Optical approach for the thermal partition function of photons

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The optical manifold method to compute the one-loop effective action in a static space-time is extended from the massless scalar field to the Maxwell field in any Feynman-like covariant gauge. The method is applied to the case of the Rindler space obtaining the same results as the point-splitting procedure. The result is free from Kabat's surface terms which instead affect the ζ-function or heat-kernel approaches working directly in the static manifold containing conical singularities. The relation between the optical method and the direct ζ-function approach on the Euclidean Rindler manifold is discussed both in the scalar and the photon cases. Problems with the thermodynamic self-consistency of the results obtained from the stress tensor in the case of the Rindler space are pointed out. [S0556-2821(97)01806-7]

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I. INTRODUCTION

In a recent paper [1] we have computed the one-loop thermal partition function of photons in the Rindler wedge employing a local ζ-function method directly in the Euclidean Rindler space. Although this approach produces thermodynamical quantities with the correct high temperature behavior requested by the statistical mechanics, the low temperature behavior seems to remain different from that obtained with other methods. This can be seen by means of a direct comparison between the free energy following from the above cited approach and the same quantity obtained by the point-splitting renormalization procedure for the stress tensor [2–4]. In particular, one sees that the direct ζ-function approach gives, for the coefficient of the term proportional to T^2, a result which is one-third of the point-splitting result. This discrepancy can be traced back to an identical discrepancy in the coefficients of the free energy of a minimally coupled massless scalar field propagating in the Rindler wedge [5–7].

It is important to remark that this problem does not arise from the particular method used in [5,1] to compute the determinant of the small fluctuations operator which appears in the one-loop free energy. In fact, the same discrepancy has also been found in [8] using a completely different method to compute the determinant. Therefore, it seems to be intrinsic of the computations made directly in the Euclidean Rindler space.

In the photon and graviton case, a further drawback of the approach in [1] is the need of a more complicated regularization procedure due to the presence of gauge depending "surface" terms [9]. Anyway, the results of [5,1] improve previous results obtained using global heat-kernel approaches [10,9] in the Rindler space, which is not able to reproduce the Planckian high temperature behavior.

There is another method which can be used to compute these one-loop quantities, and is the optical one [11–18]. In this approach, instead of computing the partition function directly in the static metric, one performs a conformal transformation in such a way that the resulting manifold has an ultrastatic metric. Then, one can compute the relevant quantities in this "optical manifold" using heat-kernel, ζ-function, or any other method and taking into account how the various quantities transform under conformal transformations. This method is particularly favorable in the Euclidean Rindler case, since this manifold has a conical singularity which can be quite tricky to deal with, whereas the related optical manifold has no singularity. However, there is more in this method than the mathematical content. In fact, it has been shown [13,14,17,18] that the canonical partition function of a quantum field in a curved background with a static metric is not directly related to the Euclidean path integral with periodic imaginary time in the static manifold, but rather it is equal to Euclidean path integral in the related optical manifold. In particular, in [14] it is shown that the statistical counting of states leads naturally to a formulation in the optical manifold. We can also notice that, as far as we know, the equivalence of the direct periodic imaginary time path integral formalism to the canonical formalism for computing finite temperature effects has been proved in ultrastatic manifolds only [19].

Therefore, the computation of the thermal partition function from Euclidean path integral in the static manifold requires the knowledge of the Jacobian of the conformal transformation. On regular manifolds this causes no trouble, since it is easily shown (see, e.g., [20]) that the Jacobian affects only the temperature-independent part of the free energy. Instead, as we will see, when in the static metric there is a conical singularity the temperature dependence of the Jacobian could be less trivial and affect the temperature-dependent part of the free energy.

Another important point is that the optical method produces thermodynamic quantities which agree with those obtained form the point-splitting procedure. This happens in the case of a massless scalar field conformally coupled in the Rindler wedge at least, but also, as we shall see, in the case of the photon field.

In the first part of this paper we shall review the compu-

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tation of the thermodynamical quantities of a massless scalar field in the Rindler wedge, comparing the point-splitting, local ζ-function, and optical results. In particular, we note that while the point-splitting approach can be applied for any coupling of the scalar field with the gravity, the optical approach is feasible only for the conformally coupled case, where it gives the same result as the point-splitting method. Moreover, the dependence on the coupling parameter, which disappears in the integrated physical quantities when the background is a regular manifold, in this case affects these quantities because of the presence of the conical singularity. On the other hand, the computations made directly in the static Euclidean Rindler manifold using the local ζ-function technique is still limited to the minimally coupled case, and even in this case the result is different from the point-splitting one.

In the second part of this work, we shall extend the optical manifold approach to the Maxwell field case. We shall show that there are two possible ways to do this, which are equivalent in the scalar case, but in the photon case could produce a different result. The difference of the two approaches is essentially in the definition of the gauge-fixing and ghost parts of the Lagrangian. In particular, we shall show that result expected counting the polarization states of the field can be obtained by defining the theory directly in the optical manifold. The other possibility is to define the partition function in the Euclidean Rindler wedge and only then perform the conformal transformation. We are not able to work out thoroughly this latter approach, because of a mathematical complication in the ghost action. Nevertheless, we argue that it gives a different coefficient of the term proportional to T^2.

In the final part, in addition to discussing the obtained results, we also show that the usual relations between thermodynamical quantities, which involve derivatives with respect to the temperature, lead to inconsistencies when applied to Rindler thermal states with temperature different from the Unruh-Hawking one.

II. THE GENERAL PROBLEM IN THE CASE OF A MASSLESS SCALAR FIELD

As we have said in the Introduction, the problem of discrepancy of the coefficient of T^2 in the free energy appears already in the case of a massless scalar field propagating in the Rindler wedge. In fact, this problem seems to be independent of the field spin. Hence, we start discussing just this case, taking into consideration the point-splitting, local ζ-function, and optical manifold approaches.

A. Point-splitting approach

We start considering the results produced by the point-splitting method. The point-splitting renormalized stress tensor reads (see, for example [6], continuing into the Rindler space the results obtained for the cosmic string)

$$\langle T^{\mu}_{\nu}\rangle^{\text{ps}}_{\beta}(\xi)=\frac{1}{1440\pi^2 r^4}\left\{\left[\left(\frac{2\pi}{\beta}\right)^4-1\right]\text{diag}(-3,1,1,1)+20(6\xi-1)\left[\left(\frac{2\pi}{\beta}\right)^2-1\right]\text{diag}\left(\frac{3}{2},-\frac{1}{2},1,1\right)\right\}. \tag{1}$$

By integrating $-\sqrt{g}T^0_0$, we get a total energy which we shall compare with those following from the other methods:

$$U^{\text{ps}}_{\beta,\xi}=\frac{L_yL_z}{2880\pi^2\epsilon^2}\left[3\left(\frac{2\pi}{\beta}\right)^4-30(6\xi-1)\left(\frac{2\pi}{\beta}\right)^2+30(6\xi-1)-3\right]. \tag{2}$$

Above, L_y and L_z are the (infinite) lengths of the transverse dimensions, and so $A_\perp=L_yL_z$ is the (infinite) area of the horizon. The parameter ξ fixes the coupling of the scalar field with the gravitation. In the (Lorentzian) Rindler space the scalar curvature R is zero everywhere and the parameter ξ remains as a relic of the fact that $T_{\mu\nu}$ is obtained by varying the metric $g_{\mu\nu}$ in the field Lagrangian [21].[1] Employing the general expression of $T_{\mu\nu}(\xi)$ [6,21] in terms of the Hadamard function, one finds that, in the case $R=0$, the global conserved quantities as total energy should not depend on the value of ξ. This is because the contributions to those quantities due to ξ are discarded into boundary surface integrals which generally vanish. However, this is not the case dealing with the Rindler wedge because such integrals diverge therein.[2] The only possibility to get a result not depending on ξ consists in taking $\beta=2\pi$ producing a trivial result. The considered ambiguity does not seem to arise from a similar ambiguity in defining the thermal quantum state. In fact, the thermal Wightman functions employed in calculating the renormalized stress tensor do not depend on ξ. Finally, it is worthwhile noting that the ξ ambiguity affects the β^{-2} term in the thermodynamical quantities and hence their low temperature behavior.

