Taub's plane-symmetric vacuum spacetime reexamined

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The gravitational properties of the *only* static plane-symmetric vacuum solution of Einstein's field equations without a cosmological term (Taub's solution, for brevity) are presented: some already known properties (geodesics, weak field limit, and pertainment to the Schwarzschild family of spacetimes) are reviewed in a physically much more transparent way, as well as new results about its asymptotic structure, possible matchings, and the nature of the source are furnished. The main results point to the fact that the solution must be interpreted as representing the exterior gravitational field due to a *negative* mass distribution, confirming previous statements to that effect in the literature. Some analogies to Kasner's spatially homogeneous cosmological model are also mentioned. [S0556-2821(97)02606-4]

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I. INTRODUCTION

The main aim of this paper is to provide a detailed interpretative investigation of the vacuum plane-symmetric spacetimes of general relativity. Some generic motivating reasons may be adduced for that.

First, the presence of so-called topological defects (monopoles, cosmic strings, domain walls, and textures), arising from cosmological phase transitions (symmetry breaking), have become quite fashionable in the study of the very early universe, particularly as concerns their deep consequences for large-scale structure formation and cosmic background radiation anisotropies [1–3]. All these objects have peculiar gravitational properties; generic walls or shells (and also textures, to that effect), however, seem to have a slightly more acceptable (mathematical) behavior in the sense that, contrary to one- or two-dimensional trapped regions, their space times can be described by curvature tensors well defined as mathematical distributions [4].

Secondly, the Casimir effect associated with the vacuum bounded by two infinite parallel plates, when treated in a fully consistent way, should take into account the proper gravitational field of the plates [5]. In this setting, it turns out to be an issue for quantum field theory in curved spacetime.

Thirdly, when trying to formalize, in a more precise way, the equivalence between inertial and gravitational effects, the problem of what the general relativistic model of a spatially homogeneous gravitational field is has to be faced. Its resolution, from a naive Newtonian point of view, suggests again the paying attention to the vacuum plane-symmetric solutions of Einstein's field equations. Of course, the determination of the Newtonian gravitational field outside a static, infinite, uniformly dense planar slab is a trivial exercise; however, as we shall see, the general relativistic situation is not so plain.

The paper is organized as follows. In Sec. II, we expose the properties of Taub's plane-symmetric vacuum model: isometries, singularity, kinematics of the observers adapted to the symmetries, timelike and null geodesics (including geodesic deviation for timelike ones). These aspects have already been dealt with in the literature [6–9]; however, not only for the sake of completeness and fixing conventions, but also for further clarification and generalization do we delve again into them. In Sec. III, we study the asymptotic structure as well as the Newtonian limit. In Sec. IV, we prove that Taub's global solution is the limit of the mirror-symmetric matching of two Taub domains, to the ''left'' and ''right'' of a negative mass planar shell. In Sec. V, we discuss the main implications of our results and present a conclusion. We also provide an Appendix where the plane-symmetric vacuum solutions are shown, via Cartan's invariant technique, to be parametric limits of a generalized Schwarzschild family of spacetimes. The signature of the metric, except for Sec. IV, is -2; also $c=8\pi G=1$.

II. PLANE-SYMMETRIC VACUUM MODELS

A. Metrics

Einstein's *vacuum* field equations, without a cosmological term, for a plane-symmetric geometry, will be satisfied by only two nontrivial distinct *single* solutions [10]: Taub's static metric and Kasner's spatially homogeneous one. Both these solutions, which we shall henceforth refer to simply as Taub's and Kasner's solutions, are particular cases of Kasner's generalized solutions [11] (see Appendix).

Taub's geometry (T)*:*

$$ds^2_{(T)}=\frac{1}{z^{2/3}}dt^2-z^{4/3}(dx^2+dy^2)-dz^2, \tag{2.1}$$

$$=\frac{1}{r}dt^2-r^2(dx^2+dy^2)-r\,dr^2, \tag{2.2}$$

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with coordinate ranges

$$-\infty<t,x,y<+\infty, \quad 0<r<+\infty \text{ and}$$

$$0<z<+\infty, \quad \text{or } -\infty<z<0; \tag{2.3}$$

Kasner's geometry (K)*:*

$$ds^2_{(K)}=dt^2-t^{4/3}(dx^2+dy^2)-\frac{1}{t^{2/3}}dz^2, \tag{2.4}$$

$$=t\,dt^2-t^2(dx^2+dy^2)-\frac{1}{t}dr^2, \tag{2.5}$$

with coordinate ranges

$$-\infty<x,y,z,r<+\infty \quad \text{and } 0<t<+\infty, \quad \text{or } -\infty<t<0. \tag{2.6}$$

The only nonvanishing Carminati-McLenaghan algebraic invariants [12] for Taub's and Kasner's solutions are

$$I_{1(T)}=\frac{64}{27z^4}, \quad I_{2(T)}=-\frac{256}{243z^6}, \tag{2.7}$$

and

$$I_{1(K)}=\frac{64}{27t^4}, \quad I_{2(K)}=\frac{256}{243t^6}, \tag{2.8}$$

where $I_1:=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ and $I_2:=R_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu}{}^{\lambda\rho}R_{\lambda\rho}{}^{\alpha\beta}$.

As concerns Taub's solution, we call attention to the following properties: (i) it is not only plane-symmetric, but also *static*; (ii) it has a timelike singularity at $z=0$ $(r=0)$; (iii) its algebraic invariants vanish when $z,r\rightarrow+\infty$ (suggesting it is asymptotically flat at spatial infinity in the z direction; see Sec. III). Kasner's solution presents "dual" properties: (i) it is not only plane-symmetric, but also *spatially homogeneous* (beware, it is *not* stationary); (ii) it has a spacelike singularity at $t=0$; (iii) its algebraic invariants vanish when $t\rightarrow+\infty$.

