

## Geometrical entropy from loop quantum gravity

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We adopt the point of view that (Riemannian) classical and (loop-based) quantum descriptions of geometry are macro- and microdescriptions in the usual statistical mechanical sense. This gives rise to the notion of geometrical entropy, which is defined as the logarithm of the number of different quantum states which correspond to one and the same classical geometry configuration (macrostate). We apply this idea to gravitational degrees of freedom induced on an arbitrarily chosen in space two-dimensional surface. Considering an “ensemble” of particularly simple quantum states, we show that the geometrical entropy  $S(A)$  corresponding to a macrostate specified by a total area  $A$  of the surface is proportional to the area  $S(A) = \alpha A$ , with  $\alpha$  being approximately equal to  $1/16\pi l_p^2$ . The result holds both for cases of open and closed surfaces. We discuss briefly physical motivations for our choice of the ensemble of quantum states. [S0556-2821(97)05006-6]

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### I. INTRODUCTION

Among the results of the loop approach to nonperturbative quantum gravity there are several which tell us that the picture of geometry on scales small as compared with our usual scales (on Planckian scales) looks quite different from the habitual picture of Riemannian geometry. The usual classical picture arises only as an approximation to a more fundamental quantum picture. The fundamental excitations of the emerging quantum geometry are one-dimensional loop excitations and the whole quantum picture is of essentially discrete, combinatorial character. (See, for instance, recent works [1,2], which also contain extensive references to the previous papers on the loop approach).

Let us assume that we can formulate a reasonable approximation criteria, and for any classical geometry configuration find the set of approximating quantum states. It is natural to expect that if the precision of approximation is chosen not too high, there will be a lot of quantum states corresponding to the same “geometry.” It is quite tempting to consider a usual Riemannian geometry description as a macroscopical one and to regard all quantum states approximating a Riemannian metric as microstates corresponding to the same macrostate. This point of view brings up a lot of interesting possibilities.

Indeed, recall that with distinguishing between macro- and microstates of a system the notion of entropy arises in statistical mechanics. Entropy is a function which depends on a macroscopic state of the system. It can be thought of as a function which for each macroscopic state gives the (logarithm of the) number of different microscopic states corresponding to this macrostate. More precisely, the space of states of the system should be divided into compartments, where all microstates belonging to the same compartment are macroscopically indistinguishable. Then the entropy of the

system in a macrostate is given<sup>1</sup> by the logarithm of the “volume” of the compartment corresponding to this macrostate.

Let us return to our quantum description of geometry. It is natural to introduce a coarse graining of the space of quantum states in such a way that quantum states approximating different “geometries” belong to different compartments. Having divided the space of states this way, it is natural to introduce the function which for any geometry configuration gives the logarithm of the “volume” of the corresponding compartment. This gives rise to the notion of *geometrical entropy*. Thus, geometrical entropy tells “how many” different quantum states there are which correspond to a given geometry configuration. To be more precise, in the cases when a compartment is itself a linear space the entropy is given by the logarithm of the dimension of the corresponding compartment.

In this paper we try to implement the idea of geometrical entropy following a very simple choice of the “ensemble” of quantum states. Although some of our results are surprising, the aim of this paper is not to argue a physical significance of the results obtained. Rather, in order to understand if the general idea of geometrical entropy makes sense, we consider a particularly simple case, in which the analysis can be accomplished, and develop a technique that may prove to be useful for future developments.

Our choice of the “ensemble” of quantum states is as follows. First, we restrict our consideration to the gravitational degrees of freedom of an arbitrary spacelike two-dimensional surface. Namely, we consider Lorentzian 3+1 general relativity in the framework of (real) Ashtekar variables, and the quantization given by the loop quantum gravity. We restrict our consideration to the gravitational system induced by the full theory on some two-surface  $S$

<sup>1</sup>In fact, the usual thermodynamical entropy is proportional to this number, but it is convenient to work in the units in which this proportionality coefficient is chosen to be unity.

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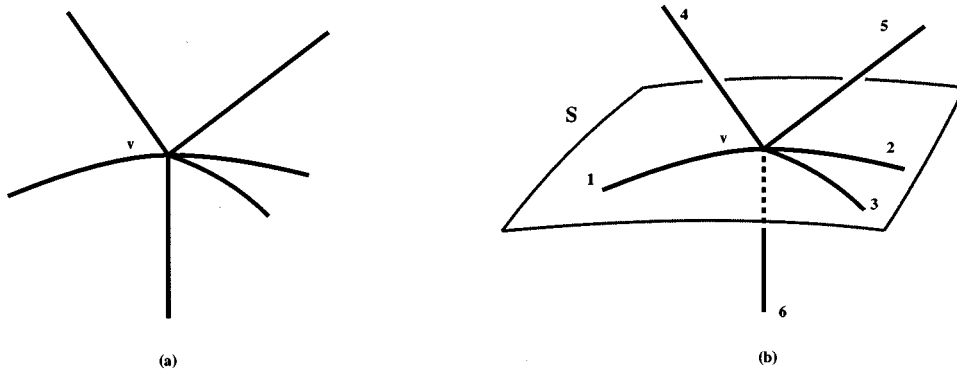


FIG. 1. Vertex of a 3D spin network (a) and its intersection with the surface  $S$  (b). Edges 1,2,3 are tangential ones.

embedded in the spatial manifold  $\Sigma$ . Thus the degrees of freedom of our system are just the degrees of freedom of general relativity that live on a surface  $S$ . Second, we specify a macrostate of our system simply fixing the total area of the surface  $S$ . In this case, as we shall see later, it is easy to pick up all quantum states which approximate a given macrostate, and the analysis becomes almost straightforward. Thus, to illustrate the general idea of geometrical entropy, we stick here to the case when a macrostate of our system (which “lives” on the surface  $S$ ) is specified simply by the total area  $A$  of  $S$ .

