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Relating CKM parametrizations and unitarity triangles

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Concise variable transformations between the four angles of the CKM matrix in the standard representation advocated by the Particle Data Group and the angles of the unitarity triangles are derived. The behavior of these transformations in various limits is explored. The straightforward extension of this calculation to other representations and more generations is indicated. [S0556-2821(97)00501-8]

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I. STANDARD PARAMETRIZATION VS UNITARITY ANGLES

The most popular model for parametrizing quark flavor-changing currents and CP violation is that of mixing between quark mass and weak interaction eigenstates, as represented by the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix [1]. Although the unitary $N\times N$ matrix for N quark generations possesses $(N-1)^2$ observable real parameters, these parameters may be (and have been) chosen in countless different ways. Even if we adopt the usual prescription of $N(N-1)/2$ Euler rotation angles and $(N-1)(N-2)/2$ phases in generation space, we are still faced with the choice of which axes to use for our rotations and in what order to perform them; this choice leads to no less than 36 distinct but equivalent parametrizations for three generations [2]. The particular form of the CKM matrix advocated by the Particle Data Group [3], as originally proposed by Chau and Keung [4], is just one of these, and is written

$$V_{\rm CKM}=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \tag{1.1}$$

where $c_{ij}\equiv\cos\theta_{ij}$ and $s_{ij}\equiv\sin\theta_{ij}$, the subscripts indicate the plane of rotation, and the Euler angles θ_{12}, θ_{23}, and θ_{13} are all chosen to lie in the first quadrant by a redefinition of (unobservable) quark field phases. The phase angle δ_{13} may not be similarly restricted:

$$0\leq\theta_{ij}\leq\frac{\pi}{2},\quad 0\leq\delta_{13}<2\pi. \tag{1.2}$$

Alternately, the CKM matrix may be described in terms of parameters invariant under choice of convention or phase redefinitions. The moduli of the elements $|V_{\alpha i}|$ fall into this category but are not always the most convenient variables in experimental measurements [5]. For example, the short-distance contribution to $B\overline{B}$ mixing, ubiquitous in neutral B decays, is proportional to $|V_{td}V_{tb}^*|^2$. Unitarity information may be more easily recovered by noting that $VV^\dagger=1$ is equivalent to the orthogonality of columns or rows in V:

$$\sum_{\alpha=u,c,t,\ldots} V_{\alpha i}V_{\alpha j}^*=\delta_{ij}, \quad \sum_{i=d,s,b,\ldots} V_{\alpha i}V_{\beta i}^*=\delta_{\alpha\beta}. \tag{1.3}$$

For two distinct columns or rows, the right-hand side is zero, and so the condition may be depicted geometrically in complex space as describing a closed polygon with one side corresponding to each quark generation. The conditions from Eq. (1.3) with 1 on the right-hand side set the scale of the polygons. For three generations one obtains triangles, which are special since knowledge of the angles is sufficient to determine their shapes uniquely; thus we concentrate on the three-generation case in this work. Equation (1.3) implies that there are six independent triangles, called the unitarity triangles [6], which are pictured in Fig. 1 and labeled by the pair of rows or columns whose orthogonality is represented.

Our chief interest in the unitarity triangles is that their angles are convention independent. One sees this by noting that for two complex numbers z_1 and z_2, the (oriented exterior) angle between them is $\arg(z_1^*z_2)$, where the argument function assumes its principal value, $-\pi<\arg(z)\leq\pi$, for all complex z. Thus angles in this case have the form

$$\omega_{\alpha\beta}^{ij}\equiv\arg(V_{\alpha i}V_{\alpha j}^*V_{\beta j}V_{\beta i}^*). \tag{1.4}$$

Each quark index in this expression appears in both a V and a V^*, so that any phase redefinition cancels in the product. Geometrically, the redefinition of a quark phase simply rotates an entire unitarity triangle by a constant angle.