B. Kinematics of adapted observers and geodesics

The motion of the observers adapted to those coordinate systems $(u^\alpha=\delta^\alpha_0/\sqrt{g_{00}})$ is covariantly characterized as: for Taub's solution, Eq. (2.1), they constitute a *rigid nonrotating accelerated frame*, whereas for Kasner's solution, Eq. (2.4), they constitute a *deforming nonrotating geodesic frame*. The results for Taub's observers are obvious since they are manifestly static, so that they remain at rest with respect to one another and with respect to the singularity too. This is a consequence of the physically more intuitive fact that the proper time, as measured by a static observer, for a photon to travel to and back from a second static observer is independent of the emission event. Half this proper time defines the so-called *radar distance* between the two observers, which is a coordinate-independent concept. Specifically, for Taub's spacetime, this radar distance between two static observers $\mathcal{O}_1$ and $\mathcal{O}_2$ with respective spatial coordinates (x,y,r_1) and (x,y,r_2), as measured by $\mathcal{O}_1$, is given by

$$\Delta s(\mathcal{O}_1,\mathcal{O}_2)=\sqrt{g_{00}(r_1)}\left|\int_{r_1}^{r_2}\sqrt{-g_{33}(r)/g_{00}(r)}dr\right|$$

$$=\frac{1}{2\sqrt{r_1}}|r_2^2-r_1^2|\neq\Delta s(\mathcal{O}_2,\mathcal{O}_1). \tag{2.9}$$

As explicitly displayed, this radar distance is *not* symmetric, a feature which already happens to Rindler's observers in Minkowski spacetime and is due to the relativity of the simultaneity for the different instantaneous inertial frames attached to each observer. We also notice that, as the radial coordinate distance $\Delta r(\mathcal{O}_1,\mathcal{O}_2):=|r_2-r_1|$ increases, so does the radar distance.

The isometries of Taub's model imply the existence of three constants of motion for the geodesics,

$$\frac{1}{r}\dot{t}=:E=\text{const}>0, \tag{2.10}$$

$$r^2\dot{x}=:p_x=\text{const}, \tag{2.11}$$

$$r^2\dot{y}=:p_y=\text{const}, \tag{2.12}$$

where a dot denotes a derivative with respect to an affine parameter τ (the proper time, for timelike geodesics). Furthermore, the invariance of the character of the geodesic $(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=\epsilon:=0 \text{ or} +1$, for null or timelike geodesics, respectively) implies now

$$\dot{r}^2=E^2-V(r), \tag{2.13}$$

with

$$V(r):=\frac{1}{r^3}(p_{||}^2+\epsilon r^2)\geqslant0 \tag{2.14}$$

and

$$p_{||}:=\sqrt{p_x^2+p_y^2}. \tag{2.15}$$

The motion in the coordinate r is thus reduced to a usual one-dimensional problem with the *effective potential* (2.14) and r *coordinate* acceleration

$$\ddot{r}=\frac{1}{2r^4}(3p_{||}^2+\epsilon r^2)\geqslant0. \tag{2.16}$$

From either Eqs. (2.13)–(2.15) or Eqs. (2.15),(2.16), we see that we have two qualitatively distinct cases, according to the mass and/or the initial conditions of the test particle: (a) massless particles in purely "perpendicular" motion $(\epsilon=p_{||}=0)$; (b) massive particles $(\epsilon=1)$ or massless particles in "transverse" motion $(\epsilon=0\neq p_{||})$. Case (a) implies $V(r)\equiv0$, $\ddot{r}\equiv0$. These null geodesics *do* attain the singularity. Case (b) implies $V(r)\rightarrow+\infty$, $\ddot{r}\rightarrow+\infty$ when $r\rightarrow0_+$ and $V(r)\rightarrow0_+$, $\ddot{r}\rightarrow0_+$ when $r\rightarrow+\infty$. These geodesics can *never attain* the singularity.

In fact, *all* massive geodesic particles or even generic $(p_{||}\neq0)$ massless geodesic particles are *repelled* from the singularity as described by the static observers $r=\text{const}$. This is clear from the following. As shown at the beginning of this section, the static observers are at rest with respect to each other and with respect to the singularity. Since the r coordinate for a geodesic particle eventually increases without bound, it will cross the static observers receding further and further away from the singularity. Furthermore, a geodesic particle initially at rest will be displaced to increasing values of r, as can be derived from Eq. (2.16). This implies that, relative to the static observers, a geodesic particle is accelerated away from the singularity. This description is consistent with the negative mass interpretation for the sources of Taub's spacetime, as discussed at the end of this subsection, in the Appendix and in Sec. IV.

Equations (2.13) and (2.14) may be trivially integrated, whereupon one can easily see that null geodesics in purely "perpendicular" motion [case (a) above] "reach" the boundary $r=+\infty$ (to be defined in Sec. III) only after an infinite interval of affine parameter.

To further interpret the model, let us now calculate, from the geodesic deviation equation, the relative acceleration between two geodesics at instantaneous rest with respect to the singularity $(\dot{x}|_0=\dot{y}|_0=\dot{r}|_0=0)$. We find

$$\left.\frac{D^2\eta^0}{D\tau^2}\right|_0=0,$$

$$\left.\frac{D^2\eta^A}{D\tau^2}\right|_0=+\frac{1}{2r^3}\eta^A, \quad (A=1,2), \tag{2.17}$$

$$\left.\frac{D^2\eta^3}{D\tau^2}\right|_0=-\frac{1}{r^3}\eta^3.$$

Note that the signs in the above equations are the opposite of the corresponding ones in the Schwarzschild spacetime. Thus, a set of two freely falling particles released from rest may behave in two characteristic ways: (a) if their initial spatial positions have equal x and y but different r, their relative distance will instantaneously decrease, while being repelled from the singularity; (b) if their initial spatial positions have equal r but different x and/or y, their relative distance will instantaneously increase, while being repelled from the singularity again. Contrary to the case of the curvature field being spatially homogeneous (independent of r), as occurs in the Newtonian theory for the gravitational field $\mathbf{g}$, the plane symmetry of the metric field does not imply homogeneity (neither in the metric nor the curvature fields): analogous spacetime measurements (such as those for the geodesic deviation) at events with different r coordinates furnish distinct results.

