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Applicability of Martin's equations in high-energy elastic hadron scattering

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The validity region of Martin's equations enabling one to determine the t dependence of the real part of the elastic hadron amplitude from its imaginary part is critically reexamined. It can be concluded on the basis of a more precise analysis that quite unjustified and in principle incorrect physical results are obtained if the equations are used outside this region, i.e., for $|t|\gtrsim 0.15\ \text{GeV}^2$. [S0556-2821(97)04905-9]

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Martin [1] showed that at asymptotic values of s and at infinitesimal values of t the real part of the elastic hadron scattering amplitude may be related to the imaginary part by the equation

$$\text{Re}F^N(s,\tau)=\text{Re}F^N(s,0)\frac{d}{d\tau}[\tau\Phi(\tau)], \tag{1}$$

where

$$\Phi(\tau)=\frac{\text{Im}F^N(s,\tau)}{\text{Im}F^N(s,0)} \tag{2}$$

and $\tau=|t|\sigma_{\text{tot}}(s)$. Here s is the square of total center-of-momentum energy, t the four-momentum transfer squared, and $\sigma_{\text{tot}}(s)$ the total cross section. Many attempts were done in the past to make use of these formulas in analyzing experimental data obtained at higher, but finite values of s and at all measured values of t. When applied to finite energies Martin's Eqs. (1) and (2) have also been called geometrical scaling equations [2].

Function $\Phi(\tau)$ may be brought to a close relation to experimentally established values of the elastic hadron differential cross section $d\sigma^N(s,\tau)/d\tau$ as it is possible to write [3,4] with the help of Eqs. (1) and (2):

$$\frac{d\Phi(\tau)}{d\tau}=\frac{1}{\rho\tau}\{-\rho\Phi(\tau)+[D(s,\tau)-\Phi^2(\tau)]^{1/2}\}, \tag{3}$$

where

$$D(s,\tau)=(1+\rho^2)\frac{d\sigma^N(s,\tau)/d\tau}{d\sigma^N(s,0)/d\tau}, \tag{4}$$

and ρ is the ratio of the real to imaginary parts of elastic hadron amplitude in forward direction [in order for $\Phi'(0)$ to be finite, only the plus sign in Eq. (3) in front of the square root should be considered]. Function $\Phi(\tau)$ can be determined in principle for any type of elastic hadron scattering from the corresponding experimental data by solving differential equation (3). Mutual consistence of Eqs. (1) and (2) requires $\Phi(\tau)$ to be real for any $\tau\geqslant 0$ and to fulfill the initial condition

$$\Phi(0)=1. \tag{5}$$

It was already shown for high-energy pp and for $\overline{pp}$ elastic scatterings that inserting the experimental data into Eq. (3) the function $\Phi(\tau)$ satisfying condition (5) is real for rather small values of τ only (corresponding to $|t|\lesssim 0.15\ \text{GeV}^2$). At higher values of $|t|$ the expression under the square root in differential Eq. (3) becomes complex. The function $\Phi(\tau)$ ceases to be real and becomes complex which contradicts the basic assumption of Martin's equations (1) and (2). It means that Martin's equations cannot be applied to the mentioned higher values of τ [4,5].

Nevertheless, the papers have been continuously published [6–11] in which Martin's equations have been applied to the region of diffractive minimum; i.e., far behind the mentioned boundary of the allowed values of momentum transfers. The starting point in the quoted papers consisted in constructing a phenomenological (and practically arbitrary) imaginary part of elastic hadron amplitude [i.e., the function $\Phi(\tau)$] vanishing at the diffractive minimum. The real part for all τ was then determined with the help of Martin's equations (1) and (2) without testing any consistency with Eq. (3).

There is, of course, a certain deficiency in determining the reality region of function $\Phi(\tau)$ in papers [4,5] as it was established with the help of an approximate approach based on an approximate interference formula [12], in which the influence of Coulomb scattering at higher values of $|t|$ was fully neglected while some non-negligible corrections should exist according to a more general and more exact approach [13,14], which might influence the previous results making them not fully certain. To give a definite answer to persistent use of Martin's equations in the region of diffractive minimum a new analysis under more exact conditions has been performed.

It was shown in our previous paper [4] that the region of τ values, where the function $\Phi(\tau)$ is real, is tightly connected with the roots of the equation

$$f^N(s,\tau)=0, \tag{6}$$

the function $f^N(s,\tau)$ being defined as

$$f^N(s,\tau)=\frac{1}{\tau}+\frac{(d/d\tau)[d\sigma^N(s,\tau)/d\tau]^{1/2}}{[d\sigma^N(s,\tau)/d\tau]^{1/2}}, \tag{7}$$

i.e., it is fully determined by the modulus of elastic hadron amplitude. Equation (6) can be rewritten also as

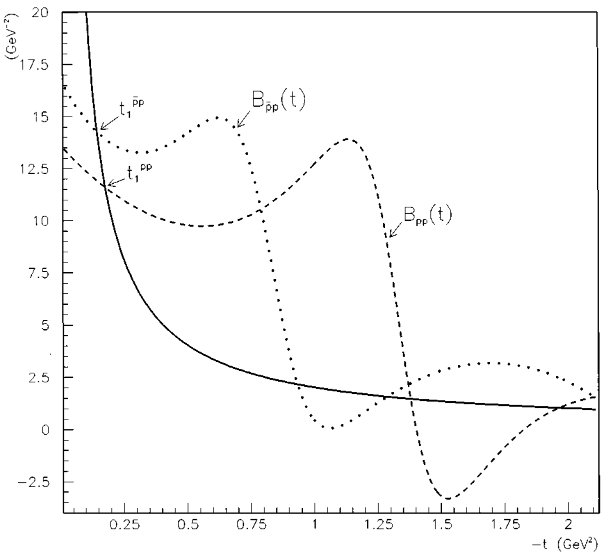

FIG. 1. The intersections of diffractive slopes $B(t)$ corresponding to pp elastic scattering at energy of 53 GeV and to $\overline{pp}$ elastic scattering at energy of 541 (546) GeV with hyperbola $2/t$.

$$B(t)=\frac{2}{t}, \tag{8}$$

where the general t-dependent diffractive slope at given energy

$$B(t)=\frac{d}{dt}\left(\ln\frac{d\sigma^N}{dt}\right) \tag{9}$$

is directly established from experimental data, which provides much easier insight into the given problem. Using now the results of paper [13] we could determine more precise roots of Eq. (6) or Eq. (8) for both previously mentioned cases of elastic scatterings; i.e., for the pp case at the energy of 53 GeV (using data from Ref. [15]) and for the $\overline{pp}$ case at the energy of 541 (546) GeV (using data from Ref. [16]). The positions of these roots being determined by the intersections of the curves illustrating the t dependence of the diffractive slopes $B(t)$ with hyperbola $2/t$ are shown in Fig. 1 for both the investigated cases of elastic scatterings.

That the function $\Phi(\tau)$ becomes complex for $\tau\geqslant\tau_1$ [τ_1 is the smallest root of Eq. (6)] was shown in paper [4] by using differential equation for the phase $\alpha_s(\tau)$ of elastic hadron amplitude and for the function $\omega(s,\tau)$ [renormalized function $\Phi(\tau)$]; for details see previous paper [4], the paragraph after Eq. (3.3) [17]. However, it was not answered in this paper whether the function $\Phi(\tau)$ does not become again real at τ higher than the other roots of Eq. (6).

That the function $\Phi(\tau)$ remains complex for all $\tau>\tau_1$ may be shown by solving Eq. (3) in the