COMMENTS

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Comment on the Kundrát-Lokajíček assertion about the applicability of the Martin formula

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(Received 29 February 1996)

Contrary to the assertion of Kundrát and Lokajíček that the Martin formula, which relates the real part of the elastic hadron-hadron scattering amplitude with its imaginary part, can be used only at small momentum transfers $|t| \lesssim 0.15(\mathrm{GeV}/c)^2$ in the CERN ISR and S$p\overline{p}$S region, the formula can be used consistently at high energies for the observed experimental data in the whole region of the momentum transfer of the experiments. Their errors are due to an inappropriate numerical approach for solving the problem. [S0556-2821(97)05005-4]

PACS number(s): 13.85.Dz, 12.40.Nn

Martin [1] derived a formula which relates the real part of the elastic scattering amplitude with its imaginary part by assuming asymptotic behavior. This formula was used in several analyses of high energy elastic processes since then [2]. Kundrát and Lokajíček [3] examined the Martin formula by analyzing the experimental data of the pp elastic differential cross sections in the CERN Intersecting Storage Rings (ISR) energy region and concluded that the formula can be used only in the region of small momentum transfers, $|t| \lesssim 0.15(\mathrm{GeV}/c)^2$. They repeated their assertion later [4]. We consider, however, their conclusion is not correct. Since the Martin formula has still been used even after their criticism [5] and no counter argument to their papers seems to have been published so far, it will be meaningful to scrutinize their points.

Neglecting spin effects, the differential cross section can be given in the center-of-mass system (c.m.s.) as

$$\frac{d\sigma}{dt} = \pi \left\{ \rho^2 \left(\frac{d}{dt}[t \mathrm{Im}\, F(s,t)] \right)^2 + [\mathrm{Im} F(s,t)]^2 \right\}, \qquad (1)$$

if the real part of the scattering amplitude $\mathrm{Re}F(s,t)$ is given from the imaginary part of the scattering amplitude $\mathrm{Im}F(s,t)$ by the Martin formula [1]

$$\mathrm{Re}F(s,t) = \rho \frac{d}{dt}[t \mathrm{Im} F(s,t)]. \qquad (2)$$

Here s is the squared c.m.s. total energy, t the squared momentum transfer, and ρ the ratio of the real part to the imaginary part of the forward scattering amplitude.

In Ref. [3] Kundrát and Lokajíček tried to solve Eq. (1) or its equivalents *numerically* as a differential equation for Im $F(s,t)$ using the experimental data of pp differential cross sections in the CERN ISR energies. They claimed that the experimental data allow $\mathrm{Im}F(s,t)$ to be real only at small momentum transfers and any application of the Martin formula to larger values of $|t|$ loses any physical sense.

In this paper we show that their assertion does not hold. There are two ways for the use of Eq. (1) to determine the scattering amplitude: (i) one is to assume some analytic form for the imaginary part and to fit the cross section by varying its parameters, and (ii) the other is to solve it as the differential equation with respect to the imaginary part for the given differential cross section data which are represented in some parametric form.

The consistency of the first method (i) with the Martin formula (2) is evident and the method can be utilized for the ISR and CERN Super Proton Synchrotron (S$p\overline{p}$S) data in the entire momentum transfer region of the experimental measurements. Here the problem is whether or not the resulting amplitude reproduces well the experimental data (the total and elastic cross sections, the differential cross section, and the real part of the forward scattering amplitude). The applicability of this approach can be refuted only on this ground, if the validity of the formula (2) is justified.

The second approach (ii) was examined by Kundrát and Lokajíček. They found that the solution becomes singular around the first zero of the real part of the scattering amplitude and they concluded that Eq. (1) has a unique solution

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only in a small momentum transfer region at ISR and $Sp\overline{p}S$ energies and beyond this region the solution becomes complex and leads to no physical consequences. However, this does not correctly state the situation: the unique physical solution exists for the given experimental data in the entire region of the momentum transfer and it is possible to have an approximate solution to the exact physical solution. The difficulty they found stems from the numerical method used to solve Eq. (1), not in the application of Eq. (1) to the experimental data at the nonasymptotic energy region. A careful consideration about the critical points of the differential equation is required to obtain the physical amplitude.