Notice that Kay and Studer [23] found an ambiguity in defining the scalar Wightman functions around a cosmic string, a background which has the same Wick-rotated metric as the thermal Euclidean Rindler manifold. However, this ambiguity is related to the time-independent modes and so, e.g., employing the ζ-function procedure, one simply finds

[1]It is worthwhile noticing that one has to consider the theory within the curved space time in order to discuss on the physics in the flat space-time. Anyhow, the extension of the theory to a curved space-time is not unique and this involves some subtleties regarding also the regularization procedure. The choice between different regularization procedures should be made on the basis of what is the physics that one is trying to describe. Obviously, the general hope is that, at the end of the complete renormalization procedure involving matter fields and gravity, all these different regularization approaches give rise to equivalent physical results. See [17,22] for a discussion on these topics.

[2]Similar problems appear working in subregions of the Minkowski space in presence of boundary conditions [21].

that this ambiguity cannot produce β^{-2} terms in the Rindler free energy. Hence, it should not be related with the ξ ambiguity.

B. Direct conical approach

One can formally define the partition function at the temperature $1/\beta$ by a Euclidean path integral

$$e^{-\beta F_\beta}=Z_\beta=\int \mathcal{D}\phi \ \exp(-S[\phi]), \tag{3}$$

where the (Euclidean) action is that of a massless scalar field coupled with the gravitation,

$$S[\phi]=-\frac{1}{2}\int d^4x\sqrt{g}\,\phi[\nabla_\mu\nabla^\mu-\xi R]\phi. \tag{4}$$

The background is the Euclidean Rindler manifold $\mathcal{C}_\beta\times R^2$ with an imaginary time period β. The Euclidean Rindler metric reads

$$ds^2=r^2d\theta^2+dr^2+dy^2+dz^2, \tag{5}$$

where $\theta\in[0,\beta]$, $r\in R^+$, $\mathbf{x}=(y,z)\in R^2$. Notice the well-known conical singularity at $r=0$ when $\beta\neq 2\pi$.

In the case $\xi=0$, the previous partition function can be explicitly computed by a local ζ-function approach recently introduced by Zerbini *et al.* [5] obtaining a Minkowski renormalized free energy $F_\beta^{\rm sub}=F_\beta-U_{\beta=2\pi}$ and a renormalized internal energy[3] $U_\beta^{\rm sub}=\partial_\beta\beta F_\beta-(\partial_\beta\beta F_\beta)|_{\beta=2\pi}$ which read

$$F_\beta^{\rm sub}=-\frac{A_\perp}{2880\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4+10\left(\frac{2\pi}{\beta}\right)^2+13\right],$$

$$U_\beta^{\rm sub}=\frac{A_\perp}{2880\pi^2\epsilon^2}\left[3\left(\frac{2\pi}{\beta}\right)^4+10\left(\frac{2\pi}{\beta}\right)^2-13\right], \tag{6}$$

where $A_\perp$ is the (infinite) event horizon area and ϵ a short-distance cutoff representing the minimal distance from the horizon [24]. We note that the above result is different from that of the point splitting with $\xi=0$ and the difference is in the coefficient of the term proportional to β^{-2}. However, as we said in the Introduction, the partition function (3) is related to the canonical one by the Jacobian of a conformal transformation: this could explain the different result. We shall come back on this point after discussing the optical approach.

It is worthwhile noticing that the Lorentz section of the Rindler space is flat and hence, as far as the real time theory is concerned, we find a complete independence on the parameter ξ. However, in calculating the partition function, one has to deal with the Euclidean section of the Rindler manifold and, considering it as a integral kernel, the curvature R takes Dirac's δ behavior at $r=0$ [25,26], thus the value of the parameter ξ could be important. The previous results have been carried out in the case $\xi=0$ in the sense that the eigenfunctions employed in computing the ζ functions properly satisfy the eigenvalue equation with no R term.

In the case $\xi\neq 0$ the problems are due to the fact that the equation for the eigenfunctions contains a Dirac δ, and so it is not mathematically clear how to treat it. In the case of a cosmic string, the Dirac δ represents a limit case, maybe unphysical, of the problem in which the string has a finite thickness, which is mathematically well defined since no Dirac δ appears. In the case of the Rindler space there is no such way out, and the only way to avoid the problem is to consider the case $\xi=0$.

C. Optical approach

Let us now consider the optical approach [11–18]. As remarked in the Introduction, this approach is not just a mathematical method to compute the functional integral in Eq. (3), but has an important physical content. In fact, as previously remarked, it fulfills the requirements of a formulation in the optical manifold following from statistical counting of states.

Let us consider a static metric $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ and perform a conformal transformation of the metric [maybe singular if $\Omega(x)=0$], $g_{\mu\nu}\to g'_{\mu\nu}=\Omega^2(x)g_{\mu\nu}$, so that

$$ds^2\to ds'^2=\Omega^2ds^2. \tag{7}$$

Choosing $\Omega^{-2}=g_{00}$, ds'^2 becomes the related ultrastatic optical metric. In the case of the Euclidean Rindler space, this conformal factor becomes singular just on the conical singularities, which are pushed away to the infinity[4] and the optical manifold is free from singularities. Under such a transformation, the massless scalar field ϕ transforms into $\phi'=\Omega^{-1}\phi$ and the Euclidean action with coupling factor ξ transforms into the following more complicated action [21]:

$$S'[\phi']=-\frac{1}{2}\int d^4x\sqrt{g'}\,\phi' \times[\nabla'_\mu\nabla'^\mu-\xi_4R'-\Omega^{-2}(\xi-\xi_4)R]\phi', \tag{8}$$

where $\xi_D=(D-2)/4(D-1)$ is the conformal invariant factor. If we consider a conformally coupled field, $\xi=\xi_D$, we see that also the transformed action is that of a conformally coupled field in the optical manifold $S^1\times H^3$. In the other

[3]As is well known, the $(\beta=2\pi)$-thermal Rindler state locally coincides with the Minkowski vacuum and, in renormalizing, we suppose that this state does not carries energy density. Notice that such a Minkowski subtraction procedure does not affect the entropy computed through F_β.

[4]The points at $r=0$ of the optical manifold $S^1\times H^3$ are infinitely far from the points of the manifold with $r>0$ taking the distance as the affine parameter along geodesics. Strictly speaking, the former points do not belong to the manifold at all.

cases, we have to keep a term proportional to R which has Dirac's δ behavior at $r=0$ and thus we have an ill-defined operator.[5]

When we compute the one-loop partition function, we may formally write [27,28]

$$e^{-\beta F_\beta}=Z_\beta=\int \mathcal{D}\phi' J[g,g',\beta]\exp(-S'[\phi'])$$
$$=J[g,g',\beta]Z'_\beta=J[g,g',\beta]e^{-\beta F'_\beta}. \tag{9}$$

We remark that is Z'_β which is equivalent to the canonical partition function [13,14,17,18]. The functional Jacobian $J[g,g',\beta]$ does not depend on ϕ' and thus it can be carried out from the integral as we have done above. When the involved manifolds are regular, it is possible to prove that such a Jacobian is in the form

$$J[g,g',\beta]=\exp(-\beta E_0). \tag{10}$$

where E_0 does not depend on β. This is substantially due to the staticity of the involved metrics [16–18] and the factor β in the exponent is due to an integration over the whole Euclidean manifold. If this holds in the presence of a conical singularity as well, one expects that F_β and F'_β differ only for the value of the renormalized zero-temperature energy. When the coupling in the Euclidean Rindler manifold is conformal, the direct computation of F'_β can be performed employing the ζ-function approach [16] (see also the Appendix of this paper). We report here the well-known final result only:

$$F'_\beta=-\frac{A_\perp}{2880\pi^2\epsilon^2}\left(\frac{2\pi}{\beta}\right)^4. \tag{11}$$

Using $U'_\beta=\partial_\beta(\beta F'_\beta)$ to compute the internal energy and performing the Minkowski renormalization $U'^{\text{sub}}_\beta=U'_\beta-U'_{\beta=2\pi}$ in order to get a vanishing internal energy at $\beta=2\pi$, we find just

$$U'^{\text{sub}}_\beta=U^{\text{ps}}_{\beta,\xi=1/6}. \tag{12}$$

Therefore, we have got a result equal to the point-splitting one by renormalizing (with respect to the Minkowski vacuum) the internal energy obtained on the optical manifold and *without* taking into account the Jacobian, whether it has the form (10) or not.

