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# Some physical consequences of abrupt changes in the multipole moments of a gravitating body

C. Barrabès*
*Department of Physics, UPRES A 6083 du CNRS, Université de Tours, 37200 France
and Département d'Astrophysique Relativiste et Cosmologie, UPR 176 du CNRS, Observatoire de Paris, 92190 Meudon, France*

G. F. Bressange†
*Department of Physics, UPRES A 6083 du CNRS, Université de Tours, 37200 France*

P. A. Hogan‡
*Department of Mathematical Physics, University College Dublin, Belfield, Dublin 4, Ireland*

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The Barrabès-Israel theory of lightlike shells in general relativity is used to show explicitly that in general a lightlike shell is accompanied by an impulsive gravitational wave. The gravitational wave is identified by its Petrov-type $N$ contribution to a Dirac $\delta$-function term in the Weyl conformal curvature tensor (with the $\delta$ function singular on the null hypersurface history of the wave and shell). An example is described in which an asymptotically flat static vacuum Weyl space-time experiences a sudden change across a null hypersurface in the multipole moments of its isolated axially symmetric source. A lightlike shell and an impulsive gravitational wave are identified, both having the null hypersurface as history. The stress-energy in the shell is dominated (at large distance from the source) by the jump in the monopole moment (the mass) of the source with the jump in the dipole moment mainly responsible for the stress being anisotropic. The gravitational wave owes its existence principally to the jump in the quadrupole moment of the source confirming what would be expected. [S0556-2821(97)04106-4]

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## I. INTRODUCTION

Very few exact solutions of Einstein's vacuum field equations exist describing gravitational waves from an isolated source having wave fronts homeomorphic to a two-sphere (we will refer to such waves loosely as ''spherical waves''). The principal example is the Robinson-Trautman [1] family of solutions. These are very special, however, because if the wave fronts are sufficiently smooth (free of conical singularities) and the field (Riemann tensor) contains no ''wire'' or ''directional'' singularities then the solutions approach a Schwarzschild limit exponentially in time [2,3]. A limiting case is Penrose's [4] spherical impulsive gravitational wave propagating through flat space-time. Another interesting class of solutions in the present context has been found by Alekseev and Griffiths [5]. When these solutions have the property that the curvature tensor of the space-time containing the history of the wave involves a Dirac $\delta$ function which is singular on the null hypersurface history of the wave, the coefficient of the $\delta$ function is singular along a generator of this null hypersurface (a wire singularity). The object of the present paper is to present an example of a wire singularity-free spherical impulsive gravitational wave propagating through a vacuum. To construct this we will use the Barrabès-Israel (BI) [6] theory of lightlike shells in general relativity. This is an extension to the null case of the usual extrinsic curvature technique [7] for studying non-null shells of matter. The Ricci tensor of the space-time in general exhibits a $\delta$ function behavior singular on the null hypersurface history of the shell (the coefficient of the $\delta$ function is constructed from the surface stress-energy tensor of the shell and this tensor is calculated using the BI approach).

The present work relies upon a property of lightlike singular hypersurfaces already announced in [6], although not fully developed there as that paper was only concerned with shells, and which we now recall. In the present context a singular hypersurface in space-time is described covariantly by the existence of finite jumps across the hypersurface in its extrinsic curvature. In the null case considered here these become jumps $\gamma_{ab}=\gamma_{ba}$, $a,b=1,2,3$ in the ''transverse extrinsic curvature'' of the hypersurface, defined in Eqs. (2.6) and (2.7) below. As a result the Ricci tensor and the Weyl tensor of the space-time contain Dirac $\delta$-function terms singular on the hypersurface and having coefficients calculated from $\gamma_{ab}$. If the singular hypersurface is lightlike the results obtained in [6] show that only four of the six components of $\gamma_{ab}$ contribute to the coefficient of the $\delta$ function in the Ricci tensor and are therefore related to the surface stress-energy tensor of the lightlike shell (this does not hold for a timelike or spacelike hypersurface since in this case there is a one to one correspondence between the surface stress-energy tensor and $\gamma_{ab}$). It follows that the coefficient of the $\delta$ function in the Weyl tensor which is linear in the full set of the $\gamma_{ab}$ can be split into two parts, one calculated with the four components of $\gamma_{ab}$ used in the calculation of the surface stress-energy tensor (the matter part) and another part calculated with the remaining two components of $\gamma_{ab}$ not used in the calculation of the matter part. This latter part is shown to be of Petrov-type $N$. It represents an impulsive gravitational

*Electronic address: barrabes@celfi.phys.univ-tours.fr
†Electronic address: bressang@celfi.phys.univ-tours.fr
‡Electronic address: phogan@ollamh.ucd.ie

 

wave and will be referred to as the wave part. The matter part mentioned above is in general Petrov-type II. Such a decomposition is made explicit in Sec. II below. Examples of space-time geometries and singular lightlike hypersurfaces where only one of the two parts of the Weyl tensor is nonzero are already known in the literature. On the one hand, one may consider the case of spherical lightlike shells described in [6] where only a shell exists. On the other hand the singular null hyperplane and the singular null cone in Minkowskian space-time studied by Penrose [4] provide examples where only an impulsive gravitational wave exists. For more complicated geometries such as the one considered here, the shell and the wave generally coexist. In addition a cataclysmic astrophysical event such as a supernova is likely to produce a burst of neutrinos travelling outward with the speed of light (modelled by the lightlike shell) accompanied by a burst of outgoing gravitational waves (modelled by the impulsive wave).

