian as in Sec. II. In $1+1$ dimensions, the appropriate rescaling is

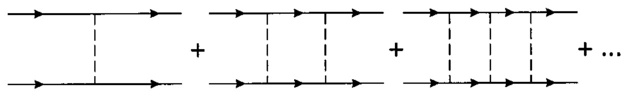

FIG. 5. Ladder graphs in the Yukawa theory.

$$x=\lambda_x X,\quad t=M\lambda_x^2 T,\quad \psi=\lambda_x^{-1/2}\Psi,\quad \phi=(M\lambda_x)^{-1/2}\Phi, \tag{3.11}$$

where $\lambda_x=1/Mv$, giving

$$\mathcal{L}_Y^R=\Psi^\dagger\left(i\partial^0+\frac{\nabla^2}{2}\right)\Psi-\frac{g}{M}v^{-3/2}\Phi\Psi^\dagger\Psi+\frac{v^2}{2}(\partial^0\Phi)^2 -\frac{1}{2}(\nabla\Phi)^2-\frac{1}{2}\frac{m^2}{M^2}v^{-2}\Phi^2. \tag{3.12}$$

The powers of v can also be obtained using modified scaling rules, as in Sec. II D. The modified scaling dimensions are

$$[x]=-1,\quad [t]=-2,\quad [\mathcal{L}]=3,\quad [\psi]=1/2,\quad [\phi]=1/2. \tag{3.13}$$

$[\psi^\dagger\psi\phi]-[\mathcal{L}]=-3/2$ and $[\phi^2]-[\mathcal{L}]=-2$, so that these terms scale as $v^{-3/2}$ and v^{-2}, respectively. Since the temporal piece of the scalar kinetic term in Eq. (3.12) is of order v^2, it may be neglected at the order at which we are working. Thus RNRY (rescaled NRY) describes a fermion coupled to a static scalar field, with coupling constant $gv^{-3/2}/M$. The Φ interactions must be summed to all orders, since the Yukawa coupling is large. Because the Φ propagator has no poles in K^0 (emission of scalars is kinematically forbidden as $v\to 0$), only ladder graphs are nonvanishing in the nonrelativistic theory. For simplicity we consider the scattering of two off-shell fermions with energy E and zero momentum, since the scattering amplitude will have the same singularities as for on-shell states. We will also treat the ψ's as distinguishable, since the additional graphs from Fermi statistics are not relevant to our arguments. The tree-level Φ exchange graph in the ladder sum in Fig. 5 gives

$$I_0^Y=i\frac{g^2}{M^2v^3}\frac{1}{\mu^2}=i\frac{g^2}{m^2v}, \tag{3.14}$$

where $\mu^2=m^2/M^2v^2$. The box graph gives

$$\begin{aligned}I_1^Y&=-i\frac{g^4}{M^4v^6}\int\frac{dK}{(2\pi)}\frac{1}{(K^2+\mu^2)^2}\frac{1}{(P_1+P_2)^0-K^2}\\&=i\frac{g^4}{M^4v^6}\frac{2\mu+\sqrt{\varepsilon}}{4\mu^3\sqrt{\varepsilon}(\mu+\sqrt{\varepsilon})^2}\\&=i\frac{g^4}{2m^4v^2\varepsilon^{1/2}}-i\frac{3g^4M}{4m^5v}+O(v^0),\end{aligned} \tag{3.15}$$

where $-\varepsilon=(P_1+P_2)^0$ is the total rescaled energy of the incoming particles. The two-loop box graph gives

$$\begin{aligned}I_2^Y&=i\frac{g^6}{M^6v^9}\int\frac{dK}{(2\pi)}\int\frac{dL}{(2\pi)}\frac{1}{(K^2+\mu^2)}\frac{1}{(L^2+\mu^2)}\\&\quad\times\frac{1}{(K-L)^2+\mu^2}\frac{1}{(P_1+P_2)^0-K^2}\frac{1}{(P_1+P_2)^0-L^2}\\&=i\frac{g^6}{M^6v^9}\frac{6\mu^2+9\mu\sqrt{\varepsilon}+2\varepsilon}{12\mu^4\varepsilon(\mu+\sqrt{\varepsilon})^2(\mu+2\sqrt{\varepsilon})(2\mu+\sqrt{\varepsilon})}\\&=i\frac{g^6}{4m^6v^3\varepsilon}-i\frac{3g^6M}{4m^7v^2\sqrt{\varepsilon}}+i\frac{41g^6M^2}{24m^8v}+O(v^0).\end{aligned} \tag{3.16}$$

