

## Detailed study of quasinormal frequencies of the Kerr black hole

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We compute the quasinormal frequencies of the Kerr black hole using a continued fraction method. The continued fraction method first proposed by Leaver is still the only known method stable and accurate for the numerical determination of the Kerr quasinormal frequencies. We numerically obtain not only the slowly but also the rapidly damped quasinormal frequencies and analyze the peculiar behavior of these frequencies at the Kerr limit. We also calculate the algebraically special frequency first identified by Chandrasekhar and confirm that it coincides with the  $n=8$  quasinormal frequency only at the Schwarzschild limit. [S0556-2821(97)01906-1]

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### I. INTRODUCTION

The linear perturbations of the Schwarzschild spacetime were first examined by Regge and Wheeler [1], who were trying to solve the stability problem of the black hole. They derived what is now called the Regge-Wheeler equation, a second order ordinary differential equation similar to the Schrödinger equation. The perturbation radiates away at the speed of light satisfying the wave equation and hence it corresponds to the gravitational wave around the black hole. Therefore, examining the perturbations of the black hole is equivalent to solving the scattering problem of gravitational waves around the black hole. Their pioneering work has led to the study of the quasinormal modes because they dominate most processes involving perturbed black holes. Quasinormal modes were first found by Vishveshwara [2] and by Press [3] through numerical computations of the time evolution of the gravitational waves around the black hole. External perturbations excite the quasinormal modes which in turn appear as damped vibrations, emitting gravitational waves with specific frequencies. This has been shown in several other investigations on the response of the black hole to the external perturbations [4–7]. There have been several algorithms proposed to determine these resonant frequencies of the black hole. The first method presented by Chandrasekhar and Detweiler [8] was not sufficient to obtain accurate values of the quasinormal frequencies because their scheme, namely, a direct numerical integration of the Regge-Wheeler equation, was not stable, especially for rapidly damped modes. However, it is remarkable that they were able to obtain for the first time values of the first few least-damped quasinormal modes.

Meanwhile, there have been some attempts to obtain the frequencies analytically. An inverse potential method by Mashhoon [9–11] was the first attempt to obtain the quasinormal frequencies using a potential for which the spectrum is analytically solvable. There is also a semianalytic approach, the WKB method used by Schutz and Will [12], and later refined by Iyer and coworkers [13,14]. They have provided relatively easy and intuitively understandable schemes

for obtaining the quasinormal frequencies. The methods have also been applied to the Kerr [15] and Reissner-Nordström black holes [16] and have provided numerical values for fundamental modes. However, these algorithms cannot provide very accurate values for the frequencies and break down completely for rapidly damped modes.

Leaver [17] has shown that a continued fraction method previously used for determining the energy spectrum of the hydrogen molecule ion [18,19] can be generalized to determine the quasinormal frequencies. This method provides extremely accurate values for the frequencies of the Schwarzschild and Kerr black hole. It was found in the Schwarzschild case that there seem to exist an infinite number of damped modes for every multipole index  $l$  (later proven in Ref. [20]) and these modes are located approximately parallel to the imaginary axis. Also for the Kerr black hole, Leaver calculated how these modes move from the Schwarzschild quasinormal frequencies as the black hole parameter changes. Later, he generalized his method to the Reissner-Nordström black hole [21], and found out how the quasinormal modes move as the charge of the black hole changes. These results for the nonrotating cases are confirmed by other independent investigations: Nollert and Schmidt [22] presented the Laplace transform method which is in excellent agreement with Leaver's results for the Schwarzschild black hole. Andersson and coworkers used the phase integral method for the Schwarzschild black hole [23–25] to obtain the consistent results. The last method was also used for the Reissner-Nordström black hole [26,27].

There are several methods to precisely obtain the quasinormal frequencies for the nonrotating cases; nevertheless, the continued fraction method by Leaver is still the only known way which can be generalized to the Kerr quasinormal frequencies. The Kerr black hole is most important astrophysically, thus its quasinormal modes should be investigated in detail. At the same time, several questions were raised by the previous detailed studies of the quasinormal frequencies of the Reissner-Nordström black hole [28–30]. In Ref. [28] we have improved a continued fraction method for the extreme case and then discovered a curious fact that the gravitational and the electromagnetic frequencies are identical. Whether this is also the case for the Kerr black hole is still unknown. In Ref. [29] we presented a detailed

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calculation of the quasinormal modes of charged black holes focusing on the rapidly damped modes for nearly extreme cases. We found that the rapidly damped modes show several peculiar features: the higher modes generally spiral into the value for the extreme black hole as the charge increases. We also found that there is a quasinormal mode that converges to the algebraically special mode [31] when the charge vanishes, even though there was no correlation between these two modes in the charged case. It is still unresolved whether this is just a coincidence or can be explained physically. There are arguments for and against the existence of the quasinormal modes on the imaginary axis. Some methods indicate that there should be such a mode and that it corresponds to the ninth quasinormal mode. The problem is that the algebraically special mode of nonrotating black holes is purely imaginary and that most existing methods do not work for purely imaginary modes. In order to resolve these arguments, it will be useful to investigate the Kerr case because the algebraically special mode moves away from the imaginary axis and thus can be compared with the quasinormal modes. It is also important to know in detail the behavior of quasinormal modes for nearly extreme Kerr black holes, because some astrophysically relevant black holes are a consequence of the coalescence of binary neutron stars and are believed to have a considerably large angular momentum. However, the behavior of these modes near the extreme limit is not yet well known.

In this paper, the quasinormal frequencies of the Kerr black hole are numerically computed using Leaver's continued fraction method beyond his original calculation and several unresolved questions about the Kerr quasinormal modes are considered. In Sec. II, we briefly review the continued fraction method. In Sec. III A, we compute the rapidly damped quasinormal modes and find that as is established in Reissner-Nordström case the spherical modes ( $m=0$ ) generally spiral into the anticipated value for  $a=M$  when the Kerr parameter approaches the maximum. In Sec. III B, we show the behavior of the twisted modes ( $m \neq 0$ ). In the maximally twisted case of  $m=l=2$ , all the modes except one converge into the critical frequency for the super-radiance phenomena. In Sec. IV, we calculate the algebraically special mode using the continued fraction method. The numerical results imply that this mode and the ninth quasinormal mode coalesce only in the Schwarzschild method. Section V is devoted to concluding remarks.