Asymptotically flat solutions of the Weyl class of static axially symmetric vacuum gravitational fields are described by metric tensor components expressed as infinite series having coefficients involving the multipole moments of the isolated source [8]. To illustrate the splitting of the Weyl tensor into a matter part and a wave part (or the coexistence of a shell and a wave) we take a future directed null hypersurface which is asymptotically a future null cone in the asymptotically flat Weyl space-time and assume a finite abrupt change takes place in these multipole moments across the null hypersurface. The surface stress-energy tensor concentrated on the null hypersurface is calculated at large distance along the null hypersurface from the history of the source and it is shown that the energy density of the shell, measured by a radially moving observer, is dominated by the jump in the monopole moment (the mass) of the source followed by terms proportional to the jumps in the dipole and quadrupole moments of the source. Jumps in higher multipole moments will contribute even less significantly and are not calculated here. There is an anisotropic surface stress dominated by the jump in the dipole moment. The $\delta$ function in the Weyl tensor has a coefficient with matter part and wave part, described above, with the matter part dominated by the jump in the dipole moment and the wave part by the jump in the quadrupole moment. In the case of the matter part this is due to the anisotropic stress in the shell while for the wave part it is a manifestation of the well-known property of gravitational waves from isolated sources that the lowest radiated multipole is the quadrupole. The Newman-Penrose components of the matter and wave parts of the coefficient of the $\delta$ function in the Weyl tensor are calculated on a null tetrad asymptotically parallel transported along the future-directed null geodesic generators of the null hypersurface. These components are nonsingular on any such generators and *so the Weyl tensor is free of wire singularities*. Since the null hypersurface chosen is asymptotically a future null cone we can say that the impulsive gravitational wave accompanying the lightlike shell is asymptotically spherical.

The paper is organized as follows: In Sec. II the BI technique is described as it applies to lightlike shells (which includes impulsive waves). The method given in BI is unified treatment applicable to hypersurfaces which are non-null as well as null and to non-null hypersurfaces which may become null in a limiting case. The identification of the matter and wave part of the coefficient of the $\delta$ function in the Weyl tensor is then presented. In Sec. III the application to the asymptotically flat Weyl solutions is initiated by introducing these solutions, describing the transformation of the line element to a form based on a family of null hypersurfaces $u=$const (say) (this is just the Bondi [9] form of the Weyl solutions; for completeness some of the original calculations by Bondi *et al.* [9] are summarized in the Appendix with comments relevant to the present application) and then describing how two such solutions with different multipole moments can be matched across one of the null hypersurfaces, $u=0$ (say). The physical properties of the boundary $u=0$ are worked out in Sec. IV. Its interpretation as the history of both a lightlike shell and an impulsive gravitational wave is verified asymptotically using the BI technique and the results derived in Sec. II. The paper ends with a discussion in Sec. V commenting in particular on the absence of conical singularities on the wave front or shell and on the asymptotic behavior (''peeling'' behavior) of the amplitude of the $\delta$ function in both the matter and wave parts of the Weyl tensor.

## II. BARRABÈS-ISRAEL TECHNIQUE: THE NULL CASE

Consider space-time $M$ to be subdivided into two halves $M^+$ and $M^-$ each with boundary a null hypersurface $\Sigma$. Let $\{x^\mu_+\}$, with Greek indices taking values 1, 2, 3, 4, be a local coordinate system in $M^+$ in terms of which the metric tensor components are $g^+_{\alpha\beta}$ and let $\{x^\mu_-\}$ be a local coordinate system in $M^-$ in terms of which the metric tensor components are $g^-_{\alpha\beta}$. Let $\{\xi^a\}$, with Latin indices taking values 1, 2, 3, be local intrinsic coordinates on $\Sigma$ and the parametric equations of $\Sigma$ have the form $x^\mu_\pm = f^\mu_\pm(\xi^a)$ say. We thus have a basis of tangent vectors $e_a = \partial/\partial\xi^a$ to $\Sigma$ and we assume that $M^+$ and $M^-$ are reattached on $\Sigma$ in such a way that the induced metrics on $\Sigma$ from $M^+$ and $M^-$ match:

$$g_{ab} := g_{\alpha\beta} e^\alpha_a e^\beta_b|_+ = g_{\alpha\beta} e^\alpha_a e^\beta_b|_- , \tag{2.1}$$

where the colon followed by an equality sign denotes a definition, $e^\alpha_a|_\pm = \partial x^\alpha_\pm/\partial\xi^a$ are the components of the tangent vectors $e_a$ to $\Sigma$ evaluated on the $M^+$ side or $M^-$ side respectively. The symbol $|_\pm$ shall mean ''evaluated on the plus or minus side of $\Sigma$.'' The manifold resulting from this reattachment of $M^+$ and $M^-$ on $\Sigma$ will be denoted by $M^+ \cup M^-$. Let $n$ be normal to $\Sigma$ with components $n^\mu_\pm$ viewed on the plus or minus sides. Thus

$$n^\mu n_\mu|_\pm = 0 \quad \text{and} \quad n_\mu e^\mu_a|_\pm = 0. \tag{2.2}$$