The $O(1/v)$ term from the box graph, Eq. (3.15), is down by a factor of κ/m relative to the $O(1/v)$ term due to tree-level exchange, and so may be neglected in the weak coupling limit, $\kappa\ll m$. In this limit, one may sum the most singular contributions of the graphs in Fig. 5 to obtain, for the ladder sum,

$$\begin{aligned}i\mathcal{A}&=i\frac{g^2}{m^2v}\left[1+\frac{g^2\varepsilon^{-1/2}}{2m^2v}+\left(\frac{g^2\varepsilon^{-1/2}}{2m^2v}\right)^2+\cdots\right]\\&=i\frac{g^2}{m^2v}\frac{1}{1-g^2\varepsilon^{-1/2}/(2m^2v)},\end{aligned} \tag{3.17}$$

which has a pole at $\varepsilon=g^4/(4m^4v^2)$. Rescaling back to physical units gives $E=-g^4M/(4m^4)$, which is the correct bound state energy for $\kappa/m\ll 1$. There is always at least one bound state pole, even for weak coupling, because the box graph diverges as $v\to 0$.

When κ/m is not small, the ladder sum is no longer a geometric series, since an n-loop ladder graph has singularities of the form

$$\left(\frac{\kappa}{m}\right)^n\frac{1}{v},\quad \left(\frac{\kappa}{m}\right)^{n-1}\frac{1}{v}\frac{1}{v\sqrt{\varepsilon}},\ \ldots,\quad \left(\frac{\kappa}{m}\right)\frac{1}{v}\frac{1}{(v\sqrt{\varepsilon})^{n-1}},\quad \frac{1}{v}\frac{1}{(v\sqrt{\varepsilon})^n} \tag{3.18}$$

(ignoring factors of g/m), as can easily be verified by a dimensional estimate of the n-loop graph. This behavior can also be seen from the explicit computation, Eqs. (3.14)–(3.16). Each term in Eq. (3.18) is as large as a term in the sum, Eq. (3.17), and cannot be neglected. The scattering amplitude will be given by the Green function for a Yukawa potential (which is not known analytically) and will have poles at the energies of all the bound states.

We now wish to integrate the Φ field out of the theory to obtain the rescaled effective four-fermion theory[3]

$$\mathcal{L}_\delta^R=\Psi^\dagger\left(i\partial^0+\frac{\nabla^2}{2}\right)\Psi+\frac{h}{v}(\Psi^\dagger\Psi)^2+\cdots. \tag{3.19}$$

The tree-level matching condition arises from Φ exchange and gives

[3]The v dependence of $(\Psi^\dagger\Psi)^2$ can also be obtained using Eq. (3.13), since $[(\Psi^\dagger\Psi)^2]-[\mathcal{L}]=-1$.

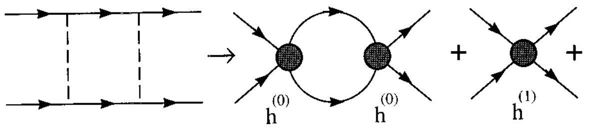

FIG. 6. One loop matching condition from NRY to NRδ.

$$h^{(0)}=\frac{g^2}{m^2}, \tag{3.20}$$

where we denote by $h^{(r)}$ the r-loop contribution to h. The one-loop matching condition is shown in Fig. 6. The box graph has already been evaluated in the full theory in Eq. (3.15). In the effective theory, the box graph is

$$\begin{aligned}I_1^\delta=&\left(\frac{h^{(0)}}{v}\right)^2\int\frac{dK^0}{(2\pi)}\frac{dK}{(2\pi)}\frac{1}{(P_1-K)^0-K^2/2+i\epsilon}\\ &\times\frac{1}{(P_2+K)^0-K^2/2+i\epsilon}\\ &=i\left(\frac{h^{(0)}}{v}\right)^2\int\frac{dK}{(2\pi)}\frac{1}{K^2+\varepsilon}\\ &=i\left(\frac{h^{(0)}}{v}\right)^2\frac{\varepsilon^{-1/2}}{2}.\end{aligned} \tag{3.21}$$

