Gauge-invariant effective stress-energy tensors for gravitational waves

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It is shown that if a generalized definition of gauge invariance is used, gauge-invariant effective stress-energy tensors for gravitational waves and other gravitational perturbations can be defined in a much larger variety of circumstances than has previously been possible. In particular it is no longer necessary to average the stress-energy tensor over a region of spacetime which is larger in scale than the wavelengths of the waves and it is no longer necessary to restrict attention to high frequency gravitational waves. [S0556-2821(97)00906-5]

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It has long been known that gravitational waves can have an effective stress energy associated with them which alters the background spacetime on which they propagate. One important example is the two-body problem in general relativity where the emission of gravitational radiation causes the orbit to decay. This effect has been observed for the binary pulsar [1]. A second example is the gravitational geon solution found by Brill and Hartle (BH) [2]. The geon consists of high frequency gravitational waves confined to a thin spherical shell by the background geometry which they create. Outside the shell the geometry is the same as that outside of a static star.

To quantify the back reaction effects of gravitational waves it is necessary to define an effective stress-energy tensor for the waves. This was first done by BH and later by Isaacson [3] for high frequency gravitational waves in a vacuum. Burnett [4] also defined an effective stress-energy tensor for high frequency waves both in a vacuum and in spacetimes containing classical matter. Efroimsky [5] extended Isaacson's definition to include lower frequency waves and spacetimes containing classical matter. In each case either some sort of averaging procedure or a procedure that gives similar results to an averaging procedure was used.

An important property of the wave equation and stress-energy tensor for gravitational waves is gauge invariance. The usual gauge transformations used for gravitational waves are related to infinitesimal coordinate transformations and gauge invariance implies invariance under these transformations. Isaacson showed that, in general, the wave equation for gravitational waves is approximately gauge invariant only for high frequency waves. He also showed that for high frequency waves the effective stress-energy tensor is gauge invariant to leading order only if it is averaged over a region of spacetime whose scale is large compared to the wavelengths of the waves. In other cases, such as low frequency gravitational waves or the gravitational geon where the averaging is over time, either the wave equation or the stress-energy tensor, or both are not gauge invariant.

In this paper it is shown that if a generalized gauge transformation which is related to arbitrary coordinate transformations is used, then it is possible to define gauge-invariant effective stress-energy tensors for gravitational waves and other gravitational perturbations in virtually all situations of interest. This includes the cases of both low and high frequency gravitational waves. The stress-energy tensor can be averaged in an arbitrary manner so long as the averaging does not affect the background geometry or it need not be averaged at all.

Two useful methods are given for defining effective stress-energy tensors. One works well for stress-energy tensors that are averaged in some way and the other works well when no averaging occurs. For the former method it is shown that there is much more freedom available than has usually been thought in defining gauge-invariant effective stress-energy tensors for gravitational waves. In the latter method there is much more freedom than usually thought in defining the wave equation for the gravitational waves. In both cases this extra freedom is due to the freedom available in separating the metric into background and perturbed parts.

In what follows the formalism used to describe gravitational waves and other gravitational perturbations in spacetimes with no classical matter is given first followed by the definition of the generalized gauge transformations. Then gauge invariance of the perturbed Einstein tensor is established followed by a description of the two methods of defining effective stress-energy tensors. Approximate gauge invariance within the context of an arbitrary perturbation expansion is next established for stress-energy tensors that have been averaged in some arbitrary way. A method of solving the resulting wave and back reaction equations is given. The case of stress-energy tensors which are not averaged is then considered. Finally, the generalization to spacetimes containing classical matter is discussed.

To begin, consider a separation of the metric into a background part $\gamma_{\mu\nu}$ and a perturbed part $h_{\mu\nu}$, such that

$$g_{\mu\nu}=\gamma_{\mu\nu}+h_{\mu\nu}. \tag{1}$$

The separation is arbitrary. Following BH, the Einstein tensor can similarly be divided into a part describing the curvature due to the background geometry and that due to the perturbation by writing

$$G_{\mu\nu}(g)=G_{\mu\nu}(\gamma)+\triangle G_{\mu\nu}(\gamma,h)=0. \tag{2}$$

Here and hereafter, indices of tensors are sometimes suppressed for notational simplicity. It is important to note that $\triangle G$ is defined by this equation.

From Eq. (2) it is seen that the quantity $\triangle G$ is conserved with respect to the background geometry γ. It can also be seen that $\triangle G$ is invariant under coordinate transformations which change the perturbed geometry but leave the background geometry alone. This property leads to the invariance of $\triangle G$ under the generalized gauge transformations which are defined below.