Next choose a ''transversal'' $N$ on $\Sigma$ and require that its projection in the plus or minus sides of $\Sigma$ be the same (continuous) so that

$$N_\mu e^\mu_a|_+ = N_\mu e^\mu_a|_- \tag{2.3}$$

and, in addition, require

$$N_\mu N^\mu|_+ = N_\mu N^\mu|_- . \tag{2.4}$$

Finally define

$$\eta^{-1} := N_\mu n^\mu. \tag{2.5}$$

Now the ''transverse extrinsic curvature'' of $\Sigma$ (a generalization of the usual extrinsic curvature [7] to the null case) is defined by

$$\mathcal{K}^{\pm}_{ab} := -N_\mu e^\mu_{a;\nu} e^\nu_b|_\pm = \mathcal{K}^{\pm}_{ba}, \tag{2.6}$$

with the semicolon denoting covariant differentiation. The jump in these components across $\Sigma$ is denoted by

$$\tfrac{1}{2}\gamma_{ab} := [\mathcal{K}_{ab}] := \mathcal{K}^+_{ab} - \mathcal{K}^-_{ab}, \tag{2.7}$$

and these quantities are independent of the choice of transversal (see [6], Sec. II). We now extend $\gamma_{ab}$ to a four-tensor by padding out with zeros in a system of coordinates having the $e_a$'s as basis vectors (the only condition required on the extension $\gamma_{\mu\nu}$ being $\gamma_{\mu\nu} e^\mu_a e^\nu_b = \gamma_{ab}$).

A calculation now of the Einstein tensor of the space-time $M^+ \cup M^-$ results in general in an energy-momentum-stress tensor concentrated on $\Sigma$ of the form [6]

$$T^{\mu\nu} = S^{\mu\nu}\delta(u), \tag{2.8}$$

where the equation of $\Sigma$ is taken to be $u(x^\mu)=0$, the normal to $\Sigma$ has the form $n_\mu = u_{,\mu}$, with the comma denoting partial differentiation, $\delta(u)$ is the Dirac $\delta$ function and $S^{\mu\nu}$ is given by [6]

$$16\pi\eta^{-1}S^{\mu\nu} = 2\gamma^{(\mu}n^{\nu)} - \gamma n^\mu n^\nu - \gamma^\dagger g^{\mu\nu}. \tag{2.9}$$

Here round brackets denote symmetrization and

$$\gamma^\mu := \gamma^{\mu\nu}n_\nu, \quad \gamma^\dagger := \gamma^\mu n_\mu, \quad \gamma := g^{\mu\nu}\gamma_{\mu\nu}, \tag{2.10}$$

with the calculation of these quantities carried out on either side of $\Sigma$. We drop the plus or minus designation in such instances. The physical interpretation of these formulas is discussed in detail in BI. The surface stress-energy tensor $S^{\mu\nu}$ in Eq. (2.9) has the property that

$$S^{\mu\nu}n_\nu = 0, \tag{2.11}$$

and thus it can be expressed on the tangent basis $e_a$ as

$$S^{\mu\nu} = S^{ab}e^\mu_a e^\nu_b, \tag{2.12}$$

with (see [6])

$$16\pi\eta^{-1}S^{ab} = 2g_*^{c(a}l^{b)}(\gamma_{cd}l^d) - g_*^{ab}(\gamma_{cd}l^c l^d) - l^a l^b(g_*^{cd}\gamma_{cd}). \tag{2.13}$$

The three-vector $l^a$ is defined via the expansion

$$n^\mu = l^a e^\mu_a, \tag{2.14}$$

remembering that since $\Sigma$ is null the normal to $\Sigma$ is also tangent to $\Sigma$. From Eq. (2.14) and the orthogonality of $n^\mu$, $e^\mu_a$ it follows that $g_{ab}l^b=0$. We note that since $\Sigma$ is null the induced metric with components $g_{ab}$ is degenerate or singular. In Eq. (2.13) $g_*^{ab}$ is a type of generalized inverse of $g_{ab}$ defined by [6]

$$g_*^{ac}g_{bc} = \delta^a_b - \eta l^a N_b, \tag{2.15}$$

with $N_b := N_\mu e^\mu_b$ which is continuous across $\Sigma$ by Eq. (2.3). The expression (2.15) determines $g_*^{ac}$ uniquely up to a multiple of $l^a l^c$. It is clear from Eq. (2.13) that if $\gamma_{ab}l^b=0$ then the surface stress-energy tensor is ''isotropic'' with

$$16\pi\eta^{-1}S^{ab} = -(g_*^{cd}\gamma_{cd})l^a l^b. \tag{2.16}$$

This is equivalent, by Eqs. (2.12) and (2.14), to $S^{\mu\nu}$ being proportional to $n^\mu n^\nu$.

The Weyl conformal curvature tensor of the space-time $M^+ \cup M^-$ has in general a part concentrated on $\Sigma$ and given by [6]

$$C^{\kappa\lambda}{}_{\mu\nu} = \left\{ 2\eta n^{[\kappa}\gamma^{\lambda]}{}_{[\mu}n_{\nu]} - 16\pi\delta^{[\kappa}_{[\mu}S^{\lambda]}_{\nu]} + \frac{8\pi}{3}S^\alpha_\alpha\delta^{\kappa\lambda}_{\mu\nu} \right\}\delta(u). \tag{2.17}$$

Here the square brackets denote skew symmetrization and $\delta^{\kappa\lambda}_{\mu\nu}$ is the usual determinant of Kronecker $\delta$'s. It has been pointed out in [6] that there is a part of the first term in the coefficient of the $\delta$ function in Eq. (2.17) that is constructed from a part of $\gamma_{\mu\nu}$ which does not contribute to the surface stress-energy tensor. This part of Eq. (2.17) therefore is decoupled from the matter part. It describes an impulsive gravitational wave propagating with the shell and having $u=0$ as the history of its wave front. To display this decomposition of Eq. (2.17) it is perhaps simplest to look first at the intrinsic form (2.13) for the stress-energy tensor. It is clear from Eq. (2.13) that a part of $\gamma_{ab}$, which we will denote by $\hat{\gamma}_{ab}$, does not contribute to the stress-energy tensor. This $\hat{\gamma}_{ab}$ satisfies

$$\hat{\gamma}_{ab}l^b = 0, \quad g_*^{ab}\hat{\gamma}_{ab} = 0. \tag{2.18}$$

This means that $\hat{\gamma}_{ab}$ has two independent components (a fact which is related to the two degrees of freedom of polarization in general present in the impulsive gravitational wave determined below by $\hat{\gamma}_{ab}$). The decomposition of Eq. (2.17) into wave and matter parts is best described by giving the components of these parts on the oblique basis $\{e^\mu_a, N^\mu\}$. Using Eq. (2.17) and having off the part of $\gamma_{ab}$ described above we find we can write

$$C_{\kappa\lambda\mu\nu} = (W_{\kappa\lambda\mu\nu} + M_{\kappa\lambda\mu\nu})\delta(u). \tag{2.19}$$