## II. CONTINUED FRACTION EQUATIONS

It is well known that perturbations of the Kerr black hole are described by the Teukolsky equations [32]. In case a  $t$  and  $\varphi$  dependence is given by  $e^{-i\omega t + im\varphi}$ , the separated differential equation for an angular part of perturbations is

$$(1-u^2)\frac{d^2 S_s}{du^2} - 2u\frac{dS_s}{du} + W(u)S_s = 0, \quad (2.1)$$

$$W = a^2\omega^2 u^2 - 2a\omega s u + A_{lm} + s - \frac{(m+su)^2}{1-u^2}, \quad (2.2)$$

and that of a radial part is

$$\Delta \frac{d^2 R_s}{dr^2} + 2(s+1)(r-M)\frac{dR_s}{dr} + V(r)R_s = 0, \quad (2.3)$$

$$V = \frac{K^2 - 2isK(r-M)}{\Delta} + 4is\omega r - A_{lm} + 2a\omega m - a^2\omega^2, \quad (2.4)$$

where

$$u = \cos\theta, \quad (2.5)$$

$$\Delta = r^2 - 2Mr + a^2, \quad (2.6)$$

$$K = (r^2 + a^2)\omega - am. \quad (2.7)$$

These equations correspond to the electromagnetic waves in the Kerr geometry when  $s=-1$  and to the gravitational waves when  $s=-2$ . In what follows, we normalize  $2M=1$  if otherwise mentioned.

In the Schwarzschild case, Eq. (2.1) can be solved analytically and hence the separation constant  $A_{lm}$  can be obtained as  $A_{lm} = l(l+1) - s(s+1)$ . However, in the Kerr case, we have to solve it numerically as follows [17]. Boundary conditions for Eq. (2.1) are that  $S_s$  is regular at the regular singular points  $u = \pm 1$ . The indices there are determined by the regularity at both singular points and are given by  $\pm(m+s)/2$  at  $u=1$  and  $\pm(m-s)/2$  at  $u=-1$ . A solution to Eq. (2.1) can be expressed as

$$S_s(u) = e^{-a\sigma u} (1+u)^{|m-s|/2} (1-u)^{|m+s|/2} \sum_{n=0}^{\infty} \hat{a}_n (1+u)^n. \quad (2.8)$$

The expansion coefficients are related by a three-term recurrence relation and the boundary condition at  $u=1$  is satisfied only by its minimal solution sequence. The recurrence relation is

$$\hat{\alpha}_0 \hat{a}_1 + \hat{\beta}_0 \hat{a}_0 = 0, \quad (2.9)$$

$$\hat{\alpha}_n \hat{a}_{n+1} + \hat{\beta}_n \hat{a}_n + \hat{\gamma}_n \hat{a}_{n-1} = 0 \quad (n \geq 1), \quad (2.10)$$

where the recurrence coefficients are, with  $k_1 = |m-s|/2$  and  $k_2 = |m+s|/2$ ,

$$\hat{\alpha}_n = -2(n+1)(n+2k_1+1), \quad (2.11)$$

$$\begin{aligned} \hat{\beta}_n &= n(n-1) + 2n(k_1+k_2+1+2a\sigma) + 2a\sigma(2k_1+s+1) \\ &+ (k_1+k_2)(k_1+k_2+1) - a^2\sigma^2 - s(s+1) - A_{lm}, \end{aligned} \quad (2.12)$$

$$\hat{\gamma}_n = 2a\omega(n+k_1+k_2+s). \quad (2.13)$$

The minimal solution sequence will be obtained if the angular separation constant  $A_{lm}$  is a root of the continued fraction equation below:

$$0 = \hat{\beta}_0 - \frac{\hat{\alpha}_0 \hat{\gamma}_1}{\hat{\beta}_1 - \frac{\hat{\alpha}_1 \hat{\gamma}_2}{\hat{\beta}_2 - \frac{\hat{\alpha}_2 \hat{\gamma}_3}{\hat{\beta}_3 - \frac{\hat{\alpha}_3 \hat{\gamma}_4}{\hat{\beta}_4 - \dots}}}. \quad (2.14)$$

The quasinormal boundary conditions of the radial Eq. (2.3) are

$$R_{-s} \sim (r-r_+)^{-s-i\sigma_+} e^{i\omega r}, \text{ as } r \rightarrow r_+, \quad (2.15)$$

$$R_{-s} \sim r^{-1-2s+i\omega} e^{i\omega r}, \text{ as } r \rightarrow \infty, \quad (2.16)$$

where  $\sigma_+ = (\omega r_+ - am)/b$  and  $b = \sqrt{1-4a^2}$ . Thus, the solution should be

$$R_{-s} = e^{i\omega r} (r-r_-)^{-1-s+i\omega+i\sigma_+} (r-r_+)^{-s-s\sigma_+} \times \sum_{n=0}^{\infty} a_n \left( \frac{r-r_+}{r-r_-} \right)^n, \quad (2.17)$$

where the expansion coefficients are again defined by a three-term recurrence relation:

$$\alpha_0 a_1 + \beta_0 a_0 = 0, \quad (2.18)$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0 \quad (n \geq 1). \quad (2.19)$$

The recursion coefficients are

$$\alpha_n = n^2 + (c_0 + 1)n + c_0, \quad (2.20)$$

$$\beta_n = -2n^2 + (c_1 + 2)n + c_3, \quad (2.21)$$

$$\gamma_n = n^2 + (c_2 - 3)n + c_4 - c_2 + 2, \quad (2.22)$$

and  $c_n$  are defined by

$$c_0 = 1 - s - i\omega - \frac{2i}{b} \left( \frac{\omega}{2} - am \right), \quad (2.23)$$

$$c_1 = -4 + 2i\omega(2+b) + \frac{4i}{b} \left( \frac{\omega}{2} - am \right), \quad (2.24)$$

$$c_2 = s + 3 - 3i\omega - \frac{2i}{b} \left( \frac{\omega}{2} - am \right), \quad (2.25)$$

$$c_3 = \omega^2(4 + 2b - a^2) - 2am\omega - s - 1 + (2+b)i\omega - A_{lm} + \frac{4\omega + 2i}{b} \left( \frac{\omega}{2} - am \right), \quad (2.26)$$

$$c_4 = s + 1 - 2\omega^2 - (2s + 3)i\omega - \frac{4\omega + 2i}{b} \left( \frac{\omega}{2} - am \right). \quad (2.27)$$

The boundary condition that Eq. (2.17) converges as  $r \rightarrow \infty$  reduces again to

$$0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \frac{\alpha_3 \gamma_4}{\beta_4 -} \dots \quad (2.28)$$

The problem is now to seek frequencies which satisfy both Eqs. (2.14) and (2.28).