D. Comparison of the results

In the previous subsection we have seen that the optical method, when applicable, gives the same result as the point splitting. On the other hand, we see that $U^{\text{ps}}_{\beta,\xi}$ does not coincide with the corresponding internal energy (6) found by the ζ-function approach at the value of coupling parameter one expects, $\xi=0$, but rather at $\xi=1/9$. Note that the discrepancy is in the term proportional to β^{-2}, while the difference in the β-independent term is not meaningful, because such terms are fixed by the subtraction procedure: they do coincide when the remaining terms are equal. Note also that we cannot compare directly the optical and the local ζ-function approaches, since they are not applicable for the same value of ξ.

In order to identify the source of the above discrepancy, we remind the reader that the thermodynamical quantities in Eq. (6) have been obtained from the Euclidean path integral in the static manifold, which differs from the canonical partition function for the Jacobian of the conformal transformation, see Eq. (9). As we have said above, on regular manifolds the logarithm of this Jacobian is simply proportional to β, thus giving a contribution only to the temperature-independent part of the free energy. However, the case of the Euclidean Rindler space could be more complicated, due to the presence of a conical singularity at $r=0$, which could yield a nonlinear dependence on β. In fact, such a singularity can be represented as an opportune Dirac δ function with a coefficient containing a factor $(2\pi-\beta)$ [25,26], and so β enters not only as integration interval, but also in the integrand. Of course, only an explicit calculation of the Jacobian can give an ultimate answer. In two dimensions, the Jacobian $J[g,g',\beta]$ is the exponential of the well-known Liouville action [27], and an easy calculation shows that the logarithm of the Jacobian is indeed proportional to β [18], regardless of the conical singularity. Unfortunately, in four dimensions the form of the Jacobian is far more complicated (see [18] and references therein) and involves also products of curvature tensors which are ill defined. Therefore, it is not clear whether the discrepancy in the term proportional to β^{-2} might be assigned to the Jacobian.

Summarizing, we have seen that the optical method has been applied to the conformally coupled case only, and in this case it gives the same result as the point-splitting method. With regard of the direct computation in the Euclidean Rindler wedge, it has to be considered as incomplete, because of our ignorance of the Jacobian and of the nonminimally coupled case. We stress that, in the case of a regular manifold, these two approaches should be equivalent.

III. OPTICAL APPROACHES IN THE CASE OF PHOTONS

In [1] the partition function of photons gas in a Rindler wedge has been computed generalizing the procedure in [5]. The found Minkowski renormalized free energy amounts to $2F^{\text{sub}}_\beta+(2-\ln\alpha)F^{\text{surface}}_\beta$, where F^{sub}_β is the scalar free energy previously discussed, Eq. (6), and the F^{surface}_β is a ''surface'' term which arises integrating a total derivative and has the form $A_\perp[(2\pi/\beta)^2-1]/(24\pi^2\epsilon^2)$ (see [9,1] for more comments), finally α is the gauge-fixing parameter. Notice that also this anomalous gauge-dependent term involves a β^{-2} dependence. We suggested dropping this latter gauge-dependent term as the simplest procedure to remove the unphysical gauge dependence. Anyway, we stressed that other procedures could also be possible. The obtained result agrees

[5]One possible way to get rid of this term is to define the action in the Lorentzian manifold, where $R=0$, perform the conformal transformation to the optical manifold, and only then use the transformed action to write to partition function with the periodic imaginary time formalism [15]. This procedure gives a result independent on the parameter ξ by nature: the coupling in the optical manifold is always conformal. However, in our opinion this procedure seems too *ad hoc*.

with the statistical mechanics request at high temperatures, but, as in the scalar case, the low temperature behavior is different from that obtained with the usual point-splitting procedure. Therefore, let us consider the point-splitting results [2,6]: the renormalized stress tensor takes a simple form

$$\langle T^{\mu}_{\nu}\rangle^{\text{phot ps}}_{\beta}=\frac{1}{720\pi^2 r^4}\left[\left(\frac{2\pi}{\beta}\right)^4+10\left(\frac{2\pi}{\beta}\right)^2-11\right]\times\text{diag}(-3,1,1,1). \tag{13}$$

The (Minkowski renormalized) internal energy corresponding to the previous photon stress tensor reads

$$U^{\text{phot ps}}_{\beta}=\frac{3A_{\perp}}{1440\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4+10\left(\frac{2\pi}{\beta}\right)^2-11\right]. \tag{14}$$

As far as the energy density is concerned, we have the following very simple relation:

$$\langle T^0_0\rangle^{\text{phot ps}}_{\beta}=2\langle T^0_0\rangle^{\text{ps}}_{\beta}(\xi=0), \tag{15}$$

where on the right the stress tensor is that of a massless scalar field. It is worthwhile noticing that $\xi=0$ takes place on the right-hand side instead of $\xi=1/6$. Hence, the energy density of the electromagnetic field does not amount to twice that of a conformally coupled scalar field, as one could naively expect considering that the electromagnetic field is conformal invariant in four dimensions. As far as the internal energy is concerned, we find the same unforeseen relation. However, as previously discussed, the integrated quantities should not have to depend on ξ in more ''regular'' theories, restoring the naively expected relation between the considered quantities.

Discussing the scalar case we have stressed the importance of the optical method: therefore, now we go to investigate whether it is possible or not to get such an energy employing the optical-manifold method. There are two possible ways to implement this method. The simplest one consists of defining the partition function directly as a functional integral on the optical manifold. However, there is another more complicated possibility: it consists of starting with a functional integral in the initial static manifold, performing the conformal transformation, and finally dropping the functional Jacobian. This is, in fact, the simplest generalization of the results obtained in the scalar case. Both methods produce the same final functional integral in the simpler conformally coupled scalar case, but in the case of the Maxwell field the two procedures do not seem to be equivalent, as we shall see, due to the presence of gauge-fixing and ghost terms.

A. Optical approaches in the case of general static manifolds

Let us start reviewing the formalism we use dealing with the photon field. The complete action for the electromagnetic field in any covariant gauge and on a general Euclidean manifold, endowed with a metric $ds^2=g_{\mu\nu}dx^\mu dx^\nu$, which we shall suppose *static* and where ∂_0 is the global (Euclidean) timelike Killing vector with closed orbits of period β. Using the in Hodge–de Rham formalism we have

$$S^{\text{em}}=\int d^4x\left[\frac{1}{4}\langle F,F\rangle+\frac{1}{2\alpha}\langle A,d\delta A\rangle\right]+S_{\text{ghost}}(\alpha)$$
$$=\frac{1}{2}\int d^4x\left[\langle A,\Delta A\rangle-\left(1-\frac{1}{\alpha}\right)\langle A,d\delta A\rangle\right]+S_{\text{ghost}}(\alpha). \tag{16}$$

In order to maintain the gauge invariance of the theory, it is important to keep the dependence on the gauge-fixing parameter of the ghost action, as one obtains by varying the gauge-fixing condition $(1/\sqrt{\alpha})\delta A=0$ [29]:

$$S_{\text{ghost}}(\alpha)=-\frac{1}{\sqrt{\alpha}}\int d^4x\sqrt{g}\,\bar{c}\,\underline{\Delta}\,c, \tag{17}$$

where $\underline{\Delta}$ is the Hodge–de Rham Laplacian for zero-forms and c, $\bar{c}$ are anticommuting scalar fields. Usually, the dependence on the gauge-fixing parameter is absorbed rescaling the ghost fields, but in the presence of a scale anomaly this rescaling gives rise to a nontrivial contribution, which is essential to maintain the gauge invariance of the theory. This is just the case here: in fact, the contribution of the action (17) to the one-loop effective action is proportional to that of a minimally coupled scalar field, which has a scale anomaly in four dimensions.

