Recursion rules for scattering amplitudes in non-Abelian gauge theories

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We present a functional derivation of recursion rules for scattering amplitudes in a non-Abelian gauge theory in a form valid to arbitrary loop order. The tree-level and one-loop recursion rules are explicitly displayed. [S0556-2821(97)04806-6]

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I. INTRODUCTION

It is by now a well-appreciated fact that non-Abelian gauge theories display many interesting and beautiful properties which are, no doubt, indicative of their deeply geometrical nature and symmetry. Already at the level of perturbation theory, many of the scattering amplitudes, in quantum chromodynamics (QCD), for example, have a simple and elegant form, although large numbers of Feynman diagrams have to be summed up to arrive at these [1]. This feature was originally understood in terms of calculational techniques based on string theory as well as recursion rules [2–4]. (A field-theoretic understanding of the string-based techniques has also emerged [5].) There are also indications that there are integrable field theories hidden in non-Abelian gauge theories which describe some aspects of these theories, such as scattering for certain specific choices of helicities as well as scattering in the Regge regime [6–9]. Given these features, it is clear that the theoretical exploration of the structure of even the perturbative scattering matrix can be quite useful.

Within a field-theoretic approach, recursion rules for scattering amplitudes have been very useful in understanding color factorization and many other properties. These rules were originally derived for tree-level amplitudes using diagrammatic analyses by Berends and Giele and have later been extended up to the one-loop level [11,12]. Elaboration and extension of this technique, we feel, can be very fruitful. In this paper, we present a derivation of the recursion rules within a functional formalism without reliance on diagrammatic analysis and in a way valid for any field theory. From the basic equations, recursion rules valid up to arbitrary loop order can be obtained, although at the expense of increasing algebraic complexity. We explicitly display the tree-level and one-loop recursion rules. Renormalization constants, which one must consider beyond the tree level, are also easily incorporated in a functional derivation.

II. S-MATRIX FUNCTIONAL

We start by considering a scalar field theory with action of the form

$$S[\phi]=S_0[\phi]+S_{\rm int}[\phi], \tag{2.1}$$

where $S_0[\phi]$ is the free action of the form $S_0[\phi]=\int\frac{1}{2}\phi K\phi$ and $S_{\rm int}$ is the interaction part of the action. Specifically, K can be taken to be of the form $K=-Z_2(\partial^2+m^2)$. The functional for the S-matrix elements can be written as [10]

$$\mathcal{F}[\varphi]=e^F\int[d\phi]e^{iS_0[\phi]+iS_{\rm int}(\phi+\varphi)}, \tag{2.2}$$

where $\varphi(x)$ can be expanded as

$$\varphi(x)=\sum_k a_k u_k(x)+a_k^* u_k^*(x). \tag{2.3}$$

$u_k(x),u_k^*(x)$ are the one-particle wave functions and are solutions of the free field equation $K\varphi=0$. F is given by $\Sigma_k a_k^* a_k$ and is introduced in Eq. (2.2) to give the matrix elements for subscattering processes where some of the particles fly by unscattered. The matrix element for a process where particles of momenta $k_1,k_2,\ldots,k_N$ scatter to particles of momenta $p_1,p_2,\ldots,p_M$ is given by

$$S_{k_1,k_2\cdots k_N\to p_1p_2\cdots p_M}=\left[\frac{\delta}{\delta a_{k_1}}\frac{\delta}{\delta a_{k_2}}\cdots\frac{\delta}{\delta a_{k_N}}\frac{\delta}{\delta a^*_{p_1}}\times\frac{\delta}{\delta a^*_{p_2}}\cdots\frac{\delta}{\delta a^*_{p_M}}\mathcal{F}[\varphi]\right]_{\varphi=0}. \tag{2.4}$$

Since F is not particularly relevant for our calculations below, we can drop it in what follows. Further, we consider $\varphi(x)$ to be an arbitrary function rather than a solution to $K\varphi=0$. Eventually, in obtaining the S-matrix elements we can choose it to be a solution to $K\varphi=0$. We thus define

$$\begin{aligned}\mathcal{F}[\varphi]&=\int[d\phi]\exp\left(iS_0[\phi]+iS_{\rm int}(\phi+\varphi)-i\frac{1}{2}\int\varphi K\varphi\right)\\&=\int[d\phi]\exp\left(iS[\phi]-i\int\varphi K\phi\right),\end{aligned} \tag{2.5}$$

where we added a term $\exp(-i/2\int\varphi K\varphi)$ for simplifications in what follows; this of course does not contribute when $K\varphi=0$. In terms of this $\mathcal{F}$, the S-matrix elements are given by

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$$S_{k_1,k_2\cdots k_N\to p_1p_2\cdots p_M}=[\alpha_{k_1}\cdots\alpha_{k_N}\alpha^*_{p_1}\cdots\alpha^*_{p_M}\mathcal{F}[\varphi]]_{\varphi=0}, \tag{2.6}$$

where

$$\alpha_k=\int u_k(x)\frac{\delta}{\delta\varphi(x)}. \tag{2.7}$$

From the second of equations (2.5), we see that $\mathcal{F}$ is the generating functional of the connected Green functions with source $K\varphi$ and Eq. (2.6) represents the well-known Lehmann-Symanzik-Zimmermann (LSZ) reduction formula.