### III. NUMERICAL RESULTS

#### A. The case of $m=0$

The case of  $m=0$  is similar to the Reissner-Nordström quasinormal modes and hence several things will be repeated. First, each  $m=0$  mode always has a corresponding mode symmetric about the imaginary axis. This is due to a symmetry of Eqs. (2.14) and (2.28). It is obvious that these continued fraction equations are equivalent even after the transformation of  $m \rightarrow -m$ ,  $i\omega \rightarrow (i\omega)^*$ , and  $A_{lm} \rightarrow A_{lm}^*$ . Therefore, when  $m=0$ , a solution always appear as a complex conjugate pair in  $(i\omega)$ . In contrast to the Reissner-Nordström black hole the mode frequencies of the first few modes have the damping rates (imaginary parts of  $\omega$ ) which decrease monotonically as  $a/M$  increases. On the other hand, the oscillation frequencies (real parts of  $\omega$ ) increase with  $a/M$ , which is the same as the Reissner-Nordström black hole. These results are clear from Fig. 1, where we show the behavior of the first three modes for  $(-s, l, m) = (1, 1, 0)$ ,  $(2, 2, 0)$ ,  $(1, 2, 0)$ , and  $(2, 3, 0)$ . The left end point of each line corresponds to the Schwarzschild mode, and the right end point corresponds to the  $a/M = 0.99$  Kerr mode. The electromagnetic modes of the Reissner-Nordström equation, which correspond to the  $s = -1$  modes here, moved very fast when the charge of the black hole was changed [29]. However, the  $s = -1$  Kerr black hole modes move slowly even when the angular momentum of the black hole is changed. This can be understood, because the electromagnetic perturbation depends more on the change of the black hole electronic charge rather than the change of the angular momentum. Even though one might expect that the gravitational and the electromagnetic mode coalesce in the extreme limit as was established in the charged black hole case [28,30], obviously, there is no such correlation between these two modes in Fig. 1.

The behavior for the first few modes was rather simple, but the rapidly damped modes show the same strange behavior as in the Reissner-Nordström black hole. As presented in Fig. 2, the first strange feature appears for  $n=3$  in the  $l=1$  electromagnetic modes and for  $n=6$  in the  $l=2$  gravitational modes. These modes go through a couple of small loops when the angular momentum increases. Similar loops are also found in the results for the higher mode number  $n$ . In these figures it is clear that the loops are getting bigger and the number of cycles increases when the mode number is getting higher, as in the case of the Reissner-Nordström black hole. These strange features typically appear when the ratio of the damping rate to the oscillation frequency reaches a certain value, approximately between 5 and 10 for any multipole index  $l$ , which is rather large compared with the charged black hole case. Since the highly damped modes are not well understood, it is difficult to explain why such a peculiar behavior occurs. In the previous paper [29] we suggested that the falloff rate of the effective potential at the horizon might be the cause. This explanation suggests itself also in this case, because for the Kerr black hole the falloff rate of the potential depends on the Kerr parameter in a similar way as that for the Reissner-Nordström black hole.

It should be noted that we have tracked the trajectory of the  $l=2$  ninth overtone ( $n=8$ ) gravitational quasinormal mode in Fig. 2. This frequency is known to coalesce with the

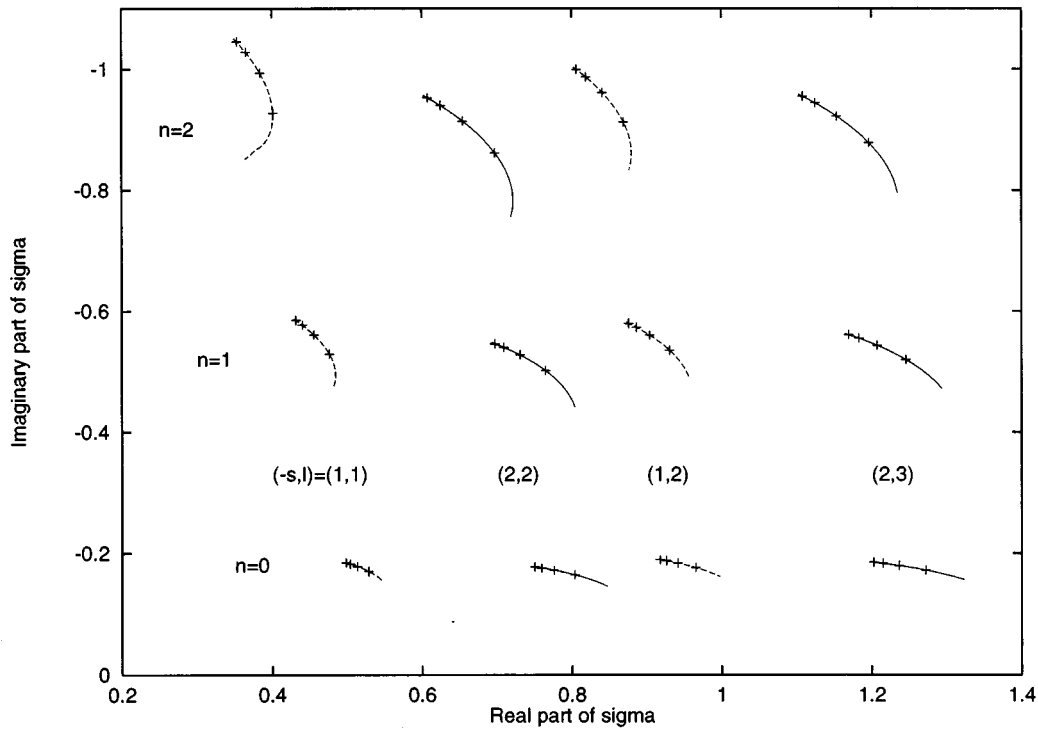


FIG. 1. Solid lines and dashed lines are trajectories of the  $m=0$  quasinormal frequencies of the gravitational and electromagnetic wave, respectively. Each right end point corresponds to the Schwarzschild quasinormal frequency, and each right end point corresponds to the frequency in the limit of nearly maximal angular momentum  $a/M=0.99$ . Ticks marked on each line are the frequencies for  $a=0.1, 0.2, 0.3,$  and  $0.4$  from the left to right.

algebraically special mode at the Schwarzschild limit which is located on the imaginary axis. The algebraically special mode is defined in a completely different way from the quasinormal mode. It is defined as a mode that is purely ingoing at both boundaries. Thus, it seems that there is no reason that these modes have any kind of relationship. We will discuss this matter in Sec. III C.