The components $W_{\kappa\lambda\mu\nu}e^\kappa_a e^\lambda_b e^\mu_c e^\nu_d$ and $W_{\kappa\lambda\mu\nu}e^\kappa_a e^\lambda_b e^\mu_c N^\nu$ of $W_{\kappa\lambda\mu\nu}$ vanish identically while

$$W_{\kappa\lambda\mu\nu}e^\kappa_a N^\lambda e^\mu_c N^\nu = -\tfrac{1}{2}\eta^{-1}\hat{\gamma}_{ab}, \tag{2.20}$$

with

$$\hat{\gamma}_{ab} = \gamma_{ab} - \tfrac{1}{2}\, g_*^{cd}\gamma_{cd}g_{ab} - 2\eta l^d\gamma_{d(a}N_{b)} + \eta^2\gamma_{cd}l^c l^d N_a N_b. \tag{2.21}$$

One easily checks that this $\hat{\gamma}_{ab}$ satisfies Eq. (2.18). Multiplying the components of $W_{\kappa\lambda\mu\nu}$ on the oblique basis by $l^a$ and using Eq. (2.14) and the first of Eq. (2.18) gives

$$W_{\kappa\lambda\mu\nu}n^\kappa = 0, \tag{2.22}$$

so that this part of Eq. (2.19) is type-$N$ in the Petrov classification with $n^\mu$ as fourfold degenerate principal null direction. Thus this part of the Weyl tensor in Eq. (2.19) describes an impulsive gravitational wave with propagation direction $n^\mu$ in space-time and with $u=0$ as the history of its wave front. Now the components of the matter part of Eq. (2.19) on the oblique basis are

$$M_{\kappa\lambda\mu\nu}e^\kappa_a e^\lambda_b e^\mu_c e^\nu_d = 8\pi(g_{a[d}S_{c]b} - g_{b[d}S_{c]a}) + \frac{16\pi}{3} S^\alpha_\alpha g_{a[c}g_{d]b}, \tag{2.23a}$$

$$M_{\kappa\lambda\mu\nu}e^\kappa_a e^\lambda_b e^\mu_c N^\nu = -8\pi g_{c[a}e^\alpha_{b]}S_{\alpha\beta}N^\beta + \frac{16\pi}{3} S^\alpha_\alpha G_{c[a}N_{b]}, \tag{2.23b}$$

$$M_{\kappa\lambda\mu\nu}e^\kappa_a N^\lambda e^\mu_c N^\nu = -8\pi S_{\alpha\beta}e^\alpha_{(a}N_{c)}N^\beta + \frac{4\pi}{3} S^\alpha_\alpha N_a N_c. \tag{2.23c}$$

Here $S_{ab} := S_{\mu\nu}e^{\ \mu}_a e^{\ \nu}_b$ which, of course, is not simply related to $S^{ab}$ in Eq. (2.13) by lowering indices because $g_{ab}$ is degenerate. We note that if $S^{\mu\nu}$ is isotropic in the sense described in Eq. (2.16), or in the sentence following Eq. (2.16), then $S_{ab}=0$ and in addition we see from Eq. (2.23) that $M_{\kappa\lambda\mu\nu}=0$ in this case. Finally we note that from the results of Sec. III of [6] one can deduce that the matter part $M_{\kappa\lambda\mu\nu}$ is in general Petrov-type-II and may specialise to type-III.

## III. ASYMPTOTICALLY FLAT WEYL SPACE TIMES

To illustrate the theory outlined in Sec. II we consider the asymptotically flat Weyl static axially symmetric solutions of Einstein's vacuum field equations [8]. The line element of these space-times has the form

$$ds^2 = -R^2 e^{-2U}(e^{2k}d\Theta^2 + \sin^2\Theta\, d\phi^2) - e^{2k-2U}dR^2 + e^{2U}dt^2, \tag{3.1}$$

where $U, k$ are functions of the coordinates $\Theta$, $R$ given by the infinite series

$$U = \sum_{n=0}^{\infty} \frac{a_n}{R^{n+1}} P_n, \tag{3.2a}$$

$$k = -\sum_{l,m=0}^{\infty} \frac{a_l a_m (l+1)(m+1)}{l+m+2} \left( \frac{P_l P_m - P_{l+1}P_{m+1}}{R^{l+m+2}} \right), \tag{3.2b}$$

where $a_n$ ($n=0,1,2,\ldots$) are constants and $P_n = P_n(\cos\Theta)$ is the Legendre polynomial of degree $n$ in the variable $\cos\Theta$. The first few terms in the series (3.2a) may be written

$$U = -\frac{m}{R} - \frac{D\cos\Theta}{R^2} - (Q + \tfrac{1}{3}m^3)\frac{P_2(\cos\Theta)}{R^3} + \cdots, \tag{3.3}$$

where, following Bondi *et al.* [9], we have written the constants $a_0, a_1, a_2$, in a form so that we can identify $m$ as the mass of the isolated source, $D$ as the dipole moment of the source and $Q$ as its quadrupole moment. The moments $D$ and $Q$ appear in Eq. (3.3) in such a way that if $D=Q=0$ then Eq. (3.3) represents the leading terms in the $1/R$ expansion of the Schwarzschild expression [8] for $U$. Corresponding to Eq. (3.3) we have, for Eq. (3.2b),

$$k = -\frac{m^2}{2R^2}\sin^2\Theta - \frac{2mD}{R^3}\cos\Theta\,\sin^2\Theta + \cdots. \tag{3.4}$$