Using the matching condition, Eq. (3.20), we see that this reproduces the leading term in Eq. (3.15). It must do this, because one cannot write down an interaction proportional to an inverse fractional power of ϵ in the low-energy RNRδ effective Lagrangian. The $O(1/v)$ term in Eq. (3.15) is reproduced in the effective theory by the one-loop matching condition to h:

$$h^{(1)}=-\frac{3}{4}\frac{g^4M}{m^5}=-\frac{3}{4}\frac{\kappa}{m}h^{(0)}. \tag{3.22}$$

At two loops, one needs to compute the matching conditions in Fig. 8. The two-loop graph in the effective theory with two insertions of $h^{(0)}$ reproduces the $1/\varepsilon$ term in Eq. (3.16), and the one-loop graph with one insertion of $h^{(0)}$ and one insertion of $h^{(1)}$ reproduces the $1/\sqrt{\varepsilon}$ term. The matching correction to h at two loops requires knowing the coefficients of operators of the form $(\Psi^\dagger\nabla\Psi)^2$. This is a complication of the (1+1)-dimensional analysis which is not present in $3+1$ dimensions.

The tree-level matching condition for h only dominates when $\kappa/m\ll 1$. In this limit, the sum of bubble graphs in NRδ shown in Fig. 7 again gives a geometric series,

$$i\frac{h^{(0)}}{v}\frac{1}{1-h^{(0)}\varepsilon^{-1/2}/(2v)}, \tag{3.23}$$

which reproduces the result, Eq. (3.15), of RNRY for $\kappa\ll m$.

When κ/m is not small, we have already seen in the full theory that the scattering amplitude is not given by the simple geometric series, Eq. (3.17); the same is, of course, true in the effective theory. Consider a four-fermion operator in the effective theory containing one time derivative,

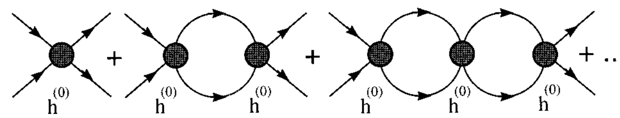

FIG. 7. The sum of ladder graphs in NRδ using the vertex $h^{(0)}$ obtained using tree-level matching.

$$\frac{c}{m}\psi^\dagger\partial_0\psi\psi^\dagger\psi+\text{H.c.} \tag{3.24}$$

Like h, higher loop matching conditions to c will be suppressed by powers of κ/m, and so the matching will have to be done to all orders when κ/m is not small. In the rescaled theory, this operator becomes

$$\frac{c}{v}\frac{Mv^2}{m}\Psi^\dagger\partial_0\Psi\Psi^\dagger\Psi+\text{H.c.} \tag{3.25}$$

In the Born approximation the matrix element of this operator may be neglected compared to the terms in the series, Eq. (3.17); however, when it is inserted into a two-loop bubble diagram, it gives a contribution to the scattering amplitude proportional to

$$\frac{h^2}{v^2}\frac{c}{v}\frac{Mv^2}{m}\sim\left(\frac{\kappa}{m}\right)^2\frac{c}{v}, \tag{3.26}$$

which, for $\kappa/m\sim 1$, is the same size as the first term in the geometric series. Similarly, operators of arbitrarily high order will contribute at all orders in $1/v$ to the scattering, suppressed only by powers of κ/m, and so the effective field theory is unable to describe bound states in the theory when this parameter is not small. An infinite number of operators is required in the effective theory to reproduce the scattering amplitude in the full theory. A simple way to see that the effective theory breaks down is to note that in RNRY, the effective mass of the Φ is $m/\sqrt{2}Mv\sim m^3/g^2M=m/\kappa$, for $v\sim g^2/m^2$. Thus, for $\kappa\ll m$ the Φ field is heavy, and may be integrated out, with higher dimensional operators suppressed by powers of $Mv/m\sim\kappa/m$. For $\kappa\sim m$, the Φ is light, and cannot be integrated out of the theory if bound states are to be properly described.