Consider now $\delta\mathcal{F}/\delta\varphi$. Differentiating with respect to φ, we find

$$iK^{-1}\frac{\delta\mathcal{F}}{\delta\varphi}=\int[d\phi]\phi\exp\left(iS[\phi]-i\int\varphi K\phi\right). \tag{2.8}$$

Effectively, ϕ's inside the integral behave as $iK^{-1}\delta/\delta\varphi$. From Eq. (2.5) we also have the equation of motion

$$\int[d\phi]\left[\frac{\delta S}{\delta\phi}-K\varphi\right]\exp\left(iS[\phi]-i\int\varphi K\phi\right)=0, \tag{2.9}$$

which can be written as

$$i\frac{\delta\mathcal{F}}{\delta\varphi}+\rho(\hat{\phi})\mathcal{F}-(K\varphi)\mathcal{F}=0, \tag{2.10}$$

where $\rho(\phi)=(\delta S_{\rm int}/\delta\phi)$ and $\hat{\phi}=iK^{-1}\delta/\delta\varphi$.

$\mathcal{F}$ itself generates connected as well as disconnected scattering processes. If we write $\mathcal{F}[\varphi]=e^{iC[\varphi]}$, where $C[\varphi]$ describes connected processes only, Eq. (2.10) reduces to

$$-\frac{\delta}{\delta\varphi}C=K\varphi-\mathcal{F}^{-1}\rho(\hat{\phi})\mathcal{F}=K\varphi-\rho\left(-K^{-1}\frac{\delta C}{\delta\varphi}+\hat{\phi}\right)1. \tag{2.11}$$

This can be regarded as a nonlinear functional differential equation for the S-matrix generating functional and can be used for deriving systematic recursion rules for scattering amplitudes. We shall do this for Yang-Mills theory in the next section.

Although it is not crucial to the discussion of recursion rules, one may also work with the quantum effective action or the generating functional for one-particle irreducible vertices $\Gamma[\Phi]$ defined by

$$\Gamma[\Phi]=C[\varphi]+\int(K\varphi)\Phi, \tag{2.12}$$

with the connecting relations

$$\frac{\delta\Gamma}{\delta\Phi}=K\varphi,\quad -K^{-1}\frac{\delta C}{\delta\varphi}=\Phi. \tag{2.13}$$

Then, Eq. (2.11) becomes

$$K\Phi=K\varphi-\rho(\Phi+\hat{\phi})1 \tag{2.14}$$

or, in terms of $\Gamma[\Phi]$,

$$\frac{\delta\Gamma}{\delta\Phi}=K\Phi+\rho(\Phi+\hat{\phi})1=\left(\frac{\delta S}{\delta\phi}\right)_{\phi=\Phi+iK^{-1}\delta/\delta\varphi}1. \tag{2.15}$$

The right-hand side involves the derivative of Φ with respect to φ which is the two-point correlator $\widetilde{G}$ given by

$$\widetilde{G}(x,y)\equiv\left(iK^{-1}\frac{\delta}{\delta\varphi}\right)_x\Phi(y)=\left(iK^{-1}\frac{\delta}{\delta\varphi}\right)_x\left(iK^{-1}\frac{\delta}{\delta\varphi}\right)_y iC. \tag{2.16}$$

It satisfies the basic Schwinger-Dyson equation for the theory, viz.,

$$\int_y\left[\frac{\delta^2\Gamma}{\delta\Phi(x)\delta\Phi(y)}\right]\widetilde{G}(y,z)=i\delta^{(4)}(x-z). \tag{2.17}$$

Equation (2.15) supplemented by Eq. (2.17) can thus be regarded as a nonlinear equation for $\Gamma[\Phi]$. From the connecting relations (2.13) we find that $\delta\Gamma/\delta\Phi=0$ for $K\varphi=0$, as is appropriate for S-matrix elements. In this case, from the definition of Γ, we have

$$C[\varphi]=\Gamma[\Phi]|_{\delta\Gamma/\delta\Phi=0}. \tag{2.18}$$

The S matrix is then given by

$$\mathcal{F}=[e^{i\Gamma[\Phi]}]_{\delta\Gamma/\delta\Phi=0}. \tag{2.19}$$

This relation gives a nonperturbative definition of the S matrix. The free data in the solutions to $\delta\Gamma/\delta\Phi=0$ are the quantities on which $\mathcal{F}$ depends. (Perturbatively, the free data are the amplitudes a_k and a_k^* in the solution for φ.) The fact that the S matrix can be obtained as the exponential of (i times) the action evaluated on solutions of the equations of motion is rather well known [10,13].