First we briefly summarize the critical points of the first order ordinary differential equation [6]. We write the differential equation in the form

$$\frac{dy}{dx}=\frac{Q(x,y)}{P(x,y)}, \tag{3}$$

then the critical points of this equation appear at the point (x,y) determined by the conditions $P(x,y)=0$ and $Q(x,y)=0$. If $P(x,y)$ and $Q(x,y)$ are expressed as

$$P(x,y)=a_{11}x+a_{12}y, \quad Q(x,y)=a_{21}x+a_{22}y, \tag{4}$$

the critical points are classified by the eigenvalues of the coefficient matrix (a_{ij}) into nodes, saddle points, foci, and centers [6].

Now we take the following equation, which is equivalent to Eq. (1):

$$\frac{d\alpha(\tau)}{d\tau}=\frac{\rho\tau f(\tau)-\tan\alpha}{\rho\tau\tan\alpha}, \tag{5}$$

where α is the phase of the scattering amplitude, $F(\tau)=i|F(\tau)|e^{-i\alpha(\tau)}$, $\tau\equiv|t|\sigma_{\rm t}$ with the total cross section $\sigma_{\rm t}$, and

$$f(\tau)\equiv\frac{1}{\tau}-\frac{B(\tau)}{2}, \quad B(\tau)=-\frac{d}{d\tau}\ln\frac{d\sigma}{d\tau}. \tag{6}$$

Here we have omitted the energy variable s. Let the roots of $f(\tau)$ be τ_i $(i=1,2,3,\ldots)$. The critical points P_i are

$$\begin{aligned} &P_0:\ \tau_0=0, \quad \alpha_0=\arctan\rho, \\ &P_i:\ \tau_i,\ \ \alpha_i=0(\mathrm{mod}\,\pi) \quad (i=1,2,3,\ldots). \end{aligned} \tag{7}$$

We examine the critical points about a realistic case. For this purpose we take the differential cross section given by the pp scattering amplitude at 52.8 GeV in Ref. [7] as the experimental data. The imaginary part of the scattering amplitude is given by

$$\begin{aligned} \mathrm{Im}F(s,t)=&A_1e^{(1/2)B_1t}+A_2e^{(1/2)B_2t} \\ &-A_3\left[1+c\left(1-\frac{t}{t_0}\right)^2\right]e^{(1/2)B_3t}, \end{aligned} \tag{8}$$

where the values of the parameters are $A_1=2.081$, $A_2=3.398$, $A_3=0.00608(\mathrm{mb})^{1/2}(\mathrm{GeV}/c)^{-1}$, $B_1=20.622$, $B_2=9.066$, $B_3=1.302(\mathrm{GeV}/c)^{-2}$, $t_0=-6.612(\mathrm{GeV}/c)^2$, and $c=3.512$. We assume $\rho=0.0701$ for the real part. In this case the function $f(\tau)$ is shown in Fig. 1, which is essentially the same as that given in Fig. 1 of Ref. [3].

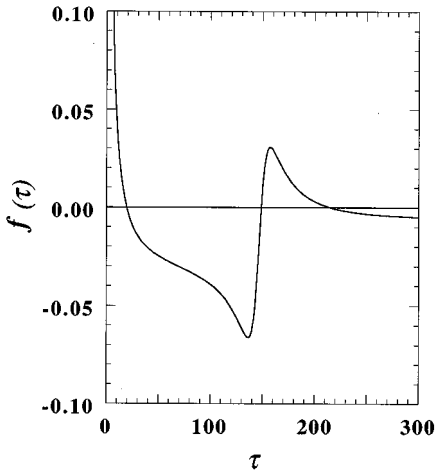

FIG. 1. The function $f(\tau)$ corresponding to the amplitude (8). The critical points appear at $\tau=0$ and the roots of $f(\tau)$.

