Improvement of the staggered fermion operators

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We present a complete and detailed discussion of the finite lattice spacing corrections to staggered fermion matrix elements. Expanding upon arguments of Sharpe, we explicitly implement the Symanzik improvement program demonstrating the absence of order a terms in the Symanzik improved action. We propose a general program to improve fermion operators to remove all $O(a)$ corrections from their matrix elements, and demonstrate this program for the examples of matrix elements of fermion bilinears and B_K. We find the former does have $O(a)$ corrections while the latter does not. Also, we give an explicit form of lattice currents which are accurate to order a^2 at the tree level. [S0556-2821(97)03301-8]

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I. INTRODUCTION

With the occurrence of a new generation of teraflop parallel supercomputers, we will be able to simulate lattice QCD with smaller and smaller statistical errors. It is now of increased importance to gain control of various kinds of systematic errors which either affect the numerical results directly or affect the way in which physical quantities are extracted. One of the most important systematic errors comes from the finite lattice spacing a which generates errors of the order of $a\Lambda_{\mathrm{QCD}}$. For the present lattice computation, this corresponds to the corrections of the order of 20–30 %. Another important systematic error comes from the choice of lattice operators. There exist a variety of lattice operators which approach the same continuum operator in the limit $a\to 0$. However, many of these operators differ from the continuum limit at $O(a)$, and a systematic formalism is needed to improve the lattice operators so as to remove these $O(a)$ corrections.

Throughout this paper, we will use such phrases as ''accurate through order a,'' ''accurate to $O(a^2)$,'' or ''no $O(a)$ corrections,'' etc., to mean that there will be no finite spacing errors of order $g_0^{2n}a$ for arbitrary order n. Whenever we are dealing with an expression that is accurate only to lowest order in g_0, we will always explicitly refer to it as ''accurate to order a^2 at the tree level'' or ''accurate to order $g_0^0 a^2$.''

For the case of Wilson fermions, the standard lattice action differs from the continuum quark action by a term of $O(a)$. So, both the action and the operators need corrections at order a. Applying the improvement program of Symanzik [1] to Wilson fermions, a procedure was proposed in [2,3] to reduce the systematic errors due to the finiteness of the lattice spacing, from terms of $O(a)$ to ones of $O(g_0^2 a)$, and it was numerically demonstrated in [4] that this procedure can reduce the finite a corrections from 30% to around 5%.

The meaning of the statement that there is no term of order a in the staggered fermion action is not clear. Let us use the free staggered fermion action as an example:

$$S_F=\sum_{x,\mu} a^4\overline{\chi(x)}\,\eta_\mu(x)\frac{1}{2a}[\chi(x+\mu)-\chi(x-\mu)] + m\sum_x a^4\overline{\chi(x)}\chi(x). \tag{1}$$

Following Golterman and Smit [5], we denote the Fourier components of the fields χ and $\overline{\chi}$ as $\widetilde{\chi}$ and $\widetilde{\overline{\chi}}$, and decompose momentum space as

$$k=p+\pi_A, \tag{2}$$

where $k_\mu\in(0,2\pi/a)$, $p_\mu\in(0,\pi/a)$, $(\pi_A)_\mu=A_\mu\pi$, in which $A_\mu=0,1$. If we define the fermion fields as

$$\widetilde{\psi}(p)=\frac{1}{8}\sum_{A,B}(-1)^{A\cdot B}\gamma_A\widetilde{\chi}(p+\pi_B), \tag{3a}$$

$$\widetilde{\overline{\psi}}(p)=\frac{1}{8}\sum_{A,B}(-1)^{A\cdot B}\gamma_A^\dagger\widetilde{\overline{\chi}}(p+\pi_B), \tag{3b}$$

where

$$\gamma_A=\gamma_1^{A_1}\gamma_2^{A_2}\gamma_3^{A_3}\gamma_4^{A_4}, \tag{4}$$

we can write the action as

$$S_F=\frac{(2\pi)^4}{\Omega}\sum_p \widetilde{\overline{\psi}}(p)\left(\sum_\mu \gamma_\mu\frac{i}{a}\sin p_\mu a+m\right)\widetilde{\psi}(p), \tag{5}$$

where Ω is the lattice volume. It is clear that there is no order a term in the action, hence the free staggered action is accurate to $O(a^2)$. However, the coordinate fields corresponding to $\widetilde{\psi}$ ($\widetilde{\overline{\psi}}$) are nonlocal superpositions of the χ's ($\overline{\chi}$'s) over all the lattice sites.

On the other hand, if we define the local hypercubic fermion fields as in [6]

$$q(y)=\frac{1}{8}\sum_A\gamma_A\chi(y+A)=\frac{1}{2}\sum_A\gamma_A\chi_A(y), \tag{6a}$$

$$\overline{q}(y)=\frac{1}{8}\sum_A\overline{\chi}(y+A)\gamma_A^\dagger=\frac{1}{2}\sum_A\overline{\chi_A}(y)\gamma_A^\dagger, \tag{6b}$$

where

$$x=y+A, \tag{7}$$

and $y_\mu=0,\pm 2,\ldots$, then the staggered fermion action in the momentum space can be written as

$$S_F=\frac{(2\pi)^4}{\Omega}\sum_p \tilde{\bar{q}}(p)\left\{\sum_\mu\left[(\gamma_\mu\otimes I)\frac{i}{2a}\sin p_\mu 2a + a(\gamma_5\otimes\xi_{5\mu})\left(\frac{1}{a}\sin p_\mu a\right)^2\right]+m\right\}\tilde{q}(p). \tag{8}$$