The numerical error in the previous work [29], possibly caused by the bad convergence property of the continued fraction, does not affect the Kerr black hole case. The convergence of the continued fraction is very fast for the Kerr black hole and thus can avoid numerical errors. When checking our code for the fundamental  $l=2$  gravitational mode, it was possible to obtain the mode accurately until  $a/M=0.999995$ . For the Reissner-Nordström black hole the same mode could be calculated only up to  $Q/M=0.9985$ . Thus, it is concluded that although the Teukolsky equations are more complicated than the Zerilli-Moncrief equations the quasinormal frequencies can be calculated very precisely through the continued fraction method even when the Kerr parameter is near its maximum value. The recurrence relations in the Reissner-Nordström black hole are four-term recurrence relations and numerical instability might be caused through the transformation of four-term relations into three-term relations.

### B. The cases of $m \neq 0$

First, we discuss the case of the  $s=-2$  gravitational perturbation with the least multipole index  $l=2$ . (The result for each higher multipole index is similar, thus we do not discuss it.) It is noted that  $m$  must be in a range of

$-l \leq m \leq l$ , thus we have four different modes for  $l=2$  with  $m \neq 0$ . Owing to the symmetry that we mentioned before, all the modes with the azimuthal multipole index  $m$  are obtained by reflecting the modes with the index  $-m$  about the imaginary axis. We therefore discuss only the modes in the right half of the complex plane. The trajectories of the  $m=+1$  modes are plotted in Fig. 3. We can find that the oscillation frequency becomes very fast when the Kerr parameter is increased; on the other hand, the damping rates become slower. The mode moves fast away from the corresponding Schwarzschild mode. However, the branch of  $m=-1$  moves slowly and stays near the corresponding Schwarzschild modes. It is surprising that the  $m=-1$  mode turns sharply at some specific Kerr parameter, approximately  $a=0.2$ , which is not yet found for any nonrotating black hole. It is also noted that the mode of  $m=\pm 1$  does not experience loops unlike a highly damped mode of the  $m=0$  mode. This distinguishes the behavior of  $m \neq 0$  modes from that of the  $m=0$  modes.

The behavior of the  $m=\pm 2$  modes is quite similar to that of the  $m=\pm 1$  modes. Figure 4 shows how these modes move as the Kerr parameter increases. As is easily seen, the  $m=+2$  modes move very fast compared with any other modes and these modes converge to a specific undamped frequency at the Kerr limit. It is already known that the modes of the  $l=2, m=+2$  branch accumulate onto the critical frequency of super radiance,  $\omega_c=2$ . Detweiler [33] has shown that when  $a=1/2$ , an infinite number of modes cluster at the critical frequency. This suggests a marginally unstable feature of the extreme Kerr black hole discussed by several authors [33–35], and this is quite interesting to investigate in

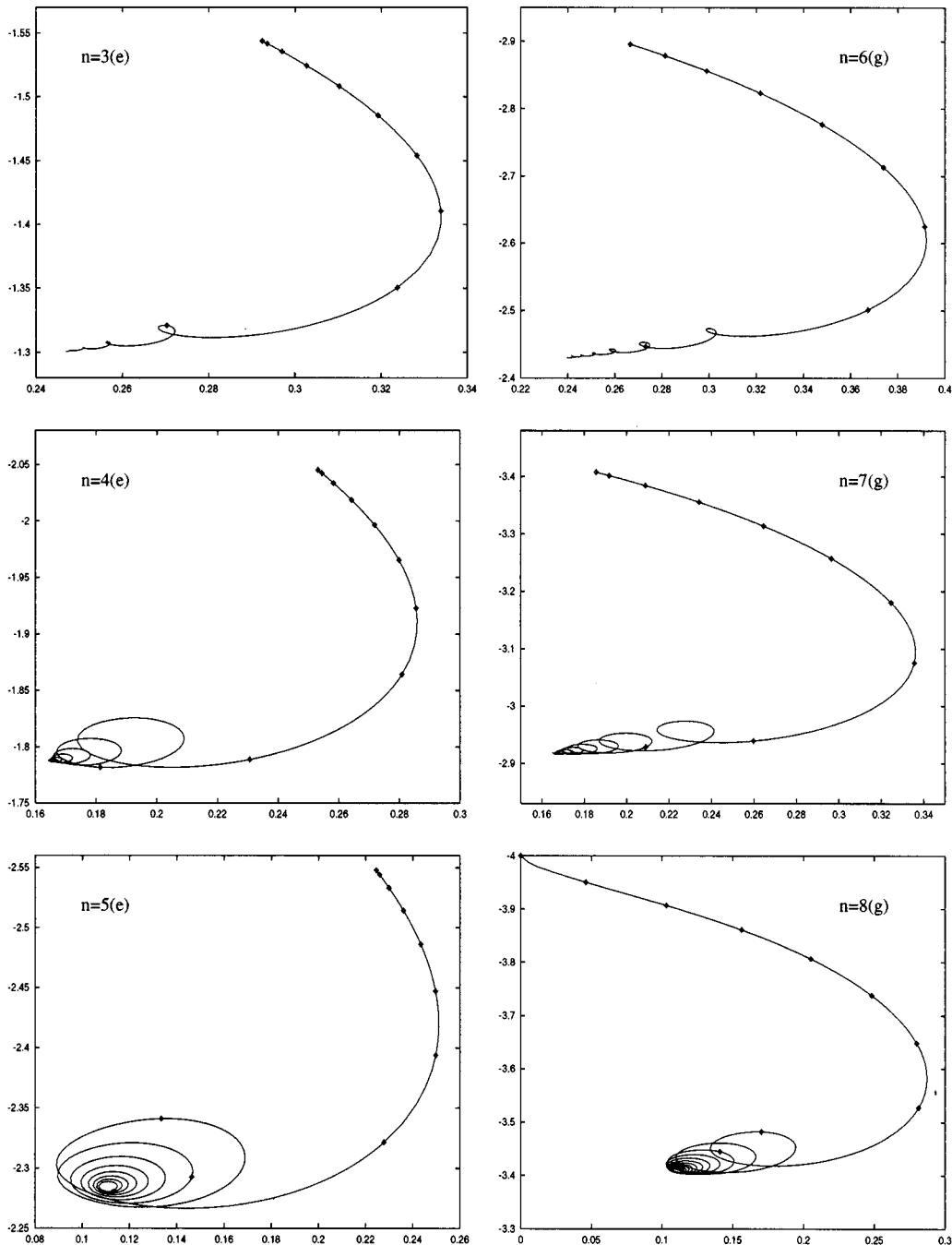


FIG. 2. The first three specific quasinormal modes for  $m=0$  are plotted. The marks correspond to the frequencies of  $a=0, 0.05, 0.10, \dots, 0.45$ . These modes are generally spiral into the values in the extreme limit. The qualitative features for the gravitational and electromagnetic frequencies are basically identical. These figures are much finer than the figures in the previous work for the charged black hole because the continued fraction scheme works fine with less numerical errors.

