String theory in curved space-time

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Intrinsic and extrinsic geometric properties of string world sheets in a curved space-time background are explored. In our formulation, the only dynamical degrees of freedom of the string are its immersion coordinates. A classical equation of motion and the space-time energy-momentum tensor of the string are obtained. One-loop divergent terms are calculated using the background field method. A divergent Euler characteristic term appears in this order. The condition for one-loop finiteness is derived. The results obtained here differ from those in the standard procedure. [S0556-2821(97)02406-5]

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I. INTRODUCTION

String theory in curved space-times is an exciting subject and, in fact, has been investigated by many authors [1–6] as a framework to study the physics of gravitation in the context of string theory. In most investigations so far, the starting point for open or closed bosonic strings propagating in a D-dimensional space-time with the metric $h_{\mu\nu}(X)$ $(0\leq\mu,\nu\leq D-1)$ is the action

$$S=\frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{g}g_{\alpha\beta}(\sigma,\tau)h_{\mu\nu}(X)\times\partial_\alpha X^\mu(\sigma,\tau)\partial_\beta X^\nu(\sigma,\tau). \tag{1}$$

Here $g_{\alpha\beta}(\sigma,\tau)(\alpha,\beta=1,2)$ is the metric on the world sheet. α' is the string tension. The metric $h_{\mu\nu}$ may be taken as either a fixed background or as dynamically generated by the string. The dynamical variables of the theory described by Eq. (1) are the immersion coordinates $X^\mu(\sigma,\tau)$ and the world sheet metric $g_{\alpha\beta}$. Let us recall that the above action is obtained from the Nambu-Goto (NG) action

$$S=\frac{1}{2\pi\alpha'}\int\sqrt{g}d\sigma d\tau, \tag{2}$$

which in turn can be written as

$$S=\frac{1}{2\pi\alpha'}\int[\sqrt{g}+\lambda^{\alpha\beta}(\partial_\alpha X^\mu\partial_\beta X^\nu h_{\mu\nu}-g_{\alpha\beta})]d\sigma d\tau, \tag{3}$$

and presuming condensation of the Lagrange multiplier field $\lambda_{\alpha\beta}$ [5],

$$\langle\lambda_{\alpha\beta}\rangle=\text{const}\times\sqrt{g}g_{\alpha\beta}. \tag{4}$$

The reasons for starting with the alternative (1) in string theory are well known. The main points being that the integration over X^μ is easily done and the full power of the theory of Riemann surfaces can be exploited when integrating over the space of all two-metrics. On the other hand, a primary reason for not working with the NG action in the original form (2) where the world sheet is treated as an immersion is that it is nonlinear in X^μ.

In investigating strings in a flat or curved background, it will be shown in this paper that it is actually convenient to work with the NG action (2), where the only dynamical variables are $X^\mu(\sigma,\tau)$ while the metric of the world sheet is induced from its immersion in background geometry. In this scheme we do not have to contend with the Weyl symmetry of the two-dimensional (2D) metric. It will also become evident below that, since the geometric properties of the world sheet are described in terms of both its first and second fundamental forms, it is natural to take into account, besides the usual NG action, an action that involves the extrinsic geometry of the world sheet. Two-dimensional surfaces that extremize the NG action have $H^i=0$, where H^i $(i=1,\ldots,D-2)$ are the scalar mean curvatures of the surface. Any solution for the NG equation of motion is also a solution of the equation of motion for the extrinsic curvature action. In addition there exist a whole new class of surfaces that extremize the extrinsic curvature action. There exists a large body of work [7–11] on strings with extrinsic curvature action in Lorentzian space-time with the hope of describing QCD strings. The necessity of including such a term for QCD was emphasized in [7]. The extrinsic geometry dependent term in the action provides rigidity to the strings, while the NG term provides tension. Rigidity effects are important also in determining the shapes of biological membranes [12,13]. In [14], the Nambu-Goto action with the square root of a metric which is a sum of the induced metric and the third fundamental form has been studied.

It is thus natural to investigate the role of extrinsic geometry on the properties of strings in curved space-time. String theory has a dimensional parameter, i.e., the string tension. Rigid strings have a dimensionless coupling which in flat background is asymptotically free. We shall see that this coupling is asymptotically free in the curved background as well.

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Although we discuss primarily the string world sheet in the curved background, many of the results obtained here may be used in the context of the (3+1) canonical formulation of gravity. It may be recalled that in the (3+1) formulation of Arnowitt, Deser, and Misner (ADM) [15,16], the Cauchy problem of general relativity can be tackled by constructing a space-time $\widetilde{M}$ out of a foliation of spacelike hypersurfaces with time coordinate t as a parameter. In this formalism the quantities needed for the description of the four-dimensional line element are the lapse function, which measures distance between nearby hypersurfaces, the shift vector that gives the relation between the spatial coordinate systems on different hypersurfaces and the 3D space metric to enable one to measure distances in spacelike, $t=\text{const}$ slices. In 4D space-time, Einstein's theory leads to three-momentum constraints, the Hamiltonian constraint, and six equations of motion. The hyperbolic form of Einstein's theory can be obtained [17,18] by imposition of a slicing condition on the space-time. One such condition involves taking $t=\text{const}$ surfaces to be constant mean curvature hypersurfaces. Thus, for example, using harmonic Gauss map, one can solve the constraints in (2+1) gravity [19–21]. The considerations in this paper offer other possibilities for choosing hypersurfaces that extremize the extrinsic curvature action. Two-dimensional surfaces of prescribed mean curvature are studied as solutions of the Einstein constraint equations on closed manifolds in [22,23].

Earlier studies in string theory in curved background, for example, in [24–28], concentrated on the classical equation of motion in order to exhibit its nonlinearity. While the authors in [24,27,28] considered Nambu-Goto action in curved background, in [25,26] the authors included an extrinsic curvature dependent term in the action. In these studies the first variation of the action giving the classical equations of motion is examined for various choices of the background spacetime. Some of the classical aspects of strings in curved background discussed in this paper have also been considered recently by Capovilla and Guven [29]. They have considered the geometry of deformations of relativistic membranes. In particular they have derived the second variation of the NG action in a general curved space and that of the extrinsic curvature action in a background Minkowski spacetime. In this paper we are interested in both the classical and the quantum aspects of strings in curved space. We calculate the second variation of both the NG and the extrinsic curvature actions in arbitrary curved space-time which is then used to calculate the renormalization of both the dimensionless extrinsic curvature coupling and the background metric. Furthermore, in the path integral formalism adopted here, where the dynamical variables of the theory are the string's immersion coordinates only, it is shown here explicitly that the volume of the two-dimensional diffeomorphism group is cancelled by the volume of the space of tangential fluctuations of the world sheet of the string. This demonstration for on shell amplitudes turns out to be nontrivial.

This paper is organized as follows. In Sec. II, we consider the classical properties of both the NG and the extrinsic curvature actions in a curved space-time. We derive their equations of motion and discuss possible solutions. The space-time energy-momentum tensor $T^{\mu\nu}$ of the string is derived in Sec. III and in the Appendix we establish that it is covariantly conserved. In Sec. IV we discuss the one-loop divergences of the theory by integrating over X^μ. As a consequence of reparametrization invariance only the fluctuations normal to the world sheet are dynamical. The longitudinal fluctuations are zero modes and their volume cancels the volume of the diffeomorphism group of the world sheet. It will be seen that a divergent Euler characteristic term appears at the one-loop level while such a term was previously known to appear only at the two-loop level in the conventional approach [5,30], where the dynamical degrees of freedom are the immersion coordinates and the metric of the world sheet. Furthermore, only the background metric gets renormalized in our scheme while the renormalization of both the background metric and the Liouville mode are needed in the conventional scheme [4,5]. The condition for one loop finiteness is also derived.

II. NG AND EXTRINSIC CURVATURE ACTIONS IN CURVED SPACE-TIME

In the previous section we discussed the NG action. If the metric $h_{\mu\nu}$ of space-time is taken as dynamical, then, we must include the Hilbert-Einstein action

$$S_{\text{HE}}=-\frac{1}{8\pi G}\int \widetilde{R}\sqrt{h}d^DX. \tag{5}$$

We denote the scalar curvature of space-time $\widetilde{M}$ by $\widetilde{R}$. We use the notation whereby geometric quantities of space-time will be denoted with a tilde and the world sheet quantities will be unadorned.