The line element (3.1) with $U$ and $k$ in the forms (3.3) and (3.4) can be transformed to the Bondi form (the procedure for doing this, due to Bondi *et al.* [9], is outlined in the Appendix)

$$ds^2 = -r^2\{f^{-1}d\theta^2 + f\sin^2\theta\, d\phi^2\} + 2g\,du\,dr + 2h\,du\,d\theta + c\,du^2, \tag{3.5}$$

with

$$f = 1 - \frac{Q}{r^3}\sin^2\theta + O(r^{-4}), \tag{3.6a}$$

$$g = 1 + O(r^{-4}), \tag{3.6b}$$

$$h = \frac{2D}{r}\sin\theta + \frac{3Q}{r^2}\sin\theta\cos\theta + O(r^{-3}), \tag{3.6c}$$

$$c = 1 - \frac{2m}{r} - \frac{2D}{r^2}\cos\theta - \frac{Q}{r^3}(3\cos^2\theta - 1) + O(r^{-4}). \tag{3.6d}$$

For our present purpose it is useful to have solutions (3.1) expressed in a coordinate system based on a family of null hypersurfaces. In the form (3.5) the hypersurfaces $u=$constant are *exactly* null (i.e., for all $r$ and not just for large $r$; this is clearly pointed out in the Appendix). Neglecting $O(r^{-4})$-terms $u=$constant are generated by the geodesic integral curves of the future-pointing null vector field $\partial/\partial r$ and $r$ is an affine parameter along them. These curves have expansion $\rho$ and shear $\sigma$ given by

$$\rho = \frac{1}{r} + O(r^{-5}), \tag{3.7a}$$

$$\sigma = \frac{3Q}{2r^4}\sin^2\theta + O(r^{-5}), \tag{3.7b}$$

demonstrating that for large values of $r$ [specifically, neglecting $O(r^{-4})$ terms] the null hypersurfaces $u=$const in the space-time with line element (3.5) are future null cones.

To illustrate the theory described in Sec. II we subdivide the space-time $M$ with line element (3.5) into two halves $M^-$ and $M^+$ having $u=0$ (say) as common boundary. To the past of $u=0$, corresponding to $u<0$, the space-time $M^-$ is given by Eqs. (3.5) and (3.6) with parameters $\{m_-, D_-, Q_-, \ldots\}$ and coordinates $x^\mu_- = (\theta_-, \phi_-, r_-, u)$, while to the future of $u=0$, corresponding to $u>0$, the space-time $M^+$ is given by Eqs. (3.5) and (3.6) with parameters $\{m_+, D_+, Q_+, \ldots\}$ and coordinates $x^\mu_+ = (\theta_+, \phi_+, r_+, u)$. We have taken $u_+ = u_- = u$ here for convenience. To save on subscripts we shall henceforth drop the minus subscript on $x^\mu_-$ and on the parameters $\{m_-, D_-, Q_-, \ldots\}$ and also use $\theta$, $\phi$, $r$ as intrinsic coordinates on $u=0$ [i.e., in the notation of Sec. II we are putting $\xi^a=(\theta,\phi,r)$]. The line element on $u=0$ induced from $M^+$ is

$$dl_+^2=-r_+^2\{f_+^{-1}d\theta_+^2+f_+\ \sin^2\theta_+ d\phi_+^2\}, \tag{3.8}$$

with

$$f_+=1-\frac{Q_+}{r_+^3}\sin^2\theta_+ + O(r_+^{-4}), \tag{3.9}$$

while the induced line element on $u=0$ from $M^-$ is

$$dl^2=-r^2\{f^{-1}d\theta^2+f\ \sin^2\theta d\phi^2\}, \tag{3.10}$$

with

$$f=1-\frac{Q}{r^3}\sin^2\theta+O(r^{-4}). \tag{3.11}$$

We now reattach $M^-$ and $M^+$ on $u=0$ requiring Eqs. (3.8) and (3.10) to be the same line element [see Eq. (2.1)] and this requirement gives the ''matching conditions''

$$\theta_+=\theta+\frac{[Q]}{r^3}\sin\theta\ \cos\theta+O(r^{-4}), \tag{3.12a}$$

$$\phi_+=\phi, \tag{3.12b}$$

$$r_+=r+\frac{[Q]}{2r^2}(1-3\cos^2\theta)+O(r^{-3}), \tag{3.12c}$$

with $[Q]:=Q_+-Q_-$. From the perspective of the space-time $M^-\cup M^+$ we have an asymptotically flat Weyl solution $M^-$ undergoing an abrupt finite jump in its multipole moments across a null hypersurface $u=0$ resulting in the Weyl solution $M^+$. We now apply the BI theory, with special reference to the results of Sec. II, to study the physical properties of the null hypersurface $u=0$.

