Lagrangian perfect fluids and black hole mechanics

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The first law of black hole mechanics (in the form derived by Wald) is expressed in terms of integrals over *surfaces*, at the horizon and spatial infinity, of a stationary, axisymmetric black hole, in a diffeomorphism-invariant Lagrangian theory of gravity. The original statement of the first law given by Bardeen, Carter, and Hawking for an Einstein-perfect fluid system contained, in addition, *volume* integrals of the fluid fields, over a spacelike slice stretching between these two surfaces. One would expect that Wald's methods, applied to a Lagrangian Einstein-perfect fluid formulation, would convert these terms to surface integrals. However, because the fields appearing in the Lagrangian of a gravitating perfect fluid are typically nonstationary (even in a stationary black-hole–perfect-fluid spacetime) a direct application of these methods generally yields restricted results. We therefore first approach the problem of incorporating general nonstationary matter fields into Wald's analysis, and derive a first-law-like relation for an arbitrary Lagrangian metric theory of gravity coupled to *arbitrary* Lagrangian matter fields, requiring only that the *metric* field be stationary. This relation includes a volume integral of matter fields over a spacelike slice between the black hole horizon and spatial infinity, and reduces to the first law originally derived by Bardeen, Carter, and Hawking when the theory is general relativity coupled to a perfect fluid. We then turn to consider a specific Lagrangian formulation for an isentropic perfect fluid given by Carter, and directly apply Wald's analysis, assuming that both the metric and fluid fields are stationary and axisymmetric in the black hole spacetime. The first law we derive contains only surface integrals at the black hole horizon and spatial infinity, but the assumptions of stationarity and axisymmetry of the fluid fields make this relation much more restrictive in its allowed fluid configurations and perturbations than that given by Bardeen, Carter, and Hawking. In the Appendix, we use the symplectic structure of the Einstein-perfect fluid system to derive a conserved current for perturbations of this system: this current reduces to one derived *ab initio* for this system by Chandrasekhar and Ferrari. [S0556-2821(97)05806-2]

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I. INTRODUCTION

The first law of black hole mechanics as stated by Bardeen, Carter, and Hawking [1] relates small changes in the mass of a stationary, axisymmetric black hole to small changes in its horizon surface area, angular momentum, and the properties of a stationary perfect fluid that might surround it: one first fixes a stationary axisymmetric Einstein-perfect fluid black hole solution with stationary killing field ξ^a (with asymptotically unit norm) and axial killing field φ^a (with closed orbits). One then defines δ to be an infinitesimal perturbation to a nearby stationary axisymmetric solution; then the first law in [1] is

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J_H - \int_\Sigma \mu' |v| \delta N_{abc} + \int_\Sigma \Omega\, \delta J_{abc} + \int_\Sigma T|v|\delta S_{abc}, \qquad (1)$$

where the spacetime is characterized by an Arnowitt-Deser-Misner (ADM) mass, M, and the black hole by its horizon surface area A, surface gravity κ, angular velocity Ω_H, and angular momentum J_H (measured at the horizon). The fields associated to the perfect fluid are its four-velocity U^a [which here is taken to be of the form $U^a = v^a/|v|$, where $v^a = \xi^a + \Omega\varphi^a$, for some (generally nonconstant) Ω], the chemical potential μ', the temperature T, stress energy T^{ab}, and number and entropy densities n and S. The three-forms $N_{abc} = nU^d \boldsymbol{\epsilon}_{abcd}$, $J_{abc} = T^d{}_e \varphi^e \boldsymbol{\epsilon}_{dabc}$, and $S_{abc} = SU^d \boldsymbol{\epsilon}_{dabc}$ represent the fluid number density, angular momentum density, and entropy density on a spacelike three-surface, Σ, that has boundaries at the black hole horizon and the two-sphere at spatial infinity. We have also set $\boldsymbol{\epsilon}_{abcd}$ to be the canonical volume element on spacetime.

Considerable effort has been spent on weakening the assumptions made in Eq. (1) on the background fields and their perturbations. For instance, consider an arbitrary diffeomorphism invariant Lagrangian theory with both metric and matter fields, and let the theory possess stationary, axisymmetric black hole solutions, which are asymptotically flat, and have a bifurcate killing horizon (for an explanation of these terms see [2,3]). Then it was shown [2,4], providing the metric and matter fields appearing in the Lagrangian were stationary and axisymmetric in the black hole background, that there existed a first law of black hole mechanics in a form only involving *surface integrals* on the sphere at spatial infinity and the bifurcation sphere of the black hole horizon. Namely, given the Lagrangian for the theory, one could algorithmically define integrals $\mathcal{E}$ and $\mathcal{J}$ over the sphere at spatial infinity, and $\mathcal{S}$ over the bifurcation sphere, satisfying the identity

$$\delta\mathcal{E} = \frac{\kappa}{2\pi}\delta\mathcal{S} + \Omega_H \delta\mathcal{J}. \qquad (2)$$

(Here δ denotes a perturbation from the background black hole solution to *any* nearby solution.) The quantity $\mathcal{E}$ was interpreted as the canonical energy of the black hole system, $\mathcal{J}$ as the canonical angular momentum and $\mathcal{S}$ as the black hole entropy.

We might therefore expect that the volume integrals in Eq. (1) involving the fluid can be converted to surface integrals in the form (2), by choosing a suitable variational form for the Einstein-perfect fluid system and using the methods of [4]. In fact, we are unable to reproduce the first law (1) in a form only containing surface integrals, using these methods; the difficulty is that at least one of the fields appearing in each of the Lagrangian formulations for a perfect fluid (that we are aware of) is generally nonstationary, even when the fluid four-velocity, number density, entropy, and functions of these fields (which we refer to collectively as the *physical* fields), are stationary. Since the methods of [4] require that all fields appearing in the Lagrangian (which we refer to henceforth as the *dynamical fields*) are stationary and axisymmetric in the black hole background, the allowed background solutions for the perfect fluid in the resulting first law are restricted.

This paper gives two results in response to this problem: we first relax all explicit symmetry assumptions on matter fields appearing in the Lagrangian, and find the consequence for the first law given in [4]. We also attempt to generate a first law of form (2) by a careful choice of an existing Lagrangian formulation for gravity coupled to a perfect fluid, directly using the methods of [4].

In Sec. II we consider an arbitrary Lagrangian theory of gravity coupled to *arbitrary* matter fields, assuming only that the metric is stationary and axisymmetric in the black hole background, but making no such assumptions about the matter dynamical fields. We then modify the methods of [4] to generate a perturbative relation, but instead of attempting to express the matter contribution to the first law (2) via surface integrals, we leave it instead as a volume integral over a hypersurface, Σ, joining the bifurcation sphere to the sphere at spatial infinity. In restricted cases (which we explain later) we can motivate an independent measurement of the ''vacuum'' black hole mass, M_g. In these cases we can also define quantities which resemble the ''vacuum'' black hole entropy, S_g, and angular momentum, J_{gH}, and having done so our perturbative relation takes the form

$$\delta M_g = \frac{\kappa}{2\pi}\delta S_g + \Omega_H \delta J_{gH} + \int_\Sigma \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab} - \delta(\boldsymbol{\epsilon}\cdot T\cdot\xi), \tag{3}$$

where T^{ab} is the stress energy of the matter fields. We will see that this relation defines a black hole entropy, S_g, which is in general not the black hole entropy defined in [4]: however, in special cases the interpretation of S_g as black hole entropy can be appropriate [for instance, as we show in Sec. IV, this relation reduces to Eq. (1) when the gravitational theory is chosen to be general relativity, and the matter source is chosen to be a perfect fluid]. Our result differs from a similar relation presented by Schutz and Sorkin [7], in that they conjectured, but did not explicitly include, the black hole entropy and angular momentum boundary terms, and so did not explicitly generalize the full form of Eq. (1). In addition, as we shall explain, the definition of our ''Noether current'' (involved in the intermediate calculations) is both less ambiguous than that presented by Schutz and Sorkin [7] and more general than the definition given by Sorkin [8]. The range of theories in which our methods are well defined is therefore larger than those addressed by their methods.

In Sec. III we define a gravitating perfect fluid and review some variational principles for it: Schutz's ''velocity-potential'' formulation [9], which uses the dynamical fields $(\phi,\alpha,\beta,\theta,\sigma)$ to define the product of the (physical) specific inertial mass and four-velocity $\mu U_a \equiv \boldsymbol{\nabla}_a\Phi + \alpha\boldsymbol{\nabla}_a\beta + \theta\boldsymbol{\nabla}_a\sigma$, and Carter's more recent ''axionic vorticity'' formulation [10] for an isentropic perfect fluid, which uses a dynamical field b_{ab} to define the number current N_{abc} [given in Eq. (1)] via $N_{abc}\equiv 3\boldsymbol{\nabla}_{[a}b_{bc]}$, and the dynamical fields $\chi^\pm$ to define the fluid vorticity via $2\boldsymbol{\nabla}_{[a}\mu U_{b]} \equiv 2\boldsymbol{\nabla}_{[a}\chi^+\boldsymbol{\nabla}_{b]}\chi^-$.

In Sec. IV we present two forms of the first law for the Einstein-perfect fluid system. The first form is derived from relation (3) and is the same as Eq. (1), with the exception that δ is now allowed to be a perturbation from the (stationary axisymmetric) background to an arbitrary nearby solution. (Note that this form of the first law contains volume integrals.) It is also of interest to know if we can construct *any* form of the first law with perfect fluids only involving surface integrals; in fact, by directly applying the methods of [4] for a metric theory of gravity coupled to a perfect fluid described using Carter's variational principle (with the potential b_{ab} for N_{abc}, and $\chi^\pm$ for ω_{ab}), we can derive a first law of the form

$$\delta M + \mu_\infty\delta\int_{S_\infty} b_{qr} - \mu_\infty\delta\int_{\mathcal{H}} b_{qr} = \frac{\kappa}{8\pi}\delta A + \Omega_H\delta J_H + \int_{\mathcal{H}} X_{qr} - \int_{S^\infty} X_{qr}, \tag{4}$$

where M is the ADM mass, A is the black hole surface area, J_H is the black hole angular momentum appearing in Eq. (1), X_{qr} is the two-form $2\xi^p b_{p[q}[\delta(\mu U_{r]}) - \boldsymbol{\nabla}_{r]}\chi^-\delta\chi^+ + \boldsymbol{\nabla}_{r]}\chi^+\delta\chi^-]$, and we have written S^∞ and $\mathcal{H}$ for the sphere at spatial infinity and the bifurcation sphere, respectively. We will see that this first law is more restrictive than Eq. (1), but it is the only nontrivial rule of type (2) involving a perfect fluid that we can currently construct.

In the Appendix we evaluate the symplectic form of the Einstein-perfect fluid system, using the variational formulation given by Schutz [9] for the perfect fluid. The symplectic form is dual to a generally conserved current, quadratic in the field perturbations [11]. We find (in parallel with Burnett and Wald's calculation for the Einstein-Maxwell system [12]) that this conserved current reduces to a current previously derived *ab initio* by Chandrasekhar and Ferrari [13] for the polar perturbations of a static axisymmetric black hole.

II. A PERTURBATIVE RELATION FOR BLACK HOLE MECHANICS WITH NONSTATIONARY MATTER FIELDS

In this section we give a perturbative relation that resembles the first law of black hole mechanics, for an arbitrary theory of gravity with a diffeomorphism invariant Lagrangian. We assume the theory possesses black hole solutions in which the metric is stationary and axisymmetric, but place no restrictions on the other fields appearing in the Lagrangian (we refer to these fields collectively as the *dynamical* fields). The motivation for this is, as we have indicated, that variational formulations for gravitating Einstein-perfect fluid systems have fluid dynamical fields which are nonstationary even when the fluid's *physical* fields (the four-velocity, number density, and entropy) are stationary and axisymmetric. We first make some necessary definitions related to the the symplectic structure of a diffeomorphism invariant Lagrangian theory. These are explained in detail in [4]; here we merely state (and, in one case, refine) the relevant definitions and results. In the following we often use boldface type to denote differential forms on spacetime, suppressing their indices when convenient.