detail. At present, people believe that when the Kerr parameter is close to the maximum, the Kerr parameter is decreased emitting the gravitational waves and losing rotational energy. Then, the black hole settles down to a stable nonextreme Kerr black hole. One can also notice in Fig. 4 that not all the modes converge to the real critical frequency, but the sixth mode turns sharply at about  $a=0.365$  and then limits to a complex value around  $\omega=(0.239, -0.242)$ . Detweiler's result was that there are an infinite number of quasinormal modes converging to the critical frequency. Our numerical

results suggest that most of modes converge into the critical frequency but some modes still stay away from the critical frequency. We so far have calculated the modes up to  $n=20$  but there was no other mode which does not converge into the critical frequency.

It is also interesting that there seems to be some relation between the  $m=\pm 2$  modes and the  $m=\pm 1$  modes. As is shown in Fig. 5, both the  $m=+1$  and  $m=+2$  modes start from the Schwarzschild mode initially in the same direction and both the  $m=-1$  and  $m=-2$  modes move in the oppo-

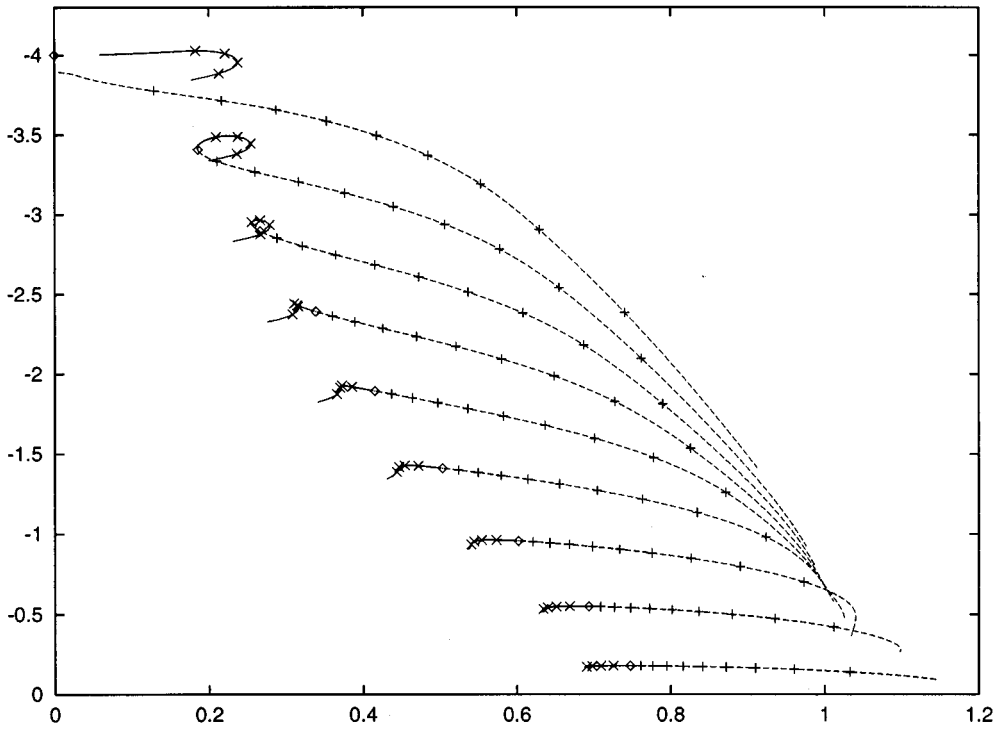


FIG. 3. The first nine  $m = -1$  and  $m = +1$  gravitational quasinormal modes for  $l=2$  are plotted. The diamonds are the Schwarzschild quasinormal frequencies and the right part starting from the Schwarzschild frequency is the branch of the  $m = +1$  modes, which is depicted by the dashed lines. The left part is the branch of the  $m = -1$ , which is depicted by the solid lines. We marked the values of the  $a = 0.05, 0.10, 0.15, \dots, 0.45$  Kerr frequencies for the  $m = +1$  branch and  $a = 0.1, 0.2, 0.3, 0.4$  for the  $m = -1$  branch. It is noticed that the ninth mode comes out of the imaginary axis that corresponds to the algebraically special mode of the Schwarzschild black hole.

site direction to the above two modes. Therefore, the trajectories of these two modes are tangential to each other. On the other hand, the  $m=0$  mode moves into a noncorrelated direction. It seems that the  $m = \pm 2$  modes move just twice as fast as the  $m = \pm 1$  mode initially. (This can be also confirmed by examining the  $l=3$  case, where the  $m=3$  mode moves three times as fast as the  $m=1$  mode.) These features can be understood, the  $m = \pm 2$  modes are more distorted than the  $m = \pm 1$  modes and hence the  $m = \pm 2$  modes are

more sensitive to the change of the Kerr parameter that corresponds how much the spacetime is twisted.