It is obvious that there are order a terms in the action and in the propagator for the fields $\tilde{q}$ and $\tilde{\bar{q}}$. In this case, we say that the fields q and $\bar{q}$ need to be improved. There exists a set of improved fields

$$\chi_A^I(y)=\left(1-a\sum_\mu A_\mu\partial_\mu^L\right)\chi_A(y), \tag{9a}$$

$$\overline{\chi}^I_A(y)=\overline{\chi_A}(y)\left(1-a\sum_\mu A_\mu\overleftarrow{\partial}^L_\mu\right), \tag{9b}$$

where

$$\partial_\mu^L f(y)=\frac{1}{4a}[f(y+2\mu)-f(y-2\mu)], \tag{10}$$

such that

$$S_F=\frac{(2\pi)^4}{\Omega}\sum_p \tilde{\bar{q}}^I(p)\left(\sum_\mu \gamma_\mu\frac{i}{a}\sin p_\mu a+m\right)\tilde{q}^I(p)+O(a^2). \tag{11}$$

Note those improved fields are still local and superior to the nonlocal fields both computationally and theoretically when gauge couplings are included. So, if we use the improved fields which remove the order a terms from the action to construct a lattice fermion operator, there will be no $O(a)$ corrections to its free field matrix elements. For the Landau gauge, Sharpe [7] proposed the following smeared operator:

$$\chi_A(y)^{\text{smeared}}=\frac{1}{4}\sum_\nu \chi_A(y+2\nu[1-2A_\nu]). \tag{12}$$

It is easy to show that

$$\chi_A(y)^{\text{smeared}}=\chi_A^I(y)+O(a^2). \tag{13}$$

The full staggered fermion action including gauge couplings is much more complicated. In Sec. II, we will give a set of improved fermion field variables in terms of which the action has no explicit order a terms, and propose a general program to remove all order a corrections from the staggered fermion operators. In Sec. III, we will expand upon the argument given by Sharpe in [8] to prove that there are no $O(a)$ terms which can be added to the staggered fermion action. Based on these two arguments, we conclude that staggered fermion action is already accurate to $O(a^2)$, and that we should use the improved field variables to construct fermion operators to reduce order a corrections from their matrix elements. We apply this program to the case of $\langle 0|\overline{s\gamma_{54}}d|K^0\rangle$ and B_K as examples. We will also determine the additional operators that must be added to improve the standard staggered fermion currents to define operators whose matrix elements are accurate to $O(a^2)$ at the tree level. We list the lattice symmetry transformation properties of the fermion fields needed in this paper in the Appendix.

II. IMPROVING THE FERMION FIELDS

The standard staggered fermion action is

$$S_F=\sum_{x,\mu} a^4\overline{\chi}(x)\frac{\eta_\mu(x)}{2a}[U(x,x+\mu)\chi(x+\mu) - U(x,x-\mu)\chi(x-\mu)]+m\sum_x a^4\overline{\chi}(x)\chi(x). \tag{14}$$

Using the hypercubic formalism [6], we define gauge-covariant hypercubic fermion fields as

$$\varphi_A(y)=\mathcal{U}_A(y)\chi_A(y), \tag{15a}$$

$$\overline{\varphi_A}(y)=\overline{\chi_A}(y)\mathcal{U}_A^\dagger(y), \tag{15b}$$

where $\mathcal{U}_A(y)$ is the average of link products along the shortest paths from y to $y+A$, and define the hypercubic matrices as

$$(\overline{\gamma_S\otimes\xi_F})_{AB}=\frac{1}{4}\text{Tr}(\gamma_A^\dagger\gamma_S\gamma_B\gamma_F^\dagger). \tag{16}$$

Rewriting the staggered fermion action by using the fields defined in Eq. (15) and decomposing Eq. (14) into terms of different dimensions (for the definition of the dimension of a lattice operator, see [9]), we obtain

$$S_F=\mathcal{O}_4+a\mathcal{O}_5+a^2\mathcal{O}_6+\cdots, \tag{17}$$

in which

$$\mathcal{O}_4=(2a)^4\sum_y\sum_{AB}\overline{\varphi_A}(y)\left[\sum_\mu(\overline{\gamma_\mu\otimes I})_{AB}D_\mu^L+m\,\delta_{AB}\right]\varphi_B(y) \tag{18}$$

and

$$\mathcal{O}_5=(2a)^4\sum_y\sum_{AB}\overline{\varphi_A}(y)\left[-\sum_\mu(\overline{\gamma_5\otimes\xi_{5\mu}})_{AB}\Delta_\mu^L + \sum_{\mu\nu}(\overline{\gamma_\mu\otimes I})_{AB}A_\nu\frac{1}{2a^2}[U_{\nu\mu}(y)-U^\dagger_{\nu\mu}(y)]\right]\varphi_B(y), \tag{19}$$

where the lattice gauge-covariant derivatives are defined as

$$D_\mu^L\chi_A(y)=\frac{1}{4a}[U_\mu(y+A)U_\mu(y+A+\mu)\chi_A(y+2\mu) - U_\mu^\dagger(y+A-\mu)U_\mu^\dagger(y+A-2\mu)\chi_A(y-2\mu)], \tag{20}$$

$$\Delta^L_\mu \chi_A(y)=\frac{1}{4a^2}[U_\mu(y+A)U_\mu(y+A+\mu)\chi_A(y+2\mu)+U^\dagger_\mu(y+A-\mu)U^\dagger_\mu(y+A-2\mu)\chi_A(y-2\mu)-2\chi_A(y)], \tag{21}$$

and $U_{\nu\mu}$ is the usual closed-loop path-ordered link product on a plaquette which lies on the (ν,μ) plane.