Let us note that such a statistical mechanical consideration of surface degrees of freedom is of a special interest because of its possible connection with the black hole thermodynamics. Indeed, there is a common belief that it is the degrees of freedom living on the horizon surface of a black hole which account for the black hole entropy. To try to reveal the connection between quantum gravity and thermodynamics that is suggested by black hole physics is one of the motivations for our investigation.

The other, and maybe even more important motivation is that the loop quantum gravity itself is a new approach. In such a situation it is necessary to apply the formalism to simple problems, just in order to see if it gives a reasonable picture. The set of problems concerning statistical properties of the theory might serve as one of such tests.

The paper is organized as follows. In the next section we recall briefly how the surface quantum states look and discuss the issue of correspondence between macro- and micro-descriptions. In Sec. III we calculate geometrical entropy of surface degrees of freedom considering the case of an open surface. Section IV contains a generalization of our result to the case of closed surfaces. We conclude with the discussion.

## II. SURFACE QUANTUM STATES

Let us recall the description of general relativity in terms of (real) Ashtekar variables. In the Hamiltonian framework general relativity can be formulated as a theory of  $SU(2)$  connection over the spatial manifold. The connection field plays the role of a configurational variable; the momentum variable is presented by the canonically conjugated field. The dynamics of the theory is determined by a set of constraint functionals.

The degrees of freedom induced on an arbitrarily chosen surface  $S$  are described by pull-backs of the connection and momentum fields into  $S$ , which we shall denote by  $a_a^{AB}$  and

$\tilde{e}^{aAB}$ , respectively ( $A, B$  stands for two component spinor indices). The “surface” momentum field  $\tilde{e}^{aAB}$  carries information about the two-metric on  $S$ .

Let us describe the quantum theory. The loop quantization of 3+1 general relativity is described in detail in [1,2]. For our purposes it is sufficient to recall that there exists an orthogonal decomposition of the Hilbert space of gauge-invariant states of quantum general relativity into subspaces, which are labeled by the so-called spin networks. Spin network states are labeled by closed graphs in  $\Sigma$  with spins (or, equivalently, with irreducible representations of the gauge group) assigned to each edge and intertwining operators assigned to each vertex of graph.

Let us now specify the space of quantum surface states. Given a two-dimensional (2D) (not necessarily closed) surface  $S$  embedded into the spatial manifold  $\Sigma$  and a 3D spin network  $\Gamma$  one can consider the intersection of this spin network with  $S$ . Let us call the intersection points vertices. Generally, there can be vertices of any valence not less than two<sup>2</sup> (valence of a vertex is the number of edges of a spin network state which meet in this vertex). Also, there can be edges lying entirely on the surface  $S$  among the edges of  $\Gamma$ ; we shall call such edges tangential (see Fig. 1). The intersection of  $\Gamma$  with the surface  $S$  defines what we shall call a surface spin network on  $S$ . A surface spin network is a graph lying entirely on the surface  $S$ , with spins assigned to each edge, and intertwining operators assigned to each vertex. The intertwiners assigned to each vertex are just those of the corresponding 3D spin network. This means that vertices of a 2D surface spin network “remember” what were the spins of the edges incident at the surface (see Fig. 2). Note that our definition of a surface spin network is not the canonical one. A “canonical” surface spin network is defined as a graph lying on the surface, with spins assigned to each edge, and intertwiners assigned to each vertex. We use the term “surface spin network” simply to denote the intersection of a 3D spin network with the surface  $S$ , or, in other words, to denote the “surface” part of information carried by a 3D spin network. To avoid confusion, we shall also use the term “gen-

<sup>2</sup>In the case of theory without fermionic degrees of freedom, which we consider here, the valence of vertices of a spin network state should be not less than two in order to have a gauge-invariant state. When fermionic degrees of freedom are present in the theory valence of vertices can be equal to one. In this case open ends of a spin network describe fermionic degrees of freedom (see [3]).

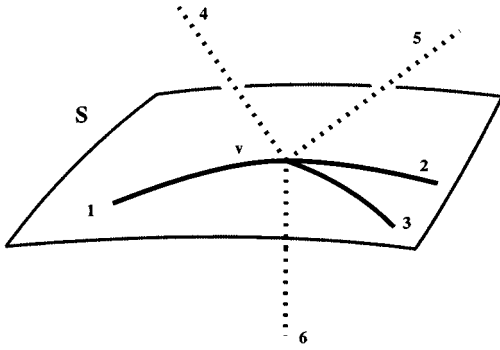


FIG. 2. Intertwining operator assigned to the vertex  $v$  of a generalized surface spin network “remembers” what were the spins  $j_4, j_5, j_6$  of the edges 4,5,6 incident at the surface.

eralized surface spin network state.”

The simplest nontrivial example of a surface spin network is that coming from a single edge intersecting the surface  $S$  (see Fig. 3). Such a spin network is simply the point  $v$  on  $S$ , with the intertwining operator being the map from the one copy of the representation space  $\rho^{(j)}$  [ $j$  here is the spin labeling the irreducible representation of  $SU(2)$ ] to the other copy of  $\rho^{(j)}$ . The intertwiner in this case is specified (up to an overall constant) simply by the spin  $j$  attached to the vertex  $v$ .