Figure 6 is a figure of the trajectories of the  $l=1, m = \pm 1$  modes for the electromagnetic perturbations ( $s = -1$ ). It is very similar to the case of the  $-s = m = 2$  modes. These modes also converge into the undamped critical frequency,  $\omega_c = 1$ . Owing to Detweiler's argument, for any  $s$  the  $l=m$  modes have a similar accumulation of an infinite number of the quasinormal modes onto the critical

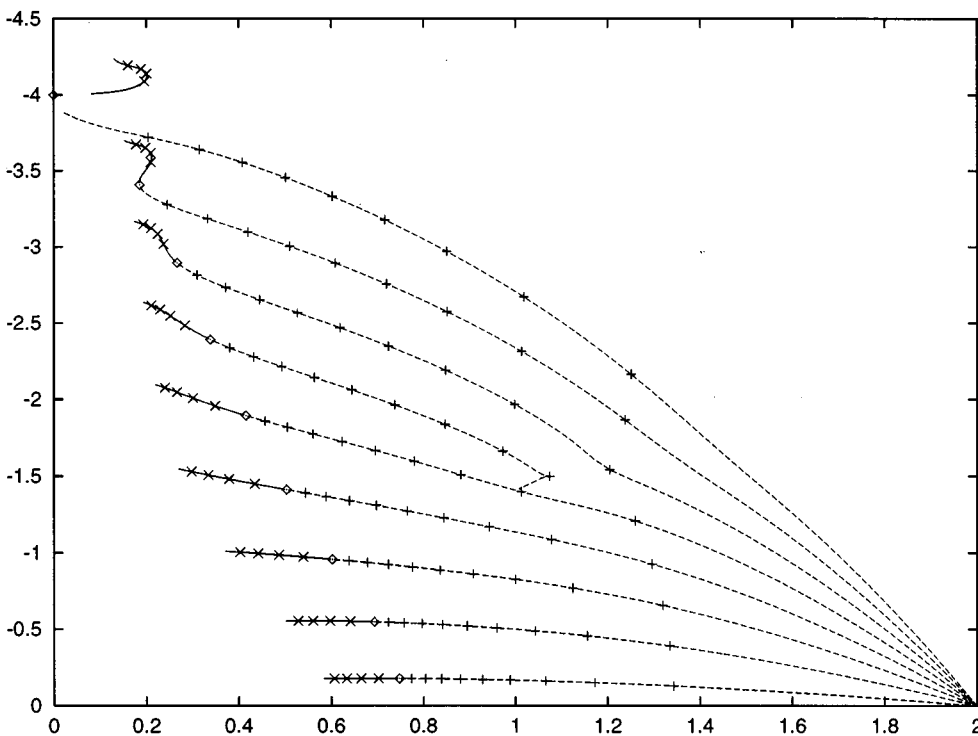


FIG. 4. The first nine  $l=2$  and  $m = -2, +2$  gravitational quasinormal modes are plotted. All the modes of the  $m = +2$  branch except the sixth overtone ( $n=5$ ) converge into the critical frequency on the real axis. This accumulation of modes is in agreement with Detweiler's result [33]. But our numerical results suggest that some of the modes do not cluster into the critical frequency.

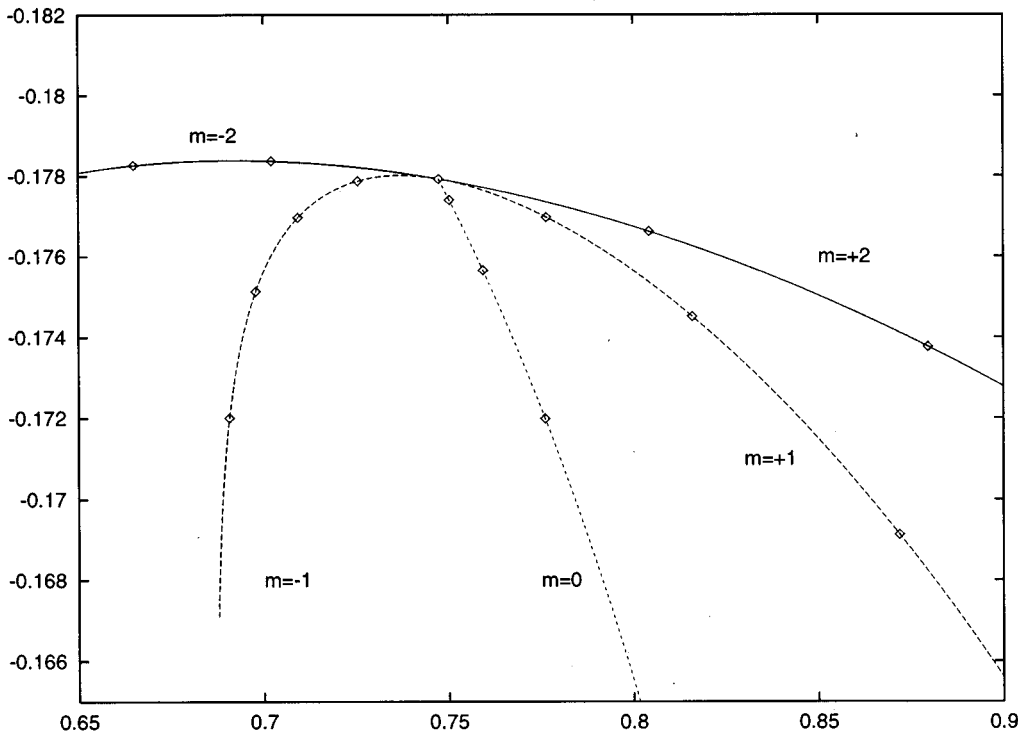


FIG. 5. The  $l=2$  gravitational fundamental quasinormal modes for any  $m$  are zoomed up. The branch of  $m=+1,-1$  is tangent to the branch of  $m=+2,-2$ , but the  $m=0$  mode is not. The marks correspond to the  $a=0.0,0.1,0.2,0.3,0.4$  values.

frequency  $\omega=l$ . There was one mode which never converged into the critical frequency for the  $s=-2$  case; however, for the  $s=-1$  case no such mode was found up to  $n=8$ .

**C. Algebraically special modes**

Here, we only consider the algebraically special mode of the  $s=-2$  gravitational perturbations because the only known implication that there is a correlation between the algebraically special mode and the quasinormal mode is that both modes for the Schwarzschild black hole coalesce for

gravitational perturbations.