In order to understand rigid strings, it is necessary to recall the structure equations for immersed surfaces in $\widetilde{M}$. First, the equation of Gauss

$$\partial_\alpha\partial_\beta X^\mu+\widetilde{\Gamma}^\mu_{\nu\rho}\partial_\alpha X^\nu\partial_\beta X^\rho-\Gamma^\gamma_{\alpha\beta}\partial_\gamma X^\mu=H^i_{\alpha\beta}N^{i\mu}, \tag{6}$$

defines the second fundamental form $H^i_{\alpha\beta}$ $(i=1,2,\ldots,D-2)$. In Eq. (6) $\widetilde{\Gamma}^\mu_{\nu\rho}$ is the connection in $\widetilde{M}$ determined by the metric $h_{\mu\nu}$, while $\Gamma^\gamma_{\alpha\beta}$ is the connection coefficient on string world sheet M determined by the induced metric $g_{\alpha\beta}$ on M. $N^{i\mu}$ $(i=1,2,\ldots,D-2)$ are the $(D-2)$ normals to the string world sheet at $X^\mu(\sigma,\tau)$. We choose the normalization

$$N^{i\mu}N^{j\nu}h_{\mu\nu}=\delta_{ij}\epsilon(i), \tag{7}$$

and

$$\partial_\alpha X^\mu N^{i\nu}h_{\mu\nu}=0 \quad (i=1,2,\ldots,D-2). \tag{8}$$

$\epsilon(i)=\pm 1$ depending on whether the world sheet M has an induced metric which is Riemannian or pseudo-Riemannian. For string theory, the world sheet metric has Lorentzian signature and hence $\epsilon(i)$ in Eq. (7) is $+1$ for $i=1,2,\ldots,D-2$. In the context of canonical (3+1) gravity, the hypersurface has induced metric which is Riemannian and so $\epsilon(i)=-1$. Note that Eq. (6) may be expressed in compact form as

$$\nabla_\alpha\nabla_\beta X^\mu=H^i_{\alpha\beta}N^{i\mu},$$

where ∇_α is the symbol for covariant derivative. Note that $\nabla_\beta X^\mu \equiv \partial_\beta X^\mu$ is a world vector as well as a two-vector and hence needs both connections $\tilde{\Gamma}^\mu_{\nu\rho}$ and $\Gamma^\gamma_{\alpha\beta}$ to define its covariant derivative.

Next, we have the equation of Weingarten

$$\partial_\alpha N^{i\mu} + \tilde{\Gamma}^\mu_{\nu\rho}\partial_\alpha X^\nu N^{i\rho} - A^{ij}_\alpha N^{j\mu} = -\epsilon(i) H^{i\beta}_\alpha \partial_\beta X^\mu. \quad (9)$$

In Eq. (9)

$$A^{ij}_\alpha = \epsilon(i)(N^{j\nu}\tilde{\nabla}_\alpha N^{i\mu}) h_{\mu\nu}, \quad (10)$$

$$\tilde{\nabla}_\alpha N^{i\mu} = \partial_\alpha N^{i\mu} + \tilde{\Gamma}^\mu_{\nu\rho}\partial_\alpha X^\nu N^{i\rho}. \quad (11)$$

A^{ij}_α is the 2D gauge field or gauge connection in the normal bundle.

We shall use the notation whereby we covariantize on all indices. μ is the world index, α,β, the two-dimensional world sheet index, and i,j the ''internal'' indices of the normal bundle. Thus

$$\nabla_\alpha N^{i\mu} \equiv \partial_\alpha N^{i\mu} + \tilde{\Gamma}^\mu_{\nu\rho}\partial_\alpha X^\nu N^{i\rho} - A^{ij}_\alpha N^{j\mu}. \quad (12)$$

Note that in D-dimensional space-time, $A^{ij}_\alpha = -A^{ji}_\alpha$ are two-dimensional gauge fields with the gauge group $\mathrm{SO}(D-2)$.

In addition to the equations of Gauss and Weingarten, there are three key equations we need in the sequel. These are the equations of Gauss, Codazzi, and Ricci. These are well known and we merely give these below for completeness [31,32].

Equation of Gauss:

$$\tilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma$$
$$= R_{\alpha\beta\gamma\delta} + \sum_i \epsilon(i)(H^i_{\beta\gamma}H^i_{\alpha\delta} - H^i_{\beta\delta}H^i_{\alpha\gamma}), \quad (13)$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature of the string world sheet.

Equation of Codazzi:

$$\nabla_\alpha H^i_{\beta\gamma} - \nabla_\beta H^i_{\alpha\gamma} = \epsilon(i)\tilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\sigma N^{i\rho}, \quad (14)$$

and equation of Ricci:

$$\tilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\mu \partial_\beta X^\nu N^{j\rho} N^{i\sigma} = F^{ij}_{\alpha\beta} + H^{i\gamma}_\alpha H^j_{\gamma\beta} - H^{i\gamma}_\beta H^j_{\gamma\alpha}, \quad (15)$$

where

$$F^{ij}_{\alpha\beta} = \partial_\alpha A^{ij}_\beta - \partial_\beta A^{ij}_\alpha + A^{ik}_\alpha A^{kj}_\beta - A^{jk}_\alpha A^{ki}_\beta. \quad (16)$$

In Eq. (14) we have used the notation

$$\nabla_\alpha H^i_{\beta\gamma} = \partial_\alpha H^i_{\beta\gamma} - \Gamma^\delta_{\alpha\beta}H^i_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma}H^i_{\beta\delta} - A^{ij}_\alpha H^j_{\beta\gamma}. \quad (17)$$

In general, these are the only relations between the curvature tensor of the space-time and the extrinsic curvature of the world sheet. In the case when M is a hypersurface in space-time there is only one normal and so $A^{ij}_\alpha \equiv 0$. Further relations may be obtained when the space-time metric is written in terms of, say, the isothermal coordinates [16,33]. The choice of the extrinsic curvature action for strings is motivated from the relation that follows from Eq. (13) when $\tilde{R}_{\mu\nu\rho\sigma}=0$ ($\tilde{M}$ is Minkowski). In this case,

$$R = \sum_i \epsilon(i)[H^i_{\alpha\beta}H^{i\alpha\beta} - (H^{i\alpha}_\alpha)^2], \quad (18)$$

where R is the scalar curvature of the world sheet. In two-dimensions,

$$\frac{1}{4\pi}\int d\sigma d\tau \sqrt{g} R = \chi, \quad (19)$$

where χ is the Euler characteristic of the two-dimensional manifold. On the other hand, each term on the right in Eq. (18) is separately reparametrization invariant and hence can be used as the extrinsic curvature action. For convenience, we take this as (subscript R for rigid strings)

$$S_R = \frac{1}{\alpha_0}\int \sqrt{g} H^2 d\sigma d\tau, \quad (20)$$

where

$$H^2 = \sum_i \left(\frac{1}{2}H^{i\alpha}_\alpha\right)^2. \quad (21)$$

We continue to take Eq. (20) as the extrinsic curvature action for rigid strings in curved background. [From now on we shall drop the factor $\epsilon(i)$ for convenience. In fact, for strings all normals will be spacelike and $\epsilon(i)=+1$.] α_0 in Eq. (20) is a dimensionless coupling. In a flat background, it is known that this coupling is asymptotically free [7,10]. Note that if we considered an action

$$S = \int \tilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma g^{\beta\gamma} g^{\alpha\delta}\sqrt{g}\, d\sigma d\tau, \quad (22)$$

it will not take into account extrinsic curvature or bending rigidity effects although such terms will probably arise in the effective action starting from the NG term.

In Minkowski space-time, Eq. (20), in addition to being reparametrization invariant on the world sheet, is scale invariant under $X^\mu \to \lambda X^\mu$. Although not usually emphasized in the literature, the action (20) is actually invariant under space-time conformal transformations [34–36]. It is not too difficult to show that Eq. (20) is invariant under four-translations, scale transformation, Lorentz transformations and special conformal transformations [37]

$$X'^\mu = \frac{X^\mu - b^\mu X^2}{1 - 2b\cdot X + b^2 X^2}, \quad (23)$$

where b^μ is a constant D vector. Invariance under the first three is manifest. It is readily verified that under infinitesimal special conformal transformation, Eq. (20) is transformed into a total divergence and hence either vanishes for closed surfaces (without boundary) or turns into a boundary contribution.