From Eq. (17), we see that, in addition to the naive dimension-four term $\mathcal{O}_4$, the staggered fermion action also contains an explicit dimension-five term $\mathcal{O}_5$ and other higher dimension terms. At first sight, the staggered fermions would appear to have $O(a)$ corrections to the continuum QCD. For example, it is obvious that the fermion propagator for the hypercubic fields φ and $\overline{\varphi}$ deviates from the continuum propagator by terms of order a. However, we notice that the dimension-five term $\mathcal{O}_5$ in Eq. (17) can be transformed away if we introduce the following improved field variables:

$$\chi^I_A(y)=\left(1-a\sum_\nu A_\nu D^L_\nu\right)\chi_A(y), \tag{22a}$$

$$\overline{\chi}^I_A(y)=\overline{\chi_A}(y)\left(1-a\sum_\nu A_\nu \overleftarrow{D}^L_\nu\right), \tag{22b}$$

and replace χ, $\overline{\chi}$ in Eq. (15) by χ^I and $\overline{\chi^I}$. In terms of the improved fermion fields, the staggered fermion action can be written as

$$S_F=(2a)^4\sum_y\sum_{AB}\overline{\varphi}^I_A(y)\left[\sum_\mu \overline{(\gamma_\mu\otimes I)}_{AB}D^L_\mu+m\,\delta_{AB}\right]\times\varphi^I_B(y)+a^2\mathcal{O}'_6+\cdots. \tag{23}$$

Using the new fermion fields, we can construct improved fermion operators. For example, the improved fermion bilinears have the form

$$\overline{\chi^I_A}(y)\overline{(\gamma_S\otimes\xi_F)}_{AB}\chi^I_B(y)=\overline{\chi_A}(y)\overline{(\gamma_S\otimes\xi_F)}_{AB}\chi_B(y)-\frac{a}{2}\sum_\nu \partial^L_\nu\{\overline{\chi_A}(y)[\overline{(\gamma_S\otimes\xi_F)}_{AB}-\overline{(\gamma_{5\nu S}\otimes\xi_{5\nu F})}_{AB}]\chi_B(y)\}-\frac{a}{2}\sum_\nu \overline{\chi_A}(y)[\overline{(\gamma_{5\nu S}\otimes\xi_{5\nu F})}_{AB}-\overline{(\gamma_{S5\nu}\otimes\xi_{F5\nu})}_{AB}]D^L_\nu\chi_B(y)+O(a^2). \tag{24}$$

How good is this improvement? We claim that the improved fermion fields defined in Eq. (22) are accurate through order a and get no correction from any order of perturbation theory. To see if this is true, we notice that in Eq. (17), none of the terms $\mathcal{O}_4,\mathcal{O}_5,\mathcal{O}_6,\ldots$ are separately invariant under the staggered fermion lattice symmetry group. Only their combination in the form given by Eq. (17) is invariant. We will prove in the next section that there exists no extra dimension-five operator which is invariant under the lattice symmetry group. Hence, to any order in perturbation theory, there is no extra dimension-five terms that can be added to the staggered fermion action when carrying out the Symanzik improvement program, and hence the relative coefficients between the $\mathcal{O}_4$ term and the $\mathcal{O}_5$ term in Eq. (17) will not change. Therefore, to any order of perturbation, the improved fields have the same form as Eq. (22). Thus, the coefficient in front of the order a term in Eq. (22) is exact to all orders of g_0, receiving no renormalization. Furthermore, any matrix elements which consist of only our improved fermion fields and contain no composite operators will be accurate through order a under the condition that the gauge fixing does not introduce an extra order a term. This argument has been checked by examining the quark propagator computed to order g_0^2 in the paper of Golterman and Smit [5] where they obtained the form required by the symmetry argument used here. For the composite operators, one should use our improved fermion fields to construct them and they are accurate through order a at tree level. To remove $O(g_0^{2n}a)$ corrections from the composite operators will require further, explicit improvement. The general program is as follows. First, using our improved fermion fields to construct the composite operator $\mathcal{O}^{(d)}(\overline{\chi}^I,\chi^I)$ which has no explicit order a term [e.g., for the continuum operator $\overline{\psi\gamma_S\otimes\xi_F\psi(y)}$, the correct lattice operator is $\overline{\chi}^I(y)\overline{(\gamma_S\otimes\xi_F)}\chi^I(y)$, but not $\overline{\chi}^I(y)\overline{(\gamma_S\otimes\xi_F)}\chi^I(y+a_\mu)$]. Second, find all one-dimensional higher operators $\mathcal{O}_i^{(d+1)}(\overline{\chi}^I,\chi^I)$ which are constructed from our improved fermion fields and have the same symmetry properties as the considered composite operator $\mathcal{O}^{(d)}(\overline{\chi^I},\chi^I)$. Then their linear combination

$$\mathcal{O}^{(d)}=\mathcal{O}^{(d)}(\overline{\chi}^I,\chi^I)+a\sum_i c_i(g_0^2)\mathcal{O}_i^{(d+1)}(\overline{\chi}^I,\chi^I)+O(a^2)$$

will be accurate to $O(a^2)$, where the coefficients $c_i(g_0^2)$ can be computed perturbatively.