C. Parametric limits

Some intuition on the nature of the source (singularity) of Taub's (global) spacetime might also be expected to arise from a study of the families of metrics to which it belongs. The problem with this argument is twofold: (i) a given metric may be a parametric limit of several disjoint families of metrics, (ii) a given family of metrics may have different limits, for the same limiting values of its parameters, when the limiting process is carried out in different coordinate systems. This was first pointed out by Geroch [13], who presented Schwarzschild's metric in three different coordinate systems, such that, as the mass tends to $+\infty$, the limiting result is either singular, Minkowski's metric or Kasner's (spatially homogeneous) metric. Because of the resemblance between Kasner's and Taub's metrics, one naturally wonders whether the latter is also a limit of the Schwarzschild family of metrics as the mass tends to $+\infty$; the answer, however, is negative, since the only Petrov type D such limits are Minkowski's and Kasner's (spatially homogeneous) spacetimes [14]. Still, coordinate systems exist, for this Schwarzschild family of metrics, such that, now in the limit $m\rightarrow-\infty$, one ends up with Taub's metric [8,15]. In the Appendix, we analyze a generalized Schwarzschild family of metrics, which is closed in the sense of Geroch [13], using an invariant approach and extending the above results.

It is expedient to mention here some other families to which Taub's metric belongs. We have two subfamilies of Weyl's static vacuum metrics [9]: the Levi-Civita metrics and the Parnovski-Papadopoulos metrics, then the Kerr-Schild solutions [10] and, as already mentioned, the generalized Kasner metrics.

III. ASYMPTOTIC STRUCTURE AND THE WEAK FIELD LIMIT

In this section, we introduce coordinate systems which render explicit the null boundary structure of the spacetime and also characterize a weak field limit to Taub's geometry.

Let us consider the static plane-symmetric Taub metric in the form

$$ds_T^2=\frac{1}{r}dt^2-r\,dr^2-r^2(dx^2+dy^2) \tag{3.1}$$

$(0<r<\infty)$ and introduce in the manifold of Eq. (3.1) the Kruskal-type coordinate system (u,v,x,y) defined in Table I.

In this coordinate system, the metric (3.1) assumes the form

$$ds_T^2=\frac{2\sqrt{2}(du^2-dv^2)}{(v^2-u^2)\sqrt{-\ln(v^2-u^2)}}+2\ln(v^2-u^2)(dx^2+dy^2). \tag{3.2}$$

We note that the curvature tensor of Eq. (3.2) tends continuously to zero as $v^2-u^2\rightarrow 0$. Asymptotic coordinates may be introduced such that Eq. (3.2) becomes regular on $v^2-u^2=0$ (see Table I). The boundaries $v^2-u^2=0$ are in fact two flat null surfaces at infinity $(r=+\infty)$, which we shall denote by J^+ and J^- (see Fig. 1), and correspond to the asymptotically flat regions of the spacetime.

If, in Eq. (3.1), we extend the domain of the coordinate r to $-\infty<r<\infty$, the resulting manifold of the geometry (3.1) is the union of a Kasner (K) spacetime to a Taub (T) spacetime, with the singular locus $r=0$ as a common boundary. The Kasner spacetime corresponds to the domain $-\infty<r<0$, its singularity $r=0$ having now a spacelike character. A typical plane (u,v) of Kasner's spacetime is represented in Fig. 2. The metric in K is given by

TABLE I. Coordinate systems used for Taub metric in this paper. Homonymous coordinates may differ by a constant and unspecified ranges are $(-\infty,+\infty)$. An * in the fourth column means the singularity. The last two coordinate systems are meaningful only in the asymptotic region. For the Kasner metric, we interchange u and v in the table; the point $(u=0_+, v=0)$ for the Kasner domain now represents the future timelike infinity i^+, instead of i^0. Replicas of these spacetimes (K and T) in the (u,v) plane can be obtained by changing $(u,v)\to(-u,-v)$.

coord.	definition	range	geometrical loci
$tzxy$		$0\leqslant z<+\infty$	$*: z=0$
$trxy$	$r=(3z/2)^2/3$	$0\leqslant r<+\infty$	$*: r=0$
$uvxy$	$u=e^{-r^2/4}\sinh(t/2)$ $v=e^{-r^2/4}\cosh(t/2)$	$0\leqslant v<+\infty$ $0\leqslant v^2-u^2\leqslant 1$	$*: v^2-u^2=1$ $J^+: v=+u\neq 0$ $J^-: v=-u\neq 0$ $i^0: u=0,\ v=0_+$
$\mu\nu xy$	$\mu=-\ln(v-u)$ $\nu=-\ln(v+u)$	$0\leqslant(\mu+\nu)<+\infty$	$*: \mu+\nu=0$ $J^+: \mu\to+\infty,\ \nu$ finite $J^-: \nu\to+\infty,\ \mu$ finite $i^0: \mu\to+\infty,\ \nu\to+\infty$
$\xi\nu xy$	$\xi=\sqrt{4\mu},\ \mu>0$	$0<\xi<+\infty$	
ξNXY	$X=\xi x,\ Y=\xi y$ $N=\nu-\xi(x^2+y^2)$		
$TZXY$	$Z=(N+\xi)/2$ $T=(N-\xi)/2$		

$$ds_K^2=\frac{2\sqrt{2}(du^2-dv^2)}{(u^2-v^2)\sqrt{-\ln(u^2-v^2)}}+2\ln(u^2-v^2)(dx^2+dy^2). \tag{3.3}$$

Expression (3.3) is obtained by interchanging u and v in the metric components of Eq. (3.2). Note that in the K region $0\leqslant u^2-v^2\leqslant 1$, and (u,v) are still timelike and spacelike coordinates, respectively.

