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Small oscillations of a chiral Gross-Neveu system

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We study the small oscillations regime (random-phase approximation) of the time-dependent mean-field equations, obtained in a previous work, which describe the time evolution of one-body dynamical variables of a uniform chiral Gross-Neveu system. In this approximation we obtain an analytical solution for the time evolution of the one-body dynamical variables. The two-fermion physics can be explored through this solution. The condition for the existence of bound states is examined. [S0556-2821(97)02906-8]

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I. INTRODUCTION

In a previous work [1] we obtained in Gaussian mean-field approximation the effective dynamics of one-fermion and pairing densities of an off-equilibrium spatially uniform (1+1)-dimensional self-interacting fermion system described by the chiral Gross-Neveu model (CGNM) [2]. These dynamical equations acquire the structure of (collisionless) kinetic equations. They determine the time evolution of all the one-fermion densities of this system for a given initial condition. Spatial uniformity (translation and reflection invariance) is assumed in our derivation.

Studying the static solutions of these equations in order to renormalize the theory [1], we found an effective potential similar to that obtained by Gross and Neveu using the $1/N$ expansion [2]. We also showed that other static results which have been discussed in the literature [2–4] such as dynamical mass generation due to chiral symmetry breaking and a phenomenon analogous to dimensional transmutation can be retrieved from this formulation. Finally, in [1], we obtained numerical solutions for the time evolution of the one-body dynamical variables initially displaced from equilibrium. The time evolution of the symmetrical and broken chiral phases of our system are discussed.

In this work we explore a particular application of the renormalized nonlinear mean-field equations obtained in [1]. We follow the recent work of Kerman and Lin [5] in order to study the near equilibrium dynamics around the stationary solution as a tool to investigate the two-fermion dynamics. In particular, the resulting equations can be solved analytically to reveal a two-(quasi)fermion bound state solution.

This paper is organized as follows. In Secs. II and III we linearize the mean-field dynamical equations which describe the time evolution of an off-equilibrium spatially uniform (1+1)-dimensional self-interacting fermion system described by the chiral Gross-Neveu model (CGNM). A self-consistent renormalization scheme is necessary [1,6]. In Secs. IV and V, making use of an analogy with scattering theory [7], we obtain a closed analytical solution for the time evolution of the one-fermion densities in this regime. Studyng the two-fermion physics in Sec. VI, we find the condition for the existence of bound states. Finally, Sec. VII is devoted to a final discussion and conclusions.

II. MEAN-FIELD KINETIC EQUATIONS

We begin this section by reviewing our approach which describes a formal treatment of the kinetics of a self-interacting quantum field. This approach was developed earlier for the nonrelativistic nuclear many-body dynamics by Nemes and de Toledo Piza [8] and was more recently applied in the quantum-field theoretical context to the self-interacting $\lambda\phi^4$ theory in 1+1 dimensions [9]. The general idea is to focus on the time evolution of the one-fermion and pairing densities. These observables are kept under direct control when one works variationally using a Gaussian functional *ansatz* and will therefore be referred to as Gaussian observables.

We consider an off-equilibrium, spatially uniform, (1+1)-dimensional system of relativistic, self-interacting fermions described by the chiral Gross-Neveu model (CGNM) [2]. The Hamiltonian density is given by

$$\mathcal{H}_{\rm CGNM}=\sum_{i=1}^{N}\{\overline{\psi^i}[-i\gamma_1\partial_1]\psi^i\}-\frac{g^2}{2}\left\{\left[\sum_{i=1}^{N}\overline{\psi^i}\psi^i\right]^2-\xi\left[\sum_{i=1}^{N}\overline{\psi^i}\gamma_5\psi^i\right]^2\right\}, \tag{1}$$

where ξ is a constant which indicates whether the model is invariant under discrete γ_5 transformation ($\xi=0$) or under the Abelian chiral U(1) group ($\xi=1$). In the form considered here, this is a massless fermion theory in 1+1 dimensions with quartic interaction. The model contains N species of fermions coupled symmetrically. In the Heisenberg picture, the ψ^i are complex Dirac spinors

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$$\psi(x)=\sum_{\mathbf{k}}\left(\frac{m}{k_0}\right)^{1/2}\left[b_{\mathbf{k},1}(t)u_1(\mathbf{k})\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{L}}+b^{\dagger}_{\mathbf{k},2}(t)u_2(\mathbf{k})\frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{L}}\right] \tag{2}$$

$$\overline{\psi(x)}=\sum_{\mathbf{k}}\left(\frac{m}{k_0}\right)^{1/2}\left[b^{\dagger}_{\mathbf{k},1}(t)\overline{u_1}(\mathbf{k})\frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{L}}+b_{\mathbf{k},2}(t)\overline{u_2}(\mathbf{k})\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{L}}\right],$$

where $b^{\dagger}_{\mathbf{k},1}$ and $b_{\mathbf{k},1}$ [$b^{\dagger}_{\mathbf{k},2}$ and $b_{\mathbf{k},2}$] are fermion creation and annihilation operators associated to positive [negative]-energy solution $u_1(\mathbf{k})$ [$u_2(\mathbf{k})$] of Dirac's equation.

This model is essentially equivalent to the Nambu–Jona-Lasinio model [10], except for the fact that in 1+1 dimensions it is renormalizable. Moreover, it is one of the very few known field theories which are asymptotically free. To leading order in $1/N$ expansion [2], the CGNM exhibits a number of interesting phenomena, like spontaneous symmetry breaking, dynamical fermion mass generation, and dimensional transmutation.