There are four critical points $P_0=(0,\arctan\rho)$, $P_1=(\tau_1,0)$, $P_2=(\tau_2,0)$, and $P_3=(\tau_3,0)$, where τ_1,τ_2, and τ_3 are 19.49, 148.85, and 214.76, respectively. We can easily see that P_0 is a saddle point, P_1 a node, P_2 a saddle point, and P_3 a node, respectively. Let us examine the solutions around these critical points. Here it is to be noted that only the analytic solution can be physical for $\tau\geqslant 0$.

P_0: By making a linear approximation for the denominator and the numerator at the right-hand side of Eq. (5), we have

$$\frac{dy}{dx}=\frac{-(1/2)\rho B(0)x-(1+\rho^2)y}{\rho^2x}, \tag{9}$$

where new variables x and y are $x=\tau$ and $y=\alpha-\arctan\rho$. The eigenvalues of the coefficient matrix are

$$\lambda_1=\rho^2, \quad \lambda_2=-(1+\rho^2). \tag{10}$$

The real eigenvalues with opposite signs imply that this critical point is a saddle point. With $\rho=0.0701$, we have $\lambda_1=0.00491$ and $\lambda_2=-1.00491$.

The general solution of this differential equation is given in terms of these eigenvalues as

$$x|2(1+2\rho^2)y+B(0)\rho x|^{\rho^2/(1+\rho^2)}=C_0, \tag{11}$$

where C_0 is the integration constant.

The solution passing the origin $(x,y)=(0,0)$ is given by $C_0=0$. This gives two equations

$$\text{(a)}\ \ x=0$$

and

$$\text{(b)}\ \ y=-\frac{B(0)\rho}{2(1+2\rho^2)}x, \tag{12}$$

which are the separatrices. Clearly the solution (b) is analytic and physical one.

P_1: We have

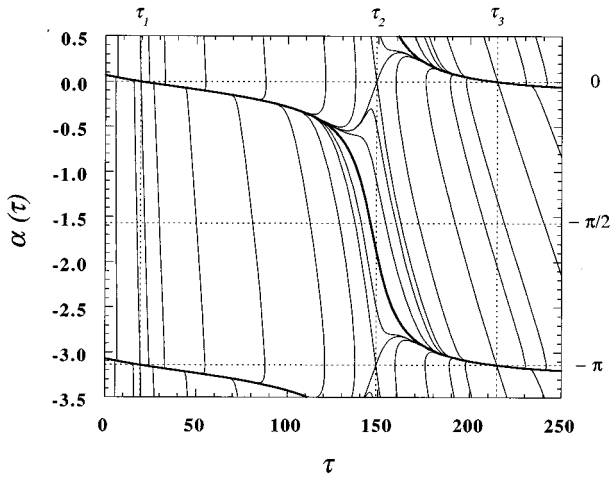

FIG. 2. Integral curves of the differential equation (5). The thick solid curve is the exact physical solution which is analytic in the entire region of the momentum transfer $\tau \geqslant 0$.

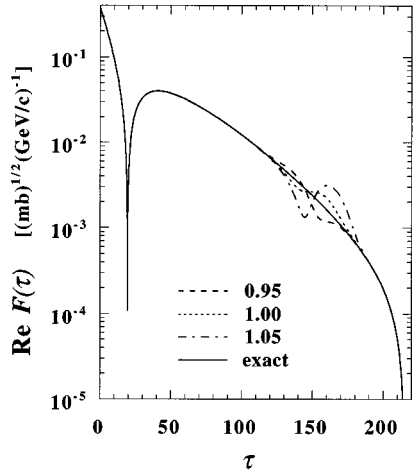

FIG. 3. The uncertainty of the real part coming from the ambiguity in the location of the zero of the imaginary part of the scattering amplitude. We show the real parts around the exact solution (the solid curve) by assuming the zero of the imaginary part at the dip location of the differential cross section $\tau_{\rm dip}$ (the dotted curve), $0.95\times\tau_{\rm dip}$ (the dashed curve), and $1.05\times\tau_{\rm dip}$ (the dot-dashed curve).