B. 3+1 dimensions

The analysis of the previous subsection can be repeated in $3+1$ dimensions. We will use the same symbols as in $1+1$ dimensions, but they now have dimension

$$[\psi]=3/2,\quad [\phi]=1,\quad [M]=1,\quad [m]=1,\quad [g]=0. \tag{3.27}$$

The three-dimensional Yukawa potential is

$$V(\mathbf{x})=-g^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{|\mathbf{q}|^2+m^2}=-\frac{g^2}{4\pi}\frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|}. \tag{3.28}$$

The Schrödinger equation with a three-dimensional Yukawa potential does not necessarily have a bound state. The bound state first appears when

$$\frac{g^2}{4\pi}\frac{M}{m}\geqslant 1.7. \tag{3.29}$$

Under the rescaling, Eq. (3.2), the NRY Lagrangian in $3+1$ dimensions becomes

$$\mathcal{L}=\Psi^\dagger\left(i\partial^0+\frac{\nabla^2}{2}\right)\Psi-\frac{g}{\sqrt{v}}\Phi\Psi^\dagger\Psi+\frac{v}{2}(\partial^0\Phi)^2-\frac{1}{2}(\nabla\Phi)^2 -\frac{1}{2}\frac{m^2}{M^2v^2}\Phi^2, \tag{3.30}$$

while the rescaled version of NRδ gives the RNRδ Lagrangian

$$\mathcal{L}=\Psi^\dagger\left(i\partial^0+\frac{\nabla^2}{2}\right)\Psi+hM^2v(\Psi^\dagger\Psi)^2. \tag{3.31}$$

The modified scaling rules of Sec. II D give

$$[x]=-1,\quad [t]=-2,\quad [\mathcal{L}]=5,\quad [\psi]=3/2,\quad [\phi]=3/2, \tag{3.32}$$

so that $[\phi\psi^\dagger\psi]-[\mathcal{L}]=-1/2$, and $[(\psi^\dagger\psi)^2]-[\mathcal{L}]=1$, which gives the v dependence of the coefficients in Eqs. (3.30) and (3.31). In three dimensions, the Yukawa interaction is a relevant operator, and the four-fermion interaction is an irrelevant operator. This is a sign that the four-fermion theory will have trouble correctly reproducing the behavior of the Yukawa theory when there is a light bound state in the spectrum.

Tree-level Φ exchange in RNRY gives the amplitude for $\Psi\Psi$ scattering,

$$I_0=i\frac{g^2M^2v}{m^2}+O(v^3), \tag{3.33}$$

and so the tree-level matching condition is the same as in $1+1$ dimensions,

$$h^{(0)}=\frac{g^2}{m^2}. \tag{3.34}$$

The one-loop box graph in RNRY is

$$\begin{aligned}I_1^Y&=-i\frac{g^4}{v^2}\int\frac{d^3\mathbf{K}}{(2\pi)^3}\frac{1}{|\mathbf{K}|^2+\mu^2}\frac{1}{|\mathbf{K}+\mathbf{P_1}-\mathbf{P_1'}|^2+\mu^2}\\&\quad\times\frac{1}{(P_1+P_2)^0-|\mathbf{P_2}-\mathbf{K}|^2/2-|\mathbf{P_1}+\mathbf{K}|^2/2}\\&=\frac{i}{8\pi}\frac{g^4}{v^2}\frac{1}{\mu(\mu+\sqrt{\varepsilon})^2}\\&=i\frac{g^4vM^3}{8\pi m^3}\left[1-\frac{2\varepsilon^{1/2}Mv}{m}+O(v^2)\right],\end{aligned} \tag{3.35}$$

for off-shell incident particles with $\mathbf{P_1}=\mathbf{P_2}=0$, where as before $\varepsilon=-(P_1+P_2)^0$. The one-loop bubble graph in the low-energy theory is

$$I_1^\delta=h^2M^4v^2\int\frac{dK^0}{(2\pi)}\frac{d^3\mathbf{K}}{(2\pi)^3}\frac{1}{(P_1-K)^0-|\mathbf{P_1}-\mathbf{K}|^2/2+i\epsilon}\times\frac{1}{(P_2+K)^0-|\mathbf{P_2}+\mathbf{K}|^2/2+i\epsilon} \tag{3.36}$$

$$=ih^2M^4v^2\int\frac{d^3\mathbf{K}}{(2\pi)^3}\frac{1}{|\mathbf{K}|^2+\varepsilon} =-ih^2M^4v^2\frac{\varepsilon^{1/2}}{4\pi}, \tag{3.37}$$

where we have evaluated the integral in dimensional regularization. Note that an integral which diverges as K^r, with r an odd integer, is finite when regulated with dimensional regularization, and needs no subtractions. The result, Eq. (3.37), correctly reproduces the leading nonanalytic term in Eq. (3.35). However, the analytic term is absent, and so one needs to add a one-loop matching contribution to h,

$$h^{(1)}=\frac{g^4M}{8\pi m^3}=\frac{g^2M}{4\pi m}\frac{1}{2}h^{(0)}. \tag{3.38}$$