The state of this system (assumed spatially uniform) is given in terms of a many-body density operator $\mathcal{F}$ of unit trace. Our implementation of the Gaussian mean-field approximation consists of approximating this object by a truncated many-body density operator $\mathcal{F}_0(t)$, also of unit trace, written as the most general Hermitian Gaussian functional of the field operators consistent with the assumed uniformity of the system [11]. It will thus be written as the exponential of a general quadratic form in the field operators, which can be reduced to diagonal form by suitable canonical transformation. The most general transformation would in general break both the chiral and charge symmetries of the CGNM. In the following development we restrict ourselves for simplicity to a special class of transformations (to be called Nambu transformations) which break the chiral symmetry only. They can be parametrized in a form that incorporates the unitarity constraints as

$$\begin{bmatrix} b_{\mathbf{k},1} \\ b_{\mathbf{k},2} \\ b^{\dagger}_{-\mathbf{k},1} \\ b^{\dagger}_{-\mathbf{k},2}\end{bmatrix}=\begin{bmatrix}\cos\varphi_{\mathbf{k}} & 0 & 0 & -e^{-i\gamma_{\mathbf{k}}}\sin\varphi_{\mathbf{k}} \\ 0 & \cos\varphi_{\mathbf{k}} & e^{-i\gamma_{\mathbf{k}}}\sin\varphi_{\mathbf{k}} & 0 \\ 0 & -e^{i\gamma_{\mathbf{k}}}\sin\varphi_{\mathbf{k}} & \cos\varphi_{\mathbf{k}} & 0 \\ e^{i\gamma_{\mathbf{k}}}\sin\varphi_{\mathbf{k}} & 0 & 0 & \cos\varphi_{\mathbf{k}}\end{bmatrix}\begin{bmatrix}\beta_{\mathbf{k},1} \\ \beta_{\mathbf{k},2} \\ \beta^{\dagger}_{-\mathbf{k},1} \\ \beta^{\dagger}_{-\mathbf{k},2}\end{bmatrix} \tag{3}$$

where reflection symmetry of the uniform system is implemented by making the parameters $\varphi_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ dependent only on the magnitude of $\mathbf{k}$.

The Gaussian truncated density operator $\mathcal{F}_0(t)$ acquires a particularly simple form when expressed in terms of the Nambu quasifermion operators which diagonalize the associated quadratic form, namely

$$\mathcal{F}_0(t)=\prod_{\mathbf{k},\lambda}\left[\nu_{\mathbf{k},\lambda}\beta^{\dagger}_{\mathbf{k},\lambda}(t)\beta_{\mathbf{k},\lambda}(t)+(1-\nu_{\mathbf{k},\lambda})\beta_{\mathbf{k},\lambda}(t)\beta^{\dagger}_{\mathbf{k},\lambda}(t)\right], \tag{4}$$

where $\nu_{\mathbf{k},\lambda}$ for $\lambda=1,2$ are the Nambu (quasifermion) occupation numbers.

With the help of Eq. (3) it is an easy task to express $\overline{\psi}(x)$ and $\psi(x)$, Eq. (2), in terms of $\beta^{\dagger}_{\mathbf{k},\lambda}(t)$ and $\beta_{\mathbf{k},\lambda}(t)$ for $\lambda=1,2$. In doing so, one finds that the plane waves of $\overline{\psi}(x)$ and $\psi(x)$ are modified by a complex, moment-dependent redefinition of m involving the Nambu parameters $\varphi_{\mathbf{k}}(t)$ and $\gamma_{\mathbf{k}}(t)$. The complex character of these parameters is actually crucial in dynamical situations, where the imaginary parts will allow for the description of time-odd (velocitylike) properties. Finally, the mean values of the Gaussian observables are parametrized in terms of the $\varphi_{\mathbf{k}}(t)$ and $\gamma_{\mathbf{k}}(t)$ and of the occupation numbers $\nu_{\mathbf{k},\lambda}(t)=\mathrm{Tr}[\beta^{\dagger}_{\mathbf{k},\lambda}(t)\beta_{\mathbf{k},\lambda}(t)\mathcal{F}(t)]$ for $\lambda=1,2$.

The next step is to obtain the mean-field time evolution for the mean values of the Gaussian observables in the context of the initial-value problem. In other words, we want the Gaussian mean-field equations of motion for the Nambu parameters $\varphi_{\mathbf{k}}(t)$, $\gamma_{\mathbf{k}}(t)$ and for the quasi-particle occupation numbers $\nu_{\mathbf{k},\lambda}(t)$. In Ref. [1] we obtained

$$\dot{\nu}_{\mathbf{k},1}=0 \quad\text{and}\quad \dot{\nu}_{\mathbf{k},2}=0; \tag{5}$$

$$[i\dot{\varphi}_{\mathbf{k}}+\dot{\gamma}_{\mathbf{k}}\sin\varphi_{\mathbf{k}}\cos\varphi_{\mathbf{k}}]e^{-i\gamma_{\mathbf{k}}}=\frac{\mathrm{Tr}([\beta_{-\mathbf{k},1}\beta_{\mathbf{k},2},H_{\mathrm{CGNM}}]\mathcal{F}_0)}{(1-\nu_{\mathbf{k},1}-\nu_{\mathbf{k},2})}. \tag{6}$$

Equation (5) shows that the occupation numbers of the Nambu quasiparticles are constant. This is the general isoentropic character of the mean-field approximation [12,13]. The complex equation of motion (6) describes the time evolution of the Nambu parameters. From the right-hand side of Eq. (6), we see that to obtain the time evolution of the Nambu parameters, we have to express the CGNM Hamiltonian in the Nambu basis.