Let us concentrate on the properties of Eq. (20) in curved background. In order to appreciate the nature of the string world sheet that satisfies the equation of motion of Eq. (20), it is instructive to start with the NG action (2). A straightforward computation shows that the equation of motion of Eq. (2) can be written as

$$\sum_i N^{i\mu} H^i = 0. \tag{24}$$

This is equivalent to $H^i = 0$, $i = 1,2, \ldots, D-2$. In other words, surfaces with zero mean curvature, i.e., minimal surfaces, extremize the NG action. This conclusion is independent of whether the background space-time is flat or curved. Note that the equation of motion for the NG action is algebraic in the scalar mean curvature. In flat space-time it reduces to

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0, \tag{25}$$

upon using isothermal coordinates on the string world sheet. In curved background it reads

$$g^{\alpha\beta} \nabla_\alpha \partial_\beta X^\mu = 0, \tag{26}$$

where $\nabla_\alpha \partial_\beta X^\mu$ is defined in Eq. (6).

In the path integral formulation, classical solutions provide a starting point for non-perturbative calculations. Thus one can integrate over minimal surfaces for the NG action. In fact the present authors have carried out such a calculation for minimal and harmonic ($H^i = \text{const}$) surfaces for rigid strings in a flat background space-time and obtained a modified Coulomb gas picture of QCD effective string action [11]. This demonstration [11] utilized the notion of the Gauss map from the world sheet into the Grassmannian manifold.

Next, consider the equation of motion of rigid string action in curved space-time. Surprisingly, the derivation is rather involved in this case. The special case of an hypersurface in D-dimensions has been discussed in [38]. We sketch below the main steps.

Starting from Eq. (20) we have, under $X^\mu \to X^\mu + \delta X^\mu$,

$$\delta S_R = \frac{1}{\alpha_0} \left(\frac{1}{2} \int \sqrt{g} g^{\alpha\beta} \delta g_{\alpha\beta} H^2 d\sigma d\tau + 2 \int H^i \delta H^i \sqrt{g} d\sigma d\tau \right). \tag{27}$$

A simple calculation yields

$$\delta g_{\alpha\beta} = \nabla_\alpha (\delta X^\mu \partial_\beta X^\nu h_{\mu\nu}) + \nabla_\beta (\delta X^\mu \partial_\alpha X^\nu h_{\mu\nu}) - 2\, \delta X^\mu h_{\mu\nu} N^{i\nu} H^i_{\alpha\beta}, \tag{28}$$

$$\delta N^{i\mu} + \delta X^\sigma N^{i\nu} \Gamma^\mu_{\sigma\nu} = -\partial_\alpha X^\mu [\nabla^\alpha (N^{i\nu} \delta X^\sigma h_{\nu\sigma}) + H^{i\alpha\beta} \partial_\beta X^\nu \delta X^\rho h_{\nu\rho}], \tag{29}$$

$$\delta H^i = -\nabla_\alpha (H^{i\alpha\beta} \delta X^\mu \partial_\beta X^\nu h_{\mu\nu}) + (\nabla_\alpha H^{i\alpha\beta}) \partial_\beta X^\nu \delta X^\mu h_{\mu\nu} + H^{i\alpha\beta} H^j_{\alpha\beta} N^{j\nu} \delta X^\mu h_{\mu\nu} + \tfrac{1}{2} g^{\alpha\beta} \delta H^i_{\alpha\beta}, \tag{30}$$

$$\begin{aligned} g^{\alpha\beta} H^i \delta H^i_{\alpha\beta} = {} & \nabla^\alpha [(\nabla_\alpha \delta X^\mu) N^{i\nu} h_{\mu\nu} H^i - \delta X^\mu h_{\mu\nu} \nabla_\alpha (N^{i\nu} H^i)] \\ & + g^{\alpha\beta} H^i \widetilde{R}_{\mu\nu\rho\sigma} \partial_\alpha X^\nu \partial_\beta X^\rho \delta X^\sigma N^{i\mu} \\ & + \delta X^\mu h_{\mu\nu} \{ H^i [-(\nabla^\alpha H^{i\gamma}_\alpha) \partial_\gamma X^\nu \\ & - H^i H^j_{\alpha\beta} H^{i\alpha\beta} N^{j\nu}] - 2 H^{i\alpha\gamma} (\partial_\gamma X^\nu) \nabla_\alpha H^i \\ & + N^{i\nu} \nabla^2 H^i \}. \end{aligned} \tag{31}$$

In Eq. (31), $\widetilde{R}_{\mu\nu\rho\sigma}$ is the curvature tensor of the background space-time defined by

$$\widetilde{R}^\mu_{\nu\rho\sigma} = \partial_\rho \widetilde{\Gamma}^\mu_{\nu\sigma} - \partial_\sigma \widetilde{\Gamma}^\mu_{\nu\rho} + \widetilde{\Gamma}^\lambda_{\nu\sigma} \widetilde{\Gamma}^\mu_{\rho\lambda} - \widetilde{\Gamma}^\lambda_{\nu\rho} \widetilde{\Gamma}^\mu_{\sigma\lambda}. \tag{32}$$

Let us note from Eq. (29) that $\delta N^{i\mu}$ is not a covariant vector, but that $\delta N^{i\mu} + \delta X^\sigma \Gamma^\mu_{\sigma\nu} N^{i\nu}$ is. Using Eqs. (28)–(31) in Eq. (27) one finds

$$\begin{aligned} \delta S_R = \frac{1}{\alpha_0} \int & (\nabla^\alpha [(H^2 g_{\alpha\beta} - 2 H^i H^i_{\alpha\beta}) \delta X^\mu \partial^\beta X^\nu h_{\mu\nu} + (\nabla_\alpha \delta X^\mu) N^{i\nu} h_{\mu\nu} H^i - \delta X^\mu h_{\mu\nu} \nabla^\alpha (N^{i\nu} H^i)] + \{[-2(\nabla^\alpha H^i) H^i \\ & + H^i \nabla_\beta H^{i\alpha\beta}] \partial_\alpha X^\nu \delta X^\mu h_{\mu\nu} + (\nabla^2 H^i - 2 H^2 H^i + H^j H^j_{\alpha\beta} H^{i\alpha\beta}) \delta X^\mu N^{i\nu} h_{\mu\nu} - g^{\alpha\beta} H^i \partial_\alpha X^\nu \partial_\beta X^\rho N^{i\mu} \delta X^\sigma \widetilde{R}_{\mu\nu\rho\sigma}\}) \sqrt{g} d\sigma d\tau. \end{aligned} \tag{33}$$

Equations of motion can be obtained by decomposing δX^μ into normal and tangential variations of the world sheet. Accordingly, we write

$$\delta X^\mu = \xi_j N^{j\mu} + \partial_\alpha X^\mu \xi^\alpha. \tag{34}$$

Then it follows from Eq. (33) that the equation of motion for normal variations is

$$\nabla^2 H^i - 2 H^2 H^i + H^j H^{j\alpha\beta} H^i_{\alpha\beta} - g^{\alpha\beta} H^j \widetilde{R}_{\mu\nu\rho\sigma} \partial_\alpha X^\nu \partial_\beta X^\rho N^{j\mu} N^{i\sigma} = 0, \tag{35}$$

where $i = 1,2, \ldots, D-2$. The tangential variations yield

$$\sum_i H^i [\nabla_\beta H^{i\alpha\beta} - 2 \nabla^\alpha H^i - \widetilde{R}_{\mu\nu\rho\sigma} \partial_\gamma X^\nu \partial_\delta X^\rho g^{\gamma\delta} \partial^\alpha X^\sigma N^{i\mu}] = 0. \tag{36}$$

The equations of motion for the action that is a sum of the NG and extrinsic curvature actions is easily seen to be

$$\nabla^2 H^i - 2 H^i (H^2 + k) + H^j H^{j\alpha\beta} H^i_{\alpha\beta} - g^{\alpha\beta} H^j \widetilde{R}_{\mu\nu\rho\sigma} \partial_\alpha X^\nu \partial_\beta X^\rho N^{j\mu} N^{i\sigma} = 0, \tag{37}$$

where $k = -\alpha_0 / 2\pi\alpha'$. Equation (36) is, however, unchanged. In Eq. (35),

$$\nabla^2 \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta,$$

$$\nabla_\alpha H^i = \partial_\alpha H^i - A_\alpha^{ij}H^j. \tag{38}$$

Equation (36) is actually not an equation of motion. It is just the structure equation of Codazzi (14) once contracted. This is a consequence of reparametrization invariance of the action. Thus the surfaces with mean curvature H^i satisfying Eq. (35) extremize the rigid string action in curved background. It is worth remarking here that the extrinsic curvature action contains a fourth derivative operator acting on X^μ. Nevertheless, its equation of motion is expressible as a second-order nonlinear partial differential equation in the mean scalar curvature H^i of the world sheet. Once one has a solution of Eq. (35) for H^i then one can solve for immersion coordinates X^μ from Eq. (6). Alternately, expressing H^i and $H^i_{\alpha\beta}$ in terms of X^μ, we find Eq. (35) can be written as

$$\nabla^4 X^\mu - 2h_{\rho\sigma}(\partial_\alpha \nabla^2 X^\rho)(\partial^\gamma X^\sigma)(\nabla_\gamma \partial^\alpha X^\mu) - h_{\rho\sigma}(\nabla^2 X^\rho)(\partial_\gamma X^\sigma)(\partial^\gamma \nabla^2 X^\mu) - \tfrac{1}{2}h_{\rho\sigma}(\nabla^2 X^\rho)(\nabla^2 X^\sigma)(\nabla^2 X^\mu)$$
$$- \tfrac{1}{2}\widetilde{R}^\mu_{\sigma\rho\nu}\partial^\alpha X^\sigma \cdot \partial_\alpha X^\rho (\nabla^2 X^\nu) = 0, \tag{39}$$

which is a fourth-order nonlinear partial differential equation for X^μ.