There is an analog between the improvements here and those originally introduced for Wilson fermions. In the case of Wilson fermions, there are two dimension-five operators which are invariant under the lattice symmetry group: one is the Wilson term; another is the SW term. A linear combination of the Wilson term and the SW term is redundant because it can be generated from the naive order a^0 action (which is dimension-four) by an ''order a'' redefinition of the fermion fields. Such transformed fermion fields have the same lattice symmetry transformations as the old ones, which means they are in the same representation. Thus, it is easy to write down an improved Wilson action accurate through order a in tree level. One simply uses a combination of the naive fermion action and the redundant term mentioned above—the transformation to an improved fermion field simply transforms away the redundant $O(a)$ term [2,3]. In order to remove all $O(g_0^{2n}a)$ errors, the coefficient in front of the SW term in this action and the coefficients in the definition of the new fermion fields are replaced by coefficients which are appropriate series in g_0^2. In the case of staggered fermions considered here, as we will prove in the next section, the fermion action is unique through order a in that there is no extra dimension-five operator which can be added to the original action. However, using the standard hypercubic formalism, there is a term of order a (which is not invariant under the lattice symmetry group) in this unique action, as Eq. (17) shows. If we allow a transformation of the hypercubic fermion fields which changes their transformation properties under the lattice symmetry group (i.e., changes their representation, but does not change the lattice symmetry group itself), then this extra order a term becomes redundant and can be transformed away. So, the staggered fermion action is similar to the improved Wilson action in that both actions are unique (no extra order a term can be added) and differ from the continuum by $O(a^2)$ in terms of the appropriate improved fermion fields. However, these improved fermion fields are different. The improved staggered fermion fields will change the transformation properties of the lattice symmetry group, but the improved Wilson fermion fields will not. Furthermore, there is another big difference: the coefficient of the order a term in Eq. (22) is accurate to all order of g_0, but the improvements in the Wilson fermions have to use more general coefficients which may be computed order by order in perturbation theory.

III. IMPROVING THE STAGGERED FERMION ACTION

In contrast to the calculation of matrix elements, the action is already accurate through order a to all orders in g_0^2, as we will now discuss. Thus physical quantities that depend only on the form of the action (for example, particle masses determined from correlation functions) will have no corrections of order $g_0^{2n}a$. This can be demonstrated by recognizing that if there were a correction of order $g_0^2 a$, we must necessarily be able to add some dimension-five operators $a\Sigma_i c_i g_0^2 O_i^{(5)}$ which must be invariant under the lattice symmetry transformations to cancel this order $g_0^2 a$ correction [1,8,9]. However, we will now prove that there exists no dimension-five operator which is invariant under the lattice symmetry group [rotations, axis reversal, translations, $U(1)\otimes U(1)$, charge conjugation], and therefore, no order a term can be added to the staggered fermion action. This discussion, included here for completeness, is intended to clarify some aspects of Sharpe's original published argument [8].

Following standard notation, we rewrite the staggered fermion action as

$$S_F=(2a)^4\sum_{y,y'}\sum_{A,B}\overline{\chi_A}(y)^a\left[\sum_{\mu}\overline{(\gamma_\mu\otimes I)}_{AB}D_\mu(y,y')_{BC}+m\,\delta(y-y')\,\delta_{AC}\right]\chi_C(y')^b U(y+A,y'+C)^{ab}, \tag{25}$$

where

$$D_\mu(y,y')_{AB}=\overline{D_\mu}(y,y')\,\delta_{AB}+a\overline{\Delta}_\mu(y,y')\overline{(\gamma_{\mu5}\otimes\xi_{\mu5})}_{AB}, \tag{26}$$

in which

$$\overline{D_\mu}(y,y')=\frac{1}{4a}[\,\delta(y+2\mu-y')-\delta(y-2\mu-y')], \tag{27}$$

$$\overline{\Delta}_\mu(y,y')=\frac{1}{4a^2}[\,\delta(y+2\mu-y')+\delta(y-2\mu-y')-2\,\delta(y-y')]. \tag{28}$$

For convenience, we will not write out the SU(3) links explicitly in the remainder of this section unless there would otherwise be confusion. Given an operator, the reader can write out the full form very easily. For example, starting with the operator $\Sigma_\mu\bar{\chi}\overline{\gamma_5\otimes\xi_{5\mu}}D_\mu^2\chi$ we would construct the corresponding gauge-invariant operator by the substitution

$$\sum_{y',y''}\sum_{\mu}\overline{\chi_A}(y)\overline{(\gamma_5\otimes\xi_{5\mu})}_{AB}D_\mu(y,y')_{BC}D_\mu(y'y'')_{CD}\chi_D(y'')$$

$$\to\sum_{y',y''}\sum_{\mu}\overline{\chi_A}(y)\overline{(\gamma_5\otimes\xi_{5\mu})}_{AB}U(y+A,y+B)D_\mu(y,y')_{BC}U(y'+B,y''+C)D_\mu(y',y'')_{CD}U(y'+C,y''+D)\chi_D(y'') \tag{29}$$

where $U(y+A,y+B)$ is the average of the products of link matrices corresponding to each of the shortest paths from point $y+A$ to $y+B$.

Using the transformation properties of the staggered fermion action (see the Appendix in detail), we try to construct all symmetrical dimension-five operators which have the general form $\overline{\chi}\overline{\gamma_S\otimes\xi_F}f(D)\chi$ where f is a homogeneous real polynomial of degree 2.