From the Hamiltonian density (1) we can explicitly evaluate the Hamiltonian of the system by integration over all one-dimensional space. This involves, in particular, choosing a representation for the γ matrices. Here we have to be careful, since a bad choice of representation can spoil manifest reflection invariance (see Appendix A of Ref. [1]). We choose the Pauli-Dirac representation: namely,

$$\gamma_0=\sigma_3,\quad \gamma_1=i\sigma_2,\quad \text{and}\quad \gamma_5=\gamma_0\gamma_1=\sigma_1. \tag{7}$$

Substituting the CGNM Hamiltonian written in Nambu basis in the dynamical equation (6), we obtain an explicit dynamical equation which describes the time evolution of the Nambu parameters. The calculation of traces is lengthy but straightforward. Taking the case $N=1$ for simplicity and splitting the resultant complex equation into real and imaginary parts we have

$$\dot{\nu}_{\mathbf{k},1}=0 \quad \text{and} \quad \dot{\nu}_{\mathbf{k},2}=0;$$

$$\dot{\varphi}_{\mathbf{k}}=\sin\gamma_{\mathbf{k}}\frac{|\mathbf{k}|}{k_0}\left[m-\left(\frac{g^2m}{4\pi}\right)(\xi+1)(I_1+I_2)\right],$$

$$\dot{\gamma}_{\mathbf{k}}\sin2\varphi_{\mathbf{k}}=\frac{2\sin2\varphi_{\mathbf{k}}}{k_0}\left[\mathbf{k}^2+\left(\frac{g^2m^2}{4\pi}\right)(\xi+1)(I_1+I_2)\right]$$
$$+2\cos2\varphi_{\mathbf{k}}\cos\gamma_{\mathbf{k}}\frac{|\mathbf{k}|}{k_0}$$
$$\times\left[m-\left(\frac{g^2m}{4\pi}\right)(\xi+1)(I_1+I_2)\right], \tag{8}$$

where I_1 and I_2 are the divergent integrals

$$I_1=\int\frac{d\mathbf{k}'}{k_0'}\cos2\varphi_{\mathbf{k}'}(1-\nu_{\mathbf{k}',1}-\nu_{\mathbf{k}',2}),$$

$$I_2=\int\frac{d\mathbf{k}'}{k_0'}\frac{|\mathbf{k}'|}{m}\sin2\varphi_{\mathbf{k}'}\cos\gamma_{\mathbf{k}'}(1-\nu_{\mathbf{k}',1}-\nu_{\mathbf{k}',2}). \tag{9}$$

A renormalization procedure is now required to deal with these divergences. In general, renormalization procedures consist in combining divergent terms with the bare mass and coupling constants of the theory to define new, finite (or renormalized) values of these quantities. In other words, the bare mass and coupling constants are chosen to be cutoff dependent in a way that will cancel the divergences. In the present case, however, the divergent integrals (9) involve the dynamical variables themselves in the integrand, so that even their degree of divergence is not directly computable. In order to handle this situation we will use a self-consistent renormalization procedure inspired in Ref. [6].

The renormalization prescription that we use is based on the consideration of the static solutions of the dynamical equations (8), which satisfy

$$\sin\gamma_{\mathbf{k}}|_{\text{eq}}\left[1-\left(\frac{g^2}{4\pi}\right)(\xi+1)(I_1+I_2)\right]=0, \tag{10}$$

$$\tan2\varphi_{\mathbf{k}}|_{\text{eq}}=\frac{-|\mathbf{k}|m[1-(g^2/4\pi)(\xi+1)(I_1+I_2)]}{[(\mathbf{k})^2+(g^2m^2/4\pi)(\xi+1)(I_1+I_2)]}\cos\gamma_{\mathbf{k}}|_{\text{eq}}. \tag{11}$$

We will show explicitly that controlling the divergences of these equations will also control the divergences which appear in the kinetic regime of the mean-field approximation [see below, Eqs. (17)].

In order to obtain the renormalization prescription we introduce a regularizing momentum cutoff Λ and begin by assuming that, in order to render the theory finite, the bare coupling constant g^2 must approach zero for large values of the Λ as (see, e.g., Refs. [2,4])

$$g^2=\frac{4\pi}{(\xi+1)}\left[\ln\left(\frac{\Lambda^2}{m^2}\right)\right]^{-1}, \tag{12}$$

where the form of the first factor is dictated by later convenience. We next *assume* that the integrals I_1 and I_2 have logarithmic divergences

$$I_1=a+b\ln\left(\frac{\Lambda^2}{m^2}\right),$$

$$I_2=c+d\ln\left(\frac{\Lambda^2}{m^2}\right), \tag{13}$$

where a, b, c, and d are finite constants. Substituting Eq. (12) and the *ansatze* (13) in static equations (10) and (11), we obtain

$$d\sin\gamma_{\mathbf{k}}|_{\text{eq}}=0,$$

$$\tan2\varphi_{\mathbf{k}}|_{\text{eq}}=\frac{-(-1)^n m|\mathbf{k}|[1-(b+d)]}{[(\mathbf{k})^2+m^2(b+d)]}. \tag{14}$$