III. *S*-MATRIX FUNCTIONAL FOR YANG-MILLS THEORY

We start with the gauge-fixed Lagrangian $\mathcal{L}$ of an SU(N)-Yang-Mills theory given by

$$\begin{aligned}\mathcal{L}&=-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}-\tfrac{1}{2}(\partial\cdot A)^2-\overline{c}(-\partial\cdot D)c\\&=\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}+\mathrm{Tr}(\partial\cdot A)^2+2\mathrm{Tr}\overline{c}(-\partial\cdot D)c,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}F_{\mu\nu}&=F^a_{\mu\nu}T^a=\partial_\mu A_\nu-\partial_\nu A_\mu+g[A_\mu,A_\nu],\\D_\mu c&=\partial_\mu c+g[A_\mu,c],\end{aligned} \tag{3.2}$$

with $[T^a,T^b]=f^{abc}T^c$ and $\mathrm{Tr}T^aT^b=-\tfrac{1}{2}\delta_{ab}$. T^a are matrices in the fundamental representation of SU(N). The free part and the interaction parts of the action are, respectively, identified as

$$\mathcal{L}_0=\mathrm{Tr}A_\mu(-\partial^2)A^\mu+2\mathrm{Tr}\overline{c}(-\partial^2)c,$$

$$\mathcal{L}^{(1)}_{\rm int}=2g\mathrm{Tr}\partial_\mu A_\nu[A^\mu,A^\nu]+\frac{g^2}{2}\mathrm{Tr}[A_\mu,A_\nu]^2, \tag{3.3}$$

$$\mathcal{L}_{\rm int}^{(2)}=2g{\rm Tr}\partial_\mu\overline{c}[A^\mu,c]. \tag{3.4}$$

Following the discussion of Sec. II, the functional $\mathcal{F}$ which gives the S matrix can be constructed as

$$\mathcal{F}[a]=\int \exp\left(iS-i\int a_\mu^a(-\partial^2)A^{\mu a}\right)[dA][dc][d\overline{c}]. \tag{3.5}$$

The transition amplitude for N gluons of momenta k_i, polarizations $\epsilon_\mu^{(i)}$, and colors labeled $a_1,\ldots,a_N$ to go to M gluons of momenta p_j, polarizations $\epsilon_\nu^{(j)}$, and color labels $a_{N+1},\ldots,a_M$ is

$$T[\{k_i,\epsilon_{\mu_i}^{(i)},a_i\}\to\{p_j,\epsilon_{\nu_j}^{(j)},a_j\}]=\int\prod_i e^{-ik_ix_i}\epsilon_{\mu_i}^{(i)}\frac{\delta}{\delta a_{\mu_i}^{a_i}}\prod_j e^{ip_jy_j}\epsilon_{\nu_j}^{(j)}\times\frac{\delta}{\delta a_{\nu_j}^{a_j}}\mathcal{F}[a]\bigg|_{a=0}. \tag{3.6}$$

From Eq. (2.10), $\mathcal{F}$ satisfies the equation

$$\begin{aligned}-i\frac{\delta\mathcal{F}}{\delta a_\mu^a}&=\partial^2a_\mu^a\mathcal{F}+\int\frac{\delta S_{\rm int}}{\delta a_\mu^a}e^{iS}\\&=\partial^2a_\mu^a\mathcal{F}+J_\mu^a(A)|_{A=iG\frac{\delta}{\delta a}}\mathcal{F}+\int 2g{\rm Tr}\partial_\mu\overline{c}[T^a,c]e^{iS},\end{aligned} \tag{3.7}$$

where

$$J_\mu(A(x))=J_\mu^aT^a\equiv\frac{\delta S_{\rm int}^{(1)}}{\delta A_\mu^a}T^a=-g\partial^\nu[A_\mu,A_\nu]+g[F_{\mu\nu},A^\nu] \tag{3.8}$$

and $G(x,y)=-\langle x|\partial^{-2}|y\rangle$. As we have written it, the term in Eq. (3.7) involving ghosts cannot immediately be replaced by derivatives on $\mathcal{F}$. For this, we proceed as follows. Integrating out the ghost fields in $\mathcal{F}$ we get the Faddeev-Popov determinant $\det(-\partial\cdot D)=e^{{\rm trln}(-\partial\cdot D)}$, which is equivalent to a term in the action $-i\,{\rm trln}(-\partial\cdot D)$. This leads to a ghost current of the form

$$J_\mu^a(A)_{\rm gh}=-i\frac{\delta{\rm trln}(-\partial\cdot D)}{\delta A_\mu^a}=i\sum_{n=1}^\infty(-g)^{n+1}\int{\rm Tr}[T^aA_{\nu_i}(y_1)\cdots A_{\nu_n}(y_n)]_{\rm adj}\partial^{\nu_1}G(x,y_1)\cdots\partial^{\nu_n}G(y_{n-1},y_n)\partial_\mu G(y_n,x), \tag{3.9}$$

where we have introduced the adjoint representation of T^a's by $(T_{\rm adj}^a)_{bc}=-f^{abc}$. ($\mathcal{L}_{\rm int}^{(2)}$ in this notation reads $\mathcal{L}_{\rm int}^{(2)}=-g\partial_\mu\overline{c}T_{\rm adj}^acA_\mu^a$.)