In the region of parameter space for which bound states exist in the Yukawa theory, this term is at least as large as the tree-level matching term $h^{(0)}$, and so all orders in the loop expansion must be calculated in order to calculate the matching for h. The complete matching condition for h will be proportional to the scattering length for a Yukawa potential, which is not analytically known in closed form. Furthermore, there is no reason for h to be of the naive size $1/m^2$. In particular, if there is a bound state near threshold, h will be much larger.

The two-loop ladder graph in the full theory is

$$\begin{aligned}I_2&=i\frac{g^6}{(4\pi)^2v^3}\frac{1}{(\mu^2-\varepsilon)^2}\ln\frac{(2\mu+\varepsilon^{1/2})^2}{3\mu(\mu+2\varepsilon^{1/2})}\\&=i\frac{g^6M^4v}{(4\pi)^2m^4}\left[\ln\frac{4}{3}-\frac{\varepsilon^{1/2}Mv}{m}+\frac{\varepsilon M^2v^2}{4m^2}\right.\\&\quad\left.\times\left(7+8\ln\frac{4}{3}\right)+O(v^3)\right],\end{aligned} \tag{3.39}$$

the two-loop bubble graph in RNRδ is

$$i\frac{g^6M^6v^3}{(4\pi)^2m^6}\varepsilon, \tag{3.40}$$

and the one-loop graph with a single insertion of $h^{(1)}$ is

$$-2ih^{(0)}h^{(1)}\frac{M^4v^2}{4\pi}\varepsilon^{1/2}=-i\frac{g^6M^5v^2}{(4\pi)^2m^4}\frac{\varepsilon^{1/2}}{m}. \tag{3.41}$$

The sum of the graphs on the right-hand side of Fig. 8 is

$$i\frac{g^6M^4v}{(4\pi)^2m^4}\left[-\frac{\varepsilon^{1/2}Mv}{m}+\frac{\varepsilon M^2v^2}{4m^2}4\right]. \tag{3.42}$$

Comparing Eq. (3.42) with Eq. (3.39), we see that the nonanalytic term of order $\varepsilon^{1/2}$ is again reproduced in the

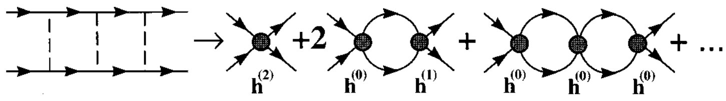

FIG. 8. Two-loop matching condition from NRY to NRδ.

effective theory, as it must be, but that the terms of order ε and ε^0 terms are not reproduced. Therefore there is a two-loop contribution to h,

$$h^{(2)}=\frac{g^6M^2}{(4\pi)^2m^4}\ln\frac{4}{3}=h^{(0)}\left(\frac{g^2M}{4\pi m}\right)^2\ln\frac{4}{3}, \qquad (3.43)$$

which once again is at least as large as $h^{(0)}$ in the region of interest, while at order ε the difference between the graphs in the two theories

$$i\frac{g^6M^4v}{(4\pi)^2m^4}\frac{\varepsilon M^2v^2}{4m^2}\left(3+8\ln\frac{4}{3}\right) \qquad (3.44)$$

contributes to the matching conditions of an operator such as $\Psi^\dagger i\partial^0\Psi\Psi^\dagger\Psi$, with coefficient

$$\frac{g^6M^6v^3}{(4\pi)^2m^6}\left(\frac{3}{4}+2\ln\frac{4}{3}\right)=\left(\frac{3}{4}+2\ln\frac{4}{3}\right)\left(\frac{h^{(0)}M^2v}{4\pi}\right)^2h^{(0)}M^2v. \qquad (3.45)$$

The important observation is that this counterterm contributes to the ε term at order $(QM)^3$, the same order as the two-loop bubble graph in the effective theory; hence, without knowing the coefficients of the dimension-8 operators in the effective theory all terms of the form $(QM)^n$ are not summed. It is clear that a similar situation exists at higher loops: Each loop graph in the effective theory is the same order as a counterterm, and so without knowing the counterterms, the graphs may not be summed. Thus, without including all higher dimensional operators, the effective theory does not correctly sum all terms of order $(QM)^n$.