We now verify that the assumed divergent character of the integrals I_1 and I_2 is consistent with the *ansatze* (13). Substituting the solution (14) into Eq. (9) we find that I_1 and I_2 indeed have the prescribed logarithmic divergence (see Appendix C of Ref. [1]). Moreover, from this calculation, we obtain the values of the constants a, b, c, and d. We find $b=1$, while d remains arbitrary. The renormalized static solution of our system in the mean-field approximation are then obtained by simply substituting these values in to Eq. (14):

$$d\sin\gamma_{\mathbf{k}}|_{\text{eq}}=0, \tag{15}$$

$$\tan2\varphi_{\mathbf{k}}|_{\text{eq}}=\frac{(-1)^n m|\mathbf{k}|d}{[\mathbf{k}^2+(1+d)m^2]}.$$

We also show in Ref. [1] that the connection between particle mass m and quasiparticle mass m_{eff} is given by

$$m_{\text{eff}}=(1+d)m. \tag{16}$$

From the redefinition of the mass scale given by Eq. (16), we note that, unlike the situation found in connection with the $1/N$ expansion, the use of the Gaussian *ansatz*, Eq. (4), parametrized by the canonical transformation leading to the quasifermion basis, allows for the direct dynamical determination of the stable equilibrium situation of the system [see Eq. (15)], including symmetry breaking [when $d\neq-1$, see also Ref. [1]] and mass generation [see Eq. (16)]. Moreover, the renormalization procedure effectively replaces the dimensionless coupling constant g^2 by the free parameter d associated to the mass scale [see Eq. (16)]. This is analogous to the phenomenon of dimensional transmutation found by Gross and Neveu [2] in the case of a $1/N$ expansion. Finally, aside from the overall mass scale (characterized by d) there are no free adjustable parameters.

Using Eqs. (8), (12), and (13), we finally write the renormalized form of the dynamical equations that describe the mean-field time evolution of the system. As mentioned before they are now also finite and read

$$\dot{\nu}_{\mathbf{k},1}=0 \quad \text{and} \quad \dot{\nu}_{\mathbf{k},2}=0,$$

$$\dot{\varphi}_{\mathbf{k}}=(-1)md\frac{|\mathbf{k}|}{k_0}\sin\gamma_{\mathbf{k}},$$

$$\dot{\gamma}_{\mathbf{k}}\sin2\varphi_{\mathbf{k}}=\frac{2\sin2\varphi_{\mathbf{k}}}{k_0}[(\mathbf{k})^2+m^2(1+d)]-2md\frac{|\mathbf{k}|}{k_0}\cos2\varphi_{\mathbf{k}}\cos\gamma_{\mathbf{k}}. \tag{17}$$

III. SMALL OSCILLATIONS REGIME

In order to study the small oscillation regime of the kinetic equations we next linearize the above kinetic equations around the static solution (15) taking $\nu_{\mathbf{k},1}=\nu_{\mathbf{k},2}=0$. We begin by introducing the displacement away from equilibrium of the dynamical variables $\varphi_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$:

$$\varphi_{\mathbf{k}}=\varphi_{\mathbf{k}}|_{\text{eq}}+\delta\varphi_{\mathbf{k}},$$

$$\gamma_{\mathbf{k}}=\gamma_{\mathbf{k}}|_{\text{eq}}+\delta\gamma_{\mathbf{k}}, \tag{18}$$

where the static solution $\varphi_{\mathbf{k}}|_{\text{eq}}$ and $\gamma_{\mathbf{k}}|_{\text{eq}}$ are obtained from Eq. (15) [see also Ref. [1]]:

$$\sin2\varphi_{\mathbf{k}}|_{\text{eq}}=\frac{m|\mathbf{k}|d}{k_0[\mathbf{k}^2+(1+d)^2m^2]^{1/2}},$$

$$\cos2\varphi_{\mathbf{k}}|_{\text{eq}}=\frac{[\mathbf{k}^2+(1+d)m^2]}{k_0[\mathbf{k}^2+(1+d)^2m^2]^{1/2}}, \tag{19}$$

$$\gamma_{\mathbf{k}}|_{\text{eq}}=0 \quad \text{with} \quad d\neq0.$$

The quantities $\delta\varphi_{\mathbf{k}}$ and $\delta\gamma_{\mathbf{k}}$,will be treated as (first-order) small displacements. Functions of the dynamical variables are expanded also to first order around the equilibrium solution (19). Therefore, we must linearize the divergent integrals (9) around equilibrium. Taking $\nu_{\mathbf{k},\lambda}=0$ we have

$$I_1=I_1^{(0)}+I_1^{(1)}+O(\delta\varphi_{\mathbf{k}})^2=\int\frac{d\mathbf{k}'}{k_0'}\cos2\varphi_{\mathbf{k}'}|_{\text{eq}}-2\int\frac{d\mathbf{k}'}{k_0'}\sin2\varphi_{\mathbf{k}'}|_{\text{eq}}\delta\varphi_{\mathbf{k}'}$$

$$I_2=I_2^{(0)}+I_2^{(1)}+O[(\delta\varphi_{\mathbf{k}})^2,(\delta\gamma_{\mathbf{k}})^2,(\delta\varphi_{\mathbf{k}}\delta\gamma_{\mathbf{k}})]=\int\frac{d\mathbf{k}'}{k_0'}\frac{|\mathbf{k}'|}{m}\sin2\varphi_{\mathbf{k}}|_{\text{eq}}+2\int\frac{d\mathbf{k}'}{k_0'}\frac{|\mathbf{k}'|}{m}\cos2\varphi_{\mathbf{k}}|_{\text{eq}}\delta\varphi_{\mathbf{k}'}. \tag{20}$$