## IV. LIGHTLIKE SHELL AND GRAVITATIONAL WAVE

To apply the BI theory as outlined in Sec. II we first calculate the tangent basis vectors $e_a=\partial/\partial\xi^a$ on the plus and minus sides of $\Sigma{:}u=0$. With $x_+^\mu$ given in terms of $x_-^\mu=x^\mu$ by Eq. (3.12) and with $\xi^a=(\theta,\phi,r)$ we find that

$$e_a^\mu|_-=\delta_a^\mu, \tag{4.1}$$

while

$$e_1^\mu|_+=\left[1+\frac{[Q]}{r^3}\cos 2\theta+O(r^{-4}),0,\ \frac{3[Q]}{2r^2}\sin 2\theta +O(r^{-3}),0\right], \tag{4.2a}$$

$$e_2^\mu|_+=(0,1,0,0), \tag{4.2b}$$

$$e_3^\mu|_+=\left[\frac{-3[Q]}{2r^4}\sin 2\theta+O(r^{-5}),0,1-\frac{[Q]}{r^3}(1-3\cos^2\theta) +O(r^{-4}),0\right]. \tag{4.2c}$$

The normal to $u=0$ is

$$n_\mu dx^\mu|_\pm=du. \tag{4.3}$$

As transversal on the minus side we can take

$${}^-N_\mu=\left[0,0,1,\tfrac{1}{2}-\frac{m}{r}-\frac{D}{r^2}\cos\theta-\frac{Q}{2r^3}(3\cos^2\theta-1) +O(r^{-4})\right], \tag{4.4}$$

and thus when viewed on the plus side we find, after insisting on Eqs. (2.3) and (2.4) being satisfied, that the components of the transversal are ${}^+N_\mu$ with

$${}^+N_1=-\frac{3[Q]}{2r^2}\sin 2\theta+O(r^{-3}), \tag{4.5a}$$

$${}^+N_2=0, \tag{4.5b}$$

$${}^+N_3=1+\frac{[Q]}{r^3}(1-3\cos^2\theta)+O(r^{-4}), \tag{4.5c}$$

$${}^+N_4=\frac{1}{2}-\frac{m_+}{r}-\frac{D_+}{r^2}\cos\theta+\frac{(2Q_+-Q)}{2r^3}(1-3\cos^2\theta) +O(r^{-4}). \tag{4.5d}$$

Now $\eta$ given by Eq. (2.5) has the form

$$\eta=1+O(r^{-3}). \tag{4.6}$$

The transverse extrinsic curvature on the plus and minus sides of $u=0$ is calculated now from Eq. (2.6). On the minus side we find

$$\mathcal{K}_{11}^-=\frac{r}{2}-m-\frac{3D}{r}\cos\theta+\frac{Q}{4r^2}(13-29\cos^2\theta)+O(r^{-3}), \tag{4.7a}$$

$$\mathcal{K}_{12}^-=0, \tag{4.7b}$$

$$\mathcal{K}_{22}^-=\frac{r}{2}\sin^2\theta-m\ \sin^2\theta-\frac{3D}{r}\cos\theta\ \sin^2\theta +\frac{Q}{4r^2}(3-19\cos^2\theta)\sin^2\theta+O(r^{-3}), \tag{4.7c}$$

$$\mathcal{K}_{13}^-=\frac{3D}{r^2}\sin\theta+\frac{6Q}{r^3}\sin\theta\ \cos\theta+O(r^{-4}), \tag{4.7d}$$

$$\mathcal{K}_{23}^-=0, \tag{4.7e}$$

$$\mathcal{K}_{33}^-=O(r^{-5}). \tag{4.7f}$$

On the plus side we have

$$\mathcal{K}^{+}_{11}=\frac{r}{2}-m_{+}-\frac{3D_{+}}{r}\cos\theta+\frac{Q_{+}}{4r^{2}}(13-29\cos^{2}\theta)+\frac{[Q]}{4r^{2}}(11-25\cos^{2}\theta)+O(r^{-3}), \tag{4.8a}$$

$$\mathcal{K}^{+}_{12}=0, \tag{4.8b}$$

$$\mathcal{K}^{+}_{22}=\frac{r}{2}\sin^{2}\theta-m_{+}\sin^{2}\theta-\frac{3D_{+}}{r}\cos\theta\sin^{2}\theta+\frac{Q_{+}}{4r^{2}}(3-19\cos^{2}\theta)\sin^{2}\theta+\frac{[Q]}{4r^{2}}(3+7\cos^{2}\theta)\sin^{2}\theta+O(r^{-3}) \tag{4.8c}$$

$$\mathcal{K}^{+}_{13}=\frac{3D_{+}}{r^{2}}\sin\theta+O(r^{-3}), \tag{4.8d}$$

$$\mathcal{K}^{+}_{23}=0, \tag{4.8e}$$

$$\mathcal{K}^{+}_{33}=O(r^{-5}). \tag{4.8f}$$

From Eqs. (4.7) and (4.8) the jump $\gamma_{ab}$ in the transverse extrinsic curvature defined by Eq. (2.7) has the following components:

$$\gamma_{11}=-2[m]-\frac{6[D]}{r}\cos\theta+\frac{[Q]}{r^{2}}(12-27\cos^{2}\theta)+O(r^{-3}), \tag{4.9a}$$

$$\gamma_{22}=-2[m]\sin^{2}\theta-\frac{6[D]}{r}\cos\theta\sin^{2}\theta+\frac{3[Q]}{r^{2}}(1-2\cos^{2}\theta)\sin^{2}\theta+O(r^{-3}), \tag{4.9b}$$