A further coordinate system is now introduced (see Table I), which will be useful in examining the asymptotic form of Taub's metric. In the coordinate system (μ,ν,x,y), Taub's geometry (3.2) assumes the form

$$ds_T^2=-\frac{2\sqrt{2}d\mu d\nu}{\sqrt{\mu+\nu}}-2(\mu+\nu)(dx^2+dy^2) \tag{3.4}$$

and Kasner's geometry (3.3) becomes

$$ds_K^2=\frac{2\sqrt{2}d\mu d\nu}{\sqrt{\mu+\nu}}-2(\mu+\nu)(dx^2+dy^2). \tag{3.5}$$

The null infinities of Taub's geometry are now expressed by

$$J^+:\mu\to\infty,\qquad J^-:\nu\to\infty \tag{3.6}$$

and analogously for Kasner's geometry. The diagrams of Fig. 3 make explicit the relation between the (u,v) and (μ,ν) coordinates.

Although we have introduced coordinate systems (see Table I) where the asymptotic boundaries of Taub's geometry are characterized and explicitly exhibited (the two flat null infinities J^+ and J^-, and the spatial infinity i^0), the metric in these coordinate systems is not well defined in the boundaries. In order to give a regular expression for Taub's metric in a neighborhood of the boundaries J^+ and J^-, we therefore introduce two new sets of asymptotic coordinates (see Table I).

We rewrite Eq. (3.4) as

$$ds_T^2=-\frac{2\sqrt{2}d\xi d\nu}{\sqrt{1+\frac{4\nu}{\xi^2}}}-\frac{1}{2}\xi^2\left(1+4\frac{\nu}{\xi^2}\right)(dx^2+dy^2). \tag{3.7}$$

Near J^+ we have $4\nu/\xi^2\to 0$, and we may expand the above line element as

$$ds_T^2\approx ds_M^2+h, \tag{3.8}$$

where

$$ds_M^2=-2\sqrt{2}d\xi d\nu-\tfrac{1}{2}\xi^2(dx^2+dy^2) \tag{3.9}$$

and

$$h=\frac{4\sqrt{2}\nu}{\xi^2}d\xi d\nu-2\nu(dx^2+dy^2). \tag{3.10}$$

We note that the line element ds_M^2 is the Minkowski one in coordinates $x^\alpha=(\xi,\nu,x,y)$. Indeed, let us realize a further coordinate transformation (see Table I), which casts ds_M^2 into the form

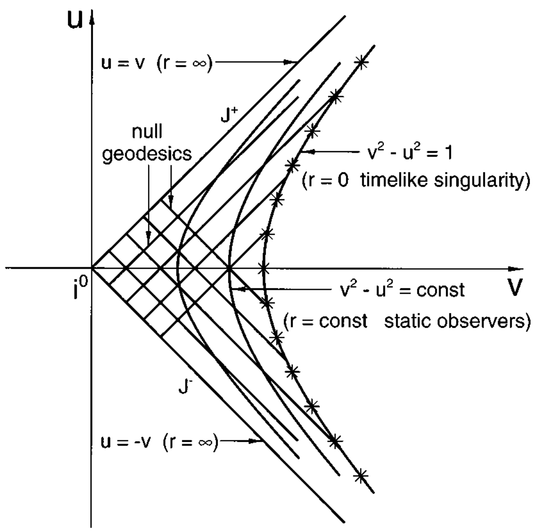

FIG. 1. The Taub spacetime, (3.2), with two coordinates (x,y) suppressed. Each point of the diagram corresponds to a two-dimensional spacelike plane. The spacetime manifold is the quasi-compact region of the (u,v) plane bounded by the timelike singularity $v^2-u^2=1$, the null boundaries J^+ and J^-, and the spatial infinity i^0 $(u=0,\ v=0_+)$. The null geodesics (with $p_{||}=0$) are forty-five degrees straight lines. With respect to the metric (3.2) u and v are timelike and spacelike coordinates, respectively.

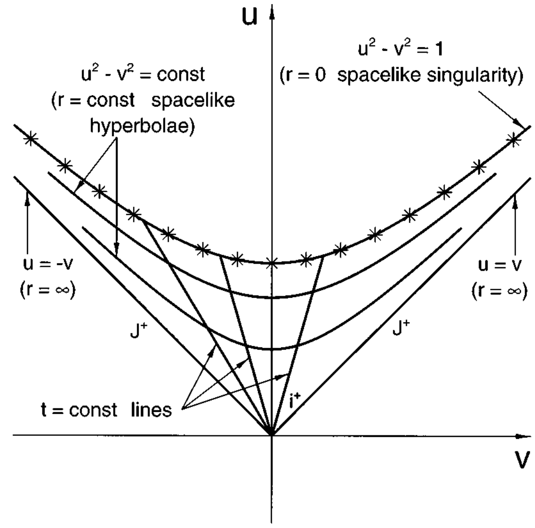

FIG. 2. The Kasner spacetime, (3.3), with coordinates (x,y) suppressed. The future flat null infinities J^+ correspond to $u^2-v^2=0$. The (u,v) coordinates are now related to the (t,r) coordinates of Eq. (3.1) with $-\infty<r\leqslant 0$ by $u=f(r)\cosh(t/2)$, $v=f(r)\sinh(t/2)$, in order that (u,v) have timelike and spacelike character, respectively. The future timelike infinity i^+ is the point $(u=0_+,v=0)$.

$$ds_M^2=-d\xi dN-(dX^2+dY^2) \tag{3.11}$$

and all components of h in this new coordinate system are at least $O(1/\xi^2)$, as well as its first and second derivatives. Therefore, in the coordinate system (ξ,N,X,Y) the metric near J^+ is regular, as well as its first and second order derivatives, and differ from the Minkowski metric by a small perturbation h which tends to zero as $\xi\to\infty$.