The linearized form of kinetic equations for $\delta\varphi_{\mathbf{k}}$ and $\delta\gamma_{\mathbf{k}}$ are then obtained as

$$\delta\dot{\varphi}_{\mathbf{k}}=-\frac{m|\mathbf{k}|d}{k_0}\delta\gamma_{\mathbf{k}}, \tag{21}$$

$$\delta\dot{\gamma}_{\mathbf{k}}\frac{m|\mathbf{k}|d}{k_0}=4[\mathbf{k}^2+(1+d)^2m^2]\delta\varphi_{\mathbf{k}}-4\left(\frac{g^2}{4\pi}\right)(\xi+1)|\mathbf{k}|\times\int d\mathbf{k}'\frac{|\mathbf{k}'|}{[(\mathbf{k}')^2+(1+d)^2m^2]^{1/2}}\delta\varphi_{\mathbf{k}'}, \tag{22}$$

where the renormalization procedure (12) controls the logarithmic divergence of the integral appearing in Eq. (22) [see below Eqs. (38) and (42)]. Substituting Eq. (22) into Eq. (21) we obtain finally

$$\delta\ddot{\varphi}_{\mathbf{k}}+4[\mathbf{k}^2+(1+d)^2m^2]\delta\varphi_{\mathbf{k}}-4\left(\frac{g^2}{4\pi}\right)(\xi+1)|\mathbf{k}|\times\int d\mathbf{k}'\frac{|\mathbf{k}'|}{[(\mathbf{k}')^2+(1+d)^2m^2]^{1/2}}\delta\varphi_{\mathbf{k}'}=0. \tag{23}$$

As usual in small oscillation treatments, this is a linear oscillator equation. Note that the last term couples different momenta. The solution to this problem involves determining the normal modes of small oscillation and their frequencies. This is done by looking for solutions of Eqs. (21) and (22), which are of the form

$$\delta\varphi_{\mathbf{k}}=\Psi_{\mathbf{k}}e^{i\omega t}$$

$$\delta\gamma_{\mathbf{k}}=\Gamma_{\mathbf{k}}e^{i\omega t}, \tag{24}$$

where $\Psi_{\mathbf{k}}$ and $\Gamma_{\mathbf{k}}$ are time-independent amplitudes. Substituting the solution (24) into Eqs. (21) and (22), we obtain equations for these amplitudes, namely

$$i\omega\Psi_{\mathbf{k}}+\frac{m|\mathbf{k}|d}{k_0}\Gamma_{\mathbf{k}}=0, \tag{25}$$

$$i\omega\frac{m|\mathbf{k}|d}{k_0}\Gamma_{\mathbf{k}}-4[\mathbf{k}^2+(1+d)^2m^2]\Psi_{\mathbf{k}}+4\left(\frac{g^2}{4\pi}\right)(\xi+1)|\mathbf{k}|\times\int d\mathbf{k}'\frac{|\mathbf{k}'|}{[(\mathbf{k}')^2+(1+d)^2m^2]^{1/2}}\Psi_{\mathbf{k}'}=0. \tag{26}$$

IV. TWO-BODY DYNAMICS FROM THE LINEARIZED MEAN-FIELD EQUATIONS

In this section we implement and discuss a reinterpretation of the small oscillations problem along the line proposed by Kerman and Lin in Ref. [5]. We begin rewriting Eqs. (25) and (26) as

$$\Gamma_{\mathbf{k}}=-\frac{i\omega}{md}\frac{k_0}{|\mathbf{k}|}\Psi_{\mathbf{k}} \tag{27}$$

$$\frac{(k_0^{\text{eff}})^2}{k_0^{\text{eff}}}\Psi_{\mathbf{k}}-\left(\frac{g^2}{4\pi}\right)(\xi+1)\frac{|\mathbf{k}|}{(k_0^{\text{eff}})}\int d\mathbf{k}'\frac{|\mathbf{k}'|}{(k_0^{\text{eff}})'}\Psi_{\mathbf{k}'}=\frac{\omega^2}{4k_0^{\text{eff}}}\Psi_{\mathbf{k}}, \tag{28}$$

where $(k_0^{\rm eff})^2=[\mathbf{k}^2+(1+d)^2m^2]=[\mathbf{k}^2+m_{\rm eff}^2]$.

The crucial point is to realize that Eq. (28) has the form of a Lippmann-Schwinger equation with separable potential

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=v(\mathbf{k})v^*(\mathbf{k}')=\left(\frac{g^2}{4\pi}\right)(\xi+1)h(\mathbf{k})h(\mathbf{k}'),\tag{29}$$

where

$$h(\mathbf{k})=\frac{|\mathbf{k}|}{k_0^{\rm eff}}.$$

In order to interpret this scattering problem we follow Ref. [5] in relating dynamical small amplitude distortions of the Gaussian vacuum to two-quasifermion states. The Gaussian vacuum $|\tilde{0}\rangle$ can be written explicitly in terms of the fermion operators $b^\dagger_{\mathbf{k},\lambda}$ and the Nambu parameters in the well-known Bardeen-Cooper-Schriffer (BCS) form

$$|\tilde{0}\rangle=\prod_{\mathbf{k}>0,\lambda\neq\bar{\lambda}}[\cos\varphi_\mathbf{k}+(-1)^\lambda\sin\varphi_\mathbf{k}e^{-i\gamma_\mathbf{k}}b^\dagger_{\mathbf{k},\lambda}b^\dagger_{\overline{\mathbf{k},\lambda}}]|0\rangle,\tag{30}$$