The algebraically special perturbations were first considered by Wald [36], who was interested in how the perturbation can be the only independent component among several Weyl scalars. (Actually, it is impossible if one considers only the real frequency.) As is well known, the perturbations have generally two independent components  $\Psi_0$  and  $\Psi_4$ , which correspond to the ingoing and the outgoing wave, respectively. Thus the algebraically special mode is the mode that has only an ingoing or an outgoing wave. Such kind of perturbations was first analytically solved by Chandrasekhar [31]. For the Schwarzschild black hole, algebraically special

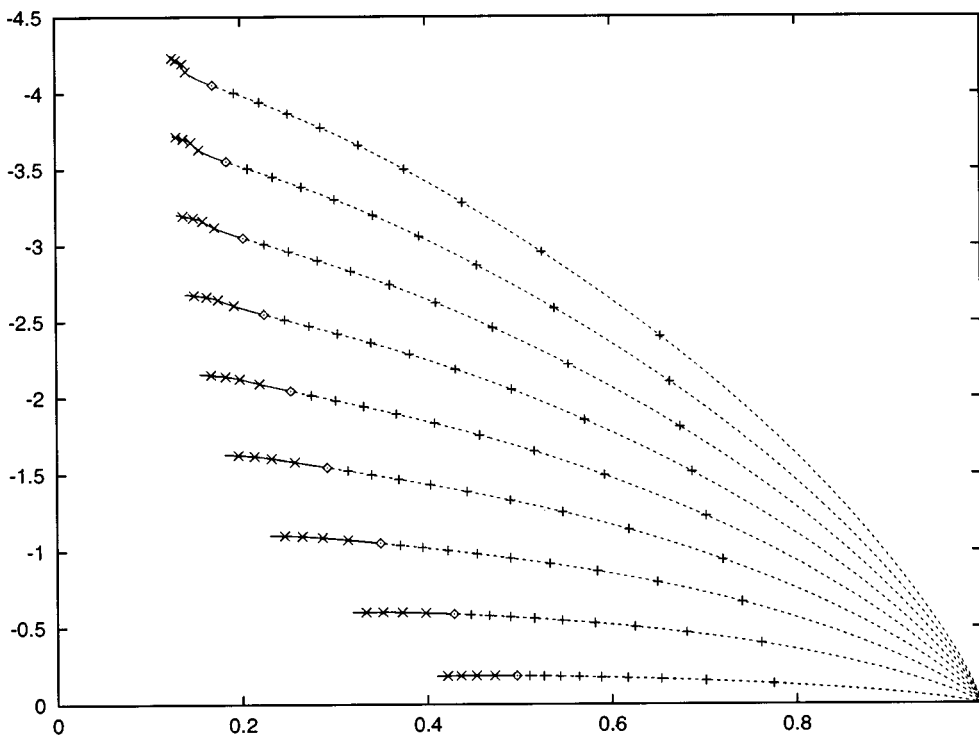


FIG. 6. The first nine  $m=-1$  and  $m=+1$  quasinormal modes for the electromagnetic waves in the Kerr geometry are shown. This is very similar to the case of  $l=m=2$  gravitational quasinormal modes; however, all the modes up to  $n=8$  converge into the critical frequency in contrast with the gravitational frequencies.

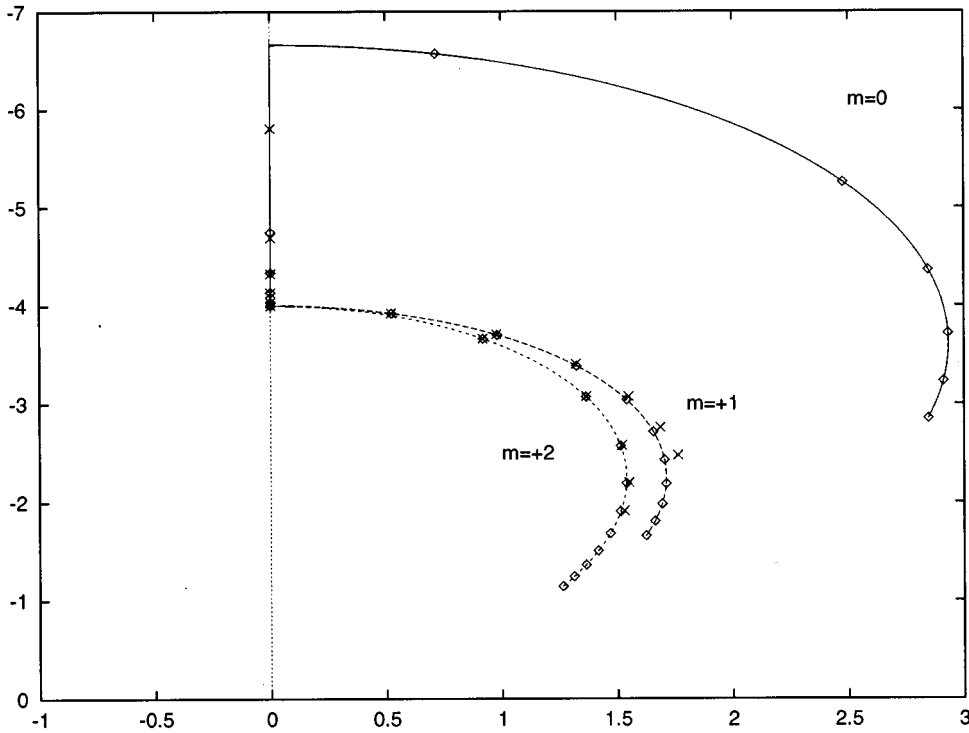


FIG. 7. The algebraically special modes of  $m=0, +1$ , and  $+2$  for  $l=2$ . There also exist another branches of modes symmetric about the real axis. The mode of  $m=-1$  and  $m=-2$  are in the left-half plane, which are obtained by reflecting the  $m=+1$  and  $m=+2$  modes about the imaginary axis, respectively. The  $m=0$  algebraically special mode first goes up vertically and then produces two branches, one in the right half of the complex plane and the other in the left half of the complex plane, which is not shown in this figure. The corresponding Kerr parameter is  $a=0.247\ 222$  and the pure imaginary frequency is about  $-6.681i$ .

frequency can be easily solved by imposing the condition that Starobinsky's constant vanishes (because the two modes are related to each other with a Starobinsky constant). It is then found that there is such a mode on the imaginary axis at  $\omega = -2i/M$ .

For the Kerr black hole it is rather difficult to obtain the algebraically special frequencies owing to the difficulty to get the separation constant  $A_{lm}$  without resorting to numerical computation. Nevertheless, if we use the continued fraction technique for the angular part of the Teukolsky equations (2.1), the separation constant is accurately obtained. The square of the Starobinsky constant is given by

$$|C|^2 = \lambda^2(\lambda+2)^2 - 8\omega^2\lambda[\alpha^2(5\lambda+6) - 12a^2] + 144\omega^2(M^2 + \sigma^2 + \alpha^4), \quad (3.1)$$

where

$$\alpha = a^2 - \frac{am}{\omega}, \quad (3.2)$$

and  $\lambda$ , the separation constant in Chandrasekhar's notation, is related to ours with the relation

$$\lambda = A_{lm} - 2a\omega m + a^2\omega^2. \quad (3.3)$$

Therefore, we only need to solve  $|C|^2=0$ , together with Eq. (2.28). In contrast with quasinormal modes, algebraically special modes can be calculated even when the black hole is extreme, because we do not need to solve Eq. (2.3) which has two confluent singularities in the extremal limit.