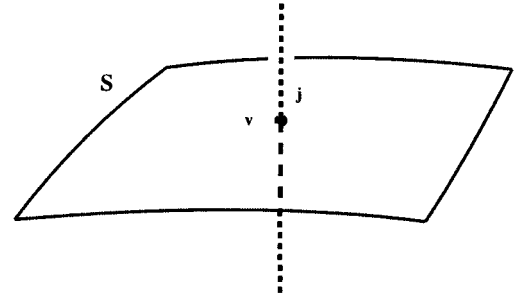


FIG. 3. The valence of the simplest vertex is two.

We define the space  $\mathcal{H}$  of surface quantum states as the space spanned by all (generalized) surface spin network states. This definition means that the basis in  $\mathcal{H}$  is formed by surface spin network states, which gives us all that we need for our counting purposes.

We can now recall that there exists a set of well defined operators  $\hat{A}_R$ , which “measure” quantum geometry of  $S$ . These operators correspond to areas of various regions  $R$  of the surface  $S$  (see [1]). It turns out that 3D spin network states are eigenstates of operators  $\hat{A}_R$ . The corresponding eigenvalues are given by

$$A_s = \sum_{v \in R} \frac{1}{2} \sqrt{2j_{(v)}^d(j_{(v)}^d + 1) + 2j_{(v)}^u(j_{(v)}^u + 1) - j_{(v)}^{u+d}(j_{(v)}^{u+d} + 1)}. \quad (1)$$

Note that we measure areas in the units of  $16\pi l_p^2$ , which is convenient in the loop quantum gravity. Here the sum is taken over all vertices  $v$  lying in the region  $R$  of the surface  $S$ ;  $j_{(v)}^u$ ,  $j_{(v)}^d$ , and  $j_{(v)}^l$  are the total spins of edges lying up, down, and tangential to the surface (see [1] for details). Although these operators are defined on 3D quantum states, the eigenvalues (1) depend only on the “surface” part of the information carried by a 3D spin network state. Therefore, we can think of  $\hat{A}_R$  as operators defined on the (generalized) surface spin network states, with eigenvalues given by Eq. (1).

As we have said in the Introduction, in this paper we are going to consider a geometrical entropy that corresponds to a macrostate specified simply by a total surface area  $A$ . For this simple case, the approximation criterion between macro and micro descriptions is straightforward. Namely, we can say that a quantum state (microstate) approximates a given macrostate if the mean value of the operator  $\hat{A}_S$  in this quantum state is approximately equal to the fixed value  $A$ . It is straightforward to see that quantum states approximating total surface area form a linear subspace in the space of all surface states. Its dimension is equal to the number of different surface spin network states approximating  $A$ .

Let us see now how many different quantum states approximate a given total area. It is easy to see that this number

is infinite. Indeed, loops on the surface, which are the simplest possible parts of a surface spin network state, do not give any contribution to the areas. Therefore, one has an infinite number of spin networks which approximate one and the same total area of  $S$  being different only in configurations of loops on the surface.

One can argue, however, that this happens because, using terminology from statistical mechanics, a macrostate of our system is not completely specified when we fix only a total area of the surface. Indeed, areas of regions on  $S$  carry information only about degrees of freedom described by  $e^{AB}$  field. But there are also degrees of freedom described by the pull-back of the connection field on the surface which one should take care of when specifying a macrostate. Our guess is that different configurations of loops on the surface from the above example correspond to different classical configurations of the connection field on  $S$ . Indeed, as we know, in classical theory loop quantities are constructed as traced holonomies of the connection, and, therefore, are just those objects which carry information about the connection field. Therefore, since in our simple treatment we want to forget about degrees of freedom described by the field  $a^{AB}$  when we specify a macrostate, to be consistent we have also to forget about those quantum states which, as we believe, contribute to  $a^{AB}$  and do not contribute to the area of  $S$ .

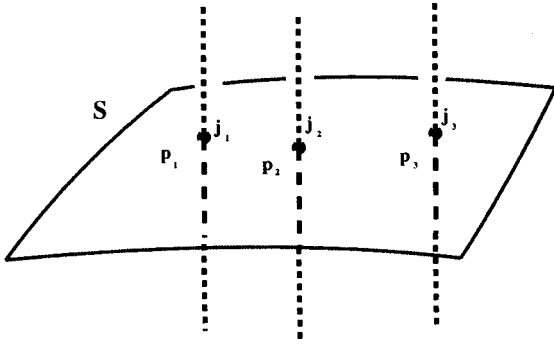


FIG. 4. The simplest surface spin network states are specified by a set of punctures on the boundary.

Let us, therefore, consider only quantum states which contribute to the areas of regions on  $S$ , but do not “contribute” to the connection field on the surface. These, according to our guess, are the states which contain no loops on the surface. We shall call the corresponding spin networks *open* spin networks. A (generalized) surface spin network is called open if the (surface) graph that labels it contains no closed paths (or no loops). The simplest open spin network state is the spin network containing a single vertex (see Fig. 3).

Now, to find the entropy which corresponds to a macrostate of a fixed total surface area  $A$  we should calculate the number of quantum states which approximate  $A$ , taking into account only open spin networks. However, let us first analyze the problem taking into account only some particularly simple spin network states, and then try to generalize the result obtained.

### III. GEOMETRICAL ENTROPY: SETS OF PUNCTURES

Let us consider spin networks whose vertices are bivalent, i.e., those specified simply by sets of points (vertices) on the surface with spins assigned to these points (see Fig. 4). Points on the surface with spins assigned are sometimes called punctures (see, for example, [4]), and we shall use this name as well. So quantum states which we consider now are specified by sets of punctures on the surface.