Invariance under $U_A(1)$ requires that $S+F$ is odd, so only the following combinations of $S\otimes F$ are valid:

$$(I,\gamma_5,\gamma_{\mu\nu},\gamma_{5\mu\nu})\otimes(\xi_\lambda,\xi_{5\lambda}), \tag{30}$$

$$(\gamma_\mu,\gamma_{5\mu})\otimes(I,\xi_5,\xi_{\lambda\tau},\xi_{5\lambda\tau}). \tag{31}$$

Under reflection with respect to a hyperplane normal to the ρ direction, we have the transformation

$$\chi\rightarrow\mathcal{I}_\rho\chi, \tag{32a}$$

$$\overline{\chi}\rightarrow\overline{\chi}\mathcal{I}_\rho^{-1}, \tag{32b}$$

and

$$\overline{D_\mu}\rightarrow(1-2\delta_{\mu\rho})\mathcal{I}_\rho\overline{D_\mu}\mathcal{I}_\rho^{-1}, \tag{33a}$$

$$\overline{\Delta_\mu}\rightarrow\mathcal{I}_\rho\overline{\Delta_\mu}\mathcal{I}_\rho^{-1}, \tag{33b}$$

$$D_\mu\rightarrow(1-2\delta_{\mu\rho})\mathcal{I}_\rho D_\mu\mathcal{I}_\rho^{-1}. \tag{33c}$$

Using the transformation formulas of $\gamma_S\otimes\xi_F$ listed in Eq. (A7) of the Appendix, we deduce that axis reversal invariance and $\mathrm{U}_A(1)$ invariance allow only the four terms

$$\gamma_5\otimes\xi_\mu D_\mu^2, \tag{34}$$

$$\gamma_5\otimes\xi_{5\mu}D_\mu^2, \tag{35}$$

$$\gamma_5[\gamma_\mu,\gamma_\nu]\otimes\xi_5(\xi_\mu+\xi_\nu)[D_\mu,D_\nu], \tag{36}$$

$$\gamma_5[\gamma_\mu,\gamma_\nu]\otimes\xi_5(\xi_\mu-\xi_\nu)\{D_\mu,D_\nu\} \tag{37}$$

where D_μ,D_ν can be replaced by $\overline{D_\mu},\overline{D_\nu}$ without affecting these operators up to order a^2.

Under a rotation around the center of a hypercube, we have

$$\chi\rightarrow\mathcal{R}^{(\rho\sigma)}\chi, \tag{38a}$$

$$\overline{\chi}\rightarrow\overline{\chi}\mathcal{R}^{(\rho\sigma)-1}, \tag{38b}$$

$$\overline{D_\mu}\rightarrow\mathcal{R}^{(\rho\sigma)}R_{\mu\nu}\overline{D_\nu}\mathcal{R}^{(\rho\sigma)-1}, \tag{39a}$$

$$\overline{\Delta_\mu}\rightarrow\mathcal{R}^{(\rho\sigma)}|R_{\mu\nu}|\overline{\Delta_\nu}\mathcal{R}^{(\rho\sigma)-1}, \tag{39b}$$

$$D_\mu\rightarrow\mathcal{R}^{(\rho\sigma)}R_{\mu\nu}D_\nu\mathcal{R}^{(\rho\sigma)-1}. \tag{39c}$$

Combining the transformation properties listed in Eq. (A12) of the Appendix, the rotational invariance will further eliminate the term in Eq. (34) but allows the remaining three terms Eqs. (35)–(37).

Finally, let us discuss invariances under translation by one lattice unit:

$$\chi\rightarrow\mathcal{S}^{(\rho)}\chi, \tag{40a}$$

$$\overline{\chi}\rightarrow\overline{\chi}\mathcal{S}^{(\rho)-1}, \tag{40b}$$

$$\overline{D_\mu}\rightarrow\mathcal{S}^{(\rho)}\overline{D_\mu}\mathcal{S}^{(\rho)-1}, \tag{41a}$$

$$\overline{\Delta_\mu}\rightarrow\mathcal{S}^{(\rho)}\overline{\Delta_\mu}\mathcal{S}^{(\rho)-1}, \tag{41b}$$

$$D_\mu\rightarrow\mathcal{S}^{(\rho)}\Lambda^{(\rho)-1}_{\gamma_\mu\otimes I}D_\mu\mathcal{S}^{(\rho)-1}, \tag{41c}$$

where

$$\Lambda^{(\rho)}_{S\otimes F}(y,y')_{AB}=\varepsilon(F)\{(-1)^{F_\rho}\delta_{AB}\delta(y-y')+a[(-1)^{F_\rho}-(-1)^{S_\rho}]\times[a\,\delta_{AB}\overline{\Delta_\rho}(y,y')+\overline{(\gamma_{5\rho}\otimes\xi_{5\rho})}_{AB}\overline{D_\rho}(y,y')]\}, \tag{42a}$$

$$\Lambda^{(\rho)-1}_{S\otimes F}(y,y')_{AB}=\varepsilon(F)\{(-1)^{F_\rho}\delta_{AB}\delta(y-y')+a[(-1)^{F_\rho}-(-1)^{S_\rho}]\times[a\,\delta_{AB}\overline{\Delta_\rho}(y,y')-\overline{(\gamma_{5\rho}\otimes\xi_{5\rho})}_{AB}\overline{D_\rho}(y,y')]\}, \tag{42b}$$

and

$$\mathcal{S}^{(\rho)-1}\overline{\gamma_S\otimes\xi_F}\mathcal{S}^{(\rho)}=\overline{\gamma_S\otimes\xi_F}\Lambda^{(\rho)}_{S\otimes F}. \tag{43}$$

From these properties, we can see that none of the terms listed in Eqs. (35)–(37) are invariant under lattice translation. So, we conclude that there is no dimension-five fermion operator which is invariant under the lattice symmetry group, and therefore no dimension-five operator can be added to the staggered fermion action.