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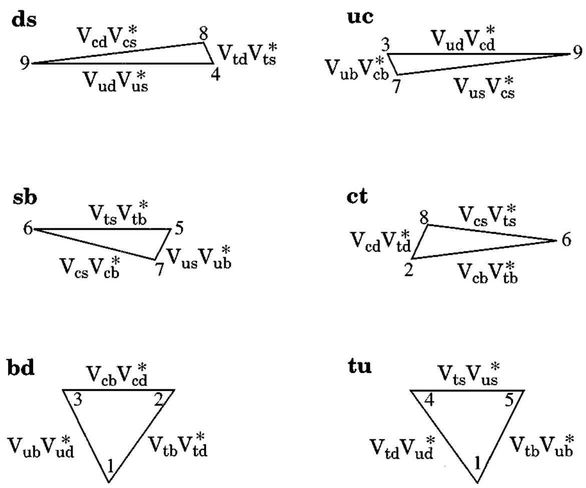

FIG. 1. The unitarity triangles for three quark generations, as presented in Ref. [9]. The triangles are labeled by the pair of rows or columns whose orthogonality is represented. The angles as numbered are $1 \equiv \alpha$; $2 \equiv \beta$; $3 \equiv \pi - (\alpha + \beta)$ (conventionally called γ if closure of the triangle is not assumed); $4 \equiv \beta + \epsilon - \epsilon'$; $5 \equiv \pi - (\alpha + \beta + \epsilon - \epsilon')$; $6 \equiv \epsilon$; $7 \equiv \alpha + \beta - \epsilon'$; $8 \equiv \pi - (\beta + \epsilon)$; $9 \equiv \epsilon'$. In all cases, the arrows on the complex vectors are oriented counterclockwise, indicating the experimental positivity of J.

It is also known that the six unitarity triangles all have the same area, given by the Jarlskog parameter J [7]. This follows first because 3×3 unitary matrices enjoy the property that a particular pattern of elements is invariant up to a sign,

$$\mathrm{Im}(V_{\alpha i} V_{\alpha j}^* V_{\beta j} V_{\beta i}^*) = J \sum_{\gamma,k} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk}, \tag{1.5}$$

which defines J for any choice of $\alpha \neq \beta$, $i \neq j$, and second because the area of a triangle with sides z_1 and z_2 is $|\mathrm{Im}(z_1^* z_2)|/2 = |z_1| \cdot |z_2| \cdot |\sin[\arg(z_1^* z_2)]|/2$. It follows that the area of each unitarity triangle is given by $|J|/2$. J is the unique convention-independent CP violation parameter of the CKM matrix, in that all measurable CP-violating quantities turn out to be proportional to J, so that vanishing area in any of the unitarity triangles indicates the vanishing of CP violation in the CKM matrix.

Phenomenologically, much is already known about the unitarity triangles. As schematically indicated in Fig. 1, four of them are nearly flat because they possess one side much shorter than the other two. Of the remaining two, the **bd** triangle is of greatest current interest to experimentalists: Its sides, which are expected to be of comparable length, represent the least well-known elements of the CKM matrix and will be accessible in several B factories currently under development. There is already a large literature dedicated to methods of measuring the sides of the **bd** triangle and extracting its angles, denoted α, β, and γ; see Ref. [8] for a recent review.

II. A COMPLETE SET OF UNITARITY ANGLES

All that has been said up to this point is already several years old. What has been appreciated only recently is that the angles of the unitarity triangles enjoy a number of elegant properties, and may be used [9] to reconstruct the full CKM matrix except for the sign of J, which has been determined by experiment to be positive [10]. The success of this approach follows from the observation that the six triangles have 18 distinct sides in all, but only nine distinct angles. One sees this by noticing that $\omega_{\alpha\beta}^{ij}$ appears in both the **ij** and $\boldsymbol{\alpha\beta}$ triangles, halving the potential number; furthermore, applying the condition that the angles of any given triangle add to π shows that these nine angles may be written as sums and differences of only four independent angles. Since this is also the number of independent parameters in the three-generation CKM matrix, the angles may be used as a basis for all convention-independent quantities.