For these simple quantum states there exists a simplified version of the formula (1). Namely, a set  $\{p, j_p\}$  of punctures gives the area of a region  $R$  on  $S$ :

$$A(\{p, j_p\}) = \sum_{p \in R} \sqrt{j_p(j_p + 1)}. \quad (2)$$

Here the sum is taken over all punctures which lie in the region  $R$  and  $j_p$  are the corresponding spins. To get the total area of the surface we just have to sum over all punctures  $p \in S$ .

With these simple states the problem of calculating the entropy becomes almost straightforward.

Let us use the standard trick. Instead of counting states which correspond to the same area we shall take a sum over *all* states but take them with different statistical weight. Namely, let us consider the sum

$$Q(\alpha) = \sum_{\Gamma} \exp[-\alpha A(\Gamma)] \quad (3)$$

over all different states  $\Gamma = \{p, j_p\}$ , where  $\alpha > 0$  is a parameter. Considering  $p_{\Gamma} = [1/Q(\alpha)] \exp[-\alpha A(\Gamma)]$  as a *probability* of our system to be found in a state  $\Gamma$ , it is easy to see that the *mean* value of the area in such statistical state is

$$A(\alpha) = - \frac{\partial \ln Q}{\partial \alpha}. \quad (4)$$

The entropy of the system in this macrostate is given by the standard formula  $S = -\sum_{\Gamma} p_{\Gamma} \ln p_{\Gamma}$  or, as it is easy to check, by

$$S(\alpha) = \alpha A(\alpha) + \ln Q(\alpha). \quad (5)$$

We see that the mean value of the surface area depends on  $\alpha$ . If the statistical “ensemble” of states is chosen properly, then one can adjust the value of  $\alpha$  in such a way that  $A(\alpha)$  acquires any prescribed value. There is some particular value of  $S$  which corresponds to the chosen value of  $A$ . Solving away  $\alpha$  from  $A(\alpha)$  and  $S(\alpha)$ , we obtain the entropy as a function of the area  $S = S(A)$ . Statistical mechanics tells us that when the density  $\eta(A)$  of states of our system grows sufficiently fast with  $A$  it is of no difference which way of calculating  $S(A)$  is chosen; one can count the logarithm of the number of different states which give one and the same area or calculate the function  $S(A)$  as described — the results will not differ. Let us note that  $S(A)$  calculated through Eq. (5) will be meaningful only for large  $A$  (as compared with unity, i.e., with the Planckian area). This is really what we need because only in this case does the notion of approximation and, therefore, the notion of entropy acquire sense.

Let us now discuss whether all the sets of  $\{p, j_p\}$  should be taken into account in Eq. (3). First of all, let us recall that we have to count not the surface spin networks themselves but the diffeomorphism equivalence classes of spin networks. This means that two surface spin networks which can be transformed one into another by a diffeomorphism on the surface should be considered as a single state in Eq. (3). Thus the continuous set of data  $\{p, j_p\}$  ( $p$  runs all over the surface  $S$ ) reduces simply to a set of spins  $\{j_p\}$  when one identifies sets of punctures which can be mapped one into another by a diffeomorphism on the surface (note, however, the discussion following the next paragraph).

Next, we note that, if the surface  $S$  is a closed one, not every set  $\{p, j_p\}$  can be obtained as a result of intersection with  $S$  of some 3D spin network in  $\Sigma$ . Namely, the sum  $\sum_p j_p$ , which is the total spin which “enters” the surface must be an integer for gauge-invariant states. This corresponds to the fact that not all eigenvalues given by the formula (1) are eigenvalues of the area of a closed surface, as it was first pointed out in [1]. Let us consider in this section only a simpler case of an open surface  $S$ ; we return to the case of a closed surface in Sec. IV.

And, finally, let us consider the states specified by the following two sets of punctures (see Fig. 5):

$$\begin{aligned} \{ \dots, j(p') = s, \dots, j(p'') = q, \dots \}, \\ \{ \dots, j(p') = q, \dots, j(p'') = s, \dots \}. \end{aligned} \quad (6)$$

These two states differ one from another only at two points  $p', p''$ ; they give the same total area of the boundary. Should

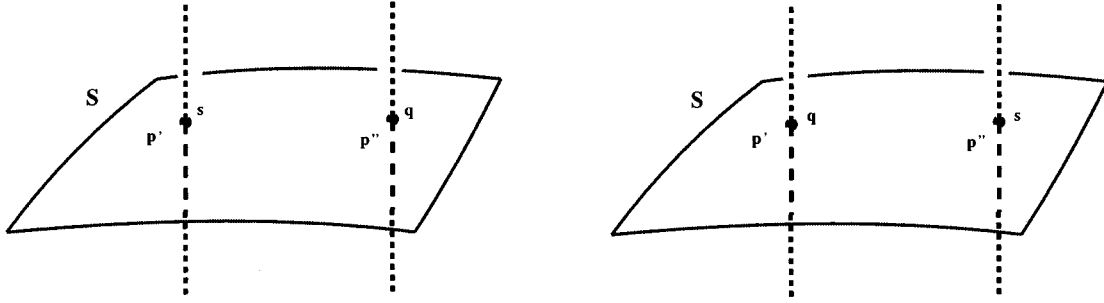


FIG. 5. Two sets of punctures which are considered as specifying different quantum states.