IV. APPLICATIONS

As we argued above, actual numerical simulation should use the improved fermion field variables. However, in most situations, we can use the improvement program proposed in this paper to remove the $O(a)$ corrections at tree level without increasing the computational work. Here, we apply this program to the calculation of the matrix element $\langle 0|\overline{s}\gamma_{54}d|K^0\rangle$ which gives f_K in the continuum, and the calculation of B_K. We will show that the former differs from its continuum counterpart by $O(m_K a)$, but B_K has no $O(a)$ corrections. We also apply the improvement program to the matrix elements of lattice currents.

A. The matrix element $\langle 0|\bar{s}\gamma_{54}d|K^0\rangle$

The axial current used in numerical simulations is

$$A_\mu(y)=\sum_{AB} \overline{\chi_A}\overline{(\gamma_{5\mu}\otimes\xi_5)}_{AB}U_\mu(y+A)\chi_B(y). \quad (44)$$

From the continuum expression

$$P(t)^{\text{cont}}=\left\langle 0\left|\sum_{\vec{x}} A_4(\vec{x},t)^{\text{cont}}\right|K^0\right\rangle=\sqrt{2}f_K m_K e^{-m_K|t|}, \quad (45)$$

we define, on the lattice,

$$P(t)=\left\langle 0\left|\sum_{\vec{x}} A_4(\vec{x},t)\right|K^0\right\rangle, \quad (46)$$

and put the wall source that creates the K^0 on the time slice at $t=0$. Then we will have

$$P(t)=\begin{cases}\sqrt{2}f_K^+ m_K e^{-m_K|t|} & (t>0),\\ \sqrt{2}f_K^- m_K e^{-m_K|t|} & (t<0),\end{cases} \quad (47)$$

where

$$f_K^\pm=f_K\pm O(m_K a). \quad (48)$$

If we do not consider $O(g_0^2 a)$ terms, we can take only the term

$$-\frac{a}{2}\sum_\nu \partial_\nu^L[\overline{\chi_A}\overline{(\gamma_{54}\otimes\xi_5)}_{AB}\chi_B]$$

in Eq. (24) because other terms contribute zero ''flavor'' trace at the tree level. So, we have

$$P(t)^{\text{imp}}=P(t)-\frac{a}{2}\partial_4^L P(t)$$
$$=\begin{cases}\sqrt{2}f_K^{+,\text{imp}} m_K e^{-m_K|t|} & (t>0),\\ \sqrt{2}f_K^{-,\text{imp}} m_K e^{-m_K|t|} & (t<0),\end{cases} \quad (49)$$

and

$$f_K^{+,\text{imp}}=\left(1+\frac{1}{2}m_K a\right)f_K^+ + O(g_0^2 a), \quad (50a)$$

$$f_K^{-,\text{imp}}=\left(1-\frac{1}{2}m_K a\right)f_K^- + O(g_0^2 a), \quad (50b)$$

$$f_K^{\pm,\text{imp}}=f_K\pm O(g_0^2 a). \quad (51)$$

So, we get

$$f_K^\pm=\left(1\mp\frac{1}{2}m_K a\right)f_K + O(g_0^2 a). \quad (52)$$

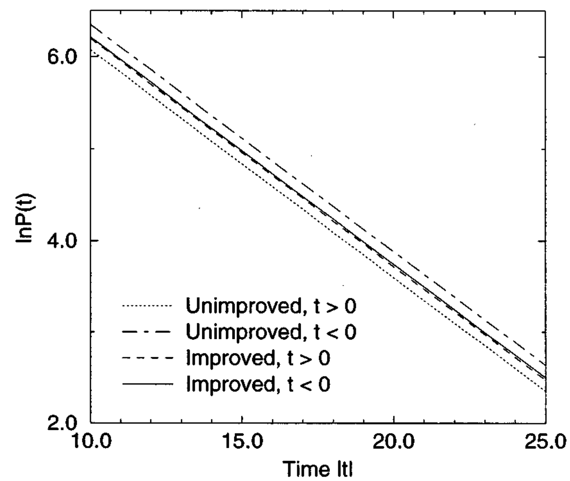

FIG. 1. The value of $\ln P(t)$ with respect to the time $|t|$. Where $m_d a=m_s a=0.01$. Calculated with the cubic wall source method. The unimproved data correspond to $\ln P(t)$, the improved data correspond to $\ln P(t)^{\text{imp}}$. Here we use an axial current similar to Eq. (44), but work in Landau gauge and replace the link variable by 1. The error bars are about the symbol size and not included in the figure.

The numerical data (from the full QCD simulation on a $16^3\times 40$ lattice with the cubic wall source, at the quark mass $m_s a=m_d a=0.01$, see [10]) for the unimproved and improved matrix elements are shown in Fig. 1, from which we see that $(f_K^- - f_K^+)/f_K\approx m_K a\sim 25\%$ and $(f_K^{-,\text{imp}} - f_K^{+,\text{imp}})/f_K\sim 5\%$ is much smaller.

From this simple example, we can see that if we do not consider $O(g^2 a)$ corrections, the improved operator is equivalent to the extrapolation:

$$P(t)=\sqrt{2}f_K^{\text{imp}} m_K e^{-m_K|t+1/2|}. \quad (53)$$

B. The weak matrix element B_K

The formula for calculating B_K is

$$B_K=\frac{\mathcal{M}_K}{(8/3)\mathcal{M}_K^V}, \quad (54)$$

where

$$\mathcal{M}_K=\langle\overline{K^0}|\bar{s}\gamma_\mu(1+\gamma_5)d\bar{s}\gamma_\mu(1+\gamma_5)d|K^0\rangle, \quad (55)$$

$$\mathcal{M}_K^V=\langle\overline{K^0}|\bar{s}\gamma_4\gamma_5 d|0\rangle\langle 0|\bar{s}\gamma_4\gamma_5 d|K^0\rangle. \quad (56)$$

The improved numerator is [omitting the $O(g_0^2 a)$ terms]

$$\mathcal{M}_K^{\text{imp}}=\mathcal{M}_K-\frac{a}{2}\partial_4^L\mathcal{M}_K+O(g_0^2 a). \quad (57)$$

Since $\mathcal{M}_K(t)$ is computed from a plateau (i.e., time independent) within the statistical error, there is no $O(a)$ corrections to the numerator.