A. Some preliminaries

All theories we consider arise from a Lagrangian, which is taken to be a diffeomorphism invariant four-form on spacetime, dependent on the metric, g_{ab}, and some arbitrary set of matter fields, ψ. (We collectively refer to all the dynamical fields by ϕ.) By this we mean that the Lagrangian has the functional dependence

$$\mathbf{L}=\mathbf{L}(g_{ab},R_{abcd},\boldsymbol{\nabla}R_{abcd},\ldots,(\boldsymbol{\nabla})^{p}R_{abcd},\psi,\boldsymbol{\nabla}\psi,\ldots,(\boldsymbol{\nabla})^{q}\psi) \tag{5}$$

(here multiple derivatives appearing in the above expression are assumed to be symmetrized—see [4] for further discussion about this dependence). In particular we require that every field appearing in the Lagrangian give rise to an equation of motion (there are no ''background'' fields). The variation of the Lagrangian defines these equations, $\mathbf{E}=0$, along with the symplectic potential $\boldsymbol{\Theta}$, by

$$\delta\mathbf{L}=\mathbf{E}\delta\phi+d\boldsymbol{\Theta}(\phi,\delta\phi). \tag{6}$$

[Here $\boldsymbol{\Theta}(\phi,\delta\phi)$ is a linear differential operator in the field variations $\delta\phi$. Because the Lagrangian is only defined up to the addition of an exact form, $\mathbf{L}\rightarrow\mathbf{L}+d\boldsymbol{\mu}$, the symplectic potential is only defined up to the following terms: $\boldsymbol{\Theta}(\phi,\delta\phi)\rightarrow\boldsymbol{\Theta}(\phi,\delta\phi)+d\mathbf{Y}(\phi,\delta\phi)+\delta\boldsymbol{\mu}(\phi)$, where $\mathbf{Y}$ and $\boldsymbol{\mu}$ are covariant forms with the same type of functional dependence as $\boldsymbol{\Theta}$ and $\mathbf{L}$, respectively. These ambiguities were discussed in [4].]

Now fix a smooth vector field, ξ^a, on spacetime. Then the Noether current $\mathbf{J}[\xi]$ associated to ξ^a is a three-form defined by

$$\mathbf{J}[\xi]\equiv\boldsymbol{\Theta}(\phi,\mathcal{L}_{\xi}\phi)-\xi\cdot\mathbf{L}, \tag{7}$$

where the centered dot denotes contraction of the vector into the first index of the form. This Noether current can be seen [4] to obey the identity

$$d\mathbf{J}[\xi]=-\mathbf{E}\mathcal{L}_{\xi}\phi, \tag{8}$$

which we now use to further elucidate its structure. (Although they appear in a different context, the calculations below have the same flavor as those in the Appendix of [4].)

Lemma 1. Fix $\mathbf{L}$ to be the Lagrangian of a diffeomorphism invariant theory of gravity and matter fields, with equation of motion $\mathbf{E}=0$ as given in Eq. (6). Without loss of generality, label each dynamical field ϕ by i, and give each field u_i upper, and d_i lower indices: also label the equations of motion for each field similarly, so that the equation of motion term in Eq. (6) becomes

$$\mathbf{E}\delta\phi=\boldsymbol{\epsilon}E_{\phi_i b_1\cdots b_{ui}}{}^{a_1\cdots a_{di}}\delta\phi_i{}^{b_1\cdots b_{u_i}}{}_{a_1\cdots a_{d_i}}. \tag{9}$$

Then for any smooth field ξ^a there exists a two-form, $\mathbf{Q}[\xi]$, called the Noether charge associated to ξ^a (which is local in the dynamical fields and ξ^a), such that the Noether current $\mathbf{J}[\xi]$, defined in Eq. (7), can be written

$$\mathbf{J}[\xi]=-(\boldsymbol{\epsilon}\cdot E\cdot\phi\cdot\xi)+d\mathbf{Q}[\xi], \tag{10}$$

where we define the three-form

$$\begin{aligned}(\boldsymbol{\epsilon}\cdot E\cdot\phi\cdot\xi)_{abc}&\\ \equiv\boldsymbol{\epsilon}_{eabc}\sum_i E_{\phi_i b_1\cdots b_{ui}}{}^{a_1\cdots a_{di}}&\big(-\phi_i{}^{e\cdots b_{ui}}{}_{a_1\cdots a_{di}}\delta^{b_1}_{p}\cdots\\ &-\phi_i{}^{b_1\cdots e}{}_{a_1\cdots a_{di}}\delta^{b_{u_i}}_{p}+\phi_i{}^{b_1\cdots b_{ui}}{}_{p\cdots a_{di}}\delta^{e}_{a_1}\cdots\\ &+\phi_i{}^{b_1\cdots b_{u_i}}{}_{a_1\cdots p}\delta^{e}_{a_{d_i}}\big)\xi^{p}.\end{aligned} \tag{11}$$

Proof. For clarity, we first consider the case where the metric is the only dynamical field: $\phi\rightarrow g_{ab}$. Then setting the metric field equations $\mathbf{E}^{ab}=\boldsymbol{\epsilon}E^{ab}$, Eq. (8) reads

$$d\mathbf{J}[\xi]=-2\boldsymbol{\epsilon}E_g^{ab}\boldsymbol{\nabla}_a\xi_b=-2\boldsymbol{\epsilon}\boldsymbol{\nabla}_a(E_g^{ab}\xi_b)+2\boldsymbol{\epsilon}\boldsymbol{\nabla}_a(E_g^{ab})\xi_b. \tag{12}$$

Therefore setting $(\boldsymbol{\epsilon}\cdot E_g\cdot\xi)_{abc}\equiv\boldsymbol{\epsilon}_{dabc}E_g^{de}\xi_e$, we have

$$d(\mathbf{J}[\xi]+2\boldsymbol{\epsilon}\cdot E_g\cdot\xi)=2\boldsymbol{\epsilon}\boldsymbol{\nabla}_a(E_g^{ab})\xi_b, \tag{13}$$

which shows that the right side of Eq. (13) is both linear in ξ^a, and exact for all ξ^a. The results of [6] now imply that the right side must vanish identically, and so $\boldsymbol{\nabla}_aE^{ab}=0$. This in turn implies that the left side of Eq. (13) must be an identically closed three-form, which (using the results of [6] again) implies the existence of a two-form, $\mathbf{Q}[\xi]$, local in the dynamical fields and ξ^a, such that

$$\mathbf{J}[\xi]+2\boldsymbol{\epsilon}\cdot E_g\cdot\xi=d\mathbf{Q}[\xi]. \tag{14}$$

We define $\mathbf{Q}[\xi]$, the Noether charge associated to ξ^a, as any two-form which is local in the dynamical fields and ξ^a, and satisfies this relation.

We can also perform this analysis for $\mathbf{L}$ with the general dependence (5). With the labels for each field and its equation of motion given in Eq. (9), the first equation in Eq. (12) becomes

$$d\mathbf{J}[\xi]=-\boldsymbol{\epsilon}\sum_i E_{\phi_i b_1\cdots b_{ui}}{}^{a_1\cdots a_{di}}\mathcal{L}_{\xi}\phi_i{}^{b_1\cdots b_{u_i}}{}_{a_1\cdots a_{di}}, \tag{15}$$

which, through a similar manipulation to Eq. (12) leads to the structure for $\mathbf{J}[\xi]$ and the definition of the Noether charge, $\mathbf{Q}[\xi]$, in Eq. (10). □

The Noether charge was defined in [4] only when $\mathbf{E}=0$, via $\mathbf{J}[\xi]=d\mathbf{Q}[\xi]$. This left open the definition of $\mathbf{Q}[\xi]$ when $\mathbf{E}\neq 0$. In the Appendix of [5], however, it was shown that $\mathbf{Q}[\xi]$ could be defined when $\mathbf{E}\neq 0$, such that there existed forms $\mathbf{C}_a$ with $\mathbf{J}[\xi]-d\mathbf{Q}[\xi]=\mathbf{C}_a\xi^a$, and where the $\mathbf{C}_a$ vanished when $\mathbf{E}=0$. At that time it was not known whether $\mathbf{Q}[\xi]$ was uniquely defined this way, nor was the explicit form of $\mathbf{C}_a$ specified. We have given this explicit form in Eq. (11). Moreover, the above analysis uniquely defines the Noether charge via Eq. (10), without imposing the field equations, up to the following ambiguities (which were discussed in detail in [4]): The ambiguity in $\mathbf{\Theta}$ described after Eq. (6) means that $\mathbf{J}[\xi]$ is only defined up to the following terms: $\mathbf{J}[\xi]\rightarrow\mathbf{J}[\xi]+d[\mathbf{Y}(\phi,\mathcal{L}_\xi\phi)-\xi\cdot\boldsymbol{\mu}]$, and so the ambiguity in $\mathbf{Q}[\xi]$ is $\mathbf{Q}[\xi]\rightarrow\mathbf{Q}[\xi]+\mathbf{Y}(\phi,\mathcal{L}_\xi\phi)-\xi\cdot\boldsymbol{\mu}$. These ambiguities will not affect the results stated in the following sections.

We now define the symplectic current $\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi)$ (a three-form on spacetime) by

$$\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi)\equiv\delta_2\mathbf{\Theta}(\phi,\delta_1\phi)-\delta_1\mathbf{\Theta}(\phi,\delta_2\phi). \quad (16)$$

Note that $\boldsymbol{\omega}$ is a function of an unperturbed set of fields, ϕ, and is bilinear and skew in pairs of variations $(\delta_1\phi,\delta_2\phi)$. It can be shown (see [11]) that this three-form is closed when ϕ is a solution of the field equations and $\delta_1\phi$ and $\delta_2\phi$ are solutions of the linearized equations of motion. (In the Appendix we examine this closed form — it is dual to a conserved vector field, which we evaluate for perturbations of an Einstein-perfect fluid system.) Moreover, if we let ξ^a be a smooth vector field, set $\delta_1\phi=\mathcal{L}_\xi\phi$ and let $\delta_2\phi=\delta\phi$ be a variation to a nearby solution (with $\delta\xi^a=0$), then $\boldsymbol{\omega}(\phi,\mathcal{L}_\xi\phi,\delta\phi)$ can be shown [4] to be exact:

$$\boldsymbol{\omega}(\phi,\mathcal{L}_\xi\phi,\delta\phi)=d[\delta\mathbf{Q}[\xi]-\xi\cdot\mathbf{\Theta}(\phi,\delta\phi)]. \quad (17)$$

Now fix a black hole spacetime with a stationary and axisymmetric metric, for the theory given by the Lagrangian in Eq. (5); let the stationary killing field with unit norm at spatial infinity be ξ^a and the axial killing field (with closed orbits) be φ^a. Let the black hole have a bifurcate killing horizon, with bifurcation sphere $\mathcal{H}$, and let it be asymptotically flat, with the two-sphere at spatial infinity S^∞. Let Σ be a three-surface with these two boundaries, and set $\delta\phi$ to be an arbitrary perturbation of the background which satisfies the linearized equations. Then the first law of black hole mechanics as stated in [4] is an interpretation of the identity

$$\int_\Sigma\boldsymbol{\omega}(\phi,\mathcal{L}_\xi\phi,\delta\phi)=\int_{S_\infty}\delta\mathbf{Q}[\xi]-\xi\cdot\mathbf{\Theta}(\phi,\delta\phi)-\int_{\mathcal{H}}\delta\mathbf{Q}[\xi]-\xi\cdot\mathbf{\Theta}(\phi,\delta\phi) \quad (18)$$

[which arises from integrating Eq. (17) over Σ]. When ξ^a Lie derives *all* the dynamical fields in the background, the left side of Eq. (18) vanishes, and one is left with a relation between surface integrals on the boundaries of Σ, which can be shown to be of the form (2). In Sec. IV we present an explicit Lagrangian for the Einstein-perfect fluid system, and, assuming that all dynamical fields are stationary and axisymmetric, compute the surface terms arising from this Lagrangian.

B. The perturbative identity

Having stated these necessary definitions we turn to construct our perturbative identity. We start by decomposing the Lagrangian $\mathbf{L}$ into a part $\mathbf{L}_g$, depending on the metric g_{ab} (which is assumed to be stationary and axisymmetric in the black hole background), and a part $\mathbf{L}_m$, dependent on both the metric and a set of matter fields ψ (on which we place no restrictions):

$$\mathbf{L}=\mathbf{L}_g(g_{ab},R_{abcd},\boldsymbol{\nabla}R_{abcd},\ldots,(\boldsymbol{\nabla})^pR_{abcd})+\mathbf{L}_m(\psi,\boldsymbol{\nabla}\psi,\ldots,(\boldsymbol{\nabla})^q\psi,g_{ab},R_{abcd},\boldsymbol{\nabla}R_{abcd},\ldots,(\boldsymbol{\nabla})^rR_{abcd}). \quad (19)$$

Since this breakup only requires that $\mathbf{L}_g$ be independent of any matter fields, it is very nonunique, and in general we have no method of controlling the ambiguity

$$\mathbf{L}_g\rightarrow\mathbf{L}_g+\boldsymbol{\lambda},\qquad \mathbf{L}_m\rightarrow\mathbf{L}_m-\boldsymbol{\lambda}, \quad (20)$$

where $\boldsymbol{\lambda}=\boldsymbol{\lambda}(g_{ab},R_{abcd},\boldsymbol{\nabla}R_{abcd},\ldots,(\boldsymbol{\nabla})^sR_{abcd})$.