An analogous expansion may be realized in a neighborhood of J^- : $\nu\to\infty$. This may be obtained from the expressions of the above paragraph by the obvious substitution $\nu\rightleftharpoons\mu$.

We can now discuss the weak field limit for Taub's geometry. Equation (3.11) suggests the introduction of a Cartesian coordinate system at infinity, namely (T,Z,X,Y), defined in Table I. Thus, when $z\to\infty$ for finite T, we have $\xi\to\infty$ and $N\to\infty$, with $N/\xi^2\to 0$. Therefore, the region described by $Z\to\infty$ for finite T is a neighborhood of the spatial infinity i^0 of Fig. 1. The asymptotic expression of Taub's metric (for X and Y finite) is then

$$ds_T^2=dT^2-dX^2-dY^2-dZ^2 + \frac{4N}{\xi^2}\left(\frac{1}{2}(dZ^2-dT^2)-dX^2-dY^2\right), \tag{3.12}$$

and, in a neighborhood of the spatial infinity i^0, Eq. (3.12) reduces to

$$ds_T^2=(1-2/Z)(dT^2-dZ^2)-(1+4/Z)(dX^2+dY^2). \tag{3.13}$$

A Newtonian-like potential at i^0 is then given by

$$\Phi_N=-1/Z. \tag{3.14}$$

If we choose a plane $Z=Z_0$ in a neighborhood of i^0, we can expand Eq. (3.14) about Z_0 as

$$\Phi_N=-1/(Z_0+\ell)\approx-(1/Z_0)(1-\ell/Z_0). \tag{3.15}$$

Then, near the plane $Z=Z_0$, we have a homogeneous field orthogonal to Z_0 with strength $1/Z_0^2$, for all (X,Y) finite.

This configuration of the gravitational field (in a neighborhood of i^0) is the nearest to a homogeneous gravitational field we may achieve in the static plane-symmetric Taub geometry. In the past literature, this problem of the nonrelativistic limit to a static plane-symmetric metric was considered in [7] (Sec. IV). The treatment there is, however, incomplete: a coordinate system is introduced where Taub's metric is put in the form $ds_T^2=ds_M^2+h$, where h is a small (in a specific region) deviation of the Minkowski metric, ds_M^2. Nevertheless, the connection and curvature tensor components are not small (of the order of h) in that region, unless he specifies that $g\to 0$, thus fixing the asymptotic region $(z\to+\infty)$ as the region of true nonrelativistic limit; in this case, the above coordinate system is well defined only in the asymptotic region (see [7], Eq. (12)). Furthermore, since Taub's metric has no essential free parameter in it, there simply does not exist a quantity which would play the role of the mass density of the source.

IV. MATCHING AND SHELL

We now implement a *mirror-symmetric* junction of two Taub domains by means of a *negative* mass shell, in order to show that Taub's original spacetime (2.1) may be naturally viewed as the limit of this matched one as the surface energy density of the shell approaches $-\infty$. To this end, we shall

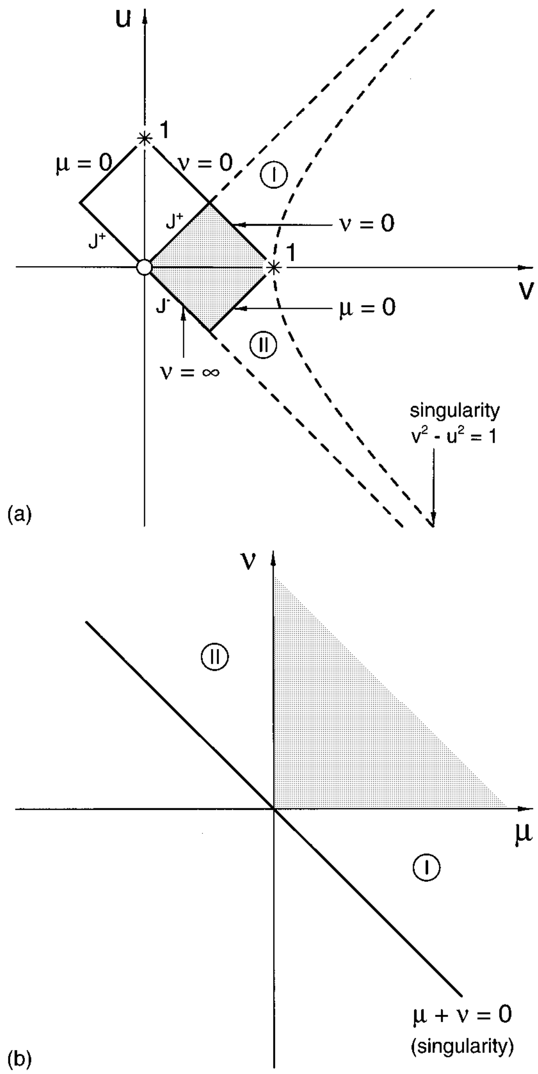

FIG. 3. The relation between the (u,v) and (μ,ν) coordinates. The gray region of (b) corresponds to the gray T region in (a) bounded by the flat null infinities J^+ and J^-, the null surfaces $\nu=0$ and $\mu=0$ plus the singular point $\mu=0=\nu$. Coordinate systems may be introduced such that on the null surfaces $\mu=0$ and $\nu=0$ the metrics ds_K^2 and ds_T^2 have the form of Minkowski metric in Cartesian coordinates, except at the singular point $\mu=0=\nu$.

take advantage of Israel's shell formalism [16], following all his conventions inclusively; in particular, the signature of the metric, in this section, is $+2$.