Consider an arbitrary coordinate transformation. It is always possible to write such a transformation in the form

$$\overline{x^\mu}=x^\mu+\xi^\mu. \tag{3}$$

If the functional form of the background geometry is not allowed to change under this coordinate transformation, that is if $\overline{g}=\gamma+h$, then h is given implicitly by the equation

$$\begin{aligned}\gamma_{\mu\nu}(x)+h_{\mu\nu}(x)=\gamma_{\mu\nu}(\overline{x})+\overline{h}_{\mu\nu}(\overline{x})+[\gamma_{\mu\alpha}(\overline{x})+\overline{h}_{\mu\alpha}(\overline{x})]\xi^\alpha{}_{,\nu}\\ +[\gamma_{\alpha\nu}(\overline{x})+\overline{h}_{\alpha\nu}(\overline{x})]\xi^\alpha{}_{,\mu}\\ +[\gamma_{\alpha\beta}(\overline{x})+\overline{h}_{\alpha\beta}(\overline{x})]\xi^\alpha{}_{,\mu}\xi^\beta{}_{,\nu}.\end{aligned} \tag{4}$$

Here, derivatives of ξ are with respect to x and not $\overline{x}$.

A generalized gauge transformation is defined as one in which the quantity $\overline{h}(x)$ is substituted for $h(x)$ into the expression of interest. If h, ξ and their derivatives are small enough, then to leading order this gauge transformation is equivalent to the usual one used for gravitational waves which is

$$\overline{h}_{\mu\nu}(x)=h_{\mu\nu}(x)-\gamma_{\mu\alpha}(x)\xi^\alpha{}_{,\nu}-\gamma_{\alpha\nu}(x)\xi^\alpha{}_{,\mu}-\gamma_{\mu\nu,\alpha}\xi^\alpha. \tag{5}$$

The quantity $\triangle G$ plays an important role in the definitions of gravitational wave stress-energy tensors which follow. To prove its invariance under generalized gauge transformations first note that, since the functional form of the background metric is not to be changed by the coordinate transformation (3),

$$\overline{G}(\overline{g(\overline{x})})=G(\gamma(\overline{x}))+\triangle\overline{G}(\gamma(\overline{x}),\overline{h}(\overline{x}))=0. \tag{6}$$

Having obtained $\triangle\overline{G}(\gamma(\overline{x}),\overline{h}(\overline{x}))$ via a coordinate transformation, it is next useful to consider it simply as a function of $\overline{x}$. If this is done and it is evaluated at $\overline{x}=x$, then combining Eq. (6) with Eq. (2) gives

$$\triangle\overline{G}(\gamma(x),\overline{h}(x))=\triangle G(\gamma(x),h(x)). \tag{7}$$

If Eq. (4) is solved for $\overline{h}(x)$ as a function of $h(x)$ and this solution is substituted for $h(x)$, then $\triangle G$ is gauge invariant if it retains its original functional form after these substitutions, that is if

$$\triangle G(\gamma(x),\overline{h}(x))=\triangle G(\gamma(x),h(x)). \tag{8}$$

To show that Eq. (8) is correct, first note that the Einstein tensor can be written in terms of a particular combination of the metric tensor and its derivatives which is the same in any coordinate system. This implies that $\triangle\overline{G}$ can be computed by direct substitution of $\overline{h}$ into $\triangle G$. Thus,

$$\triangle\overline{G}(\gamma(x),\overline{h}(x))=\triangle G(\gamma(x),\overline{h}(x)). \tag{9}$$

Combining Eqs. (7) and (9) gives Eq. (8). This proves that $\triangle G$ is invariant under gauge transformations of the form $h(x)\rightarrow\overline{h}(x)$.

The next question to be addressed is the solution of Eq. (2). It is not possible to solve Eq. (2) directly without specifying in some way the split between the background geometry and the perturbed geometry. One can, of course, either fix γ and solve for h or vice versa. However, to describe gravitational waves the most useful methods are the following.

(1) Define an effective stress-energy tensor for the perturbed geometry which is conserved with respect to the background geometry and invariant under generalized gauge transformations, but which is otherwise arbitrary. Then, one can write Eq. (2) as

$$\triangle G_{\mu\nu}(\gamma,h)=-8\pi T^G{}_{\mu\nu}(\gamma,h), \tag{10a}$$

$$G_{\mu\nu}(\gamma)=8\pi T^G{}_{\mu\nu}(\gamma,h). \tag{10b}$$

(2) Impose a gauge-invariant equation which specifies the perturbed geometry. For gravitational waves it is the wave equation. It has the general form

$$H_{\mu\nu}(\gamma,h)=0. \tag{11}$$

In this case Eq. (2) is the back reaction equation and the effective stress-energy tensor for the gravitational waves is

$$T^G{}_{\mu\nu}(\gamma,h)=-\frac{1}{8\pi}\triangle G_{\mu\nu}(\gamma,h). \tag{12}$$