Some comments on the formalism in Eq. (16) are in order. $F\equiv\partial_\mu A_\nu-\partial_\nu A_\mu=\nabla_\mu A_\nu-\nabla_\nu A_\mu$ is the two-form representing the photon strength field, ∇_μ being the covariant derivative; the brackets stand for the p-forms Hodge local product:

$$\langle G,H\rangle=G\wedge *H=\sqrt{g}g^{\mu_1\nu_1}\cdots g^{\mu_p\nu_p}G_{\mu_1\cdots\mu_p}H_{\nu_1\cdots\nu_p}.$$

For future reference we also define the internal product

$$G\cdot H=g^{\mu_1\nu_1}\cdots g^{\mu_p\nu_p}G_{\mu_1\cdots\mu_p}H_{\nu_1\cdots\nu_p}.$$

We remind the reader that $\delta=(-1)^{N(p+1)+1}*d*$ is the formal adjoint of the operator d with respect to the scalar product of p-forms induced by the integration of the previous Hodge local product; finally, $\Delta=d\delta+\delta d$ is the Hodge–de Rham Laplacian of the p-forms. In order to perform calculations through the usual covariant derivative formalism the following relations for zero-forms and one-forms are quite useful:

$$\Delta\phi=-\nabla_\mu\nabla^\mu\phi,$$

$$\delta A=-\nabla_\mu A^\mu,$$

$$(\Delta A)_\mu=-\nabla_\nu\nabla^\nu A_\mu+R^\nu_\mu A_\nu.$$

The second line of Eq. (16) represents the complete photon action now expressed in terms of the vector field A_μ and the ghost fields only and it is the one usually employed in order to compute the partition function of the photon field by means of a functional integral. The partition function of photon at the temperature $T=1/\beta$ is then formally expressed by

$$Z_\beta = \int \mathcal{D}A \, \exp\left\{ -\frac{1}{2}\int d^4x \left[\langle A, \Delta A\rangle - \left(1-\frac{1}{\alpha}\right) \times \langle A, d\delta A\rangle \right]\right\} \int \mathcal{D}c\mathcal{D}\overline{c} \, \exp[-S_{\rm ghost}(\alpha)].$$

In order to compute this partition function, we want to pass to the related optical manifold, and so we consider a conformal transformation, Eq. (7), with $\Omega^2 = g_{00}$. Notice that, since we work in four dimensions, the p-forms A and F have a vanishing mass dimension and thus they must be conformally invariant, namely, $A = A'$ and $F = F'$. Furthermore the following identity arises:

$$\langle F, F\rangle' = \langle F, F\rangle. \tag{18}$$

B. First general approach

As we said above, the way to proceed is twofold. As a first way, we can suppose to have performed the conformal transformation *before* we start with the field theory. This means that we define the partition function of photons in the static manifold as a path integral directly in the optical manifold. In such a case the expression of the partition function is defined by

$$Z_\beta^{(1)} = \int \mathcal{D}A \, \exp\left\{ -\frac{1}{2}\int d^4x \left[\langle A, \Delta' A\rangle' - \left(1-\frac{1}{\alpha}\right) \times \langle A, d\delta' A\rangle' \right]\right\} \int \mathcal{D}c'\mathcal{D}\overline{c'} \exp[-S'^{(1)}_{\rm ghost}(\alpha)], \tag{19}$$

where

$$S'^{(1)}_{\rm ghost}(\alpha) = -\frac{1}{\sqrt{\alpha}}\int d^4x \sqrt{g'}\,\overline{c'}\Delta' c',$$

and where the primed metric and variables appearing in the previous functional integral are the optical ones. In other words, for the one-loop Euclidean effective action $-\ln Z_\beta^{(1)}$ we have

$$\ln Z_\beta^{(1)} = -\frac{1}{2}\ln\det \mu^{-2}\left[\Delta' - \left(1-\frac{1}{\alpha}\right) d\delta'\right] + \ln Z^{(1)}_{\beta,\rm ghost}(\alpha). \tag{20}$$

Here μ is an arbitrary renormalization scale necessary on a dimensional ground in the above formula and denoting the presence of a scale anomaly if it does not disappear from the final formulas.

For future reference we note that the effective action of the ghosts, except for the α-dependent factor, amounts trivially to minus twice the Euclidean effective action of an uncharged massless scalar field with the Euclidean action minimally coupled with the gravitation. Therefore its contribution to the one-loop effective action can be written immediately from the ζ function of a minimally coupled scalar field, $\zeta^{\rm MCS}(s;x)$, in the same background, taking the α dependence into account:

$$\ln Z^{(1)}_{\beta,\rm ghost}(\alpha) = -\int d^4x \sqrt{g'}\left[\frac{d}{ds}\zeta^{\rm MCS}(s;x)|_{s=0} + \zeta^{\rm MCS}(s;x)|_{s=0} \ln\sqrt{\alpha}\mu^2\right]. \tag{21}$$

C. Second general approach

As a second way, we can suppose to define the partition function directly in the static manifold, adding also the gauge-fixing term and the ghost Lagrangian to the pure electromagnetic action, and only *after* perform the conformal transformation to the optical metric. In this way we have to find how all the pieces in the path integral transform under the conformal transformation. In particular, the operator $\Delta - (1-\alpha^{-1}) d\delta$ transforms into another operator Λ_α, which we are going to write shortly. As regards the functional Jacobian which arises from the functional measure, a direct generalization of the discussion made in the scalar case tells us that it has to be ignored if we are interested in computing the thermal partition function. However, we would have to take it into account if we were computing, for example, the zero-temperature effective action in a cosmic string background.

Hence, employing this second procedure, we shall assume the photon partition function to be defined by

$$Z_\beta^{(2)} = \int \mathcal{D}A' \exp\left[-\left(\frac{1}{2}\int d^4x \langle A', \Lambda_\alpha A'\rangle'\right)\right] \times \int \mathcal{D}c'\mathcal{D}\overline{c'} \exp[-S'^{(2)}_{\rm ghost}(\alpha)].$$

In other words, for the Euclidean effective action $-\ln Z_\beta^{(2)}$ we have

$$\ln Z_\beta^{(2)} = -\frac{1}{2}\ln\det(\mu^{-2}\Lambda_\alpha) + \ln Z^{(2)}_{\beta,\rm ghost}(\alpha). \tag{22}$$

The form of $S'^{(2)}_{\rm ghost}(\alpha)$ is that of Eq. (17) after a conformal transformation:

$$S'^{(2)}_{\rm ghost}(\alpha) = -\frac{1}{\sqrt{\alpha}}\int d^4x \sqrt{g'}\,\overline{c'}\left[\Delta' + \frac{1}{6}(R' - \Omega^{-2}R)\right] c', \tag{23}$$

where $c' = \Omega c$, $\overline{c'} = \Omega\overline{c}$. For future reference, we note that this effective action of the ghosts amounts trivially to minus twice the Euclidean effective action of an uncharged massless scalar field φ with the Euclidean action $(\Delta' = -\nabla'_\mu\nabla'^\mu)$ endowed by an α-depending overall factor

$$S^{(2)}(\alpha) = \frac{1}{\sqrt{\alpha}}\int d^4x \sqrt{g'}\frac{1}{2}\varphi\left[\Delta' + \frac{1}{6}(R' - \Omega^{-2}R)\right]\varphi. \tag{24}$$

When the static manifold is flat, $R = 0$, the contribution of the ghosts to the effective action can be written in terms of the ζ function of a conformally coupled scalar field:

$$\ln Z^{(2)}_{\beta,\mathrm{ghost}}(\alpha)=-\int d^4x\sqrt{g'}\left[\frac{d}{ds}\zeta^{\mathrm{CCS}}(s;x)|_{s=0}+\zeta^{\mathrm{CCS}}(s;x)|_{s=0}\ln\sqrt{\alpha}\mu^2\right]. \qquad (25)$$