The variation of the Lagrangian yields equations of motion for the metric $\mathbf{E}_g^{ab}=0$ and matter fields $\mathbf{E}_m=0$ via

$$\delta\mathbf{L}=\mathbf{E}_g^{ab}\delta g_{ab}+\mathbf{E}_m\delta\psi+d\mathbf{\Theta}(\phi,\delta\phi). \quad (21)$$

For convenience we set $\mathbf{E}_g^{ab}=\boldsymbol{\epsilon}E_g^{ab}$ and $\mathbf{E}_m=\boldsymbol{\epsilon}E_m$. As discussed above we can compute $\mathbf{J}[\xi]$ [defined by Eq. (7)], and define $\mathbf{Q}[\xi]$, for the theory described by Eq. (19): it must have the form given in Eq. (10):

$$\mathbf{J}[\xi]=-2\boldsymbol{\epsilon}\cdot E_g\cdot\xi-\boldsymbol{\epsilon}\cdot E_m\cdot\psi\cdot\xi+d\mathbf{Q}[\xi] \quad (22)$$

(the factor of 2 between the terms with equations of motion here is purely a matter of convention). We can also use the individual Lagrangians $\mathbf{L}_g$ and $\mathbf{L}_m$ to define the stress-energy tensor T^{ab}, and symplectic potentials $\mathbf{\Theta}_g(g,\delta g)$ and $\mathbf{\Theta}_m(\phi,\delta\phi)$:

$$\delta\mathbf{L}_g=\mathbf{E}'^{ab}\delta g_{ab}+d\mathbf{\Theta}_g(g,\delta g),$$

$$\delta\mathbf{L}_m=\mathbf{E}_m\delta\psi+\boldsymbol{\epsilon}\frac{1}{2}T^{ab}\delta g_{ab}+d\mathbf{\Theta}_m(\phi,\delta\phi). \quad (23)$$

Clearly $\mathbf{E}_g^{ab}=\mathbf{E}_g'^{ab}+\boldsymbol{\epsilon}\frac{1}{2}T^{ab}$, and up to the ambiguities present in the symplectic potentials, we also have $\mathbf{\Theta}=\mathbf{\Theta}_g+\mathbf{\Theta}_m$. Similarly, if we define the Noether currents for the individual Lagrangians by

$$\mathbf{J}_g[\xi]\equiv\mathbf{\Theta}_g(g,\mathcal{L}_\xi g)-\xi\cdot\mathbf{L}_g,$$

$$\mathbf{J}_m[\xi]\equiv\mathbf{\Theta}_m(\phi,\mathcal{L}_\xi\phi)-\xi\cdot\mathbf{L}_m, \quad (24)$$

then it follows that

$$\mathbf{J}[\xi]=\mathbf{J}_g[\xi]+\mathbf{J}_m[\xi]. \quad (25)$$

Now we impose the structure (10) on each of $\mathbf{J}_g$ and $\mathbf{J}_m$, in the process defining $\mathbf{Q}_g$ and $\mathbf{Q}_m$, which are the Noether charges in the theories arising from these Lagrangians:

$$\mathbf{J}_g[\xi] = -2\boldsymbol{\epsilon}\cdot E'_g\cdot\xi + d\mathbf{Q}_g[\xi],$$

$$\mathbf{J}_m[\xi] = -\boldsymbol{\epsilon}\cdot E_m\cdot\psi\cdot\xi - \boldsymbol{\epsilon}\cdot T\cdot\xi + d\mathbf{Q}_m[\xi]. \tag{26}$$

Finally we substitute Eq. (26) into the right side of Eq. (25) and Eq. (22) into the left side, obtaining

$$-2\boldsymbol{\epsilon}\cdot E_g\cdot\xi - \boldsymbol{\epsilon}\cdot E_m\cdot\psi\cdot\xi + d\mathbf{Q}[\xi] = -2\boldsymbol{\epsilon}\cdot E'_g\cdot\xi - \boldsymbol{\epsilon}\cdot T\cdot\xi - \boldsymbol{\epsilon}\cdot E_m\cdot\psi\cdot\xi + d\mathbf{Q}_g[\xi] + d\mathbf{Q}_m[\xi]. \tag{27}$$

All the terms involving equations of motion and stress-energy tensors can be seen to cancel, and the resulting identity implies

$$\mathbf{Q}[\xi] = \mathbf{Q}_g[\xi] + \mathbf{Q}_m[\xi] + d\mathbf{Z} \tag{28}$$

(where $\mathbf{Z}$ is some arbitrary covariant one-form). We therefore have a relation (independent of any field equations) between the Noether charge $\mathbf{Q}$, of the full theory given by $\mathbf{L}$, and that of the ''pure gravity'' theory $\mathbf{Q}_g$, arising from $\mathbf{L}_g$. We are now ready to state the identity.

Lemma 2. Fix $\mathbf{L}$, $\mathbf{L}_g$ (the ''vacuum'' Lagrangian), and $\mathbf{L}_m$ (the ''matter'' Lagrangian) to be diffeomorphism invariant Lagrangians related as given in Eq. (19) with the functional dependence shown there. Fix a smooth vector field ξ^a, let $\boldsymbol{\Theta}_g$ be defined by Eq. (23), let $\mathbf{Q}_g[\xi]$ be the Noether charge defined by Eq. (26) for the theory described by $\mathbf{L}_g$, and let T^{ab} be the stress-energy tensor of the matter fields defined by (23). Now consider an asymptotically flat, stationary, axisymmetric black hole solution with bifurcate killing horizon, in the theory described by $\mathbf{L}$, with stationary killing field ξ^a (with unit norm at the sphere S^∞, at spatial infinity), and axial killing field φ^a (with closed orbits), so that ξ^a and φ^a Lie derive the metric but not necessarily the matter fields. Let the horizon killing field (which vanishes on the bifurcation sphere $\mathcal{H}$) be given by $\chi^a = \xi^a + \Omega_H\varphi^a$, where Ω_H is a constant. Then for δ a perturbation to an arbitrary nearby solution, such that $\delta\xi^a = 0$,

$$\int_{S_\infty}\delta\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\Theta}_g = \int_{\mathcal{H}}\delta\mathbf{Q}_g[\chi] - \Omega_H\int_{\mathcal{H}}\delta\mathbf{Q}_g[\varphi] + \int_\Sigma \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab} - \delta(\boldsymbol{\epsilon}\cdot T\cdot\xi). \tag{29}$$

Proof. We evaluate expression (18) for the theory (19), where the background solution is a black hole with the symmetry and structure described above (29), demanding that the metric be stationary and axisymmetric in the background spacetime, but placing no restrictions on the matter fields. In this case the integrand on the left side of Eq. (18) is generally nonvanishing. Assuming that the field equations hold in background for the matter fields, $E_m = 0$, and that $\delta\psi$ is a solution to the linearized matter equations of motion off this background ($\delta E_m = 0$), we find the left side of Eq. (18) is

$$\begin{aligned}\boldsymbol{\omega}(\phi,\mathcal{L}_\xi\phi,\delta\phi) &= \delta\boldsymbol{\Theta}_g(g,\mathcal{L}_\xi g) - \mathcal{L}_\xi\boldsymbol{\Theta}_g(g,\delta g) + \delta\boldsymbol{\Theta}_m(\phi,\mathcal{L}_\xi\phi) - \mathcal{L}_\xi\boldsymbol{\Theta}_m(\phi,\delta\phi)\\ &= \delta\boldsymbol{\Theta}_m(\phi,\mathcal{L}_\xi\phi) - \mathcal{L}_\xi\boldsymbol{\Theta}_m(\phi,\delta\phi) = \delta(d\mathbf{Q}_m[\xi] - \boldsymbol{\epsilon}\cdot T\cdot\xi + \xi\cdot\mathbf{L}_m) - \mathcal{L}_\xi\boldsymbol{\Theta}_m(\phi,\delta\phi)\\ &= \delta(d\mathbf{Q}_m[\xi] - \boldsymbol{\epsilon}\cdot T\cdot\xi + \xi\cdot\mathbf{L}_m) - \xi\cdot d\boldsymbol{\Theta}_m(\phi,\delta\phi) - d(\xi\cdot\boldsymbol{\Theta}_m(\phi,\delta\phi))\\ &= d(\delta\mathbf{Q}_m[\xi] - \xi\cdot\boldsymbol{\Theta}_m(\phi,\delta\phi)) - \delta(\boldsymbol{\epsilon}\cdot T\cdot\xi) + \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab},\end{aligned} \tag{30}$$

where we used the stationarity of g_{ab} in the second line, the expression (26) for $\mathbf{J}_m$ in the third, the Lie derivative identity $\mathcal{L}_\xi\boldsymbol{\lambda} = \xi\cdot d\boldsymbol{\lambda} + d(\xi\cdot\boldsymbol{\lambda})$ (which holds for an arbitrary form $\boldsymbol{\lambda}$) in the fourth line, and the definition (23) of $\boldsymbol{\Theta}_m$ and the stress energy T^{ab} in the fifth line. Now also assuming $E_g = \delta E_g = 0$, and substituting Eq. (30) into the left side of Eq. (18) yields

$$\begin{aligned}\int_\Sigma d(\delta\mathbf{Q}_m[\xi] - \xi\cdot\boldsymbol{\Theta}_m) + \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab} - \delta(\boldsymbol{\epsilon}\cdot T\cdot\xi)\\ = \int_{S_\infty}\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\Theta}(\phi,\delta\phi)\\ - \int_{\mathcal{H}}\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\Theta}(\phi,\delta\phi),\end{aligned} \tag{31}$$

and so, cancelling the boundary terms $\delta\mathbf{Q}_m[\xi] - \xi\cdot\boldsymbol{\Theta}_m$ from both sides [and using Eq. (28)] we get

$$\begin{aligned}\int_\Sigma \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab} - \delta(\boldsymbol{\epsilon}\cdot T\cdot\xi)\\ = \int_{S_\infty}\delta\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\Theta}_g(g,\delta g)\\ - \int_{\mathcal{H}}\delta\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\Theta}_g(g,\delta g).\end{aligned} \tag{32}$$

Now writing ξ^a in terms of χ^a and φ^a at the boundary $\mathcal{H}$ (and discarding terms which vanish as a result of the vanishing of χ^a at $\mathcal{H}$, or which vanish because φ^a is tangent to $\mathcal{H}$) we get

$$\int_\Sigma \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab}-\delta(\boldsymbol{\epsilon}\cdot T\cdot\xi)$$

$$=\int_{S_\infty}\delta\mathbf{Q}_g[\xi]-\xi\cdot\boldsymbol{\Theta}_g(g,\delta g)$$

$$-\int_{\mathcal{H}}\delta\mathbf{Q}_g[\chi]+\Omega_H\int_{\mathcal{H}}\delta\mathbf{Q}_g[\varphi], \qquad (33)$$

which is what we wished to show. □

The identity (29) has physical significance when we can interpret the surface integrals appearing there as (variations of) the energy, entropy, and angular momentum of the black hole. When is this possible? If the theory had no matter fields then we could choose $\mathbf{L}_m$ to vanish, and the terms involving T^{ab} in Eq. (29) would vanish (we could also choose other breakups of $\mathbf{L}$, and we will return to this shortly). In this case we would have $\mathbf{L}_g=\mathbf{L}$, $\boldsymbol{\Theta}_g=\boldsymbol{\Theta}$, and $\mathbf{Q}_g=\mathbf{Q}$. If in addition there existed a three-form $\mathbf{B}$ (local in the dynamical fields, the flat metric η_{ab}, and its associated derivative, ∂, at spatial infinity) such that at spatial infinity, $\xi\cdot\boldsymbol{\Theta}(\phi,\delta\phi)=\xi\cdot\delta\mathbf{B}$, then Eq. (29) can be written

$$\delta\mathcal{E}=\frac{\kappa}{2\pi}\delta\mathcal{S}+\Omega_H\delta\mathcal{J}_H, \qquad (34)$$

where the varied quantities in Eq. (34) are defined below, and have well-known physical interpretations [4]. These are (i) the canonical energy of the system, which we define as

$$\mathcal{E}\equiv\int_{S_\infty}\mathbf{Q}[\xi]-\xi\cdot\mathbf{B}, \qquad (35)$$