The trace in the above equation may be written in terms of the generators in fundamental representation using $f^{abc}=-2{\rm Tr}T^a[T^b,T^c]$,

$${\rm Tr}(T^{a_1}T^{a_1}\cdots T^{a_n})_{\rm adj}=2{\rm Tr}T^a[T^b,[T^{a_1},[T^{a_2},[\ldots,[T^{a_{n-1}},[T^{a_n},T^b]]\ldots]. \tag{3.10}$$

Therefore we can finally write the ghost part of the current (as a matrix in the fundamental representation) as

$$J_\mu(A)_{\rm gh}=J_\mu^a(A)_{\rm gh}T^a=i\sum_{n=1}^\infty(-g)^{n+1}D_{\mu\mu_1\cdots\mu_n}(x,y_1,\ldots,y_n)[T^b,[A^{\mu_1}(y_1),[\ldots,[A^{\mu_{n-1}}(y_{n-1}),[A^{\mu_n}(y_n),T^b]]\ldots], \tag{3.11}$$

where

$$D_{\mu\mu_1\cdots\mu_n}(x,y_1,\ldots,y_n)\equiv\partial_{\mu_1}G(x,y_1)\cdots\partial_{\mu_n}G(y_{n-1},y_n)\partial_\mu G(y_n,x). \tag{3.12}$$

Then Eq. (3.7) finally becomes

$$-i\frac{\delta\mathcal{F}}{\delta a_\mu^a}=\partial^2a_\mu^a\mathcal{F}+[J_\mu^a(A)+J_\mu^a(A)_{\rm gh}]_{A=iG(\delta/\delta a)}\mathcal{F}. \tag{3.13}$$

For connected part $C[a]=-i\ln\mathcal{F}[a]$,

$$\hbar\frac{\delta C}{\delta a_\mu^a}=\partial^2a_\mu^a+[J_\mu^a(A)+\hbar J_\mu^a(A)_{\rm gh}]_{A=-\hbar G(\delta C/\delta a)+i\hbar G(\delta/\delta a)}1. \tag{3.14}$$

This is the basic equation for the scattering amplitudes. The $\hbar$ expansion of this equation leads to recursion rules up to arbitrary order. We have explicitly displayed the factors of $\hbar$ in Eq. (3.14). Recall that the interaction part of the action carries $1/\hbar$ and that each propagator G carries $\hbar$. The ghost terms, arising from the determinant, have $\hbar^0$. $C[a]$ in Eq. (3.14) has an expansion of the form

$$C[a]\equiv-i\ln\mathcal{F}[a]=\frac{1}{\hbar}C^{(0)}+C^{(1)}+\hbar C^{(2)}+\cdots. \tag{3.15}$$

Starting from Eq. (3.14) and using the above expansion for $C[a]$, one can, in principle, systematically derive recursion relations to any desired order in $\hbar$.

Let us begin with a tree-level recursion formula for $C^{(0)}$. This is simply given by

$$\frac{\delta C^{(0)}}{\delta a^{\mu a}}=\partial^2 a_\mu^a+J_\mu^a(A^{(0)}), \tag{3.16}$$

where

$$A_\mu^{(0)a}\equiv-\int G\frac{\delta C^{(0)}}{\delta a^{\mu a}}, \tag{3.17}$$

i.e.,

$$\partial^2 A_\mu^{(0)}=\partial^2 a_\mu+J_\mu(A_\mu^{(0)}). \tag{3.18}$$

This is just the equation of motion as it should be. (This equation has previously been obtained in [14].) $A_\mu^{(0)}$ is essentially the same object as the current that Berends and Giele [2] used to derive tree-level recursion relation for gluon scattering processes; if one expands $A_\mu^{(0)}$ in powers of a_μ, the coefficient function of each term gives the one-gluon off-shell current of Ref. [2] when multiplied by polarization vectors of on-shell external gluons. More generally, if one considers a_μ's off shell, it gives the generalized current with off-shell gluons which has been used for $q\bar{q}\to q\bar{q}g\cdots g$ process and some one-loop calculations [11,12]. Also, notice that from the form of current J_μ in Eq. (3.8), it is obvious that color factors are factorized from coefficient functions and we can write

$$A_\mu^{(0)}=\sum_{n=1}^{\infty}\int C^{(0)}_{\mu\mu_1\cdots\mu_n}(x,y_1,\ldots,y_n)a^{\mu_1}(y_1)\cdots a^{\mu_n}(y_n). \tag{3.19}$$

The coefficient functions $C^{(0)}$'s do not carry color indices. We also have $C^{(0)}_{\mu\mu_1}(x,y_1)=\delta(x-y_1)\,\delta_{\mu\mu_1}$. Using Eq. (3.19) in Eq. (3.18),

$$\begin{aligned}\partial^2\sum\int C_\mu^{(0)}(x,Y)a(Y)-\partial^2 a_\mu=J_\mu(A)=-\sum\int &[2\partial^\nu C_\mu^{(0)}(x,Y)C_\nu^{(0)}(x,Z)-2\partial_\nu C_\mu^{(0)}(x,Z)C^{\nu(0)}(x,Y)\\ &+C_\nu^{(0)}(x,Y)\overleftrightarrow{\partial}_\mu C^{\nu(0)}(x,Z)+C_\mu^{(0)}(Y)\partial_\nu C^{\nu(0)}(x,Z)-C_\mu^{(0)}(x,Z)\partial^\nu C_\nu^{(0)}(x,Y)]a(X)a(Y)\\ &-\sum\int[2C_\nu^{(0)}(x,X)C_\mu^{(0)}(x,Y)C^{\nu(0)}(x,Z)-C_\nu^{(0)}(x,X)C^{\nu(0)}(x,Y)C_\mu^{(0)}(x,Z)\\ &-C_\mu^{(0)}(x,X)C_\nu^{(0)}(x,Y)C^{\nu(0)}(x,Z)]a(X)a(Y)a(Z),\end{aligned} \tag{3.20}$$