Now, let us find the explicit form of the operator Λ_α. The following identity holds:

$$\delta A=\frac{1}{\Omega}(\delta' A-\eta\cdot A), \qquad (26)$$

provided the one-form η be defined as

$$\eta=d(\ln\Omega)\equiv\partial_\mu\ln\Omega. \qquad (27)$$

Taking into account that $\delta=d^\dagger$ and employing Eqs. (16), (18), and (27), we get the identity

$$\begin{aligned}S^{\mathrm{em}}&=\frac{1}{2}\int d^4x\left[\langle A,\Delta A\rangle-\left(1-\frac{1}{\alpha}\right)\langle A,d\delta A\rangle\right]+S_{\mathrm{ghost}}(\alpha)\\&=\frac{1}{2}\int d^4x\left[\langle A,\Delta' A\rangle'-\left(1-\frac{1}{\alpha}\right)\langle A,d\delta' A\rangle'+\frac{1}{\alpha}\langle A,\eta\eta\cdot A\rangle'-\frac{1}{\alpha}\langle A,(\eta\delta'+d\eta\cdot)A\rangle'\right]\\&\quad+S'^{(2)}_{\mathrm{ghost}}(\alpha).\end{aligned} \qquad (28)$$

Looking at the first line of Eq. (28) we find the explicit form of the operator Λ_α:

$$\Lambda_\alpha=\Delta'-\left(1-\frac{1}{\alpha}\right)d\delta'+\frac{1}{\alpha}\eta\eta\cdot-\frac{1}{\alpha}(\eta\delta'+d\eta\cdot). \qquad (29)$$

Notice that the use of such an operator is equivalent to employing an unusual gauge-fixing term in the initial photon Lagrangian which reads

$$\frac{1}{\alpha}\langle A,(d-\eta)(\delta'-\eta\cdot)A\rangle. \qquad (30)$$

IV. THE CASE OF THE RINDLER SPACE

Let us check the physical results arising from Eqs. (20) and (22) in the case of the Rindler space. Setting $\Omega^2=r^2$ in Eq. (7), the related ultrastatic optical metric reads

$$ds'^2=d\tau^2+r^{-2}(dr^2+dy^2+dz^2). \qquad (31)$$

Obviously, this is the natural metric of $S^1\times H^3$ which does not contain conical singularities. We remind one that $R'^{\mu}_{\nu}=-2\,\mathrm{diag}(0,1,1,1)$ and $R'=-6$. As for the one-form η necessary to define the operator Λ_α, we get

$$\eta_\mu=\frac{2}{r}\delta^r_\mu. \qquad (32)$$

We want to employ a local ζ-function [30,31,20] regularization technique and hence we define the determinant of an (at least) symmetric operator L through

$$-\frac{1}{2}\ln\det(\mu^{-2}L)=\frac{1}{2}\int d^4x\sqrt{g'}\times[\zeta'(s=0;x)+\zeta(s=0;x)\ln\mu^2], \qquad (33)$$

where the *local* ζ function of the operator L is defined, as usual, by means of the analytic continuation in the variable $s\in C$ of the spectral representation of the complex power of the operator L:

$$\zeta(s;x)=\sum_n \lambda_n^{-s}A_n(x)\cdot A_n^*(x). \qquad (34)$$

Above, $A_n(x)$ is a one-form eigenfunction of a suitable self-adjoint extension of the operator L and λ_n is its eigenvalue. The index n stands for all the quantum numbers, discrete or continuous, needed to specify the spectrum. The set of these modes is supposed complete and (Dirac, Kroneker) δ normalized. We will make also use of the following notation for the one-forms on $S^1\times H^3$:

$$A\equiv(a|B),$$

where a indicates a one-form on S^1 and B a one-form on H^3. All the operations between forms which appear after ''|'' are referred to the manifold H^3 and its metrical structure only. Latin indices $a,b,c,d,\ldots$ are referred to the coordinates r,y,z on H^3 only.

A suitable set of eigenfunctions of the operator $\Delta'-(1-\alpha^{-1})d\delta'$ as well as Λ_α as can be constructed using the following complete and normalized set of eigenfunction of the scalar Hodge–de Rham Laplacian on $S^1\times H^3$:

$$\phi^{(\mathbf{k},n,\omega)}(\tau,r,\mathbf{x})=\frac{e^{i\mathbf{kx}}e^{i\nu_n\tau}}{2\pi^2\sqrt{\beta}}\sqrt{2\omega\sinh(\pi\omega)}\,rK_{i\omega}(kr), \qquad (35)$$

where $\nu_n=2\pi n/\beta$, $n\in Z$, $\omega\in R^+$, $\mathbf{k}=(k_y,k_z)\in R^2$, $k=|\mathbf{k}|$ and all the previous eigenfunctions have eigenvalue $(\nu^2+\omega^2+1)$. $K_{i\omega}(x)$ is the usual MacDonald function with an imaginary index. The normalization reads

$$\int d^4x\sqrt{g'}\,\phi^{(\mathbf{k},n,\omega)*}\phi^{(\mathbf{k}',n',\omega')}=\delta^{nn'}\delta^2(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega').$$

In the following, we report some relations which are very useful in checking the results which we shall report shortly. It is convenient to define the one-form $\xi=-d(1/r)=\eta/2r$ on H^3. On H^3 we have $\delta\eta=4$, $\Delta\eta=0$, $d\eta=0$, $d\xi=0$, $\Delta\xi=-3\xi$, $\nabla^a\xi_b=-\delta^a_b/r$. Furthermore we remind one that, if f is a zero-form and ω is a one-form:

$$[\Delta(f\omega)]_a=f[\Delta\omega]_a+\omega_a\Delta f-2(\nabla_b f)\nabla^b\omega_a. \qquad (36)$$

Finally, on a three-manifold the following relation holds:

$$\begin{aligned}\Delta*(\omega\wedge\omega')&=*[(\Delta\omega)\wedge\omega']+*[\omega\wedge\Delta\omega']+R*(\omega\wedge\omega')\\&\quad-*[(R\omega)\wedge\omega']-*(\omega\wedge R\omega')\\&\quad-2*(\nabla_a\omega\wedge\nabla^a\omega'),\end{aligned} \qquad (37)$$

where obviously $[*(\nabla_a\omega\wedge\nabla^a\omega')]_e := \sqrt{g}\epsilon_{ebc}(\nabla_d\omega^b)\nabla^d\omega'^c$, ω and ω' are one-forms, and the Ricci tensor acts on one-forms trivially as $(R\omega)_a = R_a^b\omega_b$.

A. First optical approach

Let us now consider the first optical approach, in which we define the path integral directly in the optical manifold, see Eq. (19). Starting from the scalar eigenfunctions, one can obtain the following set of eigenfunctions of the operator $\Delta-(1-\alpha^{-1})d\delta$ on $S^1\times H^3$:

$$A^{(1)}=\frac{\sqrt{\omega^2+1}}{|\nu|\sqrt{\omega^2+\nu^2+1}}(\partial_\tau\phi|d\phi) =\frac{\sqrt{\omega^2+1}}{|\nu|\sqrt{\omega^2+\nu^2+1}}(\partial_\tau\phi,\partial_r\phi,\partial_y\phi,\partial_z\phi),$$

$$A^{(2)}=\frac{1}{\sqrt{\omega^2+\nu^2+1}}\left(\partial_\tau\phi\left|\frac{-\nu^2}{\omega^2+1}d\phi\right.\right),$$

$$A^{(3)}=\frac{1}{k}[0|*d(\xi\phi)]=\frac{1}{k}\left(0,0,\partial_z\frac{\phi}{r},-\partial_y\frac{\phi}{r}\right),$$

$$A^{(4)}=\frac{1}{k\omega}[0|\delta d(\xi\phi)]=\frac{r}{k\omega}\left(0,\frac{k^2}{r}\phi,\partial_r\partial_y\frac{\phi}{r},\partial_r\partial_z\frac{\phi}{r}\right).$$

The last three modes are transverse, $\delta A=0$, whereas the first one is a pure gauge mode. From a little Hodge algebra, the following normalization relations can be proved:

$$\int d^4x\langle A^{(J,\omega,n,\mathbf{k})*},A^{(J',\omega',n',\mathbf{k}')}\rangle =\delta^{JJ'}\delta^{nn'}\delta^2(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega'). \tag{38}$$