Minimal surfaces, i.e., $H^i=0$, trivially satisfy the equation of motion (39) for rigid strings. Solutions to Eq. (39) in flat Minkowski background have been studied both analytically and numerically in [8,9]. In particular, for closed strings, finite energy static configurations are found which have nonlinear Regge trajectories at the low energy end of the spectrum. Solutions to NG (nonrigid) strings in cosmological space times have been investigated in [6] (and references cited therein). They develop a method for solving Eq. (26) written in the form of the geodesic equation of motion and find both stable and unstable configurations. It would thus be interesting to investigate similar stability considerations for rigid strings. In flat space-time, solutions to Eq. (35) in three-dimensional Euclidean space are referred to as Willmore surfaces [36] and have been investigated in the context of biological membranes in [39]. This equation takes the form [after using Eq. (13)]

$$\nabla^2 H + 2H\left(H^2 - \frac{R}{2}\right) = 0, \tag{40}$$

where R is the scalar curvature of the world sheet. A sphere satisfies this equation with constant H. Solutions with nonzero genus are known to exist for Eq. (40) [36]. Note that Eq. (40) is a nonlinear wave equation for H^i in the context of string theory.

The special case of a hypersurface in curved space-time is worth considering. In this case the indices i,j take only one value and α,β take values 1,2,3. Then Eq. (35) reads

$$\nabla^2 H - 2H^3 + HH_{\alpha\beta}H^{\alpha\beta} - H\widetilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\nu \partial_\beta X^\rho g^{\alpha\beta}N^\mu N^\sigma = 0. \tag{41}$$

Using Eq. (13) to eliminate $H_{\alpha\beta}H^{\alpha\beta}$ from Eq. (41) we find

$$\nabla^2 H + 2H^3 + H[\widetilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma g^{\alpha\gamma}g^{\beta\delta} - R$$
$$- \widetilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\nu \partial_\beta X^\rho g^{\alpha\beta}N^\mu N^\sigma] = 0. \tag{42}$$

In [38] the term $\widetilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma g^{\alpha\gamma}g^{\beta\delta}$ is replaced by $\widetilde{R}$, which in their notation presumably is the scalar curvature of $\widetilde{M}$, which is given by $\widetilde{R}_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma}$. This would imply that $\partial_\alpha X^\mu \partial_\gamma X^\rho g^{\alpha\gamma} = h^{\mu\rho}$ which is, of course, wrong [see Eq. (61) below].

In string theory, it would be most interesting to investigate solutions to Eq. (35) in cosmological backgrounds. Let us however restrict here to de Sitter or anti–de Sitter space-times, i.e., space-times of constant curvature. We can write for these space-times

$$\widetilde{R}_{\mu\nu\rho\sigma} = K(h_{\mu\rho}h_{\nu\sigma} - h_{\mu\sigma}h_{\nu\rho}). \tag{43}$$

The equation of motion (35) reads [we choose $\epsilon(i)=1$]

$$\nabla^2 H^i - 2H^i(H^2+K) + H^j H^j_{\alpha\beta}H^{i\alpha\beta} = 0. \tag{44}$$

The corresponding equation of motion (26) for the Nambu-Goto action can be written, using Eq. (6), as

$$\partial_\alpha \partial_\beta X^\mu + \widetilde{\Gamma}^\mu_{\nu\rho}\partial_\alpha X^\nu \partial_\beta X^\rho - \Gamma^\gamma_{\alpha\beta}\partial_\gamma X^\mu = 0. \tag{45}$$

In de Sitter or anti–de Sitter background, the above equation has been studied by [40,41] and found to be related to sinh-Gordon, cosh-Gordon, or Liouville equations. It would be of interest to analyze solutions to Eq. (44).

Locally conserved currents arising as a consequence of the invariance of the extrinsic curvature action under local space-time conformal transformations may be readily obtained from the total divergence terms in Eq. (33). Let us discuss, in particular, only the string's space-time momenta. It is easily seen to be given by

$$(P^\alpha_\mu)_R = \frac{\sqrt{g}}{\alpha_0}[(H^2 g^{\alpha\beta} - H^i H^{i\alpha\beta})\partial_\beta X^\nu h_{\mu\nu} - (\nabla^\alpha H^i)N^{i\nu}h_{\mu\nu}]. \tag{46}$$

We notice that $(P^\alpha_\mu)_R N^{i\mu} = -(\sqrt{g}/\alpha_0)(\nabla^\alpha H^i)$ is the component of string momenta normal to its world sheet. This measures the bending energy of the rigid string. In comparison the space-time momentum of the NG string is given by

$$(P^{\alpha}_{\mu})_{\rm NG}=\frac{\sqrt{g}}{2\pi\alpha'}g^{\alpha\beta}(\partial_\beta X^\nu)h_{\mu\nu}, \tag{47}$$

which has no component normal to the world sheet.

III. STRING ENERGY-MOMENTUM TENSOR

The string's space-time energy-momentum tensor $T^{\mu\nu}$ is obtained by taking the functional derivative of the action with respect to the metric $h_{\mu\nu}$ at the space-time point X. The calculation nevertheless turns out to be rather long. We quote here the result:

$$\delta S_R=\frac{1}{2}\int\sqrt{h}T^{\mu\nu}\delta h_{\mu\nu}d^DX, \tag{48}$$

with

$$T^{\mu\nu}(X)=\frac{1}{\alpha_0}\int d\sigma d\tau\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\Big(\sqrt{g}\partial_\alpha X^\mu\partial_\beta X^\nu(H^2g^{\alpha\beta}-H^iH^{i\alpha\beta})-\frac{1}{2}\sqrt{g}g^{\alpha\beta}\nabla_\beta H^i(N^{i\nu}\partial_\alpha X^\mu+N^{i\mu}\partial_\alpha X^\nu)$$
$$+2\sqrt{g}H^iH^jN^{i\mu}N^{j\nu}\Big)-\frac{\nabla_\rho}{2\alpha_0}\int d\sigma d\tau\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}g^{\alpha\beta}H^i[\partial_\beta X^\mu(N^{i\nu}\partial_\alpha X^\rho-N^{i\rho}\partial_\alpha X^\nu)$$
$$+\partial_\beta X^\nu(N^{i\mu}\partial_\alpha X^\rho-N^{i\rho}\partial_\alpha X^\mu)]. \tag{49}$$

Note that X in Eq. (49) is just a space-time point, whereas $X(\sigma,\tau)$ stands for the string dynamical variables. One sees from the D-dimensional Dirac δ function in Eq. (49) that $T^{\mu\nu}(X)$ vanishes unless X is exactly on the string world sheet. It is instructive to compare Eq. (49) with the energy-momentum tensor of the NG strings.

$$T^{\mu\nu}_{\rm NG}(X)=\frac{1}{\alpha'}\int d\sigma d\tau\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu. \tag{50}$$

Note that the flux of $T^{\mu\nu}_{\rm NG}$ along the normal direction $T^{\mu\nu}_{\rm NG}N^i_\nu(X)=0$. This is no longer true for rigid strings as can be seen from Eq. (49). This is consistent with the remarks in the Introduction that bending of the strings costs energy.

It can be verified, although the calculations are rather long, that

$$\nabla_\mu T^{\mu\nu}=0 \tag{51}$$

when the equation of motion are taken into account. Demonstration of Eq. (51) is sketched in the Appendix.