BH used method (1) and made the following definition for their effective stress-energy tensor:

$$T^G{}_{\mu\nu}(\gamma,h)=-\frac{1}{8\pi}\langle\triangle G_{\mu\nu}(\gamma,h)\rangle. \tag{13}$$

Here, the angular brackets indicate a time average. This was useful for the problem they were considering which was the gravitational geon. Isaacson's definition is the same except that the average is over a region of spacetime which is large in scale compared to the wavelengths of the waves, but smaller in size than the scale on which the background geometry varies. The definitions of BH and Isaacson can be extended to any averaging procedure which does not affect $G(\gamma)$. It is clear from the above proof of the invariance of $\triangle G$ under generalized gauge transformations that the BH stress-energy tensor is gauge invariant.

Once a gauge is chosen $\langle\triangle G\rangle$ is still not specified. It can be set equal to any symmetric second rank tensor which does not depend on the variables which are averaged over and is conserved with respect to the background geometry. This is because Eq. (10a) ensures that $\triangle G$ will always have the correct average. One interesting choice that can be made is $\langle\triangle G\rangle=0$. In this case the background geometry is an exact solution to Einstein's equations and the gravitational waves do not alter this geometry. The choice implicitly made by BH and Isaacson is discussed below.

Method (2) can be used to define an effective stress-energy tensor without averaging. One gauge-invariant choice for the wave equation is $H = \triangle G$. This is equivalent to using method (1) and choosing $\langle \triangle G \rangle = 0$.

So far, the discussion has been formal with exact results. If an expansion of the form

$$\triangle G = \triangle_1 G + \triangle_2 G + \cdots \tag{14}$$

exists then it can immediately be seen from Eqs. (2) and (14) that to nth order it is the quantity $\triangle_1 G + \cdots + \triangle_n G$ that is conserved with respect to the background geometry. Throughout when discussing the order of the approximation, quantities whose value is of the same order as $\triangle_n G$ are considered to be nth order quantities.

Gauge invariance is more difficult. Under an arbitrary gauge transformation, h can change dramatically and in the new gauge the appropriate perturbation expansion for $\triangle G$ might be very different from that in the original gauge. This would make it impossible to usefully compare terms in the two expansions. Thus, when using perturbation expansions it is usually necessary to restrict gauge transformations to those which are small enough so that $\bar{h}$ is of the same order of magnitude as h [3]. Then, the perturbation expansions of $\triangle G$ in both gauges are similar and can be compared. Therefore, throughout this paper, whenever a perturbation expansion is used, it is assumed that gauge transformations are small enough so that $\bar{h}$ is of the same order of magnitude as h.

The proof of gauge invariance given above implies that $\triangle_1 G + \cdots + \triangle_n G$ is gauge invariant to nth order. This means that, in general, $\triangle_1 G$ by itself is only gauge invariant to first order and $\triangle_2 G$ by itself is not gauge invariant at all. There are exceptions. For the high frequency waves considered by Isaacson, $\triangle_1 G$ and $\langle \triangle_2 G \rangle$ are both gauge invariant to second order.

If, for some arbitrary type of averaging, definition (13) is used along with a perturbation expansion, then Eqs. (10a) and (10b) become

$$\triangle_1 G + \cdots + \triangle_n G = \langle \triangle_1 G + \cdots + \triangle_n G \rangle, \tag{15a}$$

$$G(\gamma) = -\langle \triangle_1 G + \cdots + \triangle_n G \rangle. \tag{15b}$$

As discussed above it is still necessary, in a particular gauge, to explicitly fix the right-hand sides of these equations. Once this is done their general form assures gauge invariance to nth order. BH and Isaacson implicitly fix the values of their stress-energy tensors by imposing the condition

$$\langle \triangle_1 G \rangle = 0. \tag{16}$$

This is a reasonable condition to impose because they are considering high frequency gravitational waves and $\triangle_1 G$ is linear in h in that case.

It is important to understand that the condition (16) (as opposed to the stress-energy tensor itself) is only gauge invariant to first order in general and even in Isaacson's case is only gauge invariant to second order. Thus, when a transformation is made from the original gauge where this condition is imposed to a new gauge, care must be taken to determine the correct form of $\langle \triangle_1 G \rangle$ in the new gauge. In general, it no longer vanishes. It is this fact which makes it possible for the combination $\langle \triangle_1 G + \cdots + \triangle_n G \rangle$ to be gauge invariant to nth order.