The four-dimensional manifolds V^+ and V^- will be taken as Taub domains; specifically, $V^+:=\{(x_+^\alpha):z_+>A>0\}$ and $V^-:=\{(x_-^\alpha):z_-<-A<0\}$, that is V^+ is "on the right side" of the original Taub singularity and V^- is "on its left side." We are assuming that the $t,z=$const surfaces have the topology of a plane ($\mathbf{R}^2$) and we can thus assign to the symmetry $z\to -z$ a natural meaningful interpretation as a mirror or specular symmetry. This interpretation seems to be unwarranted for other choices of topology of the $t,z=$const surfaces (see Sec. V).

The timelike hypersurfaces of junction Σ^+ and Σ^- are characterized by the equations $F_+(x_+^\alpha):=z_+-A=0$ and $F_-(x_-^\alpha):=z_-+A=0$, with $A>0$. The induced three-metrics on Σ^+ and Σ^- are [cf. (2.1)] identical:

$$ds_\pm^2|_{\Sigma^+}=-A^{-2/3}dt_\pm^2+A^{4/3}(dx_\pm^2+dy_\pm^2), \tag{4.1}$$

thus automatically satisfying the first junction condition.

The verification of the second junction condition requires the calculation of the extrinsic curvatures of Σ^+ and Σ^-. The unit normal vector to Σ (directed from V^- to V^+) will have components n_+^α and n_-^α, relative to the charts x_+^α and x_-^α, given by

$$n_\pm^\alpha=+\delta_3^\alpha. \tag{4.2}$$

For an intrinsic basis of tangent vectors to Σ, we will choose three orthonormal vectors $\mathbf{e}_i$ $(i=0,1,2)$ of components e_{i+}^α and e_{i-}^α, relative to the charts x_+^α and x_-^α, given by

$$e_{0\pm}^\alpha=|A|^{1/3}\delta_0^\alpha,\quad e_{1\pm}^\alpha=|A|^{-2/3}\delta_1^\alpha,\quad e_{2\pm}^\alpha=|A|^{-2/3}\delta_2^\alpha. \tag{4.3}$$

The extrinsic curvature of a hypersurface with unit normal n^α can be calculated as

$$K_{ij}=e_i^\alpha e_j^\beta n_{\alpha;\beta}, \tag{4.4}$$

which furnishes

$$K_{ij}^\pm=\pm\frac{1}{3A}\,\mathrm{diag}(1,2,2). \tag{4.5}$$

Lanczos equation [16], $\gamma_{ij}-g_{ij}\gamma_k^k=-S_{ij}$, where $\gamma_{ij}:=K_{ij}^+-K_{ij}^-$ is the jump in the extrinsic curvature, then determines the surface energy-momentum tensor of a shell:

$$S_{ij}=\frac{2}{3A}\,\mathrm{diag}(-4,1,1). \tag{4.6}$$

Thus, if we suppose the shell is a perfect fluid one, with proper four-velocity

$$u^\alpha|_\Sigma=|A|^{1/3}\delta_0^\alpha, \tag{4.7}$$

we are forced to recognize that it has a *negative* surface energy density. Furthermore, as A approaches 0, thus realizing the matching ever closer to the original Taub singularity, we are naturally lead to interpreting it as an infinite uniform *plane* distribution of negative diverging surface energy density. This interpretation is consistent with the repulsion of geodesics found in Sec. II B and the limit for infinite negative mass.

V. DISCUSSION

In this paper, we carried out an extensive investigation of the local and global properties of Taub's (static) geometry, and by extension, of Kasner's (spatially homogeneous) geometry. First, we studied the geodesic motion, by using the method of the effective potential. We showed, in a manifestly coordinate-independent way, that massive ($\epsilon=1$) free particles eventually recede from the singular locus, no matter what initial conditions are attributed to them. Only massless ($\epsilon=0$) particles can attain the singularity, even so provided their motion is purely ''perpendicular'' ($p_{||}=0$). This enlarges upon some results in the literature [7]. Second, we showed that Taub's metric is one of the limits of a family of Schwarzschild's spacetimes, which is closed in the sense of Geroch [13]. This limit corresponds to the mass parameter approaching infinitely negative values. This analysis is carried out by means of Cartan scalars, which characterize in an invariant way a given geometry. Third, we built a new plane-symmetric model, by conveniently matching two Taub domains. Fourth, we also uncovered the asymptotic structure (null, timelike, and spacelike infinities) of Taub's solution. Fifth, we obtained the complete weak field limit in asymptotically flat regions, proving that, at spatial infinity, we may have a nonrelativistic approximately homogeneous configuration; however, this cannot be associated to a truly Newtonian limit due to the absence of any essential parameters which could be related to the mass density of the source or, in other words, Taub's solution is a single one, not a family of solutions (like Schwarzschild's one, for instance). We will now take the opportunity to make some generic comments on related current literature.