we distinguish these two states or consider them as a single physical state? It turns out that this is the key question which determines the form of the dependence of the entropy  $S(A)$  on the area  $A$  of the surface. Let us now consider these states as *different* and postpone the discussion of such a choice to the last section. So let us denote by  $N(A)$  the number of states which correspond to one and the same area  $A$  in the case when all the sets (6) are regarded as specifying different states. It is easy to see that  $N(A)$  is the number of *ordered* sets of punctures which approximate the total area  $A$ . It is straightforward to compute  $N(A)$  using our method with the statistical sum.<sup>3</sup> One can easily check that the fact that we regard the sets (6) as specifying different states (or, equivalently, the fact that we count ordered sets of punctures) means that we can sum over the spin of each puncture *independently*

$$Q = 1 + \sum_{n=1}^{\infty} \sum_{j_{p_1}=1/2}^{\infty} \cdots \sum_{j_{p_n}=1/2}^{\infty} \exp - \alpha \sum_p \sqrt{j_p(j_p+1)}. \quad (7)$$

The first sum here denotes the sum over the number of possible punctures on  $S$  and the subsequent ones denote the summation over the possible spins. It is easy to see that

$$Q = \frac{1}{1 - z(\alpha)}, \quad (8)$$

where  $z(\alpha)$  is given by

$$z(\alpha) = \sum_{j=1/2}^{\infty} \exp - \alpha \sqrt{j(j+1)}. \quad (9)$$

Note that the sum here runs over all positive integers and half-integers. One can expect that  $Q(\alpha)$  will increase as  $\alpha$  gets smaller because in any case (7) diverges when  $\alpha$  goes to zero. However, we see that Eq. (8) diverges even for some finite value of  $\alpha'$  such that  $z(\alpha')=1$  (we shall see in a minute to which value of  $\alpha$  this corresponds). When  $\alpha$  gets smaller and approaches  $\alpha'$ ,  $Q(\alpha)$  increases and diverges at the point  $\alpha'$ . It can easily be checked that  $A(\alpha)$  and  $S(\alpha)$  also diverge when  $\alpha \rightarrow \alpha'$ . This means that changing  $\alpha$  slightly we will obtain substantial differences in values of

$A$  and  $S$ . What we are interested in is the dependence  $S(A)$  for large values of  $A$ . But we see that all large values of  $A$  can be obtained by small changes in  $\alpha$  when  $\alpha \rightarrow \alpha'$ . Thus, from Eq. (5) we conclude that, for  $A \gg 1$ ,

$$S \approx \alpha' A. \quad (10)$$

Here we neglected the term  $\ln Q$  which is small as compared to the main term (10).

This result tells us that entropy grows precisely as the first power of area of the boundary. Now let us see what  $\alpha'$  amounts to. One can do this numerically but it is also straightforward to find an approximate value. Note that  $z(\alpha)$  can be rewritten as the sum over integers

$$z(\alpha) = \sum_{l=1}^{\infty} \exp - \frac{\alpha}{2} \sqrt{l^2 + 2l}. \quad (11)$$

The term under the square root in the exponential can be given the form  $(l+1)^2 - 1$ . Because the sum starts from  $l=1$  we can neglect the unity as compared to a larger term. Then  $z(\alpha)$  can be easily computed

$$z(\alpha) = \frac{\exp(-\alpha)}{1 - \exp(-\alpha/2)}. \quad (12)$$

This gives for the  $\alpha': z(\alpha')=1$  the value  $\alpha' = 2 \ln[2/(\sqrt{5}-1)] \approx 0.96$ . An explicit numerical investigation of the equation  $z(\alpha')=1$  gives a close value  $\alpha' \approx 1.01$ .

So we have shown that for large values of  $A$  the entropy depends on the area as

$$\begin{aligned} S(A) &= \alpha' A; \\ \alpha' &= 1.01. \end{aligned} \quad (13)$$

#### IV. GEOMETRICAL ENTROPY: THE CASE OF CLOSED SURFACES

As we have mentioned before, in the case of closed surfaces  $S$  we have to take into account the fact that not all eigenvalues given by the formula (1) are eigenvalues of the operator  $\hat{A}_S$ , which measures the total area of the surface. Namely, in the case of a closed surface  $S$  gauge-invariant quantum states are those which satisfy the condition that the sums  $\sum_{v \in S} j_{(v)}^u$  and  $\sum_{v \in S} j_{(v)}^d$  over all vertices lying on the

<sup>3</sup>One can also do this calculation explicitly, using combinatorial methods. See [5].

surface are integers [1] [spins  $j_{(v)}^u, j_{(v)}^d$  are those defined after the formula (1)]. This condition means that for the case of a closed surface some spin network states should be excluded from the sum (3). Recall that our result essentially follows from the fact that the statistical sum  $Q(\alpha)$  diverges when  $\alpha$  approaches some *finite* value  $\alpha'$ . Because we have to drop some positive terms from the statistical sum corresponding to a closed  $S$ ,  $Q(\alpha)$  can prove to be convergent for all  $\alpha > 0$  and as a result we would obtain some other (nonlinear) dependence  $S(A)$ .<sup>4</sup>

The aim of this section is to show that geometrical entropy  $S(A)$  of (ordered) sets of punctures, considered in the previous section, for a closed surface  $S$  still depends linearly on the total surface area. So, again, our states are specified by ordered sets  $\{p, j_p\}$  of punctures on the surface. However, in the case of a closed surface, not all sets of punctures correspond to gauge-invariant physical states. Namely, in our case gauge-invariant states are those for which  $\sum_{p \in S} j_p$  is an integer. To find the number of different states which approximate one and the same total area we can again apply our trick with the statistical sum. The statistical sum  $Q(\alpha)$  will be given by the expression (7), where we have to take into account only the states satisfying the above condition.