where $\varphi_\mathbf{k}=\varphi_\mathbf{k}|_{\rm eq}$ and $\gamma_\mathbf{k}=\gamma_\mathbf{k}|_{\rm eq}$ and $|0\rangle$ is the vacuum of the b operators. It is also well known that $|\tilde{0}\rangle$ is the quasifermion vacuum, namely

$$\beta_{\mathbf{k},\lambda}|\tilde{0}\rangle=0\tag{31}$$

and that one- and two-quasifermion states can be written as

$$\beta^\dagger_{\mathbf{k}_1,\lambda_1}|\tilde{0}\rangle=b^\dagger_{\mathbf{k}_1,\lambda_1}\prod_{(\mathbf{k},\lambda)\neq(\mathbf{k}_1,\lambda_1)}[\cos\varphi_\mathbf{k}+(-1)^\lambda\times\sin\varphi_\mathbf{k}e^{-i\gamma_\mathbf{k}}b^\dagger_{\mathbf{k},\lambda}b^\dagger_{\overline{\mathbf{k},\lambda}}]|0\rangle\tag{32}$$

$$\beta^\dagger_{\mathbf{k}_1,\lambda_1}\beta^\dagger_{\overline{\mathbf{k}_1,\lambda_1}}|\tilde{0}\rangle=[(-1)^{\bar{\lambda}_1}\sin\varphi_{\mathbf{k}_1}e^{i\gamma_{\mathbf{k}_1}}+\cos\varphi_{\mathbf{k}_1}b^\dagger_{\mathbf{k}_1,\lambda_1}b^\dagger_{\overline{\mathbf{k}_1,\lambda_1}}]\times\prod_{(\mathbf{k},\lambda)\neq(\mathbf{k}_1,\lambda_1)}[\cos\varphi_\mathbf{k}+(-1)^\lambda\times\sin\varphi_\mathbf{k}e^{-i\gamma_\mathbf{k}}b^\dagger_{\mathbf{k},\lambda}b^\dagger_{\overline{\mathbf{k},\lambda}}]|0\rangle.\tag{33}$$

On the other hand, making the variations Eq. (24) in Eq. (30) leads to a first order variation of the Gaussian vacuum which, apart from the exponential time dependence, is given by

$$\delta|\tilde{0}\rangle=\Psi_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_1,\lambda_1}\beta^\dagger_{\overline{\mathbf{k}_1,\lambda_1}}|\tilde{0}\rangle-i(-1)^{\bar{\lambda}_1}\Gamma_{\mathbf{k}_1}\times\sin\varphi_{\mathbf{k}_1}|_{\rm eq}b^\dagger_{\mathbf{k}_1,\lambda_1}b^\dagger_{\overline{\mathbf{k}_1,\lambda_1}}\times\prod_{(\mathbf{k},\lambda)\neq(\mathbf{k}_1,\lambda_1)}[\cos\varphi_\mathbf{k}+(-1)^\lambda\sin\varphi_\mathbf{k}\times e^{-i\gamma_\mathbf{k}}b^\dagger_{\mathbf{k},\lambda}b^\dagger_{\overline{\mathbf{k},\lambda}}]|0\rangle.\tag{34}$$

This result just illustrates the well-known theorem by Thouless [14] when $\Gamma_{\mathbf{k}_1}=0$ (as is appropriate in a static context), in which case it says that the linear variation corresponds to two-paired quasi fermion addition to the Gaussian vacuum. Kerman and Lin have used this fact, in the context of the ϕ^4 theory, to associate the scattering problem (28) to the two-boson dynamics. A similar association can thus also be made here. Moreover, the last term of Eq. (34) indicates the presence of dynamical symmetry breaking effects, since this term, proportional to the variation of the phase $\gamma_\mathbf{k}$, vanishes unless the chiral symmetry is broken in the static vacuum $(\varphi_\mathbf{k}\neq0)$.

The small oscilation regime can thus be seen as a nonperturbative way of approaching the dynamics of paired two-quasi fermion excitations of the vacuum, including dynamical symmetry-breaking effects.

V. ANALYTICAL SOLUTION OF THE LINEARIZED EQUATIONS

We will now show how Eqs. (27) and (28) can be solved analytically. A general solution to two-fermion wave function $\Psi_\mathbf{k}$ will have two terms. The first one is the free solution ($g=0$ vanishing potential) and represents an incident wave. The second term is the nontrivial part (when $g\neq0$) which couples different momenta, and is associated with the scattered wave. Thus

$$\frac{|\mathbf{k}|}{k_0^{\rm eff}}\Psi(\mathbf{k},\mathbf{q};\omega)=\alpha\,\delta(\mathbf{q}-\mathbf{k})+\frac{1}{[(\mathbf{k}_0^{\rm eff})^2-\omega^2/4+i\epsilon]}\left(\frac{g^2}{4\pi}\right)\times(\xi+1)\frac{\mathbf{k}^2}{k_0^{\rm eff}}\int d\mathbf{k}'\frac{|\mathbf{k}'|}{(k_0^{\rm eff})'}\Psi(\mathbf{k}',\mathbf{q};\omega),\tag{35}$$

where $\mathbf{q}$ is interpreted as the relative momentum for two incident quasifermions and α is an overall phase factor. We choose the outgoing wave condition $(+i\epsilon)$ as solution of Eq. (28), but we could have chosen, e.g., the incoming wave condition $(-i\epsilon)$ or Van Kampen wave condition [15] or another condition.