(ii) the entropy $\mathcal{S}$ of the black hole; by taking the functional derivative of the Lagrangian with respect to the Riemann tensor (treated as an independent field) we know (setting $\boldsymbol{\epsilon}_{ab}$ to be the binormal to the bifurcation sphere) that

$$\delta\int_{\mathcal{H}}\mathbf{Q}[\chi]=\frac{\kappa}{2\pi}\delta\mathcal{S}, \qquad (36)$$

where

$$\mathcal{S}\equiv-2\pi\int_{\mathcal{H}}\frac{\delta\mathbf{L}}{\delta R_{abcd}}\boldsymbol{\epsilon}_{ab}\boldsymbol{\epsilon}_{cd}, \qquad (37)$$

and κ is the surface gravity of the background black hole horizon; (iii) the angular momentum of the system measured at the black hole, defined by

$$\mathcal{J}_H\equiv-\int_{\mathcal{H}}\mathbf{Q}[\varphi]. \qquad (38)$$

In fact, the angular momentum can be measured either at the black hole horizon or at spatial infinity; since the metric is axisymmetric with axial killing field φ^a, it can be seen from Eq. (7) that $\mathbf{J}[\varphi]$ vanishes, when pulled back to a slice to which φ^a is tangent. This ensures (integrating the relation $\mathbf{J}[\varphi]=d\mathbf{Q}[\varphi]$ over Σ) that, for the background solution,

$$\int_{S^\infty}\mathbf{Q}[\varphi]=\int_{\mathcal{H}}\mathbf{Q}[\varphi]. \qquad (39)$$

In addition, by considering the identity

$$\int_\Sigma\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\varphi\phi)=\int_{\partial\Sigma}\delta\mathbf{Q}[\varphi]-\varphi\cdot\boldsymbol{\Theta}, \qquad (40)$$

we see that when φ^a Lie derives all the dynamical fields, the left side of this equation vanishes. Since φ^a is tangent to the two-spheres $\mathcal{H}$ and S^∞, the pullback of the second term on the right side vanishes. It follows that

$$\delta\int_{S^\infty}\mathbf{Q}[\varphi]=\delta\int_{\mathcal{H}}\mathbf{Q}[\varphi]. \qquad (41)$$

Therefore, in spacetimes which have axisymmetric background configurations, the angular momentum measured at the black hole is equivalent to the canonical angular momentum $\mathcal{J}$, measured at spatial infinity

$$\mathcal{J}\equiv-\int_{S^\infty}\mathbf{Q}[\varphi], \qquad (42)$$

both when φ is an axial killing field (in the background solution) and for arbitrary solutions which are perturbations, $\delta\phi$, of the axisymmetric solution. This calculation also shows that the definition of $\mathcal{J}_H$ is gauge independent, for arbitrary perturbations of an axisymmetric solution. This is because $\delta\mathcal{J}_H=0$ when we choose $\delta\phi$ to be pure gauge, which we see by first setting $\hat{\delta}\phi\equiv\mathcal{L}_v\phi$ for some smooth v^a, and then replacing $\hat{\delta}\phi$ with a gauge transform $\hat{\delta}'\phi$ which coincides with $\hat{\delta}\phi$ in a neighborhood of the bifurcation sphere, but vanishes in a neighborhood of spatial infinity. Then we have, for every $\hat{\delta}\phi$ [using Eq. (41)],

$$\hat{\delta}\mathcal{J}_H=\hat{\delta}\int_{\mathcal{H}}\mathbf{Q}[\varphi]=\hat{\delta}'\int_{\mathcal{H}}\mathbf{Q}[\varphi]=\hat{\delta}'\int_{S^\infty}\mathbf{Q}[\varphi]=0. \qquad (43)$$

So we have that when T^{ab} vanishes (along with $\mathbf{L}_m$), the interpretation of the terms in Eq. (29) is straightforward and one obtains a formula (34) which (bearing in mind the equivalence of $\mathcal{J}$ and $\mathcal{J}_H$) is formula (2).

What if the set of fields ψ is nonempty? In general, the ambiguity (20) in breaking $\mathbf{L}$ into $\mathbf{L}_g$ and $\mathbf{L}_m$ stops us from meaningfully interpreting the surface terms in Eq. (29) as perturbations of mass, entropy, and angular momentum: even if the overall theory is fixed, every choice of $\mathbf{L}_g$ generates a different relation, with different choices of $\mathbf{Q}_g$, etc. We therefore seek more restrictive assumptions under which we might successfully identify the surface terms in Eq. (29). One approach is to *fix* a particular choice of $\mathbf{L}_g$ and think of it as specifying an independent theory. We assume there exists a form $\mathbf{B}_g$ such that at spatial infinity, $\delta(\xi\cdot\mathbf{B}_g)=\xi\cdot\boldsymbol{\Theta}_g$, and consider the functional M_g defined by

$$M_g\equiv\int_{S_\infty}\mathbf{Q}_g[\xi]-\xi\cdot\mathbf{B}_g. \qquad (44)$$

If we now require that the stress energy of the matter distribution falls off sufficiently rapidly at spatial infinity, such that (near spatial infinity) the metric for any solution of the $\mathbf{L}$ theory approaches a metric solution of the $\mathbf{L}_g$ theory, and M_g yields the same result on both metrics, then it makes sense to define the mass of the system as M_g. We note that if we can also find a form $\mathbf{B}(\phi)$ for the full theory, such that at spatial infinity $\delta(\xi\cdot\mathbf{B})=\mathbf{\Theta}(\phi,\delta\phi)$, then we can also define a canonical energy, $\mathcal{E}$, for the full theory given by Eq. (35), and in general $\mathcal{E}\neq M_g$.

Therefore, when the stress energy of the matter distribution falls off sufficiently rapidly, we can interpret the left side of Eq. (29) as the variation of the mass of the system. The surface terms on the right side of Eq. (29) are (variations of) the functionals that would measure the entropy and angular momentum of a stationary black hole in the $\mathbf{L}_g$ theory. We might therefore be tempted to interpret them as the black hole entropy and angular momentum; indeed, since $\mathbf{Q}_g$ is the Noether charge of the $\mathbf{L}_g$ theory, we know from [4] that one can define a quantity S_g by

$$S_g\equiv-2\pi\int_{\mathcal{H}}\frac{\delta\mathbf{L}_g}{\delta R_{abcd}}\boldsymbol{\epsilon}_{ab}\boldsymbol{\epsilon}_{cd}\,, \tag{45}$$

such that

$$\delta\int_{\mathcal{H}}\mathbf{Q}_g[\chi]=\frac{\kappa}{2\pi}\delta S_g\,. \tag{46}$$

One might also define a quantity, J_{gH}, by

$$J_{gH}\equiv-\int_{\mathcal{H}}\mathbf{Q}_g[\varphi]. \tag{47}$$

Although we made no assumptions about the axisymmetry of the matter fields, we can show, providing the support of T^{ab} does not intersect some neighborhood, U, of the bifurcation sphere, that J_{gH} is also well defined (gauge independent) for arbitrary perturbations of the axisymmetric solution. This follows by evaluating the left side of Eq. (40), using the fact that the calculation (30) also holds when ξ^a is replaced by φ^a. Taking φ^a to be tangent to the spatial slice, Eq. (40) then becomes

$$\int_{S_\infty}\delta\mathbf{Q}_g[\varphi]+\int_\Sigma\delta(\boldsymbol{\epsilon}\cdot T\cdot\varphi)=\int_{\mathcal{H}}\delta\mathbf{Q}_g[\varphi]. \tag{48}$$

Now, as before, let the perturbation in this equation be gauge, $\delta\phi=\hat{\delta}\phi$. Then we again can replace the perturbation on the right side with an equivalent gauge change, which vanishes outside U, and so intersects neither the support of T^{ab} nor spatial infinity. Then we have the left side of Eq. (48) vanishes, and so

$$\hat{\delta}J_{gH}=-\int_{\mathcal{H}}\hat{\delta}\mathbf{Q}_g[\varphi]=0. \tag{49}$$

Therefore J_{gH} is defined for arbitrary perturbations of an axisymmetric solution.

Now having defined M_g, S_g, and J_{gH}, we could write out Eq. (29) in the form

$$\delta M_g=\frac{\kappa}{2\pi}\delta S_g+\Omega_H\delta J_{gH}+\int_\Sigma\frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab}-\delta(\boldsymbol{\epsilon}\cdot T\cdot\xi), \tag{50}$$

where δ is a perturbation to an arbitrary nearby solution. However, we caution the reader that the identification of black hole entropy with S_g in general gives results in conflict with those in [4]: consider a theory of gravitation with a scalar field, for which the matter Lagrangian couples to the spacetime curvature, and which displays stationary black hole configurations in which the scalar field has sufficiently rapid spatial falloff. We can therefore write out Eq. (50) and interpret the black hole entropy as S_g. From the results of [4] we expect the entropy of the black hole to *include* contributions from the scalar field; Eq. (50), however, defines a black hole entropy S_g with only metric contributions, with the entropy contribution of the scalar field somehow distributed in the volume integral of its stress energy. These two points of view are contradictory; therefore, while there are clearly special cases (for instance, the Einstein-perfect fluid system) in which we can identify S_g as the black hole entropy, and terms in the volume integral as (variations of) the matter entropy, in general we regard the notion of the black hole entropy defined by S_g as inappropriate. Clarifying when S_g can be correctly interpreted as black hole entropy is the subject of future research.

We note parenthetically that we can write out an alternative form of Eq. (50) by replacing the stationary killing field ξ^a in Eq. (18) with the horizon killing field χ^a. [The analysis up to Eq. (32) is unchanged except for the substitution $\xi^a\rightarrow\chi^a$.] Then expanding $\chi^a=\xi^a+\Omega_H\varphi^a$ at spatial infinity and on the slice Σ, (but not at $\mathcal{H}$) and using the definitions discussed above gives the identity

$$\delta M_g=\frac{\kappa}{2\pi}\delta S_g+\Omega_H\delta J_{g\infty}+\int_\Sigma\frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T^{ab}\delta g_{ab}-\delta(\boldsymbol{\epsilon}\cdot T\cdot\xi)-\Omega_H\int_\Sigma\delta(\boldsymbol{\epsilon}\cdot T\cdot\varphi), \tag{51}$$

where $J_{g\infty}\equiv-\int_{S^\infty}\mathbf{Q}_g[\varphi]$, is the system angular momentum measured at spatial infinity. Therefore the cost we have incurred for the transfer of the angular momentum integral to spatial infinity is the appearance of an extra term in the volume integral.

A relation of the form (50), was first given by Schutz and Sorkin [7], in the case where $\mathbf{L}_g$ was fixed to be the Lagrangian for general relativity, $\mathbf{L}_m$ was any matter Lagrangian, and there was no black hole boundary $\mathcal{H}$, for the hypersurface Σ. The relation stated in [7] is correct, but we comment here on the ambiguity of the ''Noether operators'' used by Schutz and Sorkin to derive it: In its initial definition [7] the Noether operator for a Lagrangian $\mathbf{L}$ and a smooth vector field ξ^a was defined to be any (not necessarily covariant) three form $\mathbf{J}^S[\xi]$ satisfying the relation

$$\mathcal{L}_\xi\mathbf{L}=\mathbf{E}\mathcal{L}_\xi\phi+d(\mathbf{J}^S[\xi]+\xi\cdot\mathbf{L}), \tag{52}$$

for every smooth field vector field ξ^a. This definition leaves $\mathbf{J}^S[\xi]$ ambiguous by an *arbitrary* exact three-form which is a linear differential operator in ξ^a. Since we know from Eq. (10) that $\mathbf{J}^S = d\mathbf{Q}[\xi]$ when the field equations hold, this ambiguity would permit $\mathbf{J}^S = 0$ as a valid Noether operator (which, following Schutz and Sorkin's methods, would yield a correct but trivial relation). On the other hand, our definition of the Noether current admits a limited set of ambiguities [stated after Eq. (15)], which cannot be used to annihilate the Noether charge, and in particular do not change the content of the first law.

Sorkin introduced an augmented definition of the Noether operator in [8], requiring that for a variation of the dynamical fields given by $\delta\phi = f\mathcal{L}_\xi\phi$, where f is any function, the Noether operator $\mathbf{J}^{S'}$ be defined by

$$\delta\mathbf{L} = \mathbf{E} f\mathcal{L}_\xi\phi + d(f\mathbf{J}^{S'}[\xi] + f\xi\cdot\mathbf{L}). \tag{53}$$

Providing one can find a $\mathbf{J}^{S'}$ which satisfies this relation, it is easy to see that one cannot add a term to $\mathbf{J}^{S'}$ which is both exact and linear in f, for arbitrary f. For a theory with a first-order Lagrangian, finding such a $\mathbf{J}^{S'}$ is always possible: in [8] a first-order (noncovariant) Lagrangian for the Einstein-Maxwell theory was used to yield an unambiguous Noether operator. It is not clear, however, that any general Lagrangian theory has a first-order *Lagrangian* formulation, so in general, Sorkin's definition may not even yield a Noether operator. In contrast, all of our Noether currents $\mathbf{J}[\xi]$ defined above can be computed for Lagrangian theories of arbitrary derivative order, and are manifestly covariant, requiring no additional background fields (apart from the symmetry field ξ^a) for their definition. For these reasons, we feel that while our relation (50) and that in [7] coincide for an Einstein-matter system without the black hole, Eq. (50) is defined more generally.