To be specific, we follow [9] in defining the interior angles:

$$\begin{aligned} \alpha &\equiv \pi - |\omega_{tu}^{bd}|, \\ \beta &\equiv \pi - |\omega_{ct}^{bd}|, \\ \epsilon &\equiv \pi - |\omega_{ct}^{sb}|, \\ \epsilon' &\equiv \pi - |\omega_{uc}^{ds}|. \end{aligned} \tag{2.1}$$

These definitions are used to label the angles in Fig. 1. In particular, α and β (not to be confused with the indices of $\omega_{\alpha\beta}^{ij}$) are the same angles traditionally used in the literature for the **bd** triangle, so these angles form a natural and very useful basis for describing invariants of the CKM matrix. The form of these expressions is chosen so that each of α, β, ϵ, ϵ' lies in the range $(0,\pi)$. Note that the quantities $\omega_{\alpha\beta}^{ij}$ as defined in Eq. (1.4) appear different from, but are formally identical to, those in Ref. [9], since $\arg(z_1/z_2^*) = \arg(z_1 z_2)$.

Observe that, although the parameters $\omega_{\alpha\beta}^{ij}$ do indeed contain the sum total of the unitarity information, some information is lost in the definitions (2.1), since magnitudes of $\omega_{\alpha\beta}^{ij}$ are taken. In fact, all that is lost is the orientation of the angles, namely, whether they are constructed in the clockwise or counterclockwise direction. Since the angles must still form closed triangles, this formulation merely surrenders one's ability to distinguish between a particular triangle and its mirror image. This freedom corresponds to the sign of J, as indicated by Eqs. (1.4) and (1.5).

Reference [9] shows how to reconstruct all measurable phases and moduli of the CKM matrix given only the angles of the unitarity triangles. On the other hand, all of the parametrizations of the CKM matrix perform the same function, but in terms of quantities interpreted in a convention-dependent way. In the remainder of this paper, we derive a complete set of concise relations between the two parametrizations for the particular standard form of Eq. (1.1). Then we check that the transformations obey the appropriate limits for vanishing CP violation in the CKM matrix and consider the transformations in a phenomenologically useful limit. Finally, we indicate the straightforward generalization to other parametrization choices and more quark generations.

III. THE TRANSFORMATIONS

First observe that the following relation holds for all complex z:

$$\cot[\pi - |\arg(z)|] = -\frac{\mathrm{Re}(z)}{|\mathrm{Im}(z)|}. \tag{3.1}$$

Since the argument function is related to the arctangent function, it is clear that one will obtain the cleanest expressions for the unitarity angles in terms of their tangents or cotangents; why cotangents are superior for our purposes will become evident. In fact, simply inserting the elements of Eq. (1.1) into Eqs. (1.4) and (2.1) obtains the desired transformation in one direction for all angles of the unitarity triangles. We use the following compact notation: s, c, t, and $\not{c}$, respectively, represent sine, cosine, tangent, and cotangent, while the indices x, y, z, and δ, respectively, represent θ_{12}, θ_{23}, θ_{13}, and δ_{13}. A number in the index indicates a multiple angle so that, for example, $s_{2y} \equiv \sin 2\theta_{23}$. Cotangents of the nine angles (or their supplements) appearing in the unitarity triangles are

$$\cot\alpha = \frac{+\not{c}_x \not{c}_y s_z - c_\delta}{|s_\delta|}, \quad \cot(\alpha+\beta) = \frac{-\not{c}_x t_y s_z - c_\delta}{|s_\delta|},$$