where X, Y, and Z are collective indices and $a(Y)$ stands for $\Pi_i a_{\mu_i}(y_i)$, etc. If a_μ is restricted to be on shell, the terms containing $\partial^\nu C_\nu^{(0)}$ in the third line vanish because of current conservation. We can transform this equation to momentum space by writing

$$C^{(0)}_{\mu\mu_1\ldots\mu_n}(x,y_1,\ldots,y_n)=\int(2\pi)^4\left(k+\sum p_i\right)C^{(0)}_{\mu\mu_1\cdots\mu_n}(p_1,\ldots,p_n)\exp(ik\cdot x)\exp\left(i\sum p_i\cdot y_i\right), \tag{3.21}$$

where momentum conservation has been taken into account. Then Eq. (3.20) becomes

$$\begin{aligned}P_{1,n}^2C_\mu^{(0)}(1,\ldots,n)=&\sum_{m=1}^{n-1}V_{3\mu}{}^{\nu\rho}(P_{1,m},P_{m+1,n})C_\nu^{(0)}(1,\ldots,m)C_\rho^{(0)}(m+1,\ldots,n)\\ &+\sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}V_{4\mu}{}^{\nu\rho\sigma}C_\nu^{(0)}(1,\ldots,m)C_\rho^{(0)}(m+1,\ldots,k)C_\sigma^{(0)}(k+1,\ldots,n),\end{aligned} \tag{3.22}$$

where $P_{i,j}=p_i+\cdots+p_j$ and $V_3^{\mu\nu\rho}$ and $V_4^{\mu\nu\rho\sigma}$ are color-ordered vertices,

$$\begin{aligned}V_3^{\mu\nu\rho}(p,q)&=-g\{g^{\nu\rho}(p-q)^\mu+g^{\rho\mu}(2q^\nu+p^\nu)\\ &\quad-g^{\mu\nu}(2p^\rho+q^\rho)\},\\ V_4^{\mu\nu\rho\sigma}&=-g(2g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}).\end{aligned} \tag{3.23}$$

This equation is the recursion relation for currents with off-shell gluons derived in [2,11].

In deriving higher order recursion relations for S-matrix elements, we must correct Eq. (3.14) by including the appropriate renormalization constants. As usual, we interpret fields and coupling constants as renormalized ones and assume that renormalization counterterms are included in $\mathcal{L}_{\rm int}$ of Eq. (3.3). Explicitly,

$$\mathcal{L}_{\rm int}=-\tfrac{1}{2}\delta Z_3 {\rm Tr}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2-2gZ_1{\rm Tr}\partial_\mu A_\nu[A_\mu,A_\nu]$$
$$-\frac{g^2}{2}Z_1^2Z_3^{-1}{\rm Tr}[A_\mu,A_\nu]^2-2\delta\widetilde{Z}_3{\rm Tr}\overline{c(}-\partial^2)c$$
$$-2gZ_1\widetilde{Z}_3/Z_3{\rm Tr}\partial_\mu\overline{c}[A_\mu,c], \tag{3.24}$$

where $\sqrt{Z_3}A_\mu$, $\sqrt{\widetilde{Z}_3}c$, and $gZ_1Z_3^{-3/2}$ are bare quantities with $\delta Z_3=Z_3-1$ and $\delta\widetilde{Z}_3=\widetilde{Z}_3-1$. Correspondingly, J_μ becomes

$$J_\mu(A)=-g\partial_\nu[A_\mu,A_\nu]+g[F_{\mu\nu},A_\nu]-\tfrac{1}{2}\delta Z_3\partial_\nu(\partial_\mu A_\nu$$
$$-\partial_\nu A_\mu)-g(Z_1-1)(\partial_\nu[A_\mu,A_\nu]-[F_{\mu\nu},A_\nu])$$
$$+g^2Z_1(Z_1/Z_3-1)[[A_\mu,A_\nu],A_\nu]. \tag{3.25}$$

The extra counterterms contribute to one-loop or higher orders and will cancel ultraviolet divergences from loop integrals. In addition, to account for the freedom of arbitrary finite renormalization we include a finite wave function renormalization constant z_3 in the external field, i.e., $a_\mu\to\widetilde{a}_\mu\equiv a_\mu/\sqrt{z_3}$.

Now we are ready to discuss one-loop recursion relations. Keeping terms relevant up to one-loop order,

$$A_\mu=-G\frac{\delta C^{(0)}}{\delta a^\mu}-\left(G\frac{\delta C^{(1)}}{\delta a^\mu}-\frac{1}{2}\delta z_3 G\frac{\delta C^{(0)}}{\delta a^\mu}\right)+\cdots$$
$$=A_\mu^{(0)}+A_\mu^{(1)}+\cdots. \tag{3.26}$$