C. Bound states and the effective range expansion

Let us look for bound state poles in the scattering amplitude. In 1+1 dimensions, the bubble chain sum in the effective theory had contributions proportional to inverse powers of v, which diverged in the low-energy limit. Thus, for weak coupling there was a bound state at threshold, which was reproduced in the low-energy theory. In 3+1 dimensions, the full theory with a Yukawa potential has a bound state at threshold when $g^2M/4\pi m\simeq 1.7$. Since in the bubble sum in NRδ only positive powers of v occur, all higher-order graphs contributing to the scattering amplitude vanish at low energy. Therefore, there can be no bound state pole at threshold unless one of the coefficients in the effective Lagrangian NRδ diverges. In particular, there can be a bound state at threshold if $h\to\infty$. To see this in the δ-function effective theory, one must evaluate the matching condition for h to all orders, since $g^2M/4\pi m$ is not small.

For finite h the bubble sum in the effective theory may be summed to find a bound state for finite h. Neglecting the contributions from operators of dimension >6, the bubble chain sum is

$$ihM^2v\left[1-\frac{hM^2v\varepsilon^{1/2}}{4\pi}+\left(\frac{hM^2v\varepsilon^{1/2}}{4\pi}\right)^2+\cdots\right]$$

$$=ihM^2v\frac{1}{1+hM^2v\varepsilon^{1/2}/4\pi}, \qquad (3.46)$$

which is the expression given by Weinberg [6]. There is a pole at

$$\frac{hM^2v\varepsilon^{1/2}}{4\pi}=-1 \qquad (3.47)$$

or, rescaling back to physical units, at

$$\frac{hM^{3/2}(-E)^{1/2}}{4\pi}=-1, \qquad (3.48)$$

which, for $h<0$, has a bound state below threshold. The bound state energy is [6]

$$E=-\frac{16\pi^2}{h^2M^3}. \qquad (3.49)$$

When $|h|\to\infty$, the bound state approaches the threshold.

We have argued, however, that the bubble sum in the effective theory does not correctly sum the terms of order $(QM)^n$. What relation does this bound state then have to a bound state in the full theory? To answer this, it is useful to recall some results of the effective range expansion for potential scattering.

The scattering amplitude $i\mathcal{A}$ is related to the phase shift δ by

$$k\cot\delta=\frac{4\pi}{M\mathcal{A}}+ik. \qquad (3.50)$$

In the effective range expansion, $k\cot\delta$ is expanded in powers of k,

$$k\cot\delta=-\frac{1}{a}+\frac{1}{2}r_ek^2+c_4k^4+\cdots. \qquad (3.51)$$

This expansion gives us some useful additional information, because it is known to have a radius of convergence $\geqslant m/2$ [12,13]. Here a is the scattering length, and r_e is the effective range. The first two terms in this series provide a very good approximation to the measured nucleon scattering cross sections. As shown in Ref. [9], Eqs. (3.50) and (3.51) imply that the four-fermion operators in NRδ have a momentum expansion of the form

$$-\frac{4\pi}{M}\left[a+\frac{1}{2}a^2r_ek^2+\left(a^3\frac{1}{4}r_e^2-a^2c_4\right)k^4+\cdots\right]. \qquad (3.52)$$

Rescaling these coefficients, we find that h is related to the exact scattering length a of the Yukawa theory,

$$h=-\frac{4\pi}{M}a, \tag{3.53}$$

whereas a two-derivative term such as $\Psi^\dagger \boldsymbol{\nabla}^2\Psi\Psi^\dagger\Psi$ has a coefficient of order

$$2\pi M^3 v^3 a^2 r_e. \tag{3.54}$$

The scattering length diverges when there is a bound state at threshold, so that $h\to\infty$. It is known from potential scattering that the other coefficients in the effective range expansion, Eq. (3.51), do not diverge [12,13]. In this case, the coefficients of the higher dimensional operators diverge as well, as is evident in Eq. (3.52), and the divergent behavior of the various coefficients is highly correlated. It is difficult to see this behavior by studying the matching conditions to the effective theory, since one must perform the matching to all orders in the loop expansion. However, the effective range expansion gives nontrivial information on the form of the complete matching conditions.