We finally remark that we could have carried out the entire analysis leading up to Eq. (29) allowing the Lagrangian $\mathbf{L}_g$ to depend on a *set* of stationary axisymmetric fields, s_i, including the metric, and the Lagrangian $\mathbf{L}_m$ to depend on s_i and a distinct set of fields, ψ, which did not appear in $\mathbf{L}_g$, to obtain a relation very similar to Eq. (29). The resulting perturbative identity has the terms $\mathbf{Q}_g$ and $\mathbf{\Theta}_g$ in Eq. (29) replaced with the Noether charge and symplectic potential in the theory described by $\mathbf{L}_g$ (which now depends on both the metric and the other matter fields in the set s_i), and the volume term is now given by

$$\int_\Sigma \frac{1}{2}\xi\cdot\boldsymbol{\epsilon}T_{s_i}\delta s_i - \delta(\boldsymbol{\epsilon}\cdot T_s\cdot s\cdot\xi), \tag{54}$$

where the first term in the volume integral is defined by the variation of $\mathbf{L}_m$:

$$\delta\mathbf{L}_m = \mathbf{E}_m\delta\psi + \frac{1}{2}T_{s_i}\delta s_i + d\mathbf{\Theta}_m(\phi,\delta\phi). \tag{55}$$

Giving each s_i field u_i upper, and d_i lower indices, in the manner

$$s_i \rightarrow s_i{}^{b_{u_1}\cdots b_{u_i}}{}_{a_1\cdots a_{d_i}}, \tag{56}$$

the second term in the volume integral is defined by

$$\begin{aligned}(\boldsymbol{\epsilon}\cdot T_s\cdot s\cdot\xi)_{abc} &\equiv \boldsymbol{\epsilon}_{eabc}\sum_i \, [T_{s_i b_1\cdots b_{ui}}{}^{a_1\cdots a_{di}} \\ &\quad \times(-s_i{}^{e\cdots b_{ui}}{}_{a_1\cdots a_{di}}\delta^{b_1}_p\cdots - s_i{}^{b_1\cdots e}{}_{a_1\cdots a_{di}}\delta^{b_{u_i}}_p \\ &\quad + s_i{}^{b_1\cdots b_{ui}}{}_{p\cdots a_{di}}\delta^e_{a_1}\cdots + s_i{}^{b_1\cdots b_{u_i}}{}_{a_1\cdots p}\delta^e_{a_{d_i}})\xi^p].\end{aligned} \tag{57}$$

III. A REVIEW OF PERFECT FLUIDS AND THREE VARIATIONAL FORMULATIONS

In this section we recall the definition, the relevant properties, and three variational principles for a self-gravitating perfect fluid: one given by Schutz [9] (which we use in the Appendix to derive a conserved current for perturbations of Einstein-perfect fluid systems), the ''axionic vorticity'' formulation given by Carter [10] for an isentropic perfect fluid (which we use in the next section, to derive a first law), and a ''convective'' approach also described by Carter [10]. Our aim is to gather the results we need for the calculations of the following sections; detailed treatments of these variational principles can be found in [9,10,14].

From the viewpoint of black hole mechanics, we would like a stationary axisymmetric black hole configuration to be represented by a Lagrangian theory in which all the fields appearing in the Lagrangian (the *dynamical* fields) are also stationary and axisymmetric. Having stated these formulations, however, we will see that they all have fluid configurations in which the *physical* fields (the fluid four-velocity, number density, entropy, and functions of these fields) are stationary and axisymmetric, but in which the dynamical fields possibly share neither of these symmetries. The question as to whether a variational principle exists that always represents (physically) stationary axisymmetric configurations with dynamical fields that also have these properties is (as far as we are aware) open.

By a perfect fluid on a fixed spacetime background [14,15] we mean a system described by five scalar fields, (n,s,ρ,p,T), on spacetime and one (unit, timelike) vector field U^a, such that $\rho = \rho(n,s)$ is a fixed function, and the following equations hold on the fields: the first law of thermodynamics

$$d\rho(n,s) = \frac{p+\rho}{n}dn + nTds, \tag{58}$$

and the equations of motion

$$\nabla_a(nU^a) = 0 \quad \text{and} \quad \nabla_a T^{ab} = 0, \tag{59}$$

where T^{ab} is defined by

$$T^{ab} \equiv (p+\rho)U^aU^b + pg^{ab}. \tag{60}$$

The fields n,ρ,s,p,T, and U^a have physical interpretations as the number density, energy density, entropy per particle (specific entropy), pressure, temperature, and four-velocity of the fluid, respectively.

We note that Eq. (59) can be given a useful alternative form, by first defining the specific inertial mass μ:

$$\mu \equiv \frac{p+\rho}{n} \tag{61}$$

which along with Eq. (58) implies

$$dp = n d\mu - nT ds. \tag{62}$$

By using these relations in the second equation of Eq. (59), we get (see [14]) an equivalent pair of equations of motion

$$\nabla_a(nU^a)=0 \quad \text{and} \quad nU^a\omega_{ab}=nT\nabla_b s, \tag{63}$$

where the fluid vorticity two-form, ω_{ab}, is defined as

$$\omega_{ab} \equiv 2\nabla_{[a}(\mu U_{b]}). \tag{64}$$

If desired, one can define the entropy per unit volume S (entropy density), by $S=ns$. Substituting this definition of S into Eq. (58) and defining the chemical potential, μ', by

$$\mu' \equiv \frac{p+\rho-TS}{n}, \tag{65}$$

then gives the relation

$$d\rho(n,S)=\mu' dn + TdS. \tag{66}$$

We now specify three variational formulations for this perfect fluid, over a fixed spacetime background (coupling the theories to gravitation amounts to adding the appropriate metric Lagrangian, which we do later). First, we state the ''velocity-potential'' representation of Schutz [9]: here the dynamical fields of the fluid are given by scalars Φ, α, β, θ, and σ. One now defines a function m which depends on these fields via the relation

$$m^2 = -(\nabla_a\Phi+\alpha\nabla_a\beta+\theta\nabla_a\sigma)(\nabla^a\Phi+\alpha\nabla^a\beta+\theta\nabla^a\sigma), \tag{67}$$

and the fluid Lagrangian is given by

$$\mathbf{L}_f \equiv \boldsymbol{\epsilon} P(m,\sigma), \tag{68}$$

where $P(m,\sigma)$ is some fixed function. One can verify [9,14] that we recover Eq. (58), and also that the equations of motion for the fields Φ, α, β, θ, σ arising from this Lagrangian reduce to Eq. (63), provided one defines the physical fields in terms of the dynamical fields in these equations by

$$\begin{gathered} P \to p, \\ m \to \mu, \\ \sigma \to s, \\ (\partial P/\partial m)_\sigma \to n, \\ (\partial P/\partial \sigma)_m \to -nT, \\ \nabla_a\Phi+\alpha\nabla_a\beta+\theta\nabla_a\sigma \to \mu U_a . \end{gathered} \tag{69}$$

Conversely, given any configuration of the physical fields (n,ρ,s,p,T,U^a) satisfying Eqs. (58) and (59), it can be shown (see [9]) that there exist functions (P,m) and (nonunique) dynamical fields $(\Phi,\alpha,\beta,\theta,\sigma)$ related to the physical fields by Eq. (69), which satisfy the equations of motion arising from Lagrangian (68).

Next, Carter's variational formulation [10] for an *isentropic* perfect fluid (by which we mean that the fluid has an everywhere constant specific entropy s), defines the dynamical fields to be a two-form and two scalars, b_{ab} and $\chi^{\pm}$. The fluid Lagrangian is given in terms of these fields by

$$\mathbf{L}_f = \left(-r(\nu) - \frac{1}{2}\boldsymbol{\epsilon}^{abcd} b_{ab}\nabla_c\chi^+\nabla_d\chi^-\right)\boldsymbol{\epsilon}, \tag{70}$$

where the function $r(\nu)$ is fixed, and the function ν is defined in term of the potentials b_{ab} by the relation

$$\nu^2 \equiv \frac{3}{2}(\nabla_{[a}b_{bc]})(\nabla^{[a}b^{bc]}). \tag{71}$$

As shown in [10], if one defines the physical fields as

$$\begin{gathered} r \to \rho, \\ \nu \to n, \\ \nu(\partial r/\partial \nu) - r \to p, \\ 3\nabla_{[c}b_{ab]} \to N_{abc}, \end{gathered} \tag{72}$$

where the number-density three-form N_{abc} is given by

$$N_{abc} = n\boldsymbol{\epsilon}_{abcd}U^d, \tag{73}$$

then we recover Eq. (58), and the field equations for b_{ab} and $\chi^{\pm}$ yield the second equation in Eq. (63) in the case $\nabla_a s=0$, as well as the relation

$$\omega_{ab} = 2\nabla_{[a}\chi^+\nabla_{b]}\chi^-. \tag{74}$$

Given relation (72) between b_{ab} and N_{abc}, one sees that the first equation in Eq. (63) is satisfied vacuously, since it can be rewritten as

$$\nabla_{[a}N_{bcd]}=0, \tag{75}$$

but the definition of N_{abc} shows $d\mathbf{N}=dd\mathbf{b}=0$ automatically.

A third type of variational formulation given by Carter [10], and treated in more detail by Brown [14], (which is the equivalent diffeomorphism invariant version of the formalisms specified by Taub [16] or Hawking and Ellis [17]), has dynamical fields X^A for $A=1,2,3$. In this formalism one must specify two functions $r(\nu,\sigma)$, and $\sigma(X)$, where ν is defined in terms of the X^A by

$$\nu^2 \equiv 6[N_{ABC}(X)\nabla_a X^A\nabla_b X^B\nabla_c X^C] \times [N_{DEF}(X)\nabla^a X^D\nabla^b X^E\nabla^c X^F], \tag{76}$$

and $N_{ABC}(X)$ is a fixed three-form on the three-dimensional manifold which has X^A as coordinate fields. The Lagrangian is then given by

$$\mathbf{L}_f = -\boldsymbol{\epsilon} r(\nu,\sigma). \tag{77}$$

The equations resulting from this Lagrangian for the fields X^A are seen to reduce to the second equation in Eq. (63) after one has set

$$r \to \rho,$$

$$\nu \to n,$$

$$\sigma \to s,$$

$$\nu(\partial r/\partial \nu)_\sigma - r \to p,$$

$$(\partial r/\partial \sigma)_m \to nT,$$

$$N_{ABC}(X)\nabla_a X^A \nabla_b X^B \nabla_c X^C \to N_{abc}, \tag{78}$$

where N_{abc} is defined from Eq. (73). (This relation between the physical N_{abc} and the dynamical fields also ensures that N_{abc} is automatically conserved.) The X^A are interpreted as coordinates on a ''base manifold,'' obtained by treating the spacetime as a bundle with fibers given by the integral curves of the four-velocity. We will not use this formulation for two reasons: first, the assignment of the entropy s as a fixed function of the X^A only allows us to perturb it by diffeomorphisms of the base manifold (for this reason we use Schutz's formalism for the calculation in the Appendix). Second, it is unclear that there are *any* solutions in which the X^A are globally well-defined axisymmetric fields on spacetime [for this reason, in Sec. IV, we use the formulation due to Carter with Lagrangian (70)].

In order to write the first law in form (2), only involving surface integrals, we must assume that all the dynamical fields are stationary and axisymmetric in the background solution. Now even if a fluid configuration has stationary and axisymmetric *physical* fields (the fluid number density, entropy, and functions of these fields), the *dynamical* fields (the fields appearing in the Lagrangian) corresponding to these physical fields may not possess these symmetries. Therefore, the requirement of stationarity and axisymmetry on the dynamical fields may restrict the choice of background configurations. In fact, for Schutz's formulation, we see from the definition of the four-velocity (69) that physical fluid configurations with an everywhere causal four-velocity (including those which are stationary and axisymmetric) must include at least one nonstationary dynamical field. There are therefore no physically interesting fluid configurations in which all the dynamical fields in this formulation are stationary.