The current can similarly be expanded as $J_\mu(A)=J_\mu^{(0)}+J_\mu^{(1)}+\cdots$, with

$$J_\mu^{(0)}=-g\partial^\nu[A_\mu^{(0)},A_\nu^{(0)}]+g[F_{\mu\nu}(A^{(0)}),A^{(0)\nu}],$$
$$J_\mu^{(1)}=-g\partial^\nu([A_\mu^{(0)},A_\nu^{(1)}]+[A_\mu^{(1)},A_\nu^{(0)}])$$
$$+g[F_{\mu\nu}(A^{(0)}),A^{(1)\nu}]+g[D_\mu A_\nu^{(1)}-D_\nu A_\mu^{(1)},A^{(0)\nu}]$$
$$-\delta Z_1 J_\mu^{(0)}(A^{(0)})-\frac{\delta Z_3}{2}\partial^\nu(\partial_\mu A_\nu^{(0)}-\partial_\nu A_\mu^{(0)})$$
$$+g^2(\delta Z_1-\delta Z_3)[[A_\mu^{(0)},A_\nu^{(0)}],A^{(0)\nu}], \tag{3.27}$$

where $D_\mu\equiv\partial_\mu+g[A_\mu^{(0)},]$. In Eq. (3.14) we need

$$J_\mu\left(A+iG\frac{\delta}{\delta a}\right)1=J_\mu(A)-g\partial^\nu\left[iG\frac{\delta}{\delta a^\mu},A_\nu^{(0)}\right]+g\left[i\partial_\mu G\frac{\delta}{\delta a^\nu}\right.$$
$$\left.-i\partial_\nu G\frac{\delta}{\delta a^\mu},A^{(0)\nu}\right]$$
$$-g^2\left(\left[\left[iG\frac{\delta}{\delta a^\mu},A_\nu^{(0)}\right],A^{(0)\nu}\right]\right.$$
$$\left.+\left[\left[A_\mu^{(0)},iG\frac{\delta}{\delta a_\nu}\right],A_\nu^{(0)}\right]\right)+\cdots. \tag{3.28}$$

Up to one-loop order, the functional derivative term in the ghost current has no contribution:

$$J_\mu\left(A+iG\frac{\delta}{\delta a}\right)_{\rm gh}1=J_\mu(A^{(0)})_{\rm gh}+\cdots. \tag{3.29}$$

Collecting all these, we get an equation for $A_\mu^{(1)}$:

$$\partial^2A_\mu^{(1)}=J_\mu^{(1)}+J_\mu(A^{(0)})_{\rm gh}-g\partial^\nu\left[iG\frac{\delta}{\delta a^\mu},A_\nu^{(0)}\right]$$
$$+g\left[i\partial_\mu G\frac{\delta}{\delta a_\nu}-i\partial^\nu G\frac{\delta}{\delta a^\mu},A_\nu^{(0)}\right]$$
$$-g^2\left(\left[\left[iG\frac{\delta}{\delta a^\mu},A_\nu^{(0)}\right],A^{(0)\nu}\right]\right.$$
$$\left.+\left[\left[A_\mu^{(0)},iG\frac{\delta}{\delta a_\nu}\right],A_\nu^{(0)}\right]\right), \tag{3.30}$$

where $A_\mu^{(0)}$ is the solution of Eq. (3.18). Notice that, in this case, $iG(\delta/\delta a_\mu)$ terms contract color indices when acting on $A_\mu^{(0)}$ and so the color decomposition does not occur as in the tree-level case. However, it is possible to write $A_\mu^{(1)}$ as a sum of color-factorized amplitudes and the proof has been given using a string-theory argument [15] and color flow diagrams [17]. Here we give a simple proof based on Eq. (3.30).

First we study the action of $iG(\delta/\delta a_\mu)$ on $A_\mu^{(0)}$. From Eq. (3.19),

$$\left[G\frac{\delta}{\delta a^\mu},A_\nu^{(0)}\right]=\sum_{n=1}^{\infty}\sum_{m=1}^{n}\int C^{(0)}_{\nu\nu_1\cdots\nu_{m-1}\mu\nu_{m+1}\cdots\nu_n}(x,y_1,\ldots,y_{m-1},y,y_{m+1},\ldots,y_n)G(x,y)$$
$$\times[T^a,a_{\nu_1}(y_1)\cdots a_{\nu_{m-1}}(y_{m-1})T^a a_{\nu_{m+1}}(y_{m+1})\cdots a_{\nu_n}(y_n)]. \tag{3.31}$$

With the help of Fierz identity for SU(N),

$$(T^aXT^a)_{ij}=-\frac{1}{2}\left(\delta_{ij}{\rm Tr}X-\frac{1}{N}X_{ij}\right), \tag{3.32}$$

it becomes

$$\left[G\frac{\delta}{\delta a^\mu},A_\nu^{(0)}\right]=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\int\widetilde{C}_{\nu\mu}(x,1,\ldots,m;m+1\ldots,n)a(1)\cdots a(m){\rm Tr}[a(m+1)\cdots a(n)], \tag{3.33}$$

where

$$\widetilde{C}_{\nu\mu}(x,1,\ldots,m;m+1,\ldots,n)=-\frac{1}{2}\int_y G(x,y)[C^{(0)}_{\nu\nu_1\cdots\nu_m\mu\nu_{m+1}\cdots\nu_n}(x,y_1,\ldots,y_m,y,y_{m+1},\ldots,y_n) - C^{(0)}_{\nu\nu_{m+1}\cdots\nu_n\mu 1\cdots\nu_m}(x,y_{m+1},\ldots,y_n,y,y_1,\ldots,y_m)]. \quad (3.34)$$