Integrating Eq. (35) with respect to $\mathbf{k}$,

$$\int d\mathbf{k}\frac{|\mathbf{k}|}{(k_0^{\rm eff})}\Psi(\mathbf{k},\mathbf{q};\omega)=\frac{\alpha}{\{1-(g^2/4\pi)(\xi+1)\int[d\mathbf{k}'/(k_0^{\rm eff})'](({\mathbf{k}'})^2/[\{(k_0^{\rm eff})'\}^2-\omega^2/4+i\epsilon])\}}\tag{36}$$

and substituting this result back into Eq. (35) yields a general solution for $\Psi(\mathbf{k},\mathbf{q};\omega)$,

$$\frac{|\mathbf{k}|}{k_0^{\text{eff}}}\Psi(\mathbf{k},\mathbf{q};\omega)=\alpha\delta(\mathbf{q}-\mathbf{k})+\frac{\alpha k_0^{\text{eff}}}{[(\mathbf{k}_0^{\text{eff}})^2-\omega^2/4+i\epsilon]}\times\left(\frac{|\mathbf{k}|}{k_0^{\text{eff}}}\right)\frac{1}{\Delta^+(\omega)}\left(\frac{|\mathbf{k}|}{k_0^{\text{eff}}}\right), \tag{37}$$

where $\Delta^+(\omega)$ is given by

$$\Delta^+(\omega)=\left(\frac{4\pi}{g^2}\right)\frac{1}{(\xi+1)}-\int\frac{d\mathbf{k}}{(k_0^{\text{eff}})}\frac{\mathbf{k}^2}{[(k_0^{\text{eff}})^2-\omega^2/4+i\epsilon]}. \tag{38}$$

The oscillation amplitude $\Gamma(\mathbf{k},\mathbf{q};\omega)$ is obtained from Eqs. (27) and (37) and reads

$$\Gamma(\mathbf{k},\mathbf{q};\omega)=-\frac{i\omega\alpha k_0}{md}\left(\frac{k_0^{\text{eff}}}{k^2}\right)\left\{\delta(\mathbf{q}-\mathbf{k})+\frac{k_0^{\text{eff}}}{[(k_0^{\text{eff}})^2-\omega^2/4+i\epsilon]}\left(\frac{|\mathbf{k}|}{k_0^{\text{eff}}}\right)\frac{1}{\Delta^+(\omega)}\left(\frac{|\mathbf{k}|}{k_0^{\text{eff}}}\right)\right\}. \tag{39}$$

Finally, substituting Eq. (37) into Eq. (28) we obtain the oscillation frequencies

$$\omega=2q_0^{\text{eff}}=2[q^2+m_{\text{eff}}^2]^{1/2}, \tag{40}$$

where $\mathbf{q}$ is the relative momentum for two incident quasifermion with masses $m_{\text{eff}}=(1+d)m$.

We observe that we can understand the factor 2 in the frequencies of oscillation ω [see Eq. (40)], as related to the treatment of harmonic oscillators in terms of the sympletic groups given by Goshen and Lipkin [16]. It can be interpreted classically by noticing that, since for harmonic oscillators the frequency does not depend on the amplitude of the motion, if a set of independent particles in a harmonic field is symmetrically stretched out of equilibrium, it will subsequently pulsate with frequency 2ω, where ω is the frequency of oscillation of the independent particles.

VI. BOUND STATES FROM THE SMALL OSCILLATIONS REGIME

In this section we will examine the condition for existence of bound states in the small oscillation regime around the stationary solution (vacuum) of our fermionic system. From Eqs. (28) and (29) we verify that the potential term, which describes the time evolution of our system in this regime is separable. Again, in analogy with scattering theory, we can evaluate the corresponding T matrix [7]. We find

$$T(\mathbf{k},\mathbf{k}';\omega)\propto h(\mathbf{k})\frac{1}{\Delta^+(\omega)}h(\mathbf{k}') \tag{41}$$

with $h(\mathbf{k})$ given by Eq. (29) and $\Delta^+(\omega)$ given by Eq. (38). The bound states are given by the poles of the T matrix. Therefore, we search for the zeros of $\Delta^+(\omega)$. It is clear that the integral in $\Delta^+(\omega)$ contains a logarithmic divergence. To keep it under control, we use the renormalization procedure of the coupling constant given by Eq. (12). Substituting Eq. (12) into Eq. (38) we get

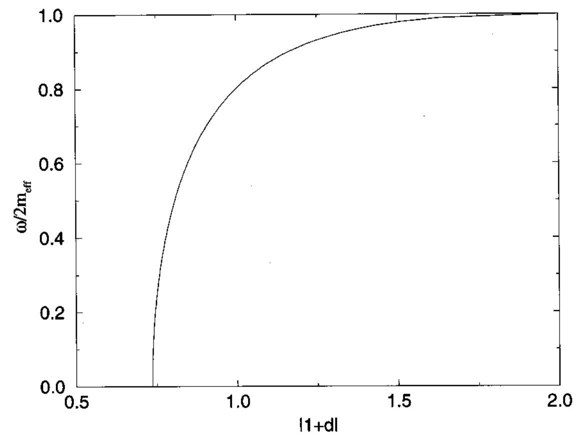

FIG. 1. The curve represents the mass ω of the two-quasifermion bound state in small oscillation regime for our system as a function of the renormalized coupling constant d.

$$\Delta^+(\omega)=\ln\left(\frac{\Lambda^2}{m^2}\right)-\int_{-\Lambda}^{+\Lambda}\frac{d\mathbf{k}}{[\mathbf{k}^2+m_{\text{eff}}^2]^{1/2}}\times\frac{\mathbf{k}^2}{[\mathbf{k}^2+m_{\text{eff}}^2-\omega^2/4+i\epsilon]}, \tag{42}$$

where we introduce the regularizing momentum cutoff Λ.