On the other hand, for Carter's formulation, it is evident that there must be *some* physically stationary fluid configurations with stationary dynamical fields (for instance, a static spherically symmetric fluid distribution could have the field b_{ab} given by $\mathbf{b} \sim f(r)^2\boldsymbol{\epsilon}$ and $\chi^\pm = 0$, where ${}^2\boldsymbol{\epsilon}$ is the volume element on the spheres of symmetry). However, we will see in the next section [in the discussion above Eq. (95)] that a stationary, axisymmetric, circular flow (in a spacetime which also has these symmetries) must be vortex-free, if $\chi^\pm$ are restricted to be stationary and axisymmetric. That is, the assumption of stationarity and axisymmetry on the vorticity potentials $\chi^\pm$ restricts the allowed stationary axisymmetric configurations a fluid can adopt. We make no attempt here to enumerate the set of physically stationary and axisymmetric configurations which also have these symmetries in the dynamical fields (or indeed, in the case of black hole spacetimes, to investigate whether this set is nonempty). Rather, in the following section we will assume the potentials are stationary and axisymmetric, and write out the resulting first law involving only surface terms, looking for any nontrivial modifications arising from the fluid fields.

We are unaware of a variational formulation for a perfect fluid which represents all stationary axisymmetric fluid configurations with stationary axisymmetric dynamical fields. If it exists, then the following argument by Schutz and Sorkin [7] shows that certain compactly supported perturbations of the physical fields must correspond to noncompactly supported perturbations of the dynamical fields. Since the calculation given in Eq. (30) does not depend on the fulfillment of the field equations for g_{ab}, it is still valid if we consider the fields ψ to be the dynamical fields for a perfect fluid over a fixed spacetime background, and we let $\delta\psi$ be a perturbation to a nearby solution of the perfect fluid equations, with $\delta g_{ab} = 0$. Now consider a formulation for a perfect fluid where, for a general configuration in which all the physical fields (and the metric of the spacetime background) are stationary, all the dynamical fields are also stationary. Then the left side of Eq. (30) vanishes, and integrating the right side over a spatial slice Σ, we are left with

$$\int_\Sigma \delta(\boldsymbol{\epsilon}\cdot T\cdot \xi) = \int_{\partial\Sigma} \delta\mathbf{Q}_m[\xi] - \xi\cdot \mathbf{\Theta}_m(\phi,\delta\psi). \tag{79}$$

This implies that for perturbations of the physical fields for which the corresponding perturbations of the dynamical fields are *compact*, we must have

$$\int_\Sigma \delta(\boldsymbol{\epsilon}\cdot T\cdot \xi) = 0, \tag{80}$$

which, for a perfect fluid, is clearly false for a general stationary background. This implies that if a variational formulation is to have dynamical fields which are always stationary when the physical fields are stationary, then perturbations of the physical fields which yield a nonzero result on the left side of (79) must correspond to spatially noncompact perturbations of the dynamical fields. This requirement rules out the existence of a variational principle in which the physical fields *are* the dynamical fields [7]. However, the existence of a variational principle for a perfect fluid in which all configurations with stationary and axisymmetric physical fields are represented by dynamical fields with these symmetries is still an open question.

IV. FIRST LAWS OF BLACK HOLE MECHANICS WITH PERFECT FLUIDS

We now present two forms of the first law of black hole mechanics which incorporate perfect fluids. The first form is a special case of the perturbative identity (50), where $\mathbf{L}_g$ is the usual Hilbert Lagrangian for general relativity, and $\mathbf{L}_m$ is any Lagrangian for a perfect fluid. This form of the first law allows nonstationary dynamical fields, at the cost of having volume integrals in the interior of the spacetime. We then compute a second form of the first law only involving surface integrals for both metric and fluid fields, using Carter's variational formulation presented above, and the methods of [4].

A. The first law with volume integrals

We now write out the perturbative relation (50), setting $\mathbf{L}_g = 1/16\pi R$, and $\mathbf{L}_m$ to be any perfect fluid Lagrangian which allows all possible perturbations of the physical fields of the perfect fluid off an arbitrary background. [From the comments below Eq. (69) it is evident that Schutz's variational formulation, with Lagrangian (68) satisfies this criterion.] As stated in lemma 2, we assume the metric of the background spacetime is asymptotically flat, stationary, and axisymmetric with a stationary killing field ξ^a and axial killing field φ^a. We also assume the existence of a bifurcate killing horizon, with horizon killing field $\chi^a = \xi^a + \Omega_H \varphi^a$, where Ω_H is the angular velocity of the horizon.

In this case (see [4]) the term M_g in Eq. (50) can be shown to be the ADM mass, S_g to be $1/4A_H$, and J_{gH} the expression J_H for black hole angular momentum given in Eq. (1). The terms involving the stress-energy tensor have been shown by Bardeen, Carter, and Hawking [1] to reduce to the fluid terms in Eq. (1), but for completeness (and to fix the signs for our choice of orientations) we briefly demonstrate this fact: in [1] the four-velocity of the fluid with angular velocity Ω (which need not be constant) was set to be $U^a = v^a/(-v\cdot v)^{1/2}$ where $v^a = \xi^a + \Omega\varphi^a$. Now using Eqs. (60), (65), and (66) (assuming, as usual, that we identify the perturbed spacetime such that $\delta\xi^a = \delta\phi^a = 0$), one obtains

$$\begin{aligned}\delta(T^a{}_b\xi^b\boldsymbol{\epsilon}_{apqr}) &= v^b\,\delta T^a{}_b\boldsymbol{\epsilon}_{apqr} - \Omega\,\delta(T^a{}_b\varphi^b\boldsymbol{\epsilon}_{apqr}) = v^a\,\delta[(\mu' n + TS)v_a(-v\cdot v)^{-1/2}U^b\boldsymbol{\epsilon}_{bpqr} + p\,\boldsymbol{\epsilon}_{apqr}] - \Omega\,\delta(T^a{}_b\varphi^b\boldsymbol{\epsilon}_{apqr})\\
&= (p+\rho)v^a\,\delta[v_a(-v\cdot v)^{-1/2}]U^b\boldsymbol{\epsilon}_{bpqr} + \frac{1}{2}pg^{cd}\,\delta g_{cd}\xi^a\boldsymbol{\epsilon}_{apqr} - \mu'(-v\cdot v)^{1/2}\delta(nU^b\boldsymbol{\epsilon}_{bpqr})\\
&\quad - T(-v\cdot v)^{1/2}\delta(SU^b\boldsymbol{\epsilon}_{bpqr}) - (n\,\delta\mu' + S\,\delta T)\xi^a\boldsymbol{\epsilon}_{apqr} + v^a\boldsymbol{\epsilon}_{apqr}\delta p - \Omega\,\delta(T^a{}_b\varphi^b\boldsymbol{\epsilon}_{apqr})\\
&= \xi^a\boldsymbol{\epsilon}_{apqr}\frac{1}{2}T^{cd}\,\delta g_{cd} + \mu'(-v\cdot v)^{1/2}\delta(nU^b\boldsymbol{\epsilon}_{pqrb}) - T(-v\cdot v)^{1/2}\delta(SU^b\boldsymbol{\epsilon}_{bpqr}) - \Omega\,\delta(T^a{}_b\varphi^b\boldsymbol{\epsilon}_{apqr}).\end{aligned}\tag{81}$$

When all these substitutions are inserted into Eq. (50), it reduces to

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H\delta J_H - \int_\Sigma \mu'|v|\,\delta N_{abc} + \int_\Sigma \Omega\,\delta J_{abc} + \int_\Sigma T|v|\,\delta S_{abc}, \tag{82}$$

which is identical to Eq. (1), except that δ now represents an arbitrary perturbation (not necessarily stationary or axisymmetric) of the background. In this sense, Eq. (82) is a generalization of Eq. (1).

B. A (restricted) first law with surface integrals

In the previous section we observed that the variational formulations we presented were constrained in the stationary axisymmetric fluid configurations they could represent, given the requirement that their dynamical fields obeyed these symmetries. One might therefore suspect that any form of the first law involving only surface integrals could not include nontrivial fluid contributions. Indeed, if we add Schutz's Lagrangian (68) to the Lagrangian of an arbitrary metric theory of gravity, and construct a first law using the analysis of [4] then we find *no* additional contributions to this first law from the fluid fields, providing the fluid's number density decays sufficiently rapidly at spatial infinity, and does not intersect the black hole horizon. It is possible, however, to convert *some* of the volume integrals in (1) into surface integrals, by choosing Carter's variational formulation (70). We do so below, finding a first law for an arbitrary metric theory of gravity coupled to an isentropic perfect fluid, in which the background configuration for the perfect fluid as well as the allowed perturbations of the physical fields are restricted. (Note that the gravitational contributions to such a first law have been considered in detail in [4]. We are interested in the fluid contributions.) We finally verify that this first law reduces to Eq. (1) when the assumptions made in the two derivations overlap. Our first law is the following result.

Lemma 3. Let $\mathbf{L}$, given by

$$\mathbf{L} = \mathbf{L}_g - \boldsymbol{\epsilon}\left(r(\nu) + \frac{1}{2}\boldsymbol{\epsilon}^{abcd}b_{ab}\boldsymbol{\nabla}_{[c}\chi^+\boldsymbol{\nabla}_{d]}\chi^-\right), \tag{83}$$

be the Lagrangian for an isentropic perfect fluid coupled to an arbitrary metric theory of gravity, where $\mathbf{L}_g = \boldsymbol{\epsilon}L_g[g_{ab}, R_{abcd}, \boldsymbol{\nabla}R_{abcd}, \ldots, (\boldsymbol{\nabla}^p)R_{abcd}]$, and the perfect fluid formulation, with dynamical fields $(b_{ab}, \chi^\pm)$, is summarized below Eq. (70). Fix an asymptotically flat black hole solution with a bifurcate killing horizon, with the spacetime structure and the killing fields described in lemma (2), with the additional assumptions that *all* the dynamical fields (not just the metric) in this theory are stationary and axisymmetric, and that all the dynamical fields are globally defined. Let $\delta\phi$ be a perturbation of the dynamical fields, from such a solution to an arbitrary nearby solution, with $\delta\xi^a = 0$. With these assumptions the following identity is the first law of black hole mechanics for this system:

$$\delta M_g = \frac{\kappa}{2\pi}\delta\mathcal{S} + \Omega_H \delta\mathcal{J}_H + \int_{\mathcal{H}} \mu_\infty \delta b_{qr}, \\ - \int_{S^\infty} \mu_\infty \delta b_{qr}, + \int_{\mathcal{H}} X_{qr} - \int_{S^\infty} X_{qr}. \quad (84)$$

Here we define the mass of the system M_g as

$$M_g \equiv \int_{S_\infty} \mathbf{Q}_g[\xi^a] - \xi\cdot\mathbf{B}_g \quad (85)$$

and the entropy $\mathcal{S}$ and angular momentum $\mathcal{J}_H$ of the system by

$$\mathcal{S} \equiv -2\pi \int_{\mathcal{H}} \frac{\delta \mathbf{L}_g}{\delta R_{abcd}} \boldsymbol{\epsilon}_{ab} \boldsymbol{\epsilon}_{cd},$$

$$\mathcal{J}_H \equiv -\int_{\mathcal{H}} \mathbf{Q}_g[\varphi], \quad (86)$$

where κ is the surface gravity of the black hole, the two-form $\mathbf{Q}_g[\xi]$ was defined in Eq. (26), and the three-form $\mathbf{B}_g$ is such that, at spatial infinity, $\delta(\xi\cdot\mathbf{B}_g) = \xi\cdot\boldsymbol{\Theta}_g$, with $\boldsymbol{\Theta}_g$ given by Eq. (23). Finally, the two-form X_{qr} is defined by

$$X_{qr} \equiv 2\xi^p b_{p[q}[\delta(\mu U_{r]}) - \boldsymbol{\nabla}_{r]}\chi^- \delta\chi^+ + \boldsymbol{\nabla}_{r]}\chi^+ \delta\chi^-]. \quad (87)$$