[Here the $1/N$ term in Eq. (3.32) does not contribute because of the commutator; notice also that the summation in Eq. (3.33) starts with $m=0$.] Thus differentiation of $A^{(0)a}_\mu$ produces terms with trace over substrings of a_μ's. Also, from Eq. (3.11), we see that the ghost current has the same structure:

$$J_\mu(A^{(0)})_{\rm gh}=i\sum_{n=1}^{\infty}(-g)^{n+1}D_\mu(x,1,\ldots,n)\sum_{k=0}^{n}\sum_{\{i_l\}}\{T^bA^{(0)}(i_1)\cdots A^{(0)}(i_k)T^bA^{(0)}(i_{k+1})\cdots A^{(0)}(i_n) - A^{(0)}(i_1)\cdots A^{(0)}(i_k)T^bA^{(0)}(i_{k+1})\cdots A^{(0)}(i_n)T^b\}, \quad (3.35)$$

where $\Sigma_{\{i_l\}}$ is the sum over all permutations of $\{1,\ldots,n\}$ such that $i_1<\cdots<i_k$ and $i_{k+1}>\cdots>i_n$. Using Eq. (3.32), we get

$$J_\mu(A^{(0)})_{\rm gh}=-\frac{i}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{\sigma\in S_{n;k}}g^{n+1}(-1)^k\widetilde{D}_\mu(x,\sigma_1,\ldots,\sigma_n)A^{(0)}(1)\cdots A^{(0)}(k){\rm Tr}[A^{(0)}(k+1)\cdots A^{(0)}(n)], \quad (3.36)$$

where $S_{n;k}$ is the set of all permutations of $\{1,2,\ldots,n\}$ that preserves the ordering of $\{\alpha\}\equiv\{1,\ldots,k\}$ and the cyclic ordering of $\{\beta\}\equiv\{n,n-1,\ldots,k+1\}$, while allowing for all possible relative orderings between elements in the two sets [for example, $(1,n,2,\ldots,k,n-1,\ldots)$ is in $S_{n;k}$, but $(2,1,\ldots,k,n,n-1,\ldots)$ is not]; $\widetilde{D}_\mu$ is given by

$$\widetilde{D}_\mu(x,1,\ldots,n)=D_\mu(x,1,\ldots,n)-(-1)^nD_\mu(x,n,n-1,\ldots,1). \quad (3.37)$$

Since Eq. (3.30) is at most linear in $A^{(1)}_\mu$, it is clear that $A^{(1)}_\mu$ can be written as

$$A^{(1)}_\mu(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\int C^{(1)}_{m\mu}(x,1,\ldots,m;m+1,\ldots,n)a(1)\cdots a(m){\rm Tr}[a(m+1)\cdots a(n)]. \quad (3.38)$$

Then in Eq. (3.30) we can separately equate terms with the same trace structure and $C^{(1)}_{m\mu}$'s with different m's do not mix with each other. The functions we need are then $C^{(1)}_{n\mu}=C^{(1)}(x,1,\ldots,n)$ corresponding to the term in Eq. (3.38) with no trace. These obey the recursion rule obtained by substituting $A^{(1)}=\Sigma_{n=1}^{\infty}\int C^{(1)}(x,1,\ldots,n)a(1)\cdots a(n)$ in Eq. (3.30). The other amplitudes can be obtained from $C^{(1)}_{n\mu}$.

The coefficient functions corresponding to different trace structures have simple relations among them (which allow us to construct the amplitudes with subtraces from $C^{(1)}_{n\mu}$). We have [15,16]

$$C^{(1)}_{m\mu}(x,1,\ldots,m;m+1,\ldots,n) =(-1)^{n-m}\sum_{\sigma\in S_{n;m}}C^{(1)}_{n\mu}(x,\sigma_1,\ldots,\sigma_n), \quad (3.39)$$

For this, we show that the right-hand side of Eq. (3.39) satisfies the same equation as that for the left-hand side. This is most easily seen for the ghost current which we shall consider first. Obviously, an equation like Eq. (3.39) holds for $\widetilde{D}_\mu$'s [with the identification of the summation indices k in Eq. (3.36) and m in Eq. (3.39)], if $A^{(0)}_\mu(x)=\int C^{(0)}_\mu(x,X)a(X)$ in (3.36) is replaced by its lowest order term a_μ. For the terms with more than one a_μ's from one $A^{(0)}$, there are potential discrepancies between both sides. This is because in Eq. (3.39) the sumation is over all $\sigma\in S_{n;m}$ while, in Eq. (3.36), we sum only over $\sigma\in S_{n;k}$ $(k<m)$, which does not include such permutations that mix indices from both $\{\alpha\}$ set and $\{\beta\}$ set within one tree structure. However, the extra terms in Eq. (3.39) cancel out thanks to the symmetry properties of color-ordered vertices:

$$V_3^{\mu\nu\rho}(p,q)+V_3^{\mu\rho\nu}(q,p)=0\,,$$

$$V_4^{\mu\nu\rho\sigma}+V_4^{\mu\rho\sigma\nu}+V_4^{\mu\sigma\nu\rho}=0\,, \quad (3.40)$$

which, when applied to Eq. (3.22) together with induction, leads to identities for $1\leq m<n$,

$$\sum_{\sigma\in S'_{n;m}}C^{(0)}_\mu(\sigma_1,\ldots,\sigma_n)=0\,, \quad (3.41)$$

where $S'_{n;m}$ is the same as $S_{n;m}$ defined above except that it does not include the cyclic permutations of $\{\beta\}$ set. (This equation can be considered as a generalization of the so-called ''dual Ward identity'' for tree amplitudes which corresponds to $m=1$ case [3,18].) Thus Eq. (3.39) holds for ghost current.