As far as the eigenvalues are concerned, we have

$$[\Delta'-(1-\alpha^{-1})d\delta']A^{(1)}=\frac{\omega^2+\nu^2+1}{\alpha}A^{(1)},$$

$$[\Delta'-(1-\alpha^{-1})d\delta']A^{(2)}=(\omega^2+\nu^2+1)A^{(2)},$$

$$[\Delta'-(1-\alpha^{-1})d\delta']A^{(J)}=(\nu^2+\omega^2)A^{(J)}$$

$$\text{if}\quad J=3,4.$$

Employing the definition in Eq. (34), the above modes and the definitions given in the Appendix, we have that (notice that ϕ^* and ϕ take the same values of $\mathbf{k}$, n, ω)

$$\zeta(s;x)=(\alpha^s+1)\sum_{n=-\infty}^{\infty}\int d^2\mathbf{k}d\omega\frac{\phi^*(x)\phi(x)}{[\omega^2+\nu^2+1]^s} +\sum_{n=-\infty}^{\infty}\int d^2\mathbf{k}d\omega\frac{2(1+\omega^{-2})\phi^*(x)\phi(x)}{[\omega^2+\nu^2]^s} =(\alpha^s+1)\zeta^{\mathrm{MCS}}(s;x)+2\zeta^{\mathrm{CCS}}(s;x)+\zeta^{\mathrm{extra}}(s;x), \tag{39}$$

where we have set

$$\zeta^{\mathrm{extra}}(s;x)=2\sum_{n=-\infty}^{\infty}\int d^2\mathbf{k}\int\frac{d\omega}{\omega^2}\frac{\phi^*(x)\phi(x)}{[\omega^2+\nu^2]^s} =\frac{\sqrt{\pi}}{\pi^2\beta}\frac{\Gamma(s-1/2)}{\Gamma(s)}\left(\frac{\beta}{2\pi}\right)^{2s-1}\zeta_R(2s-1), \tag{40}$$

so that $\zeta^{\mathrm{extra}}(s=0;x)=0$ and $\zeta'^{\,\mathrm{extra}}(s=0;x)=1/3\beta^2$. Notice that the second and third terms in Eq. (39) arise from the transverse modes $A^{(3)}$ and $A^{(4)}$. The first term in Eq. (39) is due to the modes with $J=1,2$.

In calculating Eq. (39), we encountered Kabat's surface terms similar to those we encountered in [1]. However, in the present case all these terms vanish automatically and no further regularization procedure needs. In fact, all these terms read as

$$D_r\sum_n\int d\mathbf{k}\int d\omega r^2K_{i\omega}(kr)K_{i\omega}(kr)f(\omega,\nu,s),$$

where D_r is an opportune differential operator in r. Passing from the integration variable $\mathbf{k}$ to the integration variable $r\mathbf{k}$, we see that the term after the operator does not depend on r, and so the differentiation produces a vanishing result.

In order to write the complete local ζ functions of the electromagnetic field we have to take account of the ghost contribution. We have already said that in this approach the ζ function of the ghosts is just minus two times the ζ function of a minimally coupled scalar field, but with a gauge-fixing dependent scale factor, see Eq. (21):

$$\zeta_\alpha^{\mathrm{ghosts}}(s;x)=-2\zeta^{\mathrm{MCS}}(s;\mu^{-2}\alpha^{-1/2}L_{\xi=0})(x).$$

Using this relation, Eq. (39) and reintroducing everywhere the renormalization scale μ, we can write the complete local ζ function of the electromagnetic field as

$$\zeta^{\mathrm{em}}(s;x)=(\alpha^s+1)\zeta^{\mathrm{MCS}}(s;\mu^{-2}L_{\xi=0})(x) +2\zeta^{\mathrm{CCS}}(s;\mu^{-2}L_{\xi=1/6})(x)+\zeta^{\mathrm{extra}}(s;\mu^{-2})(x) -2\zeta^{\mathrm{MCS}}(s;\mu^{-2}\alpha^{-1/2}L_{\xi=0})(x). \tag{41}$$

It follows that the one-loop effective Lagrangian density is just

$$\mathcal{L}_{\text{eff}}(x)=\frac{1}{2}\frac{d}{ds}[2\zeta^{\text{CCS}}(s;x)+\zeta^{\text{extra}}(s;x)]_{s=0}=\frac{\pi^2}{45\beta^4}+\frac{1}{6\beta^2}. \tag{42}$$

We remark the importance of keeping the α dependence of the action of the ghosts: it gives a contribution proportional to $\ln\alpha$ which cancels against the ($\ln\alpha$)-dependent term coming from $(\alpha^s+1)\zeta^{\text{MCS}}(s;x)$, restoring the gauge invariance of the theory. Note also how all the terms containing $\ln\mu^2$ cancel giving the expected scale invariant theory.

Integrating this quantity over the manifold and introducing a cutoff at a distance ϵ form $r=0$ in order to control the horizon divergence, we get the one-loop free energy:

$$\begin{aligned}F^{(1)}&=-\frac{1}{\beta}\int d^4x\sqrt{g'}\mathcal{L}_{\text{eff}}(x)\\&=-\frac{A_\perp}{1440\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4+30\left(\frac{2\pi}{\beta}\right)^2\right].\end{aligned} \tag{43}$$

Renormalizing this result in such a way that the internal energy vanishes at $\beta=2\pi$, we find just the point-splitting result

$$U_\beta^{(1)\text{sub}}=U_\beta^{\text{phot ps}}. \tag{44}$$

B. Second optical approach

Let us then consider the second optical approach. We were able to perform the calculations in the case $\alpha=1$ only, hence a complete discussion on the gauge invariance (α invariance) is not possible. However, the found result contains some interest. As before, the eigenfunctions of the operator $\Lambda_{\alpha=1}$ are constructed from the scalar eigenfunctions, Eq. (35):

$$A^{(1)}=\frac{1}{|\nu|}\left(\partial_\tau\phi\middle|+\frac{|\nu|}{2}\eta\phi\right)=\frac{1}{|\nu|}\left(\partial_\tau\phi,|\nu|\frac{\phi}{r},0,0\right)$$

$$A^{(2)}=\frac{1}{|\nu|}\left(\partial_\tau\phi\middle|-\frac{|\nu|}{2}\eta\phi\right)=\frac{1}{|\nu|}\left(\partial_\tau\phi,-|\nu|\frac{\phi}{r},0,0\right)$$

$$A^{(3)}=\frac{1}{k}[0|*d(\xi\phi)]=\frac{1}{k}\left(0,0,\partial_z\frac{\phi}{r},-\partial_y\frac{\phi}{r}\right)$$

$$A^{(4)}=\frac{1}{k}\{0|*[\xi\wedge *d(\xi\phi)]\}=\frac{1}{k}\left(0,0,\partial_y\frac{\phi}{r},\partial_z\frac{\phi}{r}\right).$$

The following normalization relations hold:

$$\begin{aligned}&\int d^4x\langle A^{(J,\omega,n,\mathbf{k})*},A^{(J',\omega',n',\mathbf{k}')}\rangle\\&=\delta^{JJ'}\delta^{nn'}\delta^2(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega').\end{aligned} \tag{45}$$

As far as the eigenvalues are concerned, we have

$$\Lambda_{\alpha=1}A^{(J)}=\{\omega^2+[(-1)^J+|\nu|]^2\}A^{(J)}\quad\text{if } J=1,2,$$

$$\Lambda_{\alpha=1}A^{(J)}=(\nu^2+\omega^2)A^{(J)}\quad\text{if } J=3,4.$$

Employing the definition in Eq. (34) and the found modes we have (notice that ϕ^* and ϕ take the same values of $\mathbf{k}$, n, ω)

$$\begin{aligned}\zeta^{(2)}(s;x)=&\sum_{n=1}^{\infty}\int d\mathbf{k}\int d\omega\frac{2\phi^*(x)\phi(x)}{[\omega^2+(\nu+1)^2]^s}\\&+\sum_{n=1}^{\infty}\int d\mathbf{k}\int d\omega\frac{2\phi^*(x)\phi(x)}{[\omega^2+(\nu-1)^2]^s}\\&+\sum_{n=1}^{\infty}\int d\mathbf{k}\int d\omega\frac{4\phi^*(x)\phi(x)}{[\omega^2+\nu^2]^s}.\end{aligned} \tag{46}$$

For simplicity, we have omitted the terms corresponding to $n=0$, which contribute only to the temperature-independent part of the free energy: this part will be changed during the renormalization process (subtraction of the Minkowski vacuum energy). We also stress that Kabat's surface terms involved during the calculations disappeared exactly as in the previous approach. The latter term in Eq. (39) is due to the modes with $J=3,4$: this term is exactly twice the ζ function of a conformally coupled Euclidean scalar field propagating in $S^1\times H^3$.