In a recent paper [9], Bonnor discusses the difficulties which arise in the interpretation of the solutions of Einstein's field equations, due to the coordinate freedom to describe the metric. Although we agree with this general idea of his papers, we do not believe that his preferred (semi-infinite line mass) interpretation of Taub's vacuum static plane-symmetric metric is the *only* tenable one. Indeed, locally isometric manifolds can be extended to global manifolds with distinct topological properties; this extension cannot be fixed by purely local (in a given coordinate system) considerations (isometries, geodesics, etc.), which are doomed to inconclusiveness. For instance, the $t,z=$const flat maximally symmetric surfaces of Eq. (2.1) can be conceived of as immersed in $\mathbf{R}^3$ either as topological planes or topological cylinders, which are, of course, locally indistinguishable. What really does seem promising, from the physical point of view, to settle the issue of interpretation is the realization of experiments which probe the large-scale structure of the spacetime. Another criterion, one of simplicity, we have advanced is the *mirror-symmetric* matching of Sec. IV, which reproduces the global Taub solution in the limit of infinitely negative surface energy density of the plane shell. In short, it seems to us that the nature of the source, from a purely mathematical point of view, is a matter of *consistent choice* of topology.

We should not conclude without recalling an elegant heuristic argument of Vilenkin [17], which might explain the unreasonableness of looking for a physically viable model for an infinite plane of constant positive surface density σ. Consider, in the alleged plane, a disk of radius R, which contains the total mass $M(R)=\pi R^2\sigma$; thus, above a critical radius given by $R_c=1/(2\pi\sigma)$, the disk will have more mass than its corresponding Schwarzschild mass, and it should collapse (incidentally, the same argument does not hold for a line or string). Of course, compelling as this argument seems, its validity rests on at least two tacit assumptions: (i) the intrinsic geometry of the plane is Euclidean, so that the disk's area is πR^2, (ii) the *hoop conjecture* [18], since we are dealing with a nonspherically symmetric system. Vilenkin's argument might then explain why all the proposed models in general relativity (GR) for an infinite homogeneous plane end up with either a flat space solution with an appropriate topology or a constant distribution of *negative* mass. In fact, in the literature, there have appeared two distinct proposals for the general relativistic problem of a static, infinite, uniformly dense plane. The first one [6] is locally isometric to Taub's plane-symmetric static vacuum spacetime, the second one [19] is a Rindler domain, locally isometric to the Minkowski spacetime. If we insist that a stationary plane massive configuration may exist in GR, we must cope with the possibility of negative mass configurations, with an exterior solution given by Taub's geometry. Another possibility is to relax the requirement of stationarity and consider time-dependent metrics with plane symmetry, as in the exact solution exhibited by Griffiths [20] (Sec. III). This solution consists of a slab of dust, with mass density a function of z and t, bounded by two Kasner regions. The mass contained within a sphere of radius R is, however, time independent and given by $M(R)=\pi R^2\sigma$, for a very thin slab with a constant planar mass density σ. This solution fits exactly Vilenkin's argument. Further consequences of this point will be presented in a forthcoming paper.

As a last remark, one might be tempted to consider the models inspected in this paper as purely academic ones; however, already in nonquantum gravitational theory, the issue of negative mass has a longstanding history, of which a relativistic landmark is afforded by Bondi's classic paper [21]. Also, from a quantum point of view, it was early recognized that the weak energy condition cannot hold everywhere [22]. Nowadays, this result has acquired a renewed interest because, besides the already mentioned cosmological interest of topological defects [1–3], of its import to the possibility of construction of time machines [23] and avoidance of singularities [24]. Thus, the best policy seems to be keeping, *cum grano salis*, an open mind to these exotic models.

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APPENDIX: THE GENERALIZED SCHWARZSCHILD FAMILY

We study a two-parameter family of metrics which includes Taub's and Kasner's ones and its limits when some parameters are taken to be 0, $+\infty$, or $-\infty$, extending the results of Sec. II C. The formalism is that of the Cartan scalars, which are the components of the Riemann tensor and its covariant derivatives calculated in a constant frame and which provide a complete local characterization of space-times [25–27]. Spinor components are used [28] and the relevant objects here are the Weyl spinor Ψ_A and its first and second symmetrized covariant derivatives $\nabla\Psi_{AB'}$ and $\nabla^2\Psi_{AB'}$. SHEEP and CLASSI [29,30] were used in the calculations.

We shall call ''the generalized Schwarzschild family'' the vacuum metrics

$$ds^2 = A dt^2 - A^{-1} dr^2 - r^2[d\theta^2 + K^2(\theta) d\varphi^2];$$

$$A = \lambda - \frac{2m}{r}, \tag{A1}$$

with λ and m constant. In the Lorentz tetrads $[\eta_{AB} = \mathrm{diag}(+1,-1,-1,-1)]$

$$\begin{aligned}&\theta^0 = A^{1/2}dt, && \theta^1 = A^{-1/2}dr, && \theta^2 = r d\theta, && \theta^3 = K r d\varphi, && A>0,\\ &\theta^0 = (-A)^{-1/2}dr, && \theta^1 = (-A)^{1/2}dt, && \theta^2 = r d\theta, && \theta^3 = K r d\varphi, && A<0\end{aligned} \tag{A2}$$

the Einstein vacuum field equations are $R_{22} = R_{33} = -r^{-2}(\lambda + K_{,\theta\theta}/K) = 0$. Using canonical null tetrads, the Cartan scalars are

$$\Psi_2 = -\frac{m}{r^3},$$

$$\nabla\Psi_{20'} = \frac{3}{\sqrt{2}}\frac{m}{|m|}\Psi_2^{4/3}\sqrt{\pm(\lambda m^{-(2/3)} + 2\Psi_2^{(1/3)})} = \mp\nabla\Psi_{31'},$$

$$\nabla^2\Psi_{20'} = \frac{4}{3}\frac{(\nabla\Psi_{20'})^2}{\Psi_2} = \nabla^2\Psi_{42'},$$

$$\nabla^2\Psi_{31'} = \mp\nabla^2\Psi_{20'} - \frac{3}{2}\Psi_2^2 \tag{A3}$$

(the upper and lower signs corresponding to $A>0$ and $A<0$, respectively). From the Cartan scalars, one infers that the metric is Petrov type D, has a one-dimensional isotropy group given by spatial rotations on the $\omega^2 - \omega^3$ plane (i.e., the $\theta - \varphi$ surface), the orbit of the isometry group is three dimensional (note that the coordinates t, θ, and ϕ do not appear in the expressions of the Cartan scalars) and, therefore, the isometry group is four dimensional.