The solutions always appear as a pair, owing to a reflection symmetry about the real axis. It is also noticed that the symmetry of Eqs. (3.1) and (2.28) allows us to get the  $m=-1, -2$  modes by reflecting the  $m=+1, +2$  modes about the imaginary axis as in the quasinormal mode. Therefore, it is enough to see the first quadrant. The solutions of

$l=2$  branch which are obtained using the continued fraction scheme are plotted in Fig. 7, together with Chandrasekhar's result [31], and these values are listed in Table I. The  $m=0$  mode increases initially along the imaginary axis as the Kerr parameter increases and then goes away from the imaginary axis when  $a=0.247\ 222$ . In Ref. [31] the value of the  $m=0$  mode for  $a=0.25$  still stays on the imaginary axis, but Chandrasekhar mentioned that this mode does not satisfy a positivity of  $\kappa$ . This contradiction was caused because his values were obtained using a separation constant with a large numerical error, and thus his value was wrong. Our values are all in agreement with this positivity check, since our scheme can avoid numerical errors. If one sees Fig. 6, there is no  $m=-1$  and  $m=-2$  mode in the first quadrant, and the  $m=+1$  and  $m=+2$  modes go horizontally from the imaginary axis such as the quasinormal mode. Despite that, there seems to be no correlation with any quasinormal mode.

From the point of view of the scattering problem, a quasinormal mode is regarded as a pole of the reflection amplitude  $R(\omega)$  and the transmission amplitude  $T(\omega)$ ; on the other hand, an algebraically special mode corresponds to a zero of  $R(\omega)$ . Therefore, these two modes will never coalesce in general, which is consistent with the above numerical results. Superficially, it might appear that these two modes coalesce at the Schwarzschild limit, but we believe that the quasinormal mode disappears at the limit owing to a cancellation with the corresponding mode in the left half of the complex plane.

#### IV. CONCLUDING REMARKS

In this paper we have presented several hints to be useful for resolving a couple of questions concerning the quasinormal mode of the black hole. We have presented the results of detailed calculations of the quasinormal modes of the Kerr black hole. This supplements previous studies in several ways. First, we take the rapidly damped mode into consider-



TABLE I. The algebraically special modes calculated through the continued fraction algorithm are compared with those obtained in Chandrasekhar's paper [31]. The upper values are obtained through the continued fraction method and the lower values are from his paper. It seems his values are not accurate for large Kerr parameters, owing to the numerical errors caused when getting the separation constants.

$a$	$m=0$	$m=1$	$m=2$
0.00	(0, -4.0000)	(0.0000, -4.0000)	(0.0000, -4.0000)
0.05	(0, -4.03107)	(0.522581, 3.92142)	(0.918857, -3.66121)
	(0, -4.0310)	(0.5218, -3.9218)	(0.9184, -3.6620)
0.10	(0, -4.13218)	(0.980889, -3.69993)	(1.36523, -3.07054)
	(0, -4.1304)	(0.9772, -3.7056)	(1.3662, -3.0750)
0.15	(0, -4.33511)	(1.32376, -3.38382)	(1.51483, -2.57005)
	(0, -4.3240)	(1.3216, -3.4026)	(1.5204, -2.5774)
0.20	(0, -4.74699)	(1.54036, -3.03936)	(1.53973, -2.19077)
	(0, -4.6906)	(1.5498, -3.0738)	(1.5508, -2.1986)
0.25	(0.715878, -6.57057)	(1.65595, -2.71464)	(1.51441, -1.90331)
	(0, -5.8038)	(1.6868, -2.7600)	(1.5304, -1.9104)
0.30	(2.47766, -5.25615)	(1.70404, -2.42938)	(1.46928, -1.68066)
		(1.7626, -2.4766)	
0.35	(2.84578, -4.36141)	(1.71091, -2.18596)	(1.41732, -1.50403)
		(1.53041, -1.9104)	
0.40	(2.93170, -3.71616)	(1.69372, -1.97997)	(1.36428, -1.36084)
0.45	(2.91142, -3.23046)	(1.66300, -1.80540)	(1.31270, -1.24256)
0.50	(2.84632, -2.85259)	(1.62505, -1.65663)	(1.26369, -1.14328)

ation. Second, we also investigate precisely how these modes move as the Kerr parameter approaches the limit. Third, we also consider the relationship between the algebraically special mode and the quasinormal mode.

As was the case for the Reissner-Nordström black hole [29], the highly damped  $m=0$  Kerr quasinormal modes generally spiral into the extreme values. Though not presented in this paper, we have found the similar behavior also in the case of the test scalar field on the Kerr background ( $s=0$ ). Therefore, these spirals must be one of the fundamental features for rapidly damped modes. The behavior of the  $m \neq 0$  modes are, on the other hand, different from the behavior of the  $m=0$  mode. The damping rates of  $0 < m \leq l$  modes decrease very fast without going through spirals. An infinite number of the  $m=l$  modes tend to accumulate onto the critical frequency,  $\omega_c=l$ , as  $a/M \rightarrow 1$ . We also found that there is one mode which does not converge to the frequency. Whether another mode such as that exists is an open question. At present, there is no evidence up to  $n=20$  for  $l=2$  gravitational perturbations. The  $-l \leq m < 0$  modes have rather simple behavior in contrast to the positive  $m$  cases. These modes move slowly around the corresponding Schwarzschild modes. An interesting fact is that some of the modes go through a quick turn. These surprising features for

$m \neq 0$  were not yet found for the spherical black holes and hence should be considered in detail. Since the  $m \neq 0$  modes do not exist for nonrotating black holes (all the  $m \neq 0$  modes degenerate into the  $m=0$  case owing to the spherical symmetry), these modes are actually specific for the rotating black holes. We also obtained the precise values of the algebraically special modes using a continued fraction method. There was no correlation between the algebraically special mode and the quasinormal mode except that these modes superficially coalesce in the Schwarzschild limit. Currently, we do not know why the coalescence occurs. However, owing to the existence of the algebraically special mode at  $\omega = -4i$ , the quasinormal mode cannot be there. Two quasinormal modes coalesce and then probably cancel at the limit. This cancellation prevents the existence of the purely damped quasinormal mode.

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