IV. ONE-LOOP QUANTUM EFFECTS

A. Nambu-Goto action

We now discuss one-loop quantum effects for strings in background space-time. In the conventional treatment, the NG action is taken in the form (1) in which the dynamical variables are $g_{\alpha\beta}$ and X^μ and in this form the action is regarded as a nonlinear σ model [5]. We show in this section that it is more convenient to perform computations by regarding the world sheet as an immersed surface and the metric on it as induced from the background space-time. The dynamical fields are just the immersion coordinates X^μ. It will become clear that in this scheme not only the metric of the world sheet but also its extrinsic geometry play a key role in the dynamics. The partition function for the string is given by

$$\mathbf{Z}=\int\frac{\mathcal{D}X^\mu}{\rm Vol}\exp\left(-\frac{1}{2\pi\alpha'}\int\sqrt{g}d\sigma d\tau\right), \tag{52}$$

where Vol is the volume of the 2D diffeomorphism group. To calculate one-loop effects we need the second variation of the action. We now evaluate this on shell, i.e., for minimal surfaces. As in Eq. (34) we decompose the fluctuations δX^μ into normal and tangential components. For the NG action the first variation of the action can be written as

$$\delta S=-\frac{1}{\pi\alpha'}\int\sqrt{g}H^i\xi_i d\sigma d\tau. \tag{53}$$

To evaluate the second variation on shell ($H^i=0$), we need only evaluate the variation of H^i. It is convenient to rewrite Eqs. (30) and (31) in the form

$$\delta H^i_{\alpha\beta}=(\nabla_\alpha\nabla_\beta\delta^i_j-H^i_{\alpha\gamma}H^{j\gamma}_\beta-\tilde{R}_{\mu\nu\rho\sigma}N^{i\mu}N^{j\sigma}\partial_\alpha X^\rho\partial_\beta X^\nu)\xi_j$$
$$+(H^{i\gamma}_\alpha\nabla_\beta+H^{i\gamma}_\beta\nabla_\alpha)\xi_\gamma+(\nabla^\gamma H^i_{\alpha\beta})\xi_\gamma. \tag{54}$$

In arriving at the result above from Eqs. (30) and (31), we have made use of the equation of Gauss (13). From above we find that δH^i can be expressed as

$$\delta H^i=\frac{1}{2}\mathbf{O}^i_j\xi^j+(\nabla^\alpha H^i)\xi_\alpha, \tag{55}$$

where

$$\mathbf{O}^i_j=\nabla^2\delta^i_j+H^i_{\alpha\beta}H^{j\alpha\beta}-\tilde{R}_{\mu\nu\rho\sigma}g^{\alpha\beta}\partial_\alpha X^\nu\partial_\beta X^\rho N^{i\mu}N^{j\sigma}. \tag{56}$$

Substituting from above we can write the second variation of Eq. (2) as follows (see also [29]):

$$\delta^2 S = -\frac{1}{2\pi\alpha'}\int \sqrt{g}\,\xi^i \mathbf{O}^i_j \xi^j d\sigma d\tau. \tag{57}$$

Note that only the normal-normal fluctuations appear in the second variation when evaluated on shell. As a consequence $\int(\mathcal{D}X^\mu/\mathrm{Vol})\rightarrow\int(\mathcal{D}\xi^\alpha/\mathrm{Vol})\mathcal{D}\xi^i\rightarrow\int\mathcal{D}\xi^i$.

The divergent part of the effective action $\Gamma^{(1)}$ can now be read off from Eqs. (56) and (57) using the method of [42]. One finds

$$\Gamma^{(1)} = \frac{I}{2}\int d\sigma d\tau \sqrt{g}\{\widetilde{R}_{\lambda\nu\rho\sigma} g^{\alpha\beta}\partial_\alpha X^\nu \partial_\beta X^\rho N^{i\lambda} N^{i\sigma} - H^i_{\alpha\beta} H^{i\alpha\beta}\}, \tag{58}$$

where

$$I = \int^\Lambda \frac{d^2k}{(2\pi)^2 k^2} \tag{59}$$

is the logarithmic divergent term. The covariant derivative ∇ in Eq. (54) contains the gauge connection A^{ij}_α. But counterterms dependent on A^{ij}_α do not arise since they can appear only through $\mathrm{Tr}(F_{\alpha\beta}F^{\alpha\beta})$ which is a dimension-four operator.

Now from Gauss' formula (13) we find

$$H^i_{\alpha\beta} H^{i\alpha\beta} = 4H^2 + R - \widetilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma g^{\beta\gamma} g^{\alpha\delta}. \tag{60}$$

R in above is the scalar curvature of the world sheet. Note that for NG action we put $H=0$ in the above formula. Substituting Eq. (60) in Eq. (58) and making use of the completeness relation

$$\sum_{i=1}^{D-2} N^{i\lambda}N^{i\sigma} + \partial_\gamma X^\lambda \partial_\delta X^\sigma g^{\gamma\delta} = h^{\lambda\sigma}, \tag{61}$$

we get

$$\Gamma^{(1)} = \frac{I}{2}\int d\sigma d\tau \sqrt{g}\{\widetilde{R}_{\nu\rho}\partial_\alpha X^\nu \partial_\beta X^\rho g^{\alpha\beta} - R\}, \tag{62}$$

where $\widetilde{R}_{\nu\rho}$ is the Ricci tensor of space-time. As claimed before, the Euler characteristic term appears in one-loop order. This is to be contrasted with the results in [5,30]. Both divergent terms in Eq. (62) can be absorbed in the renormalization of the background metric $h_{\mu\nu}$. Thus, defining renormalized $h_{\mu\nu}$ by

$$h^R_{\mu\nu} = h_{\mu\nu} + \delta h_{\mu\nu}, \tag{63}$$

we find that

$$\sqrt{g} = \det[\partial_\alpha X^\mu \partial_\beta X^\nu (h^R_{\mu\nu} - \delta h_{\mu\nu})] \simeq \sqrt{g}(1 - \tfrac{1}{2} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \delta h_{\mu\nu}). \tag{64}$$

Thus, we define

$$\delta h_{\mu\nu} = 2\pi\alpha' I(\widetilde{R}_{\mu\nu} - \tfrac{1}{2} R h_{\mu\nu}). \tag{65}$$

The divergent Euler characteristic term thus gets absorbed in the redefinition of the background metric. Alternatively, for one-loop finiteness of string theory the necessary condition is

$$\widetilde{R}_{\mu\nu}(X(\sigma,\tau)) - \tfrac{1}{2} R(X(\sigma,\tau)) h_{\mu\nu}(X(\sigma,\tau)) = 0 \tag{66}$$

which is different from the Ricci flatness condition met in the σ models. Equation (66) has a very nice interpretation. First, note that the ''Einstein-like'' equation (66) is evaluated on the world sheet. It involves the Ricci tensor of space time restricted to the world sheet and the scalar curvature R of the world sheet. Therefore, Eq. (66) makes sense only on the world sheet. Next, note that as a consequence of Eq. (66), $\widetilde{R}(X(\sigma,\tau)) = (D/2)R(X(\sigma,\tau))$. This makes sense, for, when $D=2$, $\widetilde{R}$ evaluated on the world sheet must coincide with the scalar curvature of the world sheet. Using this relation back in Eq. (66), we can rewrite it as

$$G_{\mu\nu} \equiv \widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{R}h_{\mu\nu} = \frac{2-D}{4} R h_{\mu\nu}, \tag{67}$$

which can be interpreted as Einstein equation with a cosmological term $\Lambda = [(2-D)/4]R$. The vanishing of the covariant derivative of Einstein tensor due to the contracted Bianchi identity forces the cosmological term Λ to be a constant in the Einstein equation. In contrast, here, as Eq. (67) is evaluated on the world sheet, Bianchi identity does not force R to be a constant. To see this, taking the covariant derivative ∇_α of Eq. (67) along the surface, writing $\nabla_\alpha = \partial_\alpha X^\rho \nabla_\rho$, multiplying by $g^{\alpha\beta}\partial_\beta X^\sigma$ and using Eq. (61), one finds the contracted Bianchi identity leads to,

$$N^{i\mu}N^{i\rho}\nabla_\rho G_{\mu\nu} = \frac{D-2}{4} g^{\alpha\beta}\partial_\beta X^\mu (\partial_\alpha R) h_{\mu\nu}.$$

The left-hand side need not vanish.

We now compare these results with those of [4,5]. In the usual procedure in which the world sheet metric and X^μ both are treated as dynamical, the divergent Euler characteristic term appears at the two-loop level and is absorbed in the renormalization of the Liouville mode. In our formalism in which the world sheet metric is the induced metric and only X^μ is dynamical, such a term appears at the one-loop level and is absorbed in the redefinition of the background metric. When $R=0$, Eq. (66) is the vacuum Einstein equation and this agrees with the results obtained in [4,5], where this equation was derived for a flat world sheet. The difference now is the appearence of the scalar curvature of the world sheet appearing in Eq. (66).