Note that the denominator $\mathcal{M}_K^V$ will have no $O(a)$ corrections even if our naive definition of $f_K^\pm$ is used since the order a errors will cancel in the product:

$$f_K^+ f_K^- = f_K^2 + O(g_0^2 a). \tag{58}$$

Hence, we showed that there is neither $O(a)$ nor $O(a g_0^{2n} \log^n a)$ corrections to B_K. Sharpe [8] has examined this question in greater detail and argued that in fact there are no corrections of $O(g_0^{2n} a)$ also. However, if we calculated the denominator only in one time direction and took the square of f_K^+ (or f_K^-), there would be an error of order of $O(m_K a)$.

C. Renormalization of lattice currents

The lattice currents can be written as

$$J^F_{\text{latt}} = \bar{\chi} \overline{\overline{\gamma_J \otimes \xi_F}} \chi, \tag{59}$$

and according to [11], their renormalized continuum forms can be written as

$$J^F_{\text{cont}} = Z_J \kappa_J^F J^F_{\text{latt}}, \tag{60}$$

where Z_J is the usual (divergent) renormalization constant and κ_J^F is a finite lattice renormalization constant. Using the method developed in this paper, we can explicitly determine the improved currents accurate to $O(a^2)$ at the tree level. For example, the conserved vector current and axial vector current corresponding to the $\mathrm{U}_V(1) \otimes \mathrm{U}_A(1)$ lattice symmetry can be written as

$$\begin{aligned} V_\mu^I(y) = V_\mu(y) &- \frac{a}{2} \sum_\nu \partial_\nu [\overline{\chi_A} (\overline{\overline{\gamma_\mu \otimes I}})_{AB} \chi_B] \\ &- \frac{a}{2} \partial_\mu [\overline{\chi_A} (\overline{\overline{\gamma_\mu \otimes I}})_{AB} \chi_B] \\ &- \frac{a}{4} \sum_\nu \partial_\nu [\overline{\chi_A} (\overline{\overline{\gamma_{5[\mu,\nu]} \otimes \xi_{5\nu}}})_{AB} \chi_B] + O(g_0^2 a), \end{aligned} \tag{61}$$

$$\begin{aligned} A_\mu^{\xi_5, I}(y) = A_\mu^{\xi_5}(y) &- \frac{a}{2} \sum_\nu \partial_\nu [\overline{\chi_A} (\overline{\overline{\gamma_{\mu 5} \otimes \xi_5}})_{AB} \chi_B] \\ &- \frac{a}{2} \partial_\mu [\overline{\chi_A} (\overline{\overline{\gamma_{\mu 5} \otimes \xi_5}})_{AB} \chi_B] \\ &+ \frac{a}{4} \sum_\nu \partial_\nu [\overline{\chi_A} (\overline{\overline{\gamma_{[\mu,\nu]} \otimes \xi_\nu}})_{AB} \chi_B] + O(g_0^2 a). \end{aligned} \tag{62}$$

The effect of the second term on the right-hand side is to shift the position y, labeling the current, from the corner to the center of the hypercube. The third term whose effect is to shift in the μ's direction occurs here because the currents are nonlocal operators which involve an overlap between two nearest hypercubes. The forth term is a mixing of a different spin-flavor operator and is necessary to remove all order a effects from a general matrix element. As we discussed in Sec. II, to remove all $O(a)$ corrections, we have to find out all dimension-four operators which have the same symmetry properties as these currents. This work is in preparation.

V. SUMMARY

In this paper, based on the demonstration that there is no dimension-five fermion operator which is invariant under all lattice symmetry transformations and that there exists a set of improved fermion fields with respect to which the staggered fermion action has no order a terms, we concluded that the staggered fermion action is already in fact improved to $O(a^2)$. We argued that to remove order a corrections from the matrix elements, one has to use the proposed improved fermion field variables to construct fermion operators. We applied our program to the matrix element $\langle 0 | \bar{s} \overline{\overline{\gamma_{54}}} d | K^0 \rangle$ and found that the unimproved one differs from the continuum by a factor of $O(m_K a)$. At the same time, we showed that there is no $O(a)$ corrections to B_K, which is consistent with the result of Sharpe [8]. We also discussed the matrix elements of the lattice currents, and obtained the explicit terms which should be added to the original current operators to define improved operators accurate through $O(a)$ at tree level. To improve them through order a to all order in g_0, we have to find out all dimension-four operators which are constructed from our improved fermion fields and have the same symmetry properties as the original current operators.

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APPENDIX: SYMMETRY PROPERTIES OF STAGGERED FERMION

For completeness, we collect some formulas connected with the transformation properties of the staggered fermion under the lattice symmetry group from [12].