As far as the ghost contribution is concerned, it arises from the action (23). Since the corresponding small fluctuations operator involves the curvature of the Euclidean Rindler manifold, which has a Dirac's δ singularity at $r=0$, mathematically it is not well defined and is not clear how to deal with it. However, as a try we can suppose to consider $R=0$ and see the consequences.[6] Under this hypothesis, the ghost contribution is just minus twice that of a conformally coupled scalar field [see Eq. (25)] and so it cancels against the contribution of the modes $J=3,4$.

After having added the ghost contribution, we can write the complete ζ function of the electromagnetic field as

$$\begin{aligned}\zeta^{\text{em}}(s;x)=&\sum_{n=1}^{\infty}\int d\mathbf{k}\int d\omega\frac{2\phi^*(x)\phi(x)}{[\omega^2+(\nu+1)^2]^s}\\&+\sum_{n=1}^{\infty}\int d\mathbf{k}\int d\omega\frac{2\phi^*(x)\phi(x)}{[\omega^2+(\nu-1)^2]^s}.\end{aligned} \tag{47}$$

The partition function of the photons is obtained employing the previous function opportunely continued in the variable s in Eq. (33). Dealing with it, as in the previous case, we finally find the free energy

$$F^{(2)\text{sub}}=-\frac{A_\perp}{1440\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4-30\left(\frac{2\pi}{\beta}\right)^2+29\right]. \tag{48}$$

[6] See footnote number 5.

In deriving this result we have employed the Riemann zeta function $\zeta(z,q)$ and its relation with the Bernoulli polynomials [32]. This result has the same form as that obtained with the first approach, Eq. (43), but the sign in front to the second term is opposite. The third term is fixed by the renormalization procedure. The problems arise with the β^{-2} term once again.

In this case it is easy to identify the origin of the discrepancy in our hypothesis of setting $R=0$ in the ghost action. Nevertheless, there is some evidence that the origin is not that. In particular, if we assume that the optical method gives the same results as the point-splitting one even when $\xi \neq 1/6$, then we can suppose that it is right to substitute the optical result for the ghost contribution to the above free energy with the point-splitting one for $\xi=0$. As a result, we get

$$F^{(2)\mathrm{sub}}=-\frac{A_\perp}{1440\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4-60\left(\frac{2\pi}{\beta}\right)^2+59\right], \quad (49)$$

which is different from the previous one but still different from the first optical approach result. In particular, the free energy in Eq. (48) [or Eq. (49)] would yield a negative entropy at the Unruh-Hawking temperature, which is very hard to accept on a physical ground. Summarizing, it seems to us that this second approach, which is the natural generalization of the procedure used in the scalar case, does not yield a correct result.

V. SUMMARY AND DISCUSSION

The main result of this paper is the proof that the optical method (the ''first approach'') can be used to compute one-loop quantities in the Rindler space also in the case of the photon field. The method has been developed employing a general covariant gauge choice. Furthermore, by a comparison with other methods, we have seen that this method produces the same result as the point-splitting procedure.

It is also important to stress that the partition function arising from our method is completely free from ''Kabat's'' surface terms. This is very important because, as we previously said, the approaches based on the direct computation in the Euclidean Rindler space using ζ-function or heat-kernel techniques produces such anomalous terms [9,1] and further regularization procedures seem to be necessary to get physically acceptable results.

We have also developed a general optical formalism for the Maxwell field in the covariant gauges based on Hodge–de Rham formalism which, in principle, can be used in different manifolds than the Rindler space.

However, many problems remain to be explained. In particular, both in the photon and in the scalar case the relation between the optical approach and the direct approach in the manifold with the conical singularity remains quite obscure. This is due to difficulties involved in computing the Jacobian of the conformal transformation in the presence of conical singularities. Moreover, while the optical approach can be used in the case of massless fields without particular difficulties, as soon as the fields have a mass the optical method becomes much harder to apply. In this case, the direct computation in the manifold with conical singularities could show its advantages, provided one knows how to compute the above Jacobian.

Another general point which requires further investigation is the request of self-consistency of the thermodynamics of the gas of Rindler particle, when the temperature is not the Unruh one. This is a very important point in calculating the correction to the entropy of a black hole supposing such corrections due to the fields propagating around it. We remind one that the Rindler metric approximates the region near the horizon of a Schwarzschild black hole. The entropy of the fields is computed using the relation (where β_H is the Unruh-Hawking temperature, 2π in the Rindler case): $S_{\beta_H}=\beta_H^2\partial_\beta F_\beta|_{\beta_H}$. In calculating the previous derivative at $\beta=\beta_H$, one has to consider also the partition function *off shell*, namely evaluated at $\beta\neq 2\pi$ and β *near* β_H. It is not so clear whether it is necessary or not that the thermodynamical laws hold also for $\beta\neq\beta_H$ and β near β_H in order to assure the consistency the procedure followed in calculating the entropy of the fields at $\beta=\beta_H$. Moreover, it is well known that the off-shell quantum states of a field are affected by several pathologies on the horizon event.[7] Furthermore, they are unstable states in a semiclassical approach to quantum gravity due to the divergence of the renormalized stress tensor on the horizon. Thus, it is reasonable to wonder about the thermodynamical consistency of the results when one works off shell. We conclude with a discussion on this point.

Let us first consider the point-splitting result and the annoying dependence on the parameter ξ of the massless scalar field results and its relation with the request of a consistent thermodynamics. As we have already said, on more regular manifolds the (integrated) physical quantities should not depend on the actual value of ξ, whereas in the case of Euclidean Rindler wedge the conical singularity introduces an, apparently unphysical, ξ dependence in the physical quantities.[8] A similar problem occurs even in flat spaces in presence of boundaries [21]. In those cases, one can see that the renormalized energy-momentum tensor diverges on the boundary, unless the coupling is conformal. So, one could say that the conformal coupling is, in this sense, more ''physical'' than the others. By this we mean that it behaves like a real field, such as the Maxwell field.

Inspired by this fact, we shall look for a criterion to choose a value of ξ which is more ''physical'' than the others as far as the thermodynamics is concerned. Thus we shall discuss the consistency of the thermodynamics of the point-splitting results.

From the thermodynamics, we know that we can obtain the internal energy $U^{\mathrm{ps}}_{\beta,\xi}$ [see Eq. (2)] as the derivative with

[7]Rindler thermal states with $\beta\neq 2\pi$ violates several axioms of the quantum field theory (QFT) in curved backgrounds. For example, see [33] and references therein.