B. Action with extrinsic curvature

We now calculate one-loop effects for the extrinsic curvature action. The coupling constant α_0 in Eq. (20) is asymptotically free in flat space-time. So, in this case we will encounter renormalization of both the space-time metric and the coupling of the extrinsic curvature action. As in the case of the NG action, we start by calculating the second variation of the extrinsic geometric action. This, however, turns out to be rather involved and we sketch the essential steps below. The first variation of this action given in Eq. (33) is, after ignoring the total divergence terms and the longitudinal fluctuations terms (which vanish due to the equation of Gauss),

$$\delta S_R=\int \sqrt{g}(\nabla^2 H^i-2H^2H^i+H^jH^j_{\alpha\beta}H^{i\alpha\beta} -H^j\widetilde{R}_{\lambda\nu\rho\sigma}\partial_\alpha X^\nu\partial_\beta X^\rho g^{\alpha\beta}N^{j\lambda}N^{i\sigma})\xi_i(\sigma,\tau)d\sigma d\tau, \tag{68}$$

where $\xi_i(\sigma,\tau)$ is the normal fluctuation defined in Eq. (34). Starting from the above equation we evaluate its on shell variation. Thus, it is sufficient to evaluate the variation of the terms within the brackets. This is because of the equation of motion (35). It is clear that there will be terms that involve operators that couple only normal fluctuations and a piece that couples tangential fluctuations to the normal fluctuations. It will turn out that these second types of terms vanish as a consequence of the equation of motion (35). The calculations needed to evaluate the second variation of the extrinsic curvature action are lengthy. We give below the key steps.

In addition to the results used in the second variation of the NG action, as well as Eqs. (28), (29), (54), and (55), we need the following results: The variation of $\Gamma^\gamma_{\alpha\beta}$,

$$\delta\Gamma^\gamma_{\alpha\beta}=\tfrac{1}{2}g^{\gamma\delta}(\nabla_\alpha\nabla_\beta+\nabla_\beta\nabla_\alpha)\xi_\delta-\tfrac{1}{2}g^{\gamma\delta}R^\epsilon{}_{\beta\alpha\delta}\xi_\epsilon -\tfrac{1}{2}g^{\gamma\delta}R^\epsilon{}_{\alpha\beta\delta}\xi_\epsilon-g^{\gamma\delta}\nabla_\alpha(H^j_{\delta\beta}\xi^j)-g^{\gamma\delta}\nabla_\beta(H^j_{\alpha\delta}\xi^j) +g^{\gamma\delta}\nabla_\delta(H^j_{\alpha\beta}\xi^j). \tag{69}$$

In Eq. (69), $R^\epsilon_{\beta\alpha\delta}$ is the curvature tensor of the world sheet. The longitudinal fluctuations are denoted by ξ with Greek indices while the normal fluctuations are denoted by ξ with Latin indices. Next, the variation of the normal gauge connection A^{ij}_α: From its definition

$$A^{ij}_\alpha=N^{j\nu}(\widetilde{\nabla}_\alpha N^{i\mu})h_{\mu\nu}, \tag{70}$$

where $\widetilde{\nabla}_\alpha$ is defined in Eq. (11), it can be verified that

$$\delta A^{ij}_\alpha=-F^{ij}_{\alpha\beta}\xi^\beta+(H^i_{\alpha\beta}\nabla^\beta\xi^j-H^j_{\alpha\beta}\nabla^\beta\xi^i) +\widetilde{R}_{\mu\sigma\lambda\rho}N^{i\sigma}N^{j\mu}\partial_\alpha X^\rho N^{k\lambda}\xi^k. \tag{71}$$

In Eq. (71), $F^{ij}_{\alpha\beta}$ is the field strength of A^{ij}_α defined in Eq. (16).

We now give the results for the variation of each of the terms in Eq. (68):

$$\delta(\nabla^2H^i)=[\nabla_\gamma(\nabla^2H^i)]\xi^\gamma+\tfrac{1}{2}\nabla^2(\mathbf{O}^i_j\xi^j)+2(\nabla^\alpha\nabla^\beta H^i)H^j_{\alpha\beta}\xi^j-2(\nabla^\alpha H^j)[(H^i_{\alpha\beta}\nabla^\beta\xi^j-H^j_{\alpha\beta}\nabla^\beta\xi^i) +\widetilde{R}_{\mu\sigma\lambda\rho}N^{i\sigma}N^{j\mu}\partial_\alpha X^\rho N^{k\lambda}\xi^k]-H^j\nabla^\alpha[\widetilde{R}_{\mu\sigma\lambda\rho}N^{i\sigma}N^{j\mu}\partial_\alpha X^\rho N^{k\lambda}\xi^k+(H^i_{\alpha\beta}\nabla^\beta\xi^j-H^j_{\alpha\beta}\nabla^\beta\xi^i)] +2(\nabla^\gamma H^i)[\nabla^\alpha(H^j_{\alpha\gamma}\xi^j)-\nabla_\gamma(H^j\xi^j)]. \tag{72}$$

The variation of (H^2H^i) is given by

$$\delta(H^2H^i)=[\nabla^\alpha(H^2H^i)]\xi_\alpha+(\tfrac{1}{2}H^2\delta^i_j+H^iH^j)\mathbf{O}^j_k\xi^k, \tag{73}$$

and

$$\delta(H^jH^j_{\alpha\beta}H^{i\alpha\beta})=[\nabla^\alpha(H^jH^j_{\beta\gamma}H^{i\beta\gamma})]\xi_\alpha+\tfrac{1}{2}H^j_{\alpha\beta}H^{i\alpha\beta}\mathbf{O}^j_k\xi^k+H^jH^{i\alpha\beta}\mathbf{O}^j_{k,\alpha\beta}\xi^k+H^jH^{j\alpha\beta}\mathbf{O}^i_{k,\alpha\beta}\xi^k, \tag{74}$$

where

$$\mathbf{O}^i_{k,\alpha\beta}=\nabla_\alpha\nabla_\beta\delta^i_k+H^i_{\alpha\gamma}H^\gamma_{k\beta}-\widetilde{R}_{\mu\nu\rho\sigma}N^{i\mu}N^\sigma_k\partial_\alpha X^\rho\partial_\beta X^\nu. \tag{75}$$

The above operator $\mathbf{O}^i_{k,\alpha\beta}$ satisfies

$$\mathbf{O}^i_{k,\alpha\beta}g^{\alpha\beta}=\mathbf{O}^i_k,$$

where $\mathbf{O}^i_k$ is defined in Eq. (56).

The final variation we need is given below:

$$\delta(g^{\alpha\beta}H^j\widetilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\nu\partial_\beta X^\rho N^{j\mu}N^{i\sigma})=\nabla^\alpha(H^j\widetilde{R}_{\lambda\nu\rho\sigma}\partial_\alpha X^\nu\partial_\beta X^\rho g^{\alpha\beta}N^{j\lambda}N^{i\sigma})\xi_\alpha+H^j(\nabla_\mu\widetilde{R}_{\lambda\nu\rho\sigma})\partial_\alpha X^\nu\partial_\beta X^\rho g^{\alpha\beta}N^{j\lambda}N^{i\sigma}N^{k\mu}\xi^k +\tfrac{1}{2}\widetilde{R}_{\lambda\nu\rho\sigma}\partial_\alpha X^\nu\partial_\beta X^\rho g^{\alpha\beta}N^{j\lambda}N^{i\sigma}\mathbf{O}^j_k\xi^k+H^j\widetilde{R}_{\lambda\nu\rho\sigma}N^{j\lambda}N^{i\sigma}(\partial_\alpha X^\rho N^{k\nu}+\partial_\alpha X^\nu N^{k\rho})\nabla^\alpha\xi^k +2H^j(\nabla_\gamma H^i)\nabla^\gamma\xi^j+2H^j(\nabla_\gamma H^j)\nabla^\gamma\xi^i-H^j(\nabla^\alpha H^i_{\alpha\gamma})\nabla^\gamma\xi^j-H^j(\nabla^\alpha H^j_{\alpha\gamma})\nabla^\gamma\xi^i. \tag{76}$$