$$\cot(\alpha+\beta-\epsilon') = \frac{+t_x t_y s_z - c_\delta}{|s_\delta|}, \quad \cot(\alpha+\beta+\epsilon-\epsilon') = \frac{-t_x \not{c}_y s_z - c_\delta}{|s_\delta|},$$

$$\cot\beta = \frac{(s_x^2 - c_x^2 s_z^2) s_{2y} - s_{2x} c_{2y} s_z c_\delta}{s_{2x} s_z |s_\delta|}, \quad \cot\epsilon = \frac{(c_x^2 - s_x^2 s_z^2) s_{2y} + s_{2x} c_{2y} s_z c_\delta}{s_{2x} s_z |s_\delta|},$$

$$\cot\epsilon' = \frac{(c_y^2 - s_y^2 s_z^2) s_{2x} + c_{2x} s_{2y} s_z c_\delta}{s_{2y} s_z |s_\delta|}, \quad \cot(\beta+\epsilon-\epsilon') = \frac{(s_y^2 - c_y^2 s_z^2) s_{2x} - c_{2x} s_{2y} s_z c_\delta}{s_{2y} s_z |s_\delta|},$$

$$\cot(\beta+\epsilon) = \frac{\{s_{2x}^2[\frac{1}{4} s_{2y}^2 (1+s_z^2)^2 - s_z^2] - \frac{1}{2} s_{4x} s_{4y} s_z (1+s_z^2) c_\delta - s_{2y}^2 s_z^2 (1 - s_{2x}^2 c_\delta^2)\}}{s_{2x} s_{2y} s_z c_z^2 |s_\delta|}. \tag{3.2}$$

Several comments are in order. First, it is obvious that the first four of these expressions are quite simple, the next four are of intermediate complexity, and the last is quite complicated. The origin of this distinction becomes clear with a glance at Eq. (1.1): The elements with complicated forms in the lower-left 2×2 submatrix (each the sum of a real and a complex number), respectively, appear 1, 2, or 4 times in using Eq. (1.4) to compute the corresponding first four, middle four, and final expressions in Eq. (3.2). This feature is repeated in any parametrization of the CKM matrix using Euler angles and phases. Since the angles appearing in the first four expressions in Eq. (3.2) are independent, and the cotangent function is one-to-one on $(0,\pi)$, these four simple expressions contain all the information of the CKM matrix except the sign of J. The chosen CKM parametrization picks out particular combinations of the basis angles α, β, ϵ, and ϵ' in which the transformation equations are simple. It is now clear that cotangents are chosen over tangents so that adding the quantities in Eq. (3.2) in order to invert them is simpler.

Second, as for the sign of J, note that the dependence of each expression in Eq. (3.2) on the CP-violating phase δ_{13} occurs only through the functional forms $\cos\delta_{13}$ or $|\sin\delta_{13}|$ and so is insensitive to the variable change $\delta_{13} \to (2\pi - \delta_{13})$; this is explicitly how the parametrization of Eq. (1.1) is sensitive to the sign of J but the angles of Ref. [9] are not.

Finally, important limiting cases are evident from these transformations. As is well known, CP violation does not occur in the CKM matrix if any of the following conditions hold:

$$\theta_{ij} = 0, \frac{\pi}{2} \quad \text{for any of} \quad ij = 12, 23, 13, \qquad \delta_{13} = 0, \pi. \tag{3.3}$$

In such cases our transformations (3.2) must satisfy the property that the unitarity angles collapse to zero area. To see this, note that the denominator of each expression in Eq. (3.2), as seen from Eqs. (1.4), (1.5), and (3.1), is just $|J|$. In the standard parametrization (1.1),

$$J = s_{12} c_{12} s_{23} c_{23} s_{13} c_{13}^2 s_{\delta_{13}}, \tag{3.4}$$

so that J is seen to vanish when any of the conditions in Eq. (3.3) is satisfied. If one could ignore the numerators in Eq. (3.1), each expression in Eq. (3.2) would become singular under the conditions (3.3), making each unitarity angle 0 or π so that the unitarity triangles would collapse. However, in some cases the numerator factors cancel factors in the denominator, and so some of the angles continue to assume finite values even when certain conditions in Eq. (3.3) are satisfied. For example, for $\theta_{23} \to \pi/2$, $\cot\alpha \to -\cos\delta_{13}/|\sin\delta_{13}| \neq \infty$. In such cases, however, the unitarity triangles may still be seen to collapse. For, in the given example, consider the **bd** triangle in Fig. 1. In the same limit,