[8]Notice that the case of the cosmic string theory is quite different because different values of ξ correspond to different internal structures of the string. This is obvious by considering a string with a finite thickness, which has a nonvanishing curvature within itself. In the limit of a vanishing thickness, the curvature R gets a Dirac δ behavior along the string in the Lorentzian manifold.

respect to β of an appropriate free energy $F^{\rm ps}_{\beta,\xi}$ multiplied by β, possibly corrected by an suitable energy-subtraction procedure. Taking the space homogeneity along the y and z directions into account, we found the form of the free energy

$$F^{\rm ps}_{\beta,\xi}=-\frac{L_yL_z}{2880\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4-30(6\xi-1)\left(\frac{2\pi}{\beta}\right)^2-30(6\xi-1)+3\right]-\frac{L_yL_z}{2880\pi^2\epsilon^2}\left[\mathcal{U}(\xi,\epsilon)+\frac{f(\xi,\epsilon)}{\beta}\right]. \quad (50)$$

The unknown function $f(\xi,\epsilon)$ can be dropped by requiring that the entropy $S_{\beta,\xi}=\beta^2\partial_\beta F^{\rm ps}_{\beta,\xi}$ vanishes at $\beta\to+\infty$. The function $\mathcal{U}$, which does not depend on β but can depend on the geometry background, is necessary due to the fact that the energy in Eq. (2) is the Minkowski renormalized one, but we want to remain on a more general ground in order to use the thermodynamical laws. In other words, we may notice that the energy in Eq. (2) becomes negative if the temperature is sufficiently low, for example, in the most interesting range $0\leqslant\xi\leqslant 1/6$, and hence such an energy cannot directly arise from a statistical partition function but a further subtraction procedure must have taken place. The function $\mathcal{U}$ takes into account this energy subtraction procedure.

From statistical thermodynamical laws, one expects that the y and z principal pressure, namely T_{yy} and T_{zz} in Eq. (1), integrated over $dzdr\sqrt{g}$ and $dydr\sqrt{g}$, respectively, can be obtained taking the L_y (L_z) derivative of the previous free energy, with the sign changed and $\mathcal{U}$ opportunely chosen. An easy computation shows that, due to the terms containing β^{-2}, this does not hold for any value of ξ, but only in the conformally coupled case, $\xi=1/6$. After the Minkowskian energy subtraction, the corresponding free energy reads

$$F^{\rm ps}_{\beta}:=F^{\rm ps\ sub}_{\beta,\xi=1/6}=-\frac{A_\perp}{2880\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4+3\right]. \quad (51)$$

This is just the free energy obtained by the optical method after the Minkowski renormalization. Therefore, it seem that only in the conformally coupled case the stress tensor (1) yields a consistent thermodynamics, at least as far as the relation between energy and pressures is concerned.

Now, let us consider the photon case. In such a case we have not the freedom to adjust a parameter in the stress tensor in order to agree with the thermodynamics. The free energy we find from the total energy in the case of the photon stress tensor of Eq. (15) reads

$$F^{\rm phot\ ps}_{\beta}=-\frac{L_yL_z}{1440\pi^2\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^4+30\left(\frac{2\pi}{\beta}\right)^2-33\right]-\frac{L_yL_z}{1440\pi^2\epsilon^2}\left[\mathcal{U}(\epsilon)+\frac{f(\epsilon)}{\beta}\right]. \quad (52)$$

As before, we can drop the term containing the undetermined function $f(\epsilon)$ by requiring a vanishing entropy in the limit $\beta\to+\infty$. The above free energy produces the point-splitting internal energy and, after the Minkowski renormalization, it coincides with the free energy obtained by renormalizing that obtained by the optical approach, Eq. (43).

The point is that if we apply the above procedure to compute the integrated principal pressures along the y and z directions to the above photon free energy, there is no way to choose $\mathcal{U}$ in such a way to get the same result as integrating the yy and zz components of the photon stress tensor in Eq. (15). This is due to the presence of a term proportional to β^{-2} and the independence on β of the function $\mathcal{U}$.

In order to get the ''correct'' pressures (but a wrong internal energy) employing the derivatives as previously pointed out, one should take a free energy which is twice that in Eq. (50) with $\xi=1/9$, $f=0$ and $\mathcal{U}$ opportunely chosen.

Hence, it seems that the point-splitting stress tensor of photons in the Rindler wedge does not give a consistent thermodynamics.[9] It is very important to remark that the above thermodynamical argument cannot be applied to the cosmic string theory, since in that case the stress tensor in Eq. (15) is the zero-temperature one, and β is not the inverse of the temperature.

In a pessimistic view, this problem and the ξ dependence of the integrated quantities in the scalar case could be considered as another proof of the inconsistency of the Rindler theory (and maybe of the Schwarzschild theory) when one works at temperatures different from the Unruh-Hawking one, and a discouraging result for the attempt to evaluate the correction to the Bekenstein-Hawking entropy through the ''off-shell'' procedure.[10]

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APPENDIX

In computing the photon ζ function on $S^1\times H^3$ one meets the ζ function of a scalar field in the same background, both in conformal and minimal coupling. Therefore, it is useful to report here these ζ functions. The small fluctuations operator for a scalar field in the optical metric is

$$L_\xi=\Delta-6\xi=-[\partial_\tau^2-r\partial_r+r^2\partial_r^2+6\xi],$$

where Δ is the Hodge–de Rham Laplacian on $S^1\times H^3$. A complete set of eigenfunctions has been given in the main text, Eq. (35), with eigenvalue $[\nu_n^2+\omega^2+1-6\xi]$. Therefore, the local ζ function is

[9]This problem arises also dealing with the massless spinorial field as it simply follows from the point-splitting renormalized stress tensor obtained in [6] (analytically continued from the cosmic string to the Rindler space).

[10]However, it could be possible to interpret the entropy formula at Hawking's temperature, without making use of thermodynamical laws off shell. Maybe, possible ways could arise studying the *geometrical* entropy employing the replica trick [10,22].

$$\zeta(s|L_\xi)(x)=\sum_{n=-\infty}^{\infty}\int_0^\infty d\omega\int d^2\mathbf{k}[\nu_n^2+\omega^2+1-6\xi]^{-s}\phi^*(x)\phi(x)$$
$$=\frac{\sqrt{\pi}}{8\pi^2\beta}\frac{\Gamma(s-3/2)}{\Gamma(s)}\sum_{n=-\infty}^{\infty}\int_0^\infty d\omega\,\omega^2[\nu_n^2+\omega^2+1-6\xi]^{-s}$$
$$=\frac{\sqrt{\pi}}{8\pi^2\beta}\frac{\Gamma(s-3/2)}{\Gamma(s)}\left(\frac{2\pi}{\beta}\right)^{3-2s}\left[2E\left(s-\frac{3}{2};\frac{\beta}{2\pi}\sqrt{1-6\xi}\right)-\left(\frac{\beta}{2\pi}\sqrt{1-6\xi}\right)^{3-2s}\right],$$

where $E(s;a)=\Sigma_{n=0}^{\infty}[n^2+a^2]^{-s}$ is the Epstein ζ function. In the conformally coupled case, the Epstein function becomes a Riemann ζ function and so

$$\zeta^{\mathrm{CCS}}(s;x)\equiv\zeta(s|L_{\xi=1/6})(x)$$
$$=\frac{\sqrt{\pi}}{4\pi^2\beta}\left(\frac{\beta}{2\pi}\right)^{2s-3}\frac{\Gamma(s-3/2)}{\Gamma(s)}\zeta_R(2s-3).$$

One can easily check that $\zeta^{\mathrm{CCS}}(s;x)|_{s=0}=0$ and

$$\frac{d}{ds}\zeta^{\mathrm{CCS}}(s;x)|_{s=0}=\frac{\pi^2}{45\beta^4}.$$

Another important case is the minimally coupled one, $\xi=0$, for which there is not a more explicit form. However, using the identity

$$E(s;a)=\frac{1}{2a^{2s}}+\frac{\sqrt{\pi}}{2}\frac{\Gamma(s-1/2)}{\Gamma(s)}a^{1-2s}$$
$$+\frac{2\sqrt{\pi}}{\Gamma(s)}\sum_{n=1}^{\infty}\left(\frac{\pi n}{a}\right)^{s-1/2}K_{s-1/2}(2\pi na)$$

and the fact that the MacDonald function $K_\nu(x)$ is analytic in the index ν and decays exponentially as $|x|\to\infty$ so that the third term in the previous expansion is analytic in s (and vanishes as $s\to 0$), we find that the ζ function does not vanish in $s=0$:

$$\zeta^{\mathrm{MCS}}(s;x)|_{s=0}=\frac{1}{32\pi^2}.$$

We do not know the value in zero of the derivative, but it is not required in our computations.

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