Making use of the above results we can write down the second variation of the extrinsic curvature action in background space-time. In arriving at the form given below, we have used the equation of motion (35) satisfied by the background configuration. Thus the on shell variation does not contain terms coupling normal fluctuations to the tangential ones. We thus find, for δ^2S,

$$\delta^2 S = \frac{1}{\alpha_0}\int (\xi^i \tilde{\mathbf{O}}^i_k \xi^k)\sqrt{g}\, d\sigma d\tau, \tag{77}$$

where

$$\begin{aligned}\tilde{\mathbf{O}}^i_k =& \frac{1}{2}\mathbf{O}^i_j\mathbf{O}^j_k - (H^2\delta^i_j + 2H^iH^j)\mathbf{O}^j_k + H^jH^{i\alpha\beta}\mathbf{O}^j_{k,\alpha\beta} + H^jH^{j\alpha\beta}\mathbf{O}^i_{k,\alpha\beta} + 2(\nabla^\alpha\nabla^\beta H^i)H^k_{\alpha\beta} - 2\nabla^\alpha H^j(H^i_{\alpha\beta}\nabla^\beta\delta^j_k - H^j_{\alpha\beta}\nabla^\beta\delta^i_k)\\ &- 2(\nabla^\alpha H^j)\tilde{R}_{\mu\sigma\lambda\rho}N^{i\sigma}N^{j\mu}\partial_\alpha X^\rho N^{k\lambda} - H^j(\nabla^\alpha[H^i_{\alpha\beta}\nabla^\beta\delta^j_k - H^j_{\alpha\beta}\nabla^\beta\delta^i_k + \tilde{R}_{\mu\sigma\lambda\rho}N^{i\sigma}N^{j\mu}\partial_\alpha X^\rho N^{k\lambda}]) + 2\nabla^\gamma H^i[\nabla^\alpha H^j_{\alpha\gamma} + H^j_{\alpha\gamma}\nabla^\alpha\\ &- \nabla^\gamma H^j - H^j\nabla^\gamma]\delta^j_k - H^j(\nabla_\mu\tilde{R}_{\lambda\nu\rho\sigma})\partial_\alpha X^\nu\partial_\beta X^\rho g^{\alpha\beta}N^{j\lambda}N^{k\mu} - 2H^j(\nabla_\gamma H^i)\nabla^\gamma\delta^j_k - 2H^j(\nabla_\gamma H^j)\nabla^\gamma\delta^i_k + H^j(\nabla^\alpha H^i_{\alpha\gamma})\nabla^\gamma\delta^j_k\\ &+ H^j(\nabla^\alpha H^j_{\alpha\gamma})\nabla^\gamma\delta^i_k.\end{aligned} \tag{78}$$

It is straightforward to add the second variation of the NG action to the above. Again, because of the equation of motion for the combined action given in Eq. (37), the second variation is simply the sum of the two variations.

One-loop divergent terms can be readily read off from Eq. (78). The bare propagator for the normal fluctuations is given by

$$\langle \xi^i \xi^j \rangle = \delta^i_j \frac{1}{p^4 + (2\pi\alpha')^{-1}p^2}. \tag{79}$$

As observed in the case of the NG action, divergent terms involving A^{ij}_α do not arise since they can only appear through $\mathrm{Tr}(F^{\alpha\beta}F_{\alpha\beta})$ which is a dimension-four operator. There are two types of divergent terms which are readily computed to give the following result:

$$\begin{aligned}\Gamma^1 =& -DI\int H^2\sqrt{g}\,d\sigma d\tau\\ & -I\int\left(\frac{1}{2}Rh_{\nu\rho} - \tilde{R}_{\nu\rho}\right)\partial_\alpha X^\nu\partial_\beta X^\rho g^{\alpha\beta}\sqrt{g}\,d\sigma d\tau,\end{aligned} \tag{80}$$

where I is as given in Eq. (59). Divergent extrinsic curvature terms arise from terms 1–4 and 8 in Eq. (78), whose sum yields the factor D in Eq. (80). In writing Γ^1 in the form (80), we have made use of the equation of Gauss given in Eq. (60).

It is seen that the first term in Eq. (80) can be absorbed in the redefinition of the extrinsic curvature coupling α_0 according as [7,43]

$$\alpha_R = \frac{\alpha_0}{1-\alpha_0 DI}. \tag{81}$$

Notice that the factor D appears in the denominator in Eq. (81), which is the dimensionality of the background space-time, rather than $D-2$ that would be expected of a nonlinear σ model in which the σ model fields are the $D-2$ normals to the surface. We believe that this is the first clear demonstration of the factor D appearing in Eq. (81). We thus find that the coupling for the extrinsic curvature action in an arbitrary curved background is asymptotically free. We have avoided using the σ model formalism and worked directly in terms of the immersion coordinates X^μ as the only dynamical degrees of the theory. The other one-loop divergent term in Eq. (80) can be absorbed, as in the case of the NG action, by renormalizing the background metric exactly as before. We have not calculated the finite contributions to the partition function in this paper. The simplest case would be to integrate over minimal surfaces. We can take minimal surfaces with punctures which can be interpreted as locations of instanton quarks [11].

V. SUMMARY

In this paper we have studied both the intrinsic and extrinsic geometric properties of strings immersed in a D-dimensional curved space-time. In our approach, only the string coordinates X^μ are dynamical, while the metric on the world sheet is induced from the string's immersion in space-time. The geometrical structure of the embedding is completely determined by the structure equations (6), (9), (13), (14), and (15). Minimal surfaces ($H^i=0$) embedded in curved space-time satisfy the classical equation of motion of the NG action. Any solution of Eq. (26) for the NG string is also a solution of Eq. (35). The classical equation of motion (35) for the rigid string admits besides $H^i=0$, a wider class of surfaces. In some special cases, notably in flat backgrounds, the solutions to Eq. (35) are known. For instance, in flat background three-dimensional space, surfaces of constant mean curvature are solutions to Eq. (35). In the case of embedding in de Sitter or anti–de Sitter space, Eq. (44) can be reduced to an algebraic equation for surfaces of constant mean curvature.

The space-time energy momentum tensor $T^{\mu\nu}$ is evaluated for rigid strings. It is found that $T^{\mu\nu}N^i_\nu \neq 0$, where $N^{i\mu}$ are the normals to the surface. Physically this interpreted as a measure of the bending energy of rigid strings. In contrast, for the NG string $T^{\mu\nu}_{\mathrm{NG}}N^i_\nu = 0$. That tension and rigidity are two important ingredients for QCD strings is well known. We have shown elsewhere [44] that these two effects can have an important role also in determining deviation from the area/4 law of black-hole entropy [45].

We have evaluated the divergent parts of one-loop correction to effective action for both the nonrigid and rigid strings in curved background. In the case of nonrigid strings, the divergent part of the quantum one-loop correction is evaluated by finding the second variation of the Nambu-Goto action on shell ($H^i=0$). It is important to note here that although the first variation gives $H^i=0$, Eq. (24), the second variation involves the components of the second fundamental form [see Eqs. (56) and (57)]. Furthermore, the second variation (57) involves only the normal-normal fluctuations and consequently the integration over tangential fluctuations cancel the volume of the diffeomorphism group. This feature persists for the rigid strings also and is a result of our approach of treating the string world sheet as an immersed surface with the immersion coordinates $X^\mu(\sigma,\tau)$ only as dynamical variables. Another feature is the appearance of a divergent Euler characteristic term of the world sheet at the one-loop level. The orgin of this term can be traced back to Eq. (60) which follows from the Gauss equation (13) for immersion. Such a term in the conventional approach [4,5,30] occurs at the two-loop level. Moreover, this divergent term is absorbed in the redefinition of the background metric in our approach, while in the Polyakov approach it renormalizes the Liouville mode of the two-dimensional metric.

The one-loop finiteness of the string theory leads to

$$\widetilde{R}_{\mu\nu}(X(\sigma,\tau))-\tfrac{1}{2}R(X(\sigma,\tau))h_{\mu\nu}(X(\sigma,\tau))=0,$$

which is to be compared with $\widetilde{R}_{\mu\nu}=0$ result in the usual approach. The above equation can be rewritten as Einstein equation with $[(2-D)/4]Rh_{\mu\nu}$ on the right-hand side. In other words, as the right-hand side involves R, the scalar curvature of the world sheet, it has support only on the world sheet. This equation is eminently reasonable as the Einstein equation when X^μ is restricted to the world sheet should reflect the intrinsically curved nature of the world sheet. The right-hand side is thus a cosmological term, and contrary to the Einstein equation in space-time, is not forced to be a constant.

In the case of rigid strings in flat background, the coupling constant α_0 (20) has been shown to be asymptotically free [5,10]. The one-loop correction is found to involve two divergent terms (80), in contrast to the situation in nonrigid strings (62). The first term in Eq. (80) renormalizes the coupling constant α_0 and the renormalized coupling α_R is given in (81). We find that this coupling constant is asymptotically free in a curved background. This generalizes Polyakov's result. It is gratifying to note that the finiteness condition for the rigid string produces the same equation (67) as the NG string.