It is not hard to calculate the statistical sum  $Q(\alpha)$  for our case of a closed surface  $S$ . Let us divide the function  $z(\alpha)$  given by Eq. (12) into two parts  $z(\alpha) = \tilde{z}(\alpha) + \tilde{\tilde{z}}(\alpha)$ . Function  $\tilde{z}(\alpha)$  is the sum over all integer values  $l$  of spin  $j$ :

$$\tilde{z} = \sum_{l=1}^{\infty} \exp - \alpha \sqrt{l(l+1)}. \quad (14)$$

Function  $\tilde{\tilde{z}}(\alpha)$  is the sum over all half-integers  $j = l - 1/2$ :

$$\tilde{\tilde{z}}(\alpha) = \sum_{l=1}^{\infty} \exp - \alpha \sqrt{(l-1/2)(l+1/2)}. \quad (15)$$

Let us rewrite the statistical sum  $Q(\alpha)$  over all sets of punctures in terms of the functions  $\tilde{z}, \tilde{\tilde{z}}$  introduced:

$$Q(\alpha) = \frac{1}{1 - (\tilde{z} + \tilde{\tilde{z}})} = \frac{1}{1 - \tilde{z}} \sum_{n=0}^{\infty} \left( \frac{\tilde{\tilde{z}}}{1 - \tilde{z}} \right)^n. \quad (16)$$

A moment of reflection shows that in order to get the statistical sum corresponding to the case of a closed surface, we have to drop from Eq. (16) all terms which contain odd powers of  $\tilde{\tilde{z}}$ . Thus we get

$$Q(\alpha)_{\text{closed}} = \frac{1}{1 - \tilde{z}} \sum_{n=0}^{\infty} \left( \frac{\tilde{\tilde{z}}}{1 - \tilde{z}} \right)^{2n} = \frac{1}{1 - \tilde{z}} \frac{1}{1 - \left( \frac{\tilde{\tilde{z}}}{1 - \tilde{z}} \right)^2}. \quad (17)$$

Here  $\tilde{z}, \tilde{\tilde{z}}$  are functions of  $\alpha$ . Recall now that our statistical mechanical system has a regime in which the entropy depends linearly on the area in the case when the statistical sum

diverges for some finite value of  $\alpha$ . Let us, therefore, investigate the behavior of  $Q(\alpha)_{\text{closed}}$  when  $\alpha$  goes to zero. First, let us note that

$$Q(\alpha)_{\text{closed}} = \frac{1 - \tilde{z}}{(1 - \tilde{z} + \tilde{\tilde{z}})(1 - \tilde{z} - \tilde{\tilde{z}})}. \quad (18)$$

Also, we note that  $\tilde{\tilde{z}} > \tilde{z}$ , because the sum in  $\tilde{\tilde{z}}(\alpha)$  starts from  $j = 1/2$ , whereas the sum in  $\tilde{z}$  is taken over all integer values of spin and starts from  $j = 1$ . We see, therefore, that the statistical sum  $Q(\alpha)_{\text{closed}}$  diverges for  $\alpha' : \tilde{z}(\alpha') + \tilde{\tilde{z}}(\alpha') = 1$ . But  $\tilde{z} + \tilde{\tilde{z}} = z$ , so the value of  $\alpha'$  here is the solution of equation  $z(\alpha') = 1$  obtained in the previous section.

Thus we get that

$$S_{\text{closed}}(A) = \alpha' A, \quad (19)$$

where  $\alpha'$  is the same as in Eq. (13). Thus we have proved that, although the statistical sum  $Q_{\text{closed}}(\alpha)$  for the case of a closed surface is different from  $Q(\alpha)$  corresponding to the case of an open  $S$ , it implies the linear dependence  $S(A) = \alpha' A$  of the entropy on the surface area, with the proportionality coefficient  $\alpha'$  being the same as in the case of an open surface. One can say that, although we excluded some states from the statistical sum, there are still ‘‘enough’’ states to give the same linear dependence of the entropy on  $A$  as in the case of an open surface.

## V. DISCUSSION

The result that we have obtained considering some particularly simple surface quantum states is that the entropy corresponding to a macrostate which is specified by a total area  $A$  of the surface is proportional precisely to the area  $A$ . It is important that we have obtained precisely the same dependence  $S(A)$  both for the case of open and closed surfaces  $S$ . The entropy was defined as the logarithm of the number of different quantum states that approximate one and the same area of the surface. The states which we considered were specified simply by the sets of punctures on the surface. It is crucial that the states which are different only up to a permutation of spins (see Fig. 5) were considered as different quantum states.

One of the reasons why we postponed the discussion of the key point of distinguishing the states (6) was to emphasize the importance of this choice. This is the choice which implies the linear dependence of entropy  $S(A)$  on the surface area. As we show in the Appendix, in the case when states (6) are considered as indistinguishable, the dependence of the entropy on the areas is different from the linear one (in fact, the entropy turns out to be proportional to the square root of the area).

Before discussing this key point, let us note that our aim in this paper is not to give a physical motivation for some particular choice of the ensemble of quantum states. Rather we wanted to show how the entropy arises naturally when one considers correspondence between classical and quantum pictures of geometry. We also wanted to illustrate this idea following some simple choice of the ensemble of quantum states, and to present a technique which turns out to be

<sup>4</sup>In fact, in this case we would obtain that  $S(A)$  grows *slower* than  $A$ .

useful. Having this in mind, let us discuss our choice of considering the states (6) as distinguishable ones.