$$\gamma_{12}=0,\quad \gamma_{23}=0, \tag{4.9c}$$

$$\gamma_{13}=\frac{3[D]}{r^{2}}\sin\theta+O(r^{-3}), \tag{4.9d}$$

$$\gamma_{33}=(r^{-5}), \tag{4.9e}$$

where, in keeping with earlier notation, we have put $[m]:=m_{+}-m$ and $[D]:=D_{+}-D$. Finally $\hat{\gamma}_{ab}$ in Eq. (2.21) turns out to have all but two components vanishing identically. These two components are

$$\hat{\gamma}_{11}=\tfrac{1}{2}(\gamma_{11}-\csc^{2}\theta\,\gamma_{22})+O(r^{-3}), \tag{4.10a}$$

$$\hat{\gamma}_{22}=-\hat{\gamma}_{11}\sin^{2}\theta+O(r^{-3}). \tag{4.10b}$$

Now the leading terms in the surface stress-energy tensor $S^{\mu\nu}$ are calculated from Eqs. (2.9) or (2.13). In the system of coordinates of the $M^{-}$ side, we find $S^{\mu 4}=S^{12}=S^{23}=0$ while

$$S^{11}=O(r^{-7}),\quad S^{22}=O(r^{-7}), \tag{4.11a}$$

$$16\pi S^{13}=-\frac{3[D]}{r^{4}}\sin\theta+O(r^{-5}), \tag{4.11b}$$

$$16\pi S^{33}=-\frac{4[m]}{r^{2}}-\frac{12[D]}{r^{3}}\cos\theta+\frac{3[Q]}{r^{4}}(5-11\cos^{2}\theta)+O(r^{-5}). \tag{4.11c}$$

Thus the stress in the lightlike shell with history $u=0$ is anisotropic due primarily (for large $r$) to the jump in the dipole moment $[D]$ [on account of (4.11b)]. The surface energy density of the shell measured by a radially moving observer (see [6]) is, by (4.11c), a positive multiple of

$$\sigma:=-\frac{1}{4\pi r^{2}}\left\{[m]+\frac{3[D]}{r}\cos\theta-\frac{3[Q]}{4r^{2}}(5-11\cos^{2}\theta)+O(r^{-3})\right\}. \tag{4.12}$$

This is dominated by the jump in the monopole moment (the mass of the source). It would be natural to assume that $[m]<0$ so that the source suffers a mass loss. In this case also the energy density (4.12) is positive throughout the shell. It is clear from Eqs. (4.11a) and (4.11b) that the stress in the shell is smaller in magnitude than the energy density.

It is convenient to define, in view of the line element (3.5), a null tetrad

$$M^{\mu}=\left(-\frac{1}{r\sqrt{2}}f^{1/2},-\frac{i}{r\sqrt{2}\sin\theta}f^{-1/2},0,0\right), \tag{4.13a}$$

$$\overline{M}^{\mu}=\left(-\frac{1}{r\sqrt{2}}f^{1/2},\frac{i}{r\sqrt{2}\sin\theta}f^{-1/2},0,0\right), \tag{4.13b}$$

$$n^{\mu}=[0,0,1+O(r^{-4}),0], \tag{4.13c}$$

$$N^{\mu}=\left(O(r^{-3}),0,-\frac{c}{2},1\right), \tag{4.13d}$$

with $f,c$ given Eqs. (3.6a) and (3.6d), respectively. This tetrad is asymptotically parallel transported along the integral curves of $\partial/\partial r$. On this tetrad the Newman-Penrose components of the matter part of the $\delta$ function (2.19) in the Weyl tensor (which we denote by ${}^{M}\Psi_{A}$ with $A=0,1,2,3,4$) are given, using Eq. (2.23), by

$${}^{M}\Psi_{0}=0,\quad {}^{M}\Psi_{1}=0,\quad {}^{M}\Psi_{2}=O(r^{-5}), \tag{4.14a}$$

$${}^{M}\Psi_{3}=\frac{3\sqrt{2}[D]}{4r^{3}}\sin\theta+O(r^{-4}), \tag{4.14b}$$

$${}^{M}\Psi_{4}=0. \tag{4.14c}$$

This is thus predominantly Petrov type-III (with $n^{\mu}$ as generate principal null direction) due to the anisotropy in the stress (4.11b) (which in turn is due to $[D]\neq 0$). The leading term in (4.14b) has clearly no singularity for $0\leqslant\theta\leqslant\pi$ (and thus no wire singularity). All Newman-Penrose components on the null tetrad (4.13) of the wave part of the $\delta$ function

(2.19) in the Weyl tensor (which we denote by ${}^{W}\Psi_A$) vanish with the exception, calculated from Eqs. (2.20) and (4.13), of

$$ {}^{W}\Psi_4 = -\frac{3[Q]}{4r^4}(3-7\cos^2\theta)+O(r^{-5}). \quad (4.15)$$

This impulsive gravitational wave clearly owes its existence primarily to the jump in the quadrupole moment of the source across $u=0$ and also is manifestly free of wire singularities.

## V. DISCUSSION

The degenerate metric induced on $u=0$, the history of an outgoing lightlike shell and of an impulsive gravitational wave (as has been verified asymptotically using the BI technique in Sec. IV), is given asymptotically by Eqs. (3.8) and (3.9). The line element (3.9) can be written, putting $\cos\theta = x$, as

$$dl^2 = -r^2\{G^{-1}dx^2+G\,d\phi^2\}, \quad (5.1)$$

with

$$G=(1-x^2)\left\{1-\frac{Q(1-x^2)}{r^3}+O(r^{-4})\right\}, \quad (5.2)$$

which, for each $r$, is a standard form for the line element on a two surface of revolution embedded in three-dimensional Euclidean space (see, for example [10]) with $-1\leqslant x\leqslant +1$ and $0\leqslant\phi<2\pi$. The Gaussian curvature is $K/r^2$ with

$$K=-\frac{1}{2}G''=1+\frac{4Q}{r^3}P_2(x)+O(r^{-4}). \quad (5.3)$$

Here the prime denotes differentiation with respect to $x$. Neglecting $O(r^{-4})$-terms, we see that $G'(+1)+G'(-1)=0$ and this together with $\phi$ ranging from 0 to $2\pi$ means (see [10]) that there are no conical singularities at the north or south poles of the two surface. In fact, by the Gauss-Bonnet theorem it is clear from the form of the line element (5.1) and $K$ in Eq. (5.3) that neglecting $O(r^{-4})$-terms the two surface is topologically spherical. Hence the lightlike shell and the impulsive gravitational wave can be considered asymptotically spherical in this sense.