1. The axial symmetry $\mathrm{U}(1)_A$

$$\chi_A(y) \to e^{i\alpha\varepsilon(A)} \chi_A(y), \tag{A1a}$$

$$\overline{\chi_A}(y) \to e^{i\alpha\varepsilon(A)} \overline{\chi_A}(y), \tag{A1b}$$

where

$$\varepsilon(A) = (-1)^{\Sigma_\mu A_\mu}. \tag{A2}$$

2. Reflection with respect to a hyperplane

$$I_H^\rho : \quad \begin{aligned} x'_\rho &= -x_\rho + 1, \\ x'_\mu &= x_\mu, \quad \mu \neq \rho. \end{aligned} \tag{A3}$$

The transformation of the fermion fields are

$$\chi_A(y) \to \sum_B \sum_{y'} \mathcal{I}_\rho(y,y')_{AB} \chi_B(y'), \tag{A4a}$$

$$\overline{\chi_A}(y)\rightarrow\sum_B\sum_{y'}\overline{\chi_B}(y')\mathcal{I}_\rho^{-1}(y',y)_{BA}, \tag{A4b}$$

where

$$\mathcal{I}_\rho(y,y')_{AB}=\overline{(\gamma_{\rho5}\otimes\xi_5)}_{AB}\delta(I_\rho y-y'), \tag{A5a}$$

$$\mathcal{I}_\rho^{-1}(y,y')_{AB}=\overline{(\gamma_{5\rho}\otimes\xi_5)}_{AB}\delta(I_\rho y-y'), \tag{A5b}$$

and

$$(I_\rho y)_\mu=\begin{cases} y_\mu & (\mu\neq\rho),\\ -y_\rho & (\mu=\rho).\end{cases} \tag{A6}$$

The spin-flavor matrices transform as

$$\overline{(\gamma_S\otimes\xi_F)}\rightarrow\overline{(\gamma_{\rho5}\otimes\xi_5)}\cdot\overline{(\gamma_S\otimes\xi_F)}\cdot\overline{(\gamma_{5\rho}\otimes\xi_5)}. \tag{A7}$$

3. Rotations by $\pi/2$ around the center of a hyperplane

$$R_H^{(\rho\sigma)}:\quad \begin{aligned} &x'_\rho=x_\sigma,\\ &x'_\sigma=-x_\rho+1,\\ &x'_\mu=x_\mu \quad (\mu\neq\rho,\sigma).\end{aligned} \tag{A8}$$

Define $R_{\mu\nu}$ such that

$$(Ry)_\mu=R_{\mu\nu}y_\nu=\begin{cases} y_\sigma & (\mu=\rho),\\ -y_\rho & (\mu=\sigma),\\ y_\mu & (\mu\neq\rho,\sigma).\end{cases} \tag{A9}$$

Then, we have the transformation

$$\chi_A(y)\rightarrow\sum_B\sum_{y'}\mathcal{R}^{(\rho\sigma)}(y,y')_{AB}\chi_B(y'), \tag{A10a}$$

$$\overline{\chi_A}(y)\rightarrow\sum_B\sum_{y'}\overline{\chi_B}(y')\mathcal{R}^{(\rho\sigma)}(y,y')^{-1}_{AB}, \tag{A10b}$$

where

$$\mathcal{R}^{(\rho\sigma)}(y,y')_{AB}=\frac{1}{2}\overline{[(1-\gamma_{\rho\sigma})\otimes(\xi_\sigma-\xi_\rho)]}_{AB}\delta(R^{-1}y-y'), \tag{A11a}$$

$$\mathcal{R}^{(\rho\sigma)}(y,y')^{-1}_{AB}=\frac{1}{2}\overline{[(1+\gamma_{\rho\sigma})\otimes(\xi_\sigma-\xi_\rho)]}_{AB}\delta(Ry-y'). \tag{A11b}$$

The spin-flavor matrices transform as

$$\overline{(\gamma_S\otimes\xi_F)}\rightarrow\frac{1}{2}\overline{[(1-\gamma_{\rho\sigma})\otimes(\xi_\sigma-\xi_\rho)]}\cdot\overline{(\gamma_S\otimes\xi_F)}\cdot\frac{1}{2}\overline{[(1+\gamma_{\rho\sigma})\otimes(\xi_\sigma-\xi_\rho)]}. \tag{A12}$$

4. Translations by one lattice unit

$$T_\rho:\quad \begin{aligned} &x'_\rho=x_\rho+1,\\ &x'_\mu=x_\mu \quad (\mu\neq\rho)\end{aligned} \tag{A13}$$

lead to the following transformation on the χ_A, $\overline{\chi_A}$'s:

$$\chi_A(y)\rightarrow\sum_B\sum_{y'}\mathcal{S}^{(\rho)}(y,y')_{AB}\chi_B(y'), \tag{A14a}$$

$$\overline{\chi_A}(y)\rightarrow\sum_B\sum_{y'}\overline{\chi_B}(y')\mathcal{S}^{(\rho)}(y,y')^{-1}_{AB}, \tag{A14b}$$

where

$$\mathcal{S}^{(\rho)}(y,y')_{AB}=\frac{1}{2}[(\overline{I\otimes\xi_\rho}-\overline{\gamma_{\rho5}\otimes\xi_5})_{AB}\delta(y-y') + (\overline{I\otimes\xi_\rho}+\overline{\gamma_{\rho5}\otimes\xi_5})_{AB}\delta(y+2\rho-y')], \tag{A15a}$$

$$\mathcal{S}^{(\rho)}(y,y')^{-1}_{AB}=\frac{1}{2}[(\overline{I\otimes\xi_\rho}-\overline{\gamma_{\rho5}\otimes\xi_5})_{AB}\delta(y-2\rho-y') + (\overline{I\otimes\xi_\rho}+\overline{\gamma_{\rho5}\otimes\xi_5})_{AB}\delta(y-y')]. \tag{A15b}$$

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