Isaacson developed a practical method of solving Eqs. (15a) and (15b) for the perturbation expansion that he used. It is easily extended to a general perturbation expansion. One first expands h in the following way:

$$h = h^{(1)} + h^{(2)} + \cdots. \tag{17}$$

Here, the terms on the right-hand side are defined such that $\triangle_1 G(\gamma, h^{(2)})$ is of the same order as $\triangle_2 G(\gamma, h^{(1)})$, and so forth. Then, to second order Eqs. (15a) and (15b) can be written

$$\triangle_1 G(\gamma, h^{(1)}) = \langle \triangle_1 G(\gamma, h^{(1)}) \rangle, \tag{18a}$$

$$\triangle_1 G(\gamma, h^{(2)}) + \triangle_2 G(\gamma, h^{(1)}) = \langle \triangle_1 G(\gamma, h^{(2)}) + \triangle_2 G(\gamma, h^{(1)}) \rangle, \tag{18b}$$

$$G(\gamma) = -\langle \triangle_1 G(\gamma, h^{(1)}) + \triangle_1 G(\gamma, h^{(2)}) + \triangle_2 G(\gamma, h^{(1)}) \rangle. \tag{18c}$$

The extension to higher orders is straightforward. This method, along with a condition which fixes the stress-energy tensor in a particular gauge, results in a consistent set of equations that can be solved.

As an example, consider the case of gravitational waves which have small amplitudes, frequencies, and momenta. For these waves the Einstein tensor can be expanded in powers of h and its derivatives with the result that $\triangle_1 G(\gamma, h)$ is first order in h, $\triangle_2 G(\gamma, h)$ is second order, and so forth [2,3]. The perturbed metric is expanded as in Eq. (17) and the equations to second order are given in Eqs. (18a)–(18c). Since $\triangle_1 G$ is linear in h, Eq. (18b) implies that $h^{(2)}$ is of order $(h^{(1)})^2$. The condition (16) can be imposed by requiring that $\langle \triangle_1 G(\gamma, h^{(n)}) \rangle = 0$ for all n.

Method (2) results in a gauge-invariant effective stress-energy tensor for gravitational waves when no averaging occurs. If a perturbation expansion of the form (14) is used, it is natural to define the approximate wave equation to be $\triangle_1 G = 0$. Unfortunately, this does not lead to a gauge-invariant stress-energy tensor. However, if the expansion (17) is used then the wave equation can be defined to be

$$\triangle_1 G(\gamma, h^{(1)}) = 0. \tag{19a}$$

The resulting back reaction equation to second order is

$$G(\gamma) = -\triangle_1 G(\gamma, h^{(2)}) - \triangle_2 G(\gamma, h^{(1)}). \tag{19b}$$

A proof similar to Isaacson's proof of the approximate gauge invariance of the wave equation shows that, in this case, the wave and back reaction equations are gauge invariant to second order. The stress-energy tensor is also conserved with respect to the background geometry to second order. Thus, the wave and back reaction equations are consistent to this order.

It is useful to generalize the above results to the case in which matter is present [6]. If the matter fields can be described by a covariant action, then both the wave equation and the stress-energy tensor for the matter can, in any coordinate system, be written as some particular combination of the matter fields, the metric tensor, and their derivatives. This makes it possible to split the wave equation and the stress-energy tensor into background and perturbed parts just as was done for the Einstein tensor in Eq. (2). If the matter fields are denoted by ϕ, the wave equation by W, and the stress-energy tensor for the matter by T, then the wave equation and Einstein's equations can be written as

$$W(\gamma,\phi) = -\triangle W(\gamma,h,\phi), \tag{20a}$$

$$G(\gamma) = 8\pi T(\gamma,\phi) + 8\pi \triangle T(\gamma,h,\phi) - \triangle G(\gamma,h). \tag{20b}$$

From Eq. (20b) it is clear that the combination $\triangle G(\gamma,h) - 8\pi\triangle T(\gamma,h,\phi)$ is conserved with respect to the background geometry. Two proofs of exactly the same type as that establishing the gauge invariance of $\triangle G$ when no matter is present, show that the quantities $\triangle W(\gamma,h,\phi)$ and $\triangle G(\gamma,h) - 8\pi\triangle T(\gamma,h,\phi)$ are invariant under generalized gauge transformations.

It is not difficult to show that all of the results found for the vacuum case go over in a straightforward manner to the case when matter is present if the substitution $\triangle G \rightarrow \triangle G - 8\pi\triangle T$ is made. Thus, regardless of whether or not averaging occurs, it is possible to derive a self-consistent set of equations which describe the behavior of the gravitational waves and the matter fields as well as their effects upon the background geometry.

In conclusion, use of the generalized gauge transformation implicitly given by Eq. (4) makes it possible to virtually always define an effective stress-energy tensor for gravitational waves and other perturbations which is both conserved and gauge invariant. The stress-energy tensor may be averaged in some arbitrary manner that does not affect the background geometry or it need not be averaged at all.

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