$\cot\beta \to +\cos\delta_{13}/|\sin\delta_{13}|$ and $\cot(\alpha+\beta)\to-\infty$, so that α and β add to π. Thus the sides $|V_{ub}V_{ud}^*|$ and $|V_{cb}V_{cd}^*|$ are parallel, requiring $|V_{tb}V_{td}^*|$ to have zero length for the triangle to close, and the triangle collapses. All limiting cases from Eq. (3.3) lead to trivial unitarity triangles using similar observations.

The angles α, β, ϵ, and ϵ' as defined in Eq. (2.1) are bounded between 0 and π. However, not all values for all of the angles are simultaneously allowed. In order for the unitarity triangles to close, their allowed ranges must be correlated. If one chooses them in the order α, β, ϵ, ϵ', one requires

$$0<\alpha<\pi,\quad 0<\beta<\pi-\alpha,\quad 0<\epsilon<\pi-\beta,$$
$$\max(0,\alpha+\beta+\epsilon-\pi)<\epsilon'<\beta+\min(\alpha,\epsilon). \qquad (3.5)$$

We now derive the inverse transformations to Eq. (3.2). As pointed out above, four combinations of the angles are particularly convenient to work with for this purpose, and for convenience we use the following notation for them:

$$A\equiv\cot\,\alpha,\quad B\equiv\cot(\alpha+\beta),$$
$$C\equiv\cot(\alpha+\beta-\epsilon'),\quad D\equiv\cot(\alpha+\beta+\epsilon-\epsilon'). \qquad (3.6)$$

From Eq. (3.2) one sees that differences of A, B, C, and D eliminate the $\cos\delta_{13}$ factor, and quotients eliminate the $|\sin\delta_{13}|$. In particular,

$$A-B=\not c_x s_z(\not c_y+t_y)/|s_\delta|,\quad A-D=\not c_y s_z(\not c_x+t_x)/|s_\delta|,$$
$$C-D=t_x s_z(\not c_y+t_y)/|s_\delta|,\quad C-B=t_y s_z(\not c_x+t_x)/|s_\delta|. \qquad (3.7)$$

The quantities given here are manifestly non-negative, as may be checked using the range constraints (3.5) and the monotonic decrease of cotangent over $(0,\pi)$. One other combination is particularly simple:

$$BD-AC=s_z c_\delta(\not c_x+t_x)(\not c_y+t_y)/|s_\delta|^2. \qquad (3.8)$$

From here it is trivial to obtain expressions for the parameters of the standard form (1.1):

$$\cot\theta_{12}\equiv\not c_x=\sqrt{\frac{A-B}{C-D}},\quad \cot\theta_{23}\equiv\not c_y=\sqrt{\frac{A-D}{C-B}},$$
$$\sin\theta_{13}\equiv s_z=\sqrt{\frac{(A-B)(A-D)(C-B)(C-D)}{(A-B+C-D)^2+(BD-AC)^2}},$$
$$\cos\delta_{13}\equiv c_\delta=\frac{BD-AC}{\sqrt{(A-B+C-D)^2+(BD-AC)^2}}. \qquad (3.9)$$