Proof. The first law of black hole mechanics in [4] is essentially given by the right side of Eq. (18), when the left side vanishes because of the assumed symmetries of the background fields. We therefore compute the quantities appearing on the right side of Eq. (18): Varying the dynamical fields in $\mathbf{L}$ [and performing the substitutions (72) where applicable] yields the equations of motion and the symplectic potential $\boldsymbol{\Theta}$:

$$\delta\mathbf{L} = \boldsymbol{\epsilon}\left(\frac{\delta L_g}{\delta g_{ab}} + \frac{1}{2}T^{ab}\right)\delta g_{ab} \\ + \frac{1}{6}\boldsymbol{\epsilon} N_{abc}\boldsymbol{\epsilon}^{abcd}(\boldsymbol{\nabla}_d\chi^+ \delta\chi^- - \boldsymbol{\nabla}_d\chi^- \delta\chi^+) \\ + \boldsymbol{\epsilon}\left[\boldsymbol{\nabla}_c\left(\frac{\mu}{2n}N^{abc}\right) - \frac{1}{4}\boldsymbol{\epsilon}^{abcd}\omega_{cd}\right]\delta b_{ab} + d\boldsymbol{\Theta}, \quad (88)$$

with the stress-energy tensor

$$T^{ab} = \frac{\mu}{2n}N^a{}_{cd}N^{bcd} - \rho g^{ab}, \quad (89)$$

and the symplectic potential

$$\boldsymbol{\Theta}_{pqr}(\phi,\delta\phi) = \boldsymbol{\Theta}_{gpqr}(g,\delta g) - \frac{\mu}{2n}N^{abc}\delta b_{bc}\boldsymbol{\epsilon}_{apqr} \\ + \frac{1}{2}\boldsymbol{\epsilon}_{apqr} b_{cd}\boldsymbol{\epsilon}^{abcd}(\boldsymbol{\nabla}_b\chi^+ \delta\chi^- - \boldsymbol{\nabla}_b\chi^- \delta\chi^+). \quad (90)$$

It can be verified that the equations of motion for the fluid fields reduce to Eq. (63) using definitions (72) and (73). The stress-energy tensor (89) is also seen to reduce to the usual form (60) by expanding its first term:

$$\frac{\mu}{2n}N^a{}_{cd}N^{bcd} = \frac{\mu}{2n} n U^e \boldsymbol{\epsilon}^a{}_{cde}\boldsymbol{\epsilon}^{bcdf} n U_f = \mu n(g^{ab} + U^a U^b).$$

The Noether current associated to ξ^a is

$$\mathbf{J}_{pqr}[\xi] = \mathbf{J}_{gpqr}[\xi] - \left(\frac{\mu}{2n}N^{dbc}N_{ebc}\xi^e - \rho\xi^d\right)\boldsymbol{\epsilon}_{dpqr} \\ - \boldsymbol{\nabla}_b\left(\frac{\mu}{n}N^{dbc}\xi^e b_{ec}\boldsymbol{\epsilon}_{dpqr}\right). \quad (91)$$

Therefore, the integrand on the right side of Eq. (18) evaluates to

$$(\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\Theta})_{qr} = \delta\mathbf{Q}_{gqr}[\xi] - \delta\left(\frac{\mu}{2n}N^{abc}b_{ec}\xi^e\boldsymbol{\epsilon}_{abqr}\right) \\ - \xi^p\left(\boldsymbol{\Theta}_{gpqr} - \frac{\mu}{2n}N^{abc}\delta b_{bc}\boldsymbol{\epsilon}_{apqr} + \boldsymbol{\epsilon}_{apqr}\frac{1}{2}\boldsymbol{\epsilon}^{abcd}b_{cd}(\boldsymbol{\nabla}_b\chi^- \delta\chi^+ - \boldsymbol{\nabla}_b\chi^+ \delta\chi^-)\right) \\ = \delta\mathbf{Q}_{gqr}[\xi] - \xi^p\boldsymbol{\Theta}_{gpqr} - \xi^p U_p \mu\,\delta b_{qr} - \frac{1}{2}b_{qr}\xi^p(\boldsymbol{\nabla}_p\chi^- \delta\chi^+ - \boldsymbol{\nabla}_p\chi^+ \delta\chi^-) + X_{qr}, \quad (92)$$

where we define the two-form X_{qr} by Eq. (87), and we used the identification $N_{abc} \equiv \boldsymbol{\epsilon}_{abcd} n U^d$ to obtain the second line of Eq. (92).

When the background solution is a black hole with the structure and symmetries specified in the statement of the lemma, the fourth term in the second equation of Eq. (92) vanishes because the dynamical fields are stationary: $\xi\cdot\boldsymbol{\nabla}\chi^\pm = 0$. Now given the definition of vorticity (64) and its relation to the potentials (74), it is evident that (locally) there exists some function f such that μU_a can be rewritten

$$\mu U_a = \boldsymbol{\nabla}_a f + \chi^+ \boldsymbol{\nabla}_a \chi^-. \quad (93)$$

Let t be a function such that $\xi^a dt_a = 1$. Then the requirements that the four-velocity be causal, stationary, and axisymmetric, along with the assumed stationarity and axisymmetry of $\chi^{\pm}$ force f to be a sum of terms, one of which is strictly linear in t (we define the constant of proportionality to be $-\mu_\infty$). For the same reason the φ dependence of f must also be linear, but this dependence can be ruled out because the occurrence of such a term would force U^a to be acausal near spatial infinity. We therefore have that the form of f is

$$f=-\mu_\infty t+g, \tag{94}$$

where $\xi\cdot\nabla g=\varphi\cdot\nabla g=0$. Therefore we see that the assumption of stationarity and axisymmetry on the dynamical fields (taking the four-velocity to be everywhere causal) has restricted us to a very narrow range of allowed background four-velocities; for instance, we must have $\varphi^a U_a=0$. Moreover, when the vacuum theory is general relativity, with the flow assumed to be *circular* (tangent to the $\xi-\varphi$ subspaces), there is only one possible solution: for this theory the subspaces orthogonal to ξ^a and φ^a are integrable, and the resulting submanifolds can be endowed with coordinates (x^1,x^2), such that the metric is ''block diagonal'' with no ''cross terms'' between the subspace spanned by ξ^a,φ^a, and its orthogonal complement (see Chap. 7 of [3]). Now the assumption of circular flow forces $g=0$ and $\chi^+ d\chi^-=0$, leaving us with only

$$\mu U_a=-\mu_\infty dt_a. \tag{95}$$

In any case, using just the form of f in Eq. (94), we see

$$\xi^a\mu U_a=-\xi^a\mu_\infty dt_a=-\mu_\infty, \tag{96}$$

and the boundary term (92) reduces to

$$\delta\mathbf{Q}_{qr}[\xi]-\xi^p\mathbf{\Theta}_{pqr}=\delta\mathbf{Q}_{gqr}[\xi]-\xi^p\mathbf{\Theta}_{gpqr}+\mu_\infty\delta b_{qr}+X_{qr}. \tag{97}$$

We now assume the existence of a form $\mathbf{B}_g$ such that at spatial infinity $\xi\cdot\mathbf{\Theta}_g=\delta(\xi\cdot\mathbf{B}_g)$, and write out the first law of black hole mechanics by substituting Eq. (97) into the surface integrals on the right side of Eq. (18), observing that the left side of Eq. (18) vanishes due to the symmetries assumed on the dynamical fields. If we expand $\xi^a=\chi^a-\Omega_H\varphi^a$ at the bifurcation sphere for the first two terms of Eq. (97), then we obtain Eq. (84) which is what we wished to show. □

The results of [4] predicted that the first law (84) would only contain surface integrals, and we see this is indeed the case. Note, however, that the assumptions made about the symmetry of the dynamical fields restricted the allowed background fluid configurations for the fluid fields. Moreover, by perturbing the local form of μU_a in Eq. (93) we see that the restriction to stationary and axisymmetric $\chi^{\pm}$ in background also prevents us from achieving all possible perturbations of μU_a, by perturbing only the dynamical fields b_{ab} and $\chi^{\pm}$. Finally both the background and the perturbed configurations must be restricted such that the integral $\int_{S^\infty}X_{qr}$ converges. (This, along with the following result relating this term to the fluid angular momentum will guarantee the convergence of the corresponding boundary term at the bifurcation sphere.)

We finally show that Eq. (84) reduces to Eq. (1) when the assumptions made in the two derivations overlap. From our discussion in the last section we know that M_g, $\mathcal{S}$, and $\mathcal{J}_H$ reduce to their values for general relativity given in Eq. (1), when $L_g=(1/16\pi)R$. We start by considering the fluid contribution in our first law (84) from the integral

$$\delta\int_\infty\mu_\infty b_{qr}-\delta\int_{\mathcal{H}}\mu_\infty b_{qr}=\delta\int_\Sigma\mu_\infty N_{pqr}=\int_\Sigma\mu|v|\,\delta N_{pqr}, \tag{98}$$

where the last line follows because the fluid flow in [1] is assumed to be tangent to the subspaces spanned by ξ^a and φ^a: so taking the velocity to be $U^a=v^a/|v|$ where $v^a=\xi^a+\Omega\varphi^a$, we see from the discussion above Eq. (95) that $\mu=-\mu U\cdot U=\mu_\infty v\cdot dt/|v|=\mu_\infty/|v|$, and so $\mu|v|=\mu_\infty$. Our first law now takes the form

$$\delta M=\frac{\kappa}{8\pi}\delta A+\Omega_H\delta J_H-\int_\Sigma\mu|v|\,\delta N_{abc}+\int_{\mathcal{H}}X_{qr}-\int_{S^\infty}X_{qr}. \tag{99}$$

We now concentrate on the original form of the first law in Eq. (1) and show that it agrees with Eq. (99). By repeating the calculation (81) using relation (58) instead of Eq. (66) along with the assumption $\delta s=0$ (as befits an isentropic fluid), we find the form of Eq. (1) for an isentropic fluid:

$$\delta M=\frac{\kappa}{8\pi}\delta A+\Omega_H\delta J_H-\int_\Sigma\mu|v|\,\delta N_{abc}+\int_\Sigma\Omega\,\delta J_{abc}. \tag{100}$$

Next, we demonstrate that the pullback to Σ of the angular momentum density given in Eq. (100) reduces to the exterior derivative of the two form X_{qr} defined in Eq. (87), given the assumption that the dynamical fields are stationary and axisymmetric, i.e.,

$$\Omega\,\delta J_{pqr}=-(dX)_{pqr}, \tag{101}$$

where both sides are assumed pulled back to Σ. To do this we compute the exterior derivative of Eq. (87), finding

$$(dX)_{pqr}=3\xi^e N_{e[pq}(\delta(\mu U_{r]})-\nabla_{r]}\chi^-\delta\chi^++\nabla_{r]}\chi^+\delta\chi^-), \tag{102}$$

where we have assumed $\mathcal{L}_\xi b_{ab}=0$. Pulling this form back to Σ by contracting with $1/6\boldsymbol{\epsilon}^{spqr}n_s$ (where n_s is the unit normal to Σ) yields

$$\begin{aligned}\overline{dX}=&-{}^3\boldsymbol{\epsilon}(2nn_e\delta(\mu U_r)U^{[e}\xi^{r]}\\&+2\xi^{[e}U^{r]}n_e(\nabla_r\chi^-\delta\chi^+-\nabla_r\chi^+\delta\chi^-)),\end{aligned} \tag{103}$$

where ${}^3\boldsymbol{\epsilon}$ is the volume form induced on Σ: ${}^3\boldsymbol{\epsilon}_{bcd}\equiv n^a\boldsymbol{\epsilon}_{abcd}$. Now using the axisymmetry of the $\chi^{\pm}$, and writing U^a as $U^a=v^a/|v|$ with angular velocity Ω as given above Eq. (99), we have (using $\delta\xi^a=\delta\phi^a=0$)

$$\begin{aligned}\overline{dX}&=-{}^3\boldsymbol{\epsilon}2nn_e\Omega\,\delta(\mu U_r)\varphi^{[e}\xi^{r]}/|v|\\&={}^3\boldsymbol{\epsilon}(p+\rho)\Omega(n_e\xi^e)\,\delta(U_r)\varphi^r/|v|\\&=-{}^3\boldsymbol{\epsilon}\Omega(p+\rho)\,\delta(U_r)\varphi^r=\Omega\,\delta({}^3\boldsymbol{\epsilon}n_aT^a{}_b\varphi^b)\\&=-\Omega\,\delta\overline{J},\end{aligned} \tag{104}$$

where $\overline{J}$ is the pullback of J_{abc} to Σ. Therefore Eq. (99) now matches Eq. (100) and so the first law (84) now agrees with the first law given in Eq. (1).

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APPENDIX: THE CHANDRASEKHAR-FERRARI CONSERVED CURRENT

The symplectic form $\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi)$ defined in Eq. (16) is closed when $\delta_{1,2}\phi$ satisfy the linearized equations. Its dual $\omega^d(\phi,\delta_1\phi,\delta_2\phi)$, defined by $\boldsymbol{\omega}_{abc}=\omega^d\boldsymbol{\epsilon}_{dabc}$, is therefore a covariantly conserved current for the Einstein-perfect fluid system. Chandrasekhar and Ferrari [13] have, from first principles, also derived a conserved current, $\mathcal{E}^a(\phi,\delta\phi)$, for the Einstein-perfect fluid system. Their current is quadratic in the (complex) perturbations $\delta\phi$, and is restricted to the case where ϕ is a static axisymmetric solution, and $\delta\phi$ is a ''polar'' (even parity) perturbation with harmonic time dependence (we will define this below). We now show the equivalence of the $\omega^a(\phi,\delta\phi,\delta\phi^*)$ and $\mathcal{E}^a$ for the Einstein-perfect fluid system. This calculation is the analogue for the Einstein-perfect fluid system of the calculation by Burnett and Wald [12] for the Einstein-Maxwell system.