Finally with ${}^{M}\Psi_3=O(r^{-3})$ and ${}^{W}\Psi_4=O(r^{-4})$ we see an unfamiliar Peeling behavior. This is due to (a) the conventional Peeling behavior occurring asymptotically in the field of an isolated source *with history confined to a timelike world tube* of compact cross section whereas the source of ${}^{M}\Psi_A$ and ${}^{W}\Psi_A$ is a light shell and a wave with the null hypersurface $u=0$ as history in space-time and (b) since in our case the radiation part of the field is in direct competition with the matter part it is no surprise that, in terms of $r^{-1}$, the amplitude of the matter part dominates that of the radiation part.

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## APPENDIX: TRANSFORMATION OF WEYL SOLUTION TO BONDI FORM

To make the present paper as self-contained as possible we briefly outline and discuss here the transformation of the Weyl solution given by Eq. (3.1) and (3.2) to the Bondi [9] form [Eqs. (3.5) and (3.6)]. This transformation is given in Appendix 4 of [9]. We wish to emphasize aspects of the procedure which are particularly relevant to the topic under consideration in the present paper.

Starting with Eq. (3.1) the coordinate transformation

$$t=u+F(R,\theta), \quad (A1)$$

$$\Theta=\Theta(R,\theta), \quad (A2)$$

is made with the functions $F,\Theta$ chosen so that no $dr\,d\theta$ or $dr^2$ terms appear in the line element. This will be achieved provided $F,\Theta$ satisfy the partial differential equations

$$e^{2U}F_RF_\theta = R^2e^{2k-2U}\Theta_R\Theta_\theta, \quad (A3)$$

$$e^{2U}F_R^2=e^{2k-2U}(1+R^2\Theta_R^2), \quad (A4)$$

with the subscripts on $F,\Theta$ indicating partial derivatives with respect to $R,\theta$ as appropriate. At this point the line element reads

$$\begin{aligned} ds^2 &= (-R^2e^{2k-2U}\Theta_\theta^2+e^{2U}F_\theta^2)d\theta^2-R^2e^{-2U}\sin^2\Theta\,d\phi^2 \\ &\quad +2e^{2U}F_R\,du\,dR+2e^{2U}F_\theta\,du\,d\theta+e^{2U}du^2. \end{aligned} \quad (A5)$$

We emphasize now that the hypersurfaces $u=$const are *null*. Using $F_{R\theta}=F_{\theta R}$ and $\Theta_{R\theta}=\Theta_{\theta R}$ in (A3) and (A4) we can, following [9], eliminate $F$ from (A3) and (A4) and arrive at

$$\Theta_\theta\left[\frac{R^4e^{2k-4U}\Theta_R^2}{1+R^2\Theta_R^2}\right]_R = R^2(e^{2k-4U})_\theta\Theta_R. \quad (A6)$$

This equation is now solved approximately for large values of $R$ by (see [9])

$$\Theta=\theta+\frac{p'}{4R^2}+\frac{1}{R^3}\left(\frac{1}{12}q'-\frac{1}{2}mp'\right)+\cdots, \quad (A7)$$

with

$$p=4D\cos\theta+m^2(7+\cos^2\theta), \quad (A8)$$

$$\begin{aligned} q &= 2Q(3\cos^2\theta-1)+4mD\cos\theta(3+\cos^2\theta) \\ &\quad +6m^3(1+\cos^2\theta), \end{aligned} \quad (A9)$$

and the primes in Eq. (A7) on $p,q$ indicating derivatives with respect to $\theta$. Now $F$ is obtained from (A3) and (A4) as

$$\begin{aligned} F &= R+2m\ln R+\frac{1}{R}(2m^2-\tfrac{1}{2}p) \\ &\quad +\frac{1}{R^2}(-2m^3+\tfrac{1}{2}mp-\tfrac{1}{4}q)+\cdots. \end{aligned} \quad (A10)$$

Finally we make the transformation $R=R(r,\theta)$ given by

$$R^2e^{-2U}\sin^2\Theta(R^2e^{2k-2U}\Theta_\theta^2-e^{2U}F_\theta^2)=r^4\sin^2\theta. \tag{A11}$$

This leads, for large values of $r$, to

$$R=r-m-\frac{m^2}{2r^2}\sin^2\theta-\frac{m}{2r^2}(2D\cos\theta+m^2)\sin^2\theta+\cdots. \tag{A12}$$

Putting this in $F$ given by Eq. (A10) and $\Theta$ given by Eq. (A7) we construct the transformation (A1), (A2) for large $r$ leading from Eqs. (3.1), (3.2) and (3.5) and (3.6). We emphasize that although the differential Eqs. (A3) and (A4) for $F$, $\Theta$ and the algebraic equation (11) for $R$ have been solved only for large values of $r$, the null hypersurfaces $u$=const are exactly null (for all values of $r$).


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