It is permissible to use only the positive branches of square roots since the angles θ_{12}, θ_{23}, θ_{13} in Eq. (1.1) are chosen to lie in the first quadrant, and cotangent and sine are one-to-one on $(0,\pi/2)$. Note that only $\cos\delta_{13}$ can be determined, reflecting the discrete ambiguity between δ_{13} and $(2\pi-\delta_{13})$. One may be troubled by the fact that parameters specific to a particular representation of the CKM matrix are written here in terms of its invariants, but this indicates only that the interpretation of a given parameter as a phase or rotation about particular axes, not its value, is convention dependent. The situation is analogous to Lorentz invariance: The norm of the momentum four-vector of a free particle is most easily computed in the rest frame, where the zero component of the vector has the interpretation of rest mass m, but m is also numerically the norm of the vector in any frame.

One can also see explicitly from the transformations (3.9) how the CP-vanishing cases of (3.3) are recovered when the unitarity triangles collapse. If the angles α, β, ϵ, or ϵ' approach 0 or π, then the cotangents either become singular, as for A when $\alpha\to0,\pi$, or two of them become degenerate, such as A and B when $\beta\to0,\pi$. In such cases, the expressions in (3.9) are seen to assume the values of Eq. (3.3). For completeness, we present the Jarlskog parameter in these variables, which is seen to satisfy the same degeneracy constraints:

$$|J|=\frac{(A-B)(A-D)(C-B)(C-D)[(A-B)(1+CD)+(C-D)(1+AB)]}{[(A-B+C-D)^2+(BD-AC)^2]^2}. \qquad (3.10)$$

Finally, we consider the transformations in the phenomenologically interesting case of $\epsilon'\ll\epsilon\ll1$. It is straightforward to show from Eqs. (3.9) that

$$\theta_{12}=\sqrt{\cot\beta-\cot(\alpha+\beta)}\,\epsilon^{1/2}+O(\epsilon^{3/2}),$$
$$\theta_{23}=\sqrt{\cot\beta-\cot(\alpha+\beta)}\,\epsilon'^{\,1/2}+O(\epsilon'^{\,1/2}\epsilon),$$
$$\theta_{13}=\frac{(\epsilon'\epsilon)^{1/2}}{\sin(\alpha+\beta)}+O(\epsilon'^{\,1/2}\epsilon^{3/2}),$$
$$\delta_{13}=\pi-\mathrm{sgn}(J)(\alpha+\beta-\epsilon')+O(\epsilon'\epsilon). \qquad (3.11)$$

Note the particular vanishing behaviors of the angles as $\epsilon,\epsilon'\to0$. Also note that in this limit $\delta_{13}=\pi-\alpha-\beta=\gamma$, using the experimentally determined positive sign of J.

IV. OTHER PARAMETRIZATIONS

From the construction detailed above, it should be clear that an analogous program can be carried out for any parametrization using Euler angles and phases, leading to concise transformation expressions. The key point is that in such parametrizations elements more complicated than a real number times a phase, which complicate the calculation in Eq. (1.4), are relegated to a 2×2 minor submatrix. The unique element of the CKM matrix that does not share a row or column index with this 2×2 minor is thus distinguished; in the case of the standard form (1.1), this element is V_{ub}. The angles of the unitarity triangles for which the analogous expressions to Eq. (3.2) are simple are exactly those adjacent to a side containing the distinguished element of V, as is clear for the case we have considered, from Fig. 1. As another example, consider the original CKM parametrization of Kobayashi and Maskawa [1]:

$$V_{\rm KM}=\begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3+c_2s_3e^{i\delta} & c_1s_2s_3-c_2c_3e^{i\delta} \end{pmatrix}. \tag{4.1}$$

Here the distinguished element is V_{ud}, and the simplest expressions to use will be the cotangents of α, $(\alpha+\beta)$, ϵ', and $(\beta+\epsilon-\epsilon')$.

A similar treatment for the Wolfenstein parametrization [11], defined by

$$V_W=\begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{4.2}$$

is not immediately possible, since the matrix is only unitary to corrections of order λ^4. Only once an extension rendering it fully unitary, of which there are many possible choices, is agreed upon can the full conversion between A, λ, ρ, and η and the unitarity angles be carried out [12].