We start by choosing the Lagrangian for the Einstein-perfect fluid system to be

$$\mathbf{L}_{pqrs}=\boldsymbol{\epsilon}_{pqrs}\left(-\frac{1}{4}R+P(m,\sigma)\right), \tag{A1}$$

where we have set the constant in front of the Ricci scalar to give the field equations in [13], and used Schutz's velocity-potential representation, with Lagrangian (68). The symplectic potential $\boldsymbol{\Theta}$ arising from this Lagrangian is [after substituting Eq. (69) where applicable],

$$\begin{aligned}\boldsymbol{\Theta}_{pqr}=&-\frac{1}{4}\boldsymbol{\epsilon}_{apqr}(\nabla^b\gamma^a{}_b-\nabla^a\gamma)\\&-nU^a\boldsymbol{\epsilon}_{apqr}(\delta\Phi+\alpha\delta\beta+\theta\delta s),\end{aligned} \tag{A2}$$

where $\gamma_{ab}\equiv\delta g_{ab}$ and $\gamma\equiv g^{ab}\gamma_{ab}$. The resulting presymplectic form is [from Eq. (16)]

$$\begin{aligned}\boldsymbol{\omega}_{pqr}(\phi,\delta_1\phi,\delta_2\phi)=&-\frac{1}{8}\boldsymbol{\epsilon}_{apqr}[(\gamma_2{}^{cd}-g^{cd}\gamma_2)\nabla^a\gamma_{1cd}\\&-(2\gamma_2{}^{cd}-\gamma_2g^{cd})\nabla_c\gamma_{1d}{}^a+\gamma_2{}^{ad}\nabla_d\gamma_1]\\&-\delta_2(\boldsymbol{\epsilon}_{apqr}nU^a)(\delta_1\Phi+\alpha\delta_1\beta+\theta\delta_1 s)\\&-\boldsymbol{\epsilon}_{apqr}nU^a(\delta_2\alpha\delta_1\beta+\delta_2\theta\delta_1 s)\\&-(1\leftrightarrow 2).\end{aligned} \tag{A3}$$

This form is dual to a generally conserved current: it can be shown [11] that for ω^a defined above, we have (for perturbations $\delta_1\phi$ and $\delta_2\phi$ satisfying the linearized field equations)

$$\nabla_a\omega^a=0. \tag{A4}$$

We now relate this conserved current to the current presented in [13], by fixing a coordinate system with derivative operator ∂_a, and writing the volume element $\boldsymbol{\epsilon}$ in terms of the coordinate volume element $\mathbf{e}$ of this system,

$$\boldsymbol{\epsilon}_{abcd}=\sqrt{-g}\,\mathbf{e}_{abcd}, \tag{A5}$$

then the vector field w^a defined by $\boldsymbol{\omega}_{pqr}=w^a\mathbf{e}_{apqr}$ is conserved in the sense $\partial_a w^a=0$. If we follow Chandrasekhar and Ferrari [13] and specialize to the case where the background spacetime is static (with static Killing field t^a) and axisymmetric (with axial Killing field φ^a), and the perturbations are time and angle dependent only ''harmonically'' (that is, there are constants σ and ω such that

$$\begin{aligned}\mathcal{L}_t\delta\eta&=i\sigma\delta\eta,\\ \mathcal{L}_\varphi\delta\eta&=i\omega\delta\eta,\end{aligned} \tag{A6}$$

for all the dynamical fields η) then (following [12]) it is easy to see that for complex $\delta\phi$, $w^t(\phi,\delta\phi,\delta\phi^*)$ and $w^\varphi(\phi,\delta\phi,\delta\phi^*)$ are independently conserved: $\partial_t w^t+\partial_\varphi w^\varphi=0$. We can therefore restrict our attention to the vector components (w^2,w^3). Moreover, Eq. (A6) allows us to substitute the variations of the fluid potentials $\delta(\Phi,\beta,s)$ for variations of their time derivatives: we do this and [recalling Eq. (69)] find

$$\begin{aligned}w^a=&w^a_{gr}-\delta_2(\sqrt{-g}nU^a)\frac{1}{i\sigma}\delta_1(\mu t^bU_b)\\&-\sqrt{-g}\frac{1}{i\sigma}nU^a[\delta_2(t\cdot\nabla\alpha)\delta_1(t\cdot\nabla\beta)\\&+\delta_2(t\cdot\nabla\theta)\delta_1(t\cdot\nabla s)]-(1\leftrightarrow 2),\end{aligned} \tag{A7}$$

where we labeled the contribution from the first two lines of Eq. (A3) by w^a_{gr}.

Our aim is now to show the equality of $[w^2(\phi,\delta\phi,\delta\phi^*),w^3(\phi,\delta\phi,\delta\phi^*)]$ and $(\mathcal{E}^2,\mathcal{E}^3)$. To do this we first specialize the background and perturbations in w^a to those used by Chandrasekhar and Ferrari. In the coordinates given in [13] the metric is written

$$\begin{aligned}g_{ab}=&-e^{2\nu}dt_a dt_b+e^{2\psi}d\varphi_a d\varphi_b\\&+e^{2\mu_2}dx^2_a dx^2_b+e^{2\mu_3}dx^3_a dx^3_b,\end{aligned} \tag{A8}$$

and the nonvanishing (polar) metric perturbations are taken to be

$$\gamma_{1tt} = -2e^{2\nu}\delta\nu,$$

$$\gamma_{1\varphi\varphi} = 2e^{2\psi}\delta\psi,$$

$$\gamma_{122} = 2e^{2\mu_2}\delta\mu_2,$$

$$\gamma_{133} = 2e^{2\mu_3}\delta\mu_3. \tag{A9}$$

We then set $\gamma_{2ab} = \gamma_{1ab}^*$, where the perturbed functions, $\delta\nu$, etc., are complex, but the unperturbed functions are real.

A direct substitution of these perturbations into w_{gr}^a yields the result already known from [12],

$$w_{gr}^2 = -\frac{1}{2}e^{\nu+\psi-\mu_2+\mu_3}[\delta\nu_{,2}\delta(\psi+\mu_3)^*\delta\psi_{,2}\delta(\nu+\mu_3)^* + \delta\mu_{3,2}\delta(\nu+\psi)^* + \nu_{,2}\delta(\psi+\mu_3)^*\delta(\nu-\mu_2) + \psi_{,2}\delta(\nu+\mu_3)^*\delta(\psi-\mu_2) + \mu_{3,2}\delta(\mu+\psi)^*\delta(\mu_3-\mu_2)] - \text{c.c.} \tag{A10}$$

We now turn to the fluid contributions w_f^a to the conserved current, defined by $w_f^a = w^a - w_g^a$. We set the four-velocity of the background to be $U^a = e^{-\nu}t^a$ and (following [13]) denote the perturbations of the *orthonormal frame* components of U^a by $i\sigma\xi^a$: $\delta U_{\hat{a}} = i\sigma\xi_a$. We then find the ''two''-component of w_f^a given by

$$w_f^2 = -dx_a^2\frac{1}{i\sigma}[\delta_2(e^{\nu+\psi+\mu_2+\mu_3}nU^a)\delta_1(\mu t^c U_c)] - (1\leftrightarrow 2)$$
$$= \frac{1}{i\sigma}e^{2\nu+\psi+\mu_3}n\delta_2(U^{\hat{2}})(\delta_1\mu + \mu\delta_1\nu - \mu\delta_1 U_{\hat{0}}) - (1\leftrightarrow 2)$$
$$= -e^{2\nu+\psi+\mu_3}(n\delta\mu + n\mu\delta\nu)\xi_2^* - \text{c.c.}, \tag{A11}$$

where we have set $\delta_1 = \delta$ and $\delta_2 = \delta^*$, and used the result (see [13]) that $\delta U_{\hat{0}} = 0$. We can also put $n\delta\mu = \delta p + nT\delta s$, and bearing in mind that δs must also have harmonic time dependence, we can write

$$i\sigma nT\delta s = nTt\cdot\nabla\delta s$$
$$= nTe^{\nu}U\cdot\nabla\delta s$$
$$= nTe^{\nu}[\delta(U\cdot\nabla s) - \delta(U)\cdot\nabla s]. \tag{A12}$$

Referring to Eq. (63) we see that the first term on the right side of Eq. (A12) vanishes whenever the perturbation satisfies the linearized equations. Since the background is vortex-free, we see that the second term also vanishes as a consequence of Eq. (63). Adding the resulting fluid contribution to the gravitational terms, Eq. (A10) yields

$$w^2 = -\frac{1}{2}e^{\nu+\psi-\mu_2+\mu_3}[\delta\nu_{,2}\delta(\psi+\mu_3)^*\delta\psi_{,2}\delta(\nu+\mu_3)^* + \delta\mu_{3,2}\delta(\nu+\psi)^* + \nu_{,2}\delta(\psi+\mu_3)^*\delta(\nu-\mu_2) + \psi_{,2}\delta(\nu+\mu_3)^*\delta(\psi-\mu_2) + \mu_{3,2}\delta(\nu^*+\delta\psi^*)\delta(\mu_3-\mu_2)] - e^{2\nu+\psi+\mu_3}[\delta p + (p+\rho)\delta\nu]\xi_2^* - \text{c.c.} \tag{A13}$$

Now using the appropriate linear combinations of the linearized Einstein constraint,

$$\delta(\psi+\mu_3)_{,2} + \psi_{,2}\delta(\psi-\mu_2) + \mu_{3,2}\delta(\mu_3-\mu_2) - \nu_{,2}\delta(\psi+\mu_3) = 2e^{\nu+\mu_2}(\rho+p)\xi^2, \tag{A14}$$

to replace the third, fourth, and fifth lines of Eq. (A13) we get

$$w^2 = -\frac{1}{2}e^{\nu+\psi+\mu_3-\mu_2}\{\delta\nu_{,2}\delta(\psi+\mu_3)^* + \delta\mu^*(\delta\psi+\delta\mu_3)_{,2} - [\delta\psi,\delta\psi^*]_{,2} - [\delta\mu_3,\delta\mu_3^*]_{,2} + 2e^{\nu+\mu_2}[(\rho+p)\delta(\psi+\mu_3-\mu_2)^* - \delta p^*]\xi_2\} - \text{c.c.}, \tag{A15}$$

where we define $[A,A^*]_{,i} \equiv A_{,i}A^* - AA^*_{,i}$. This is seen to agree (up to an overall constant) with $\mathcal{E}^2$ of the conserved current in [13]. A similar calculation for w^3 yields $\mathcal{E}^3$ (which is obtained from $\mathcal{E}^2$ by interchanging $2\leftrightarrow 3$), and so we find $(w^2,w^3) = (\mathcal{E}^2,\mathcal{E}^3)$, and our symplectic current w^a for the Einstein-perfect fluid system agrees with the Chandrasekhar-Ferrari current for this system.

We make two final comments. First, from the comment following Eq. (69), we know that every configuration of the physical fields of a perfect fluid has a corresponding equivalence class of configurations of the dynamical fields, and as a consequence, every perturbation of the physical fields has a corresponding perturbation of the dynamical fields. Now, two distinct perturbations of the physical fields off the same background (physical field) configuration will each select a corresponding perturbation of the dynamical fields. The background *dynamical* field configuration for each of these perturbations will certainly lie within the equivalence class corresponding to the given background physical field configuration: however, in general, these background dynamical field configurations will be *distinct* elements of this equivalence class. In using symplectic methods to derive $\mathcal{E}^a$ we have implicitly restricted ourselves to those pairs of perturbations of the physical fields where the corresponding pairs of dynamical field perturbations $(\delta_1\phi,\delta_2\phi)$ have *identical* background configurations. In fact, as we have seen above, the resulting conserved current agrees with the Chandrasekhar-Ferrari current for *all* pairs of perturbations of the physical fields, not just those restricted in this way.

Second, we notice from Eq. (A7) that as long as the U^a of the background solution lies in a plane tangent to the subspace spanned by t^a and φ^a, the last term in Eq. (A7) vanishes for the components of interest. This, in turn, yields a conserved current (w^2,w^3) which only depends on perturbations of the *physical* fields, without the explicit appearance of the fluid potentials, for any stationary background configuration in which the fluid velocity is tangent to the $t-\varphi$ subspaces. Of course, we know that ω^a is a conserved current off *any* background; this observation suggests only that a current similar in style to that presented by Chandrasekhar and Ferrari also exists for a background with a fluid in circular motion, as well as the static case considered in [13].

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