If we now consider $\psi\psi$ scattering, the two-bubble graph from the dimension-6 operator $(\psi^\dagger\psi)^2$ gives a term of order

$$\frac{a^3M^3v^3}{4\pi}, \tag{3.55}$$

which is to be compared with the counterterm contribution, Eq. (3.54). The terms have the same explicit dependence on M and v, as we have already argued, and so the bubble graph is the same order as the counterterm using the power-counting scheme of [6–9]. Furthermore, both terms are suppressed by two powers of $Q=Mv$ relative to the contact term, and so even when M is large it is not necessary to resum the bubble graphs in this scheme. However, the bubble graph will dominate when

$$\frac{a}{4\pi r_e}\gg 1. \tag{3.56}$$

Summing bubble graphs in the effective range expansion therefore corresponds to an expansion in powers of $a/4\pi r_e$, not Q/M. Therefore, when the scattering length is large (and therefore the bound state is nearly at threshold) the sum of the bubble graphs in NRδ correctly describes the bound state, and Eq. (3.49) is valid, up to corrections of order $4\pi r_e/a$. However, as discussed in [9], when the scattering length is large the effective theory breaks down not at $k\sim m\sim r_e^{-1}$, but at a much lower scale

$$k\sim\sqrt{\frac{2}{ar_e}}, \tag{3.57}$$

which, in the region where the bubble graphs dominate, is much less than m. The region of convergence is not controlled by the scalar mass, and vanishes as the bound state approaches threshold. Thus, we still do not have the situation we had in weakly bound $(1+1)$-dimensional theory in which the bound state was correctly predicted in the low-energy theory but the radius of convergence of the theory was set by the heavy scalar mass.

D. Implications for chiral perturbation theory

The results are unchanged when pions are included in the chiral Lagrangian, since pion exchange scales with M and v in the same way as the dimension-6 contact interactions. We may therefore conclude from this simple model that chiral perturbation theory cannot sum all terms of order $(QM)^n$ without including an infinite number of higher-dimensional operators. Using the effective range expansion, we see that the bubble graphs only correctly describe the low-energy bound state when the effective field theory breaks down at a scale much less than the symmetry-breaking scale $\Lambda_{\chi\mathrm{SB}}$ (the analogue of the scalar mass in our simple model). Therefore the standard form of heavy nucleon chiral perturbation theory does not provide the correct description of nucleon-nucleon scattering up to energies set by the scale of the heavy excitations which have been integrated out of the theory. This does not mean that the effective field theory idea is useless: It does mean, however, that the usual chiral Lagrangian is not the correct effective field theory. If the appropriate low-energy degrees of freedom are introduced by hand as new fields in the effective Lagrangian, it should be possible to correctly describe low-energy nucleon-nucleon scattering in an effective field theory.[4] However, the properties of the low-energy degrees of freedom are not determined by the parameters in the standard chiral Lagrangian, unless all higher dimensional operators are included.

IV. CONCLUSIONS

Rescaled nonrelativistic effective field theories simplify the study of weakly bound states in quantum field theory. The use of RNRQCD makes the v power counting of NRQCD manifest, and reorganizes the perturbation expansion in a more systematic way. Rescaled effective theories can also be used to study Yukawa bound states due to scalar exchange. In $1+1$ dimensions, a bound state occurs at weak coupling because the n-loop ladder graph diverge is proportional to $1/(-E)^{n/2}$. In this limit one may sum the diagrams of Fig. 7 and obtain the properties of the bound state, without worrying about higher-loop matching conditions. In three dimensions, one needs a critical coupling before there exists a bound state, and the problem is intrinsically strongly coupled. One needs to include the full Yukawa interaction to study the bound state. Equivalently, the matching from NRY to NRδ must be performed to all orders in the loop expansion, and the higher-order matching terms are relevant for the bound state. We have shown by explicit computation that the power-counting scheme of Refs. [6–9] that involves summing powers of QM does not hold for Yukawa theory. One cannot, in general replace the Yukawa potential by a δ-function potential to study even weakly coupled bound states in three dimensions, except for a very limited region in

[4]We thank D. Kaplan for discussions on this point.

momentum space $|\mathbf{k}| < \sqrt{2/ar_e}$, which vanishes as the bound state approaches threshold.

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