$$\frac{dy}{dx}=\frac{\rho\tau_1 f'(\tau_1)x-y}{\rho\tau_1 y}, \tag{13}$$

where new variables x and y are $x=\tau-\tau_1$, $y=\alpha$. The eigenvalues of the coefficient matrix are

$$\lambda_{1,2}=\frac{1}{2}\{-1\pm[1+4\rho^2\tau_1^2 f'(\tau_1)]^{1/2}\}. \tag{14}$$

From $f'(\tau_1)=-0.002233$ we have $\lambda_1=-0.004184$ and $\lambda_2=-0.9958$. The real eigenvalues with the same signs imply that the critical point is a node. The general solution of Eq. (14) is

$$\rho\tau_1 y-\lambda_1 x=C_1|\rho\tau_1 y-\lambda_2 x|^{\lambda_2/\lambda_1}, \tag{15}$$

where C_1 is the integration constant.

All the integral curves pass the origin $x=y=0$ and their slopes (y') take a common value $\lambda_1/\rho\tau_1$ at the origin since $\lambda_2/\lambda_1>1$, except for the value $C_1=\infty$ which gives the solution $\rho\tau_1 y-\lambda_2 x=0$. The physical solution is given by the integration constant $C_1=0$.

P_2, P_3: In the same way we can calculate the solutions around these points. For the lack of space we omit the results.

In order to obtain the analytic solution, or more precisely, to have a solution which is a good approximation to the analytic solution by the numerical method, we may start the calculation from τ_s near $\tau=0$, since at $\tau=0$ the equation is critical. We choose τ_s as small as possible and set the initial value $\alpha(\tau_s)$ arbitrary but preferably near $\arctan(\rho)$ and solve the differential equation (5) by some method such as the Runge-Kutta one. The integral curve very quickly approaches the exact solution and ends at $\tau=\tau_1$ where all the solutions contact as seen in Fig. 2. There is practically no problem to obtain a solution with reasonable accuracy to the physical solution in the region $0\leqslant\tau\leqslant\tau_1$.

For $\tau>\tau_1$ we can no longer solve *numerically* the differential equation toward larger τ. If we dare try to continue the calculation, there would be only vanishing chance for obtaining the physical solution even for a very short interval of τ, as easily understood by the node structure of the P_1 point. Any solution, if acquired small errors, easily turns aside from the analytic solution and moves nearly vertical direction. This explains what has happened in the calculation of Kundrát and Lokajíček [3] .

How do we obtain a reasonable solution in the region $\tau>\tau_1$? This would be simply carried out if we could locate the zero of the imaginary part of the scattering amplitude correctly: what we have to do is to solve numerically the differential equation starting from the zero point of the imaginary part, with $\alpha(\tau_{{\rm Im}F=0})=-\pi/2$ backwardly to τ_1 and forwardly to τ_3. This works quite well. Unfortunately, however, we cannot locate the zero of the imaginary part fairly accurately for a given differential cross section unless the real part around the zero of the imaginary part is so small that a very sharp dip structure appears in the differential cross section. The present example at 52.8 GeV gives $\tau_{{\rm Im}F=0}=145.62$ and the dip of the differential cross section appears at $\tau_{\rm dip}=149.48$. The ambiguity in locating the zero of the imaginary part brings some uncertainties to the real part around the zero of the imaginary part. We show the situation in Fig. 3. The exact physical solution behaves most *smoothly* in the dip region.

The first method (i) is clearly of better utility than the second one (ii), since the former automatically satisfying Eq. (1) needs no additional procedure of solving the differential equation which induces uncertainties to the real part, while the fitting procedures to the experimental data, either through the imaginary part or directly, require almost the same amount of computational work.

Finally it should be noted that at finite energy there exists the problem of the validity of the Martin formula which is obtainable under some asymptotic assumptions, but this is a question different from that Kundrát and Lokajíček have raised. If the experimental data do not show the geometrical scaling behavior [9], the application of the Martin formula will not be suitable. We may use the derivative dispersion relation [7,8,10] which reduces to the Martin formula if the geometrical scaling holds.

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