We shall now show that the relation Eq. (3.39) connecting amplitudes with different trace structures holds for the non-ghost terms in Eq. (3.38) as well. Towards this, consider the terms with differentiation $G(\delta/\delta a)$ in Eq. (3.30). Diagra-

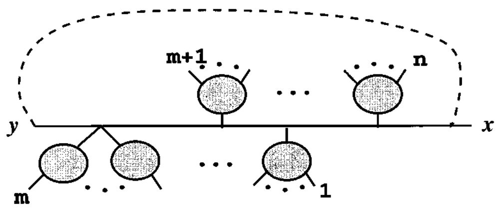

FIG. 1. A typical term in $C_{\mu}^{(0)}(x,1,\ldots,m,y,m+1,\ldots,n)$. y is connected to x and the loop encloses legs $m+1,\ldots,n$ which are under a trace.

matically, those terms correspond to one-loop gluon diagrams with the external leg x directly connected to the loop. In our setting, we can proceed as follows. Applying Eq. (3.22) repeatedly to $C_{\mu}^{(0)}(x,1,\ldots,m,y,m+1,\ldots,n)$ we can identify the internal line for each term from Eq. (3.22). It connects x and y, and the other external legs are ordered in a clockwise direction in such a way that $1,\ldots,m$th legs are below the line while $m+1,\ldots,n$th legs are above the line (Fig. 1). Then we draw a line connecting x and y which enclose $m+1,\ldots,n$th legs so that it represents that the legs inside the loop are traced. Then, it is easy to see which terms generate which color structures. Obviously, given such a diagram, we get the same subtrace structure from diagrams made by altering the relative order of trees belonging to different sets while keeping the order of $\{1,\ldots,m\}$ and the cyclic order of $\{m+1,\ldots,n\}$ (Fig. 2). Moreover, they are trivially related to terms which contribute to $C_{n\mu}^{(1)}$; i.e., we can simply pull out the trees inside the loop to outside with a minus sign, using the symmetry property of vertices, Eq. (3.40). Notice that during this procedure the order of $\{m+1,\ldots,n\}$ is also reversed. Thus, essentially we have the same situation as in the case of ghost current and a relation of the type (3.39) holds in this case as well.

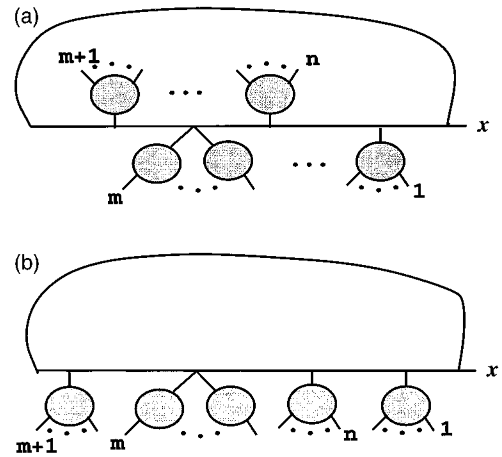

FIG. 2. (a) A diagram obtained by changing the relative order of traced legs and the others from Fig. 1. It contributes to $C_{m\mu}^{(1)}(x,1,\ldots,m;m+1,\ldots,n)$. (b) A diagram obtained by pulling the legs out of the loop in (a). It contributes to $C_{n\mu}^{(1)}(x,1,\ldots,n)$.

Now it remains to consider the term from $J^{(1)}$ which contains $C_{\mu}^{(1)}$'s with less number of legs. It corresponds to the gluon-loop diagram with the external leg x attached to a tree. This case, however, is not much different from the previous one and we can argue in the same way with the help of Eq. (3.41) if needed. Finally, there are counterterm contributions for the $C_{n\mu}^{(1)}(x,1,\ldots,n)$ case in contrast to $C_{m\mu}^{(1)}(x,1,\ldots,m;m+1,\ldots,n)$ $(m<n)$. But again the identity Eq. (3.41) guarantees that those terms cancel out when the summation in Eq. (3.39) is done. This completes the proof that both the left- and the right-hand sides of Eq. (3.39) satisfy the same recursion relation. Since Eq. (3.39) trivially holds for $n=3$, they are indeed equal to each other.

Beyond one loop, it is clear from Eq. (3.14) that $A_{\mu}^{(k)}$ will in general have terms with k subtraces,

$$A_{\mu}^{(k)}(x)=\sum_{n=1}^{\infty}\sum_{\{m_l\}}\int C_{m_1m_2\ldots m_k\mu}^{(k)}(x,1,\cdots,m_1;m_1+1,\ldots,m_2;\ldots;m_k+1\ldots n)$$
$$\times a(1)\cdots a(m_1)\mathrm{Tr}[a(m_1+1)\cdots a(m_2)]\cdots\mathrm{Tr}[a(m_k+1)\cdots a(n)], \tag{3.42}$$

and each $C_{m_1m_2\cdots m_k\mu}^{(k)}$ with different k will satisfy its own equation. It might be possible to find simple relations between them as in the one-loop case.

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