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APPENDIX

From $T^{\mu\nu}$ defined in Eq. (47) we demonstrate below that $\nabla_\mu T^{\mu\nu}=0$, when the equations of motion (35) are satisfied. A direct calculation shows that

$$\begin{aligned}\nabla_\mu T^{\mu\nu}=&\int d\sigma d\tau\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}\bigg([-\nabla^2H^i+2H^2H^i-H^jH^i_{\alpha\beta}H^{j\alpha\beta}]N^{i\nu}+H^i\partial_\beta X^\nu[2g^{\alpha\beta}\nabla_\alpha H^i-\nabla_\alpha H^{i\alpha\beta}]\\&+\frac{1}{2}[(\nabla^2H^i)N^{i\nu}-(\nabla_\alpha H^i)H^{i\alpha\beta}\partial_\beta X^\nu]\bigg)-\frac{1}{2}\nabla_\mu\int d\sigma d\tau\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}g^{\alpha\beta}(\nabla_\beta H^i)N^{i\mu}\partial_\alpha X^\nu\\&+2\nabla_\mu\int d\sigma d\tau\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}H^iN^{i\mu}H^jN^{j\nu}-[\nabla_\mu,\nabla_\rho]\int\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}g^{\alpha\beta}H^i\partial_\beta X^\nu\partial_\alpha X^\rho N^{i\mu}\\&-\frac{1}{2}\nabla_\rho\nabla_\mu\int\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}g^{\alpha\beta}H^i\partial_\beta X^\mu(N^{i\nu}\partial_\alpha X^\rho-N^{i\rho}\partial_\alpha X^\nu).\end{aligned}\tag{A1}$$

The last term in Eq. (A1) cancels the second and third terms. Using $H^iN^{i\mu}=\nabla^2X^\mu$, the fourth term becomes a boundary term. Considering the two-dimensional integral in the commutator term as a tensor $A^{\nu\rho\mu}(X)$ and applying the rule

$$[\nabla_\mu,\nabla_\rho]A^{\nu\rho\mu}=R^\nu_{\sigma\mu\rho}A^{\sigma\rho\mu}+R^\rho_{\sigma\mu\rho}A^{\nu\sigma\mu}+R^\mu_{\sigma\mu\rho}A^{\nu\rho\mu}\tag{A2}$$

and taking advantage of the Dirac δ function under the integral, we can write $\nabla_\mu T^{\mu\nu}$ as

$$\begin{aligned}\nabla_\mu T^{\mu\nu}=&\int d\sigma d\tau\frac{\delta^D(X-X(\sigma,\tau))}{\sqrt{h}}\sqrt{g}\{[-\nabla^2H^i+2H^2H^i-H^jH^i_{\alpha\beta}H^{j\alpha\beta}]N^{i\nu}\\&+H^i[2\nabla^\beta H^i-\nabla_\alpha H^{i\alpha\beta}]\partial_\beta X^\nu+H^i\widetilde{R}^\nu_{\sigma\rho\mu}\partial_\alpha X^\sigma\partial_\beta X^\rho g^{\alpha\beta}N^{i\mu}\}.\end{aligned}\tag{A3}$$

Requiring $\nabla_{\mu}T^{\mu\nu}=0$, we find that

$$(\nabla^2 H^i - 2H^2 H^i + H^j H^i_{\alpha\beta} H^{j\alpha\beta}) N^{i\nu} - H^i \widetilde{R}^{\nu}_{\sigma\rho\mu} \partial_\alpha X^\sigma \partial_\beta X^\rho g^{\alpha\beta} N^{i\mu} - H^i [2\nabla^\beta H^i - \nabla_\alpha H^{i\alpha\beta}] \partial_\beta X^\nu = 0. \tag{A4}$$

Multiplying Eq. (A4) by $N^{k\mu}h_{\mu\nu}$ we find this condition is the equation of motion (35) while the tangential projection yields the Codazzi equation (36).

[1] C. Lovelace, Phys. Lett. **135B**, 75 (1984).
[2] C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, Nucl. Phys. **B262**, 593 (1985).
[3] A. Sen, Phys. Rev. D **32**, 2102 (1985).
[4] M. B. Green, J. H. Schwarz, and E. Witten, *String Theory* (Cambridge University Press, Cambridge, England, 1987), Vols. I and II.
[5] A. M. Polyakov, in *Gauge Fields and Strings* (Harwood Academic, Chur, Switzerland, 1987).
[6] H. J. de Vega and N. Sanchez, Report No. hep-th/9512074, 1995 (unpublished).
[7] A. M. Polyakov, Nucl. Phys. **B268**, 406 (1986).
[8] T. L. Curtright, G. I. Ghandour, and C. K. Zachos, Phys. Rev. D **34**, 3811 (1986).
[9] T. Curtright, G. Ghandour, C. Thorn, and C. Zachos, Phys. Rev. Lett. **57**, 799 (1986).
[10] K. S. Viswanathan, R. Parthasarathy, and D. Kay, Ann. Phys. (N.Y.) **206**, 237 (1991).
[11] K. S. Viswanathan and R. Parthasarathy, Phys. Rev. D **51**, 5830 (1995).
[12] W. Helfrich, Z. Naturforsch. **28**, 693 (1973).
[13] L. Peliti and S. Leibler, Phys. Rev. Lett. **54**, 1690 (1985).
[14] U. Lindström, M. Rocek, and P. van Nieuwenhuizen, Phys. Lett. B **199**, 219 (1987); U. Lindström and M. Rocek, **201**, 63 (1988).
[15] R. Arnowit, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962).
[16] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1971).
[17] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J. W. York, Phys. Rev. Lett. **75**, 3377 (1995).
[18] A. A. Abrahams and J. W. York, Report No. gr-qc/9601031, 1996 (unpublished).
[19] A. Hosoya and Ken-ichi Nakao, Class. Quantum Grav. **7**, 63 (1990).
[20] A. Hosoya and Ken-ichi Nakao, Prog. Theor. Phys. **84**, 739 (1990).
[21] K. S. Viswanathan and R. Parthasarathy, IMSc report 1996 (unpublished).
[22] W. Krivan and H. Herald, Class. Quantum Grav. **12**, 2297 (1995).
[23] J. Isenberg, Class. Quantum Grav. **12**, 2249 (1995).
[24] P. S. Letelier, Phys. Rev. D **41**, 1333 (1990).
[25] B. Boisseau and P. S. Letelier, Phys. Rev. D **46**, 1721 (1992).
[26] B. Carter, Phys. Rev. D **48**, 4835 (1993); Class. Quantum Grav. **11**, 2677 (1994).
[27] J. Guven, Phys. Rev. D **48**, 4604 (1993); **48**, 5562 (1993).
[28] A. L. Larsen and V. P. Frolov, Nucl. Phys. **B414**, 129 (1994).
[29] R. Capovilla and J. Guven, Phys. Rev. D **51**, 6736 (1995).
[30] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **158B**, 316 (1985).
[31] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1926).
[32] K. Yano and M. Kon, *Structures on Manifolds* (World Scientific, Singapore, 1984).
[33] A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, Princeton, New Jersey, 1975).
[34] W. Blaschke, *Vorlesungen über Differential Geometrie III* (Springer, Berlin, 1929).
[35] J. H. White, Proc. Am. Math. Soc. **38**, 162 (1973).
[36] T. J. Willmore, *Total Curvature in Riemannian Geometry* (Ellis Horwood, England, 1982).
[37] S. Pokorski, *Gauge Field Theories* (Cambridge University Press, Cambridge, England, 1987).
[38] T. J. Willmore and C. S. Jhaveri, Q. J. Math. **23**, 319 (1972).
[39] A. L. Kholodenko, Clemson University report, 1992 (unpublished).
[40] A. L. Larsen and N. Sanchez, Phys. Rev. D **54**, 2801 (1996).
[41] F. Combes, H. J. de Vega, A. V. Mikhailov, and N. Sanchez, Phys. Rev. D **50**, 2754 (1994).
[42] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, Ann. Phys. (N.Y.) **134**, 85 (1981).
[43] S. Ichinose, Nucl. Phys. **B311** 313 (1988/89).
[44] R. Parthasarathy and K. S. Viswanathan, Report No. hep-th/9610186 (unpublished).
[45] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); S. Hawking, Nature (London) **248**, 30 (1974); Commun. Math. Phys. **43**, 199 (1975).