We were counting the number of different diffeomorphism equivalence classes of (simple) surface spin networks approximating one and the same total surface area. This means that two surface spin networks which can be transformed one into another by a diffeomorphism on the surface should have been considered as specifying the same microstate. Let us now consider the two states (6). It is easy to see that there exists a diffeomorphism which maps one state into the other; this diffeomorphism simply rearranges two points  $p', p''$  on the surface. Therefore, on the first sight, the states (6) should have been considered as a single state.

Does this mean that the result obtained is physically meaningless and is simply an exercise in statistical mechanics? There are some reasons to believe that the result obtained is more than that. So let us give some possible motivations for our strange choice of considering states (6) as different quantum states.

One possibility, which is argued by Rovelli [5], is that points on the surface are physically distinguishable, and so are the states (6). This, in fact, happens in the case of some systems for which the surface  $S$  plays the role of the boundary. It might be the case that *boundary conditions* partly (or even completely) break diffeomorphism invariance on the boundary. This would mean that some spin networks which are usually considered as specifying one and the same quantum state, should, in fact, be considered as different quantum states. This case of systems with boundaries is subtle and deserves special attention. Let us only note the possible connection of our result, viewed from this point, with the results of Carlip [6].

The other possibility, which has also been argued in [7], is that some other (in fact, loop) states on the surface make some of the states (6) belong to different diffeomorphism equivalence classes of spin networks. Indeed, the surface loop states which we considered as giving no contribution to the areas, and, thus, forgot about, may affect largely the number of different diffeomorphism equivalence classes of states which approximate one and the same area of  $S$ . To see this, let us introduce a loop configuration on the surface. This loop configuration divides the surface into regions. It is clear that some of the states (6) will belong to different equivalence classes, for there will no longer be a diffeomorphism ‘‘connecting’’ different regions. Thus the number of different diffeomorphism equivalence classes which approximate one and the same total area  $A$  in this case is larger than in the case when there are no loops on the surface. So loops on the surface may allow one to distinguish states of the form (6) (for more details see [7]).

Of course, neither of these motivations gives a final physical justification of the result obtained. But let us repeat that this is not what we aimed at in this paper. We hope, however, that the above discussion shows at least that the issue deserves a further investigation.

Finally, let us discuss the possibility to generalize the result obtained considering arbitrary open surface spin network states. First, let us take into account surface spin networks which have no tangential edges, allowing, however, vertices of arbitrary valence. In this case we have to use a general formula (1) for eigenstates of area operators (with all

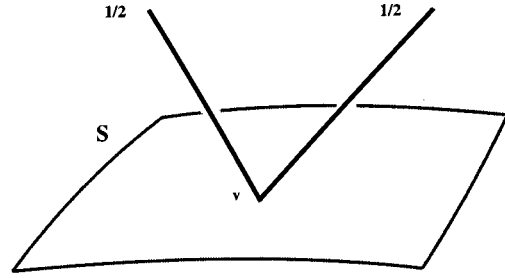


FIG. 6. Surface state which does not contribute to the surface area, thus, producing the degeneracy.

$j_{(v)}^{(u+d)}$  being equal to zero because of the fact that we consider spin networks with no tangential edges). We would like to generalize our result counting all (open) surface spin networks which approximate one and the same total surface area. However, we face the problem trying to consider all states. Namely, eigenvalues given by the formula (1) are degenerate and we have to take this degeneracy into account when calculating the entropy. Let us consider, for example, a simple state, which contains one vertex of valence two (see Fig. 6). For this state we get  $j^u = j^d = 0$ , and, therefore,  $A_S = 0$ . A moment of consideration shows that there are, in fact, an infinite number of similar surface states that do not give any contribution to the area of  $S$ . Thus we find that eigenvalue  $A_S = 0$  is infinitely degenerate. Similarly, we find that *all* eigenvalues given by the formula (1) are infinitely degenerate. Therefore, if we would like to take into account all different surface states we would get an infinite value for our geometrical entropy.

Let us note, however, that the states which we have just considered are rather pathological. Namely, we observe that small deformations of the surface  $S$  (see Fig. 6) (with the spin network state being not deformed) cause a change in the ‘‘quantum area’’ of  $S$ . Let us consider a one parameter family of surfaces  $S_\epsilon$ ,  $\epsilon \in [0, 1]$  such that  $S_\epsilon \rightarrow S$  when  $\epsilon \rightarrow 0$ . For our example (see Fig. 6), if surfaces  $S_\epsilon$  approach the surface  $S$  from below we have  $\lim_{\epsilon \rightarrow 0} A_{S_\epsilon} = A_S = 0$  (here by  $A_S$  we denote an eigenvalue of the operator  $\hat{A}_S$  on the quantum state we consider). However, if we choose the family of surfaces approaching  $S$  from above we have  $\lim_{\epsilon \rightarrow 0} A_{S_\epsilon} = \sqrt{3} \neq A_S = 0$  (we measure areas in the units  $16\pi l_p^2$ ). Thus we see that states which cause the degeneracy are ‘‘pathological’’ when considered as surface states, for ‘‘quantum area’’ of  $S$  in these states behaves noncontinually under small deformations of  $S$ . This observation suggests that we have to exclude these states when we consider an ensemble of surface states. A natural way to do it would be to change the approximation criterion between macro- and microdescriptions. Namely, let us strengthen our criterion in the following way. We fix a value  $A$  of the total area of  $S$ , which defines our macrostate. We choose one parameter family  $S_\epsilon$  of two surfaces such that  $S_\epsilon \rightarrow S$ ,  $\epsilon \rightarrow 0$ . Let us now say that a spin network state  $\Gamma$  approximates our macrostate if  $\lim_{\epsilon \rightarrow 0} A_{S_\epsilon}(\Gamma) = A$ , where  $A_{S_\epsilon}(\Gamma)$  is the eigenvalue of operator  $\hat{A}_{S_\epsilon}$  corresponding to the eigenstate  $\Gamma$ .