Since $K(\theta)$ is not present in the Cartan scalars, for each λ, any solution for $K(\theta)$ of the Einstein equations can be transformed into each other by a suitable coordinate transformation. Thus, the line element is determined by the values of λ and m only. Moreover, since Ψ_2 depends on the r coordinate, by a suitable coordinate transformation, its dependence on m can be eliminated. Thus, a member of this family of metrics is characterized by the sign of m, and by $\lambda m^{-(2/3)}$. Therefore, one can always make a coordinate transformation such that λ becomes 1, 0, or -1 and $K = \sin\theta$, 1, or $\sinh\theta$, respectively. Accordingly, the generalized Schwarzschild family may be divided into the following subfamilies.

(1) $\lambda>0$.

One can make $\lambda=1$ and $K(\theta)=\sin\theta$, corresponding to the Schwarzschild line element with $-\infty<m<\infty$.

(2) $\lambda<0$.

One can make $\lambda=-1$ and $K(\theta)=\sinh\theta$. This we shall call the anti-Schwarzschild line element with $-\infty<m<\infty$.

(3) $\lambda=0$.

One can make $K=1$ and $|m|=1/2$ (for $m=0$ the line element becomes singular). The only parameter is the sign of m.

[3(a)] $\lambda=0$ and $m=1/2>0$.

Kasner's metric (2.5) with

$$\Psi_2 = -(1/2r^3), \quad \nabla\Psi_{20'} = 3\Psi_2^{(3/2)} = \nabla\Psi_{31'},$$

$$\nabla^2\Psi_{20'} = -12\Psi_2^2 = \nabla^2\Psi_{42'}, \quad \nabla^2\Psi_{31'} = -\frac{27}{2}\Psi_2^2. \tag{A4}$$

[3(b)] $\lambda=0$ and $m=-1/2<0$.

Taub's metric (2.2) with

$$\Psi_2 = 1/2r^3, \quad \nabla\Psi_{20'} = -3\Psi_2^{3/2} = -\nabla\Psi_{31'},$$

$$\nabla^2\Psi_{20'} = 12\Psi_2^2 = \nabla^2\Psi_{42'}, \quad \nabla^2\Psi_{31'} = -\frac{27}{2}\Psi_2^2. \tag{A5}$$

The last two line elements are special cases of the Kasner-type metric [11], $ds^2 = e t^{2a_1}dt^2 - e t^{2a_2}dx^2 - t^{2a_3}dy^2 - t^{2a_4}dz^2$, where $a_2+a_3+a_4 = a_1+1$, $(a_2)^2+(a_3)^2+(a_4)^2 = (a_1+1)^2$ and $e=\pm1$. To recover Eq. (A4) we may choose $a_1=\frac{1}{2}$, $a_2=-\frac{1}{2}$, $a_3=a_4=1$, and $e=+1$; and to recover Eq. (A5) we may choose $a_1=\frac{1}{2}$, $a_2=-\frac{1}{2}$, $a_3=a_4=1$, and $e=-1$.

Therefore, the generalized Schwarzschild family (A1) may be divided into two one-parameter families, namely the Schwarzschild family and the anti-Schwarzschild family plus the Kasner's metric (2.5) and the Taub's metric (2.2). Taub's metric arises when $\lambda=0$ and $m<0$. Although $|m|$ is arbitrary, it can be absorbed away by a coordinate transformation, therefore Taub's geometry is independent of parameters: it is a *single* solution, not a family. A consequence of this fact emerges in the weak field limit (3.14), which has no essential parameter.

From the Cartan scalars (A3), we see that as $m\to\pm\infty$, the limits depend only on the sign of m and not on λ, therefore the Schwarzschild and the anti-Schwarzschild families have the same limits. The limiting procedure we adopt is that of [14], where we choose limits for Ψ_2 and find the limits of the other Cartan scalars. The possible limits for Ψ_2 are (i) 0, (ii) a nonzero constant, and (iii) an arbitrary function of the coordinates. (i) If $\Psi_2 \to 0$, then all Cartan scalars (A3) tend to zero, therefore both families tend to Minkowski spacetime (which is Petrov type 0). (ii) As discussed by [14], Petrov type D metrics cannot have Ψ_2 constant while the other components of the Weyl spinor are zero. (iii) If Ψ_2 tends to an arbitrary function of the coordinates, then as $m \to +\infty$ both families tend to the Kasner metric (2.5) and as $m \to -\infty$ both families tend to the Taub (2.2) as can be easily seen comparing the the Cartan scalars obtained in the limit with those of Kasner (A4) and Taub (A5). As discussed in [14], different functional forms of the limit of Ψ_2 do not lead to different limits, since coordinate transformations can take one form to the others. This analysis covers all Petrov type D and 0 limits.

Geroch [13] introduced the concept of a closed family of metrics, i.e., a family that contains all its limits. Considering the family of metrics defined by (A1), we can say that it is closed under Petrov type D and 0 limits as m tends to $+\infty$ or $-\infty$.

As a matter of completeness, we mention that the limits of the generalized Schwarzschild family as $\lambda \to 0$ and $m > 0$ or $m < 0$ are exactly the same as the limits as $m \to \pm\infty$. This can be easily seen from the Cartan scalars (A3), noting specially the expression of $\nabla\Psi_{20'}$.

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