In the interval $0\leqslant\omega\leqslant 2m_{\text{eff}}$ the integral of $\Delta^+(\omega)$ is well defined and we can set $\epsilon=0$. A straightforward calculation yields

$$\Delta^+(\omega)=2[f(\omega)-j(d)], \tag{43}$$

where

$$f(\omega)=\left[\frac{4m_{\text{eff}}^2}{\omega^2}-1\right]^{1/2}\arctan\left\{\left[\frac{4m_{\text{eff}}^2}{\omega^2}-1\right]^{-1/2}\right\}$$

$$j(d)=\ln\left[\frac{2}{|1+d|}\right]. \tag{44}$$

Figure 1 shows the zero of the $\Delta^+(\omega)$ as a function of d. In this calculation $\mathbf{q}=0$, therefore ω is the mass of the bound state. Obviously, when $(1+d)=0$ or $m_{\text{eff}}=0$ (free system,see Ref. [1]) there is no bound state. We see from Fig. 1 that a bound state of quasifermions occurs when $0.74\leqslant(1+d)\leqslant 2$, and that the mass of this bound state will vary in the interval $0\leqslant\omega\leqslant 2m_{\text{eff}}$. Gross and Neveu obtain $M_\sigma=2M_F$ [2] for the mass of the σ particle in leading-$1/N$ aproximation, where M_F is equivalent to m_{eff}. They argue that in higher order they might find that

$$M_\sigma=2M_F[1+O(1/N)].$$

From Fig. 1 we verify that $\omega=2m_{\text{eff}}$ corresponds to $(1+d)=2$. Observing that $j(1+d=2)=0$, we may conclude that $j(d)$ can be see as a contribution of higher order to the Gross-Neveu result.

We believe that in the limit $N \rightarrow \infty$ the function $j(d) \rightarrow 0$. On the other hand, when N is finite [$N=1$ in our calculation — see Eq. (8)], the mass ω of the bound state depends on the renormalized coupling constant d as shown in Fig. 1. This dependence cannot be obtained from $1/N$ approximation.

Therefore, we can conclude that to $N \rightarrow \infty$ the $1/N$ approximation and our mean-field approximation are equivalent. On the other hand, when N is finite our approximation permits to obtain the higher order contribution to the Gross-Neveu result [2].

It is important to observe that the higher order term obtained in Eqs. (43) and (44) in this approach does not contain all terms of order $1/N$, since the mean-field approximation is not a systematic expansion in the parameter $1/N$.

Surprisingly, we have obtained for the function $\Delta^{+}(\omega)$ a structure which entirely reproduces that which has been found by Kerman and Lin [5] in their study of the *bosonic* $\lambda \phi^4$ theory in terms of a Gaussian time-dependent variational approach.

Finally, when $\omega > 2m_{\rm eff}$ the integrand of $\Delta^{+}(\omega)$ has a singularity at $\mathbf{k} = \pm \sqrt{\omega^2/4 - m_{\rm eff}^2}$. From the theory of residues we obtain

$$\Delta^{+}(\omega) = \left(1 - \frac{4m_{\rm eff}^2}{\omega^2}\right)^{1/2} \ln\left[\frac{1+(1-4m_{\rm eff}^2/\omega^2)^{1/2}}{1-(1-4m_{\rm eff}^2/\omega^2)^{1/2}}\right] - 2\ln\left(\frac{2}{|1+d|}\right) - i\pi\left[1 - \frac{4m_{\rm eff}^2}{\omega^2}\right]^{1/2}. \tag{45}$$

Now $\Delta^{+}(\omega)$ does not have any zeros. The interesting point here is to observe that

$$\lim_{\omega \rightarrow \infty} \Delta^{+}(\omega) \rightarrow \ln\left(\frac{\omega^2}{m_{\rm eff}^2}\right) \rightarrow \infty, \tag{46}$$

so that in the large frequency limit ($\omega \rightarrow \infty$) the T matrix goes asymptotically to zero. We thus recover, in the present approximation, the asymptotically free character of the CGNM.

VII. DISCUSSION AND CONCLUSIONS

In Ref. [1] we described a treatment of the initial-values problem in a quantum field theory of self-interacting fermions in the Gaussian approximation. Although the procedure is quite general, we implemented it for the vacuum of an uniform (1+1)-dimensional relativistic many-fermion system described by the chiral Gross-Neveu model (CGNM). We obtained the renormalized kinetic equations which describe the effective dynamics of the Gaussian observables in the mean-field approximation for this system.

In this work, we have considered the linearized form of the mean-field kinetic equations obtained in Ref. [1] around the stationary (vacuum) solution. The two-quasifermion physics can be analytically investigated in this approach. In particular, we have solved these equations completely. From the solutions, we have reinterpreted the near equilibrium physics of our system as a problem of quasifermion scattering and have found the condition for the existence of a quasifermion bound state.

We verify that for N finite (in this work $N=1$), the bound state mass obtained from our approach contains a term which depends on the renormalized coupling constant as can be seen in Fig. 1. In the case of a $1/N$ expansion [2] this dependence cannot be found, so in the limit $N \rightarrow \infty$ this term goes to zero. Therefore, to small N, our approach permits to obtaining the higher order contribution to the $1/N$ expansion. Finally, it is important to observe that the higher order term obtained in the bound state mass from our approach contains not necessarily all terms of $1/N$ order, since the mean-field approximation is not a $1/N$ expansion.

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