The new approximation criterion states that we have to consider only “good” surface quantum states. By definition, “quantum area” of  $S$  in “good” surface quantum states does not change under small deformations of the surface  $S$ . It is easy to see that “good” quantum states are those which have only bivalent vertices, i.e., precisely those states which we considered in this paper. Thus we conclude that the result obtained above gives the geometrical entropy  $S(A)$  of a macrostate of a fixed total area  $A$ , the quantum states that account for this entropy being all states which approximate  $A$  in the strong sense, i.e., those for which “quantum area” does not change under small deformations of the surface  $S$ . Furthermore, the entropy  $S(A)$  is the same both for open and for closed surfaces. Thus the result obtained is general in the sense that we consider all quantum surface states which approximate  $A$  in the strong sense.

Let us conclude by saying that the notion of geometrical entropy is, presumably, valid not only in the form explored here (when we fixed only one macroscopic parameter — the surface area), but also in a more general context. For example, it is of interest to calculate the entropy  $S(g)$  which corresponds to a given two-metric on the surface, which is a genuine geometrical entropy.

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#### APPENDIX: COUNTING UNORDERED SETS OF PUNCTURES

In this appendix we would like to show that the entropy  $S(A)$  in the case when one considers sets (6) as undistinguishable is proportional to the square root of the area.

Since we do not distinguish between sets (6), all states  $\Gamma$  which enter Eq. (3) are just unordered sets  $\{j_1, j_2, \dots\}$  of spins. One can describe each such state by a set  $\{n_{j_1}, n_{j_2}, \dots\}$  of numbers of spins  $j_1, j_2, \dots$  in this state. It is convenient to imagine each state defined by a set  $\{j_1, j_2, \dots\}$  as a collection of different particles. Particles of a particular sort  $j$  correspond to punctures carrying the spin  $j$ . There can be many particles of the same sort in a state and we denote the corresponding number of such particles by  $n_j$ . Having this analogy in mind it is easy to see that in our case the statistical sum  $Q(\alpha)$  is given by

$$Q(\alpha) = \sum_{\{n_j\}} \exp\left(-\alpha \sum_{j=1/2}^{\infty} n_j A(j)\right). \quad (\text{A1})$$

Here by  $A(j)$  we denoted the contribution to the area from a single “particle” of the sort  $j$ :

$$A(j) = \sqrt{j(j+1)}. \quad (\text{A2})$$

Note that the sum over  $j$  starts from  $j=1/2$  and runs over all positive integers and half-integers. It is straightforward to take the sum over numbers  $n_j$  of different “particles” in Eq. (A1); this gives

$$Q(\alpha) = \prod_j \frac{1}{1 - \exp[-\alpha A(j)]}. \quad (\text{A3})$$

The area as a function of parameter  $\alpha$  is given by the formula (4). In our case it gives

$$A(\alpha) = \sum_j \frac{A(j) \exp[-\alpha A(j)]}{1 - \exp[-\alpha A(j)]}. \quad (\text{A4})$$

To see what dependence of  $A$  on  $\alpha$  this implies we replace the sum over spins  $j$  by the sum over integers (colors)  $l=2j$ . Then  $A(l) = 1/2\sqrt{l^2+2l}$ . For values of  $l > 1$  we have the approximation

$$A(l) \approx (l+1)/2. \quad (\text{A5})$$

Let us now approximate the sum over  $j$  in Eq. (A4) by the following integral over  $l$ :

$$A(\alpha) \approx \int_{l=1}^{\infty} dl \frac{\frac{(l+1)}{2} \exp\left(-\alpha \frac{(l+1)}{2}\right)}{1 - \exp\left(-\alpha \frac{(l+1)}{2}\right)}. \quad (\text{A6})$$

It is easy to see that this integral is equal to

$$A(\alpha) = \frac{2}{\alpha^2} \int_{\alpha}^{\infty} dx \frac{x \exp -x}{1 - \exp -x}. \quad (\text{A7})$$

Recall now that we are interested in the dependence  $S(A)$  for large values of  $A$  (as compared with unity). We see from Eq. (A7) that large values of  $A$  correspond to small  $\alpha$ . In the limit  $\alpha \rightarrow 0$  the integral in Eq. (A7) can easily be calculated:

$$\int_0^{\infty} dx \frac{x \exp -x}{1 - \exp -x} = \frac{\pi^2}{6}. \quad (\text{A8})$$

So, finally, for large values of  $A$  (or, equivalently, for small  $\alpha$ ), we have the following dependence:

$$A(\alpha) = \frac{\pi^2}{3\alpha^2}. \quad (\text{A9})$$

This gives the dependence  $S(A)$  of the entropy on the area. Namely, in the limit of large  $A$  we can neglect the logarithmic term in Eq. (5) as compared with the larger first term. Thus we get



$$S = \frac{\pi}{\sqrt{3}} \sqrt{A}. \quad (\text{A10})$$

So, in the case when one considers unordered sets of punctures, the geometrical entropy  $S(A)$  is proportional to the square root of the area.

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