Finally, we show how any four independent moduli $|V_{\alpha i}|$ form an equivalent set to the four independent unitarity angles or standard CKM angles modulo the sign of J. Showing they are equivalent to the unitarity angles requires using the normalization of rows and columns of the CKM matrix to compute the other five moduli, and using the full set to compute the lengths of the sides (and hence the angles) of the unitarity triangles. Starting instead with the standard form (1.1), one computes the remaining five moduli as before and then extracts θ_{13}, θ_{23}, and θ_{12}, respectively, from $|V_{ub}|$, $|V_{cb}|/|V_{tb}|$, and $|V_{us}|/|V_{ud}|$. δ_{13} may be extracted from any of the four moduli in the lower-left 2×2 submatrix, but because a number and its complex conjugate have the same norm, this process is insensitive to the transformation $\delta_{13}\to(2\pi-\delta_{13})$, or equivalently the sign of J. Similar remarks apply to any particular representation.

If it turns out that there are more than three generations of quarks, the angles of the unitarity polygons are no longer sufficient to determine the entire structure of the CKM matrix. In four generations, for example, a square and a rectangle have the same angles but are not similar figures. Nor are moduli alone enough, as one sees from comparing rhombi and squares. It is clear that one then requires both unitarity angles and moduli. Nevertheless, generalizations of Euler forms like Eq. (1.1) to more than three generations must continue to have ''distinguished'' elements for which expressions relating convention-dependent CKM angles to convention-independent angles of unitarity polygons remains usefully succinct.

ACKNOWLEDGMENT

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[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 552 (1973).

[2] C. Jarlskog, in *CP Violation*, edited by C. Jarlskog (World Scientific, Singapore, 1989), p. 3.

[3] Particle Data Group, R. M. Barnett *et al.*, Phys. Rev. D **54**, 1 (1996).

[4] L.-L. Chau and W.-Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984).

[5] Nevertheless, it is possible to reconstruct all four invariant quantities of the CKM matrix (but not the sign of the Jarlskog parameter) using only four independent moduli $|V_{\alpha i}|$: see C. Jarlskog and R. Stora, Phys. Lett. B **208**, 268 (1988). Another interesting set of four invariants consists of the Jarlskog parameter, two off-diagonal asymmetries, and the modulus $|V_{cs}|$, which form a hierarchy of increasing magnitude: see Z.-Z. Xing, Nuovo Cimento A **109**, 115 (1996); and in *Proceedings of the International Europhysics Conference on High Energy Physics*, Brussels, Belgium, 1995, edited by J. Lemonne, C. Vander Velde, and F. Verbeure (World Scientific, Singapore, 1996), p. 181.

[6] The idea of picturing unitarity conditions in this way appears to have originated independently with J. D. Bjorken and C. Jarlskog (see first reference in [5]).

[7] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).

[8] A. Ali, lectures given at the 20th International Nathiagali Summer College on Physics and Contemporary Needs, Bhurban, Pakistan, 1995 (unpublished), DESY Report No. DESY 96-106, hep-ph/9606324 (unpublished).

[9] R. Aleksan, B. Kayser, and D. London, Phys. Rev. Lett. **73**, 18 (1994).

[10] In fact, the sign of J appears to be already experimentally determined as $+1$. In the Wolfenstein parametrization $J\approx A^2\lambda^6\eta$, and η as determined through $K\bar{K}$ mixing is positive, assuming the bag parameter $\hat{B}_K$ from short-distance calculations is also positive (see, for example, Ref. [8]). η could have been negative, for example, if the neutral K eigenstate mass difference Δm had been negative.

[11] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).

[12] One such extension is that of M. Kobayashi, Prog. Theor. Phys. **92**, 287 (1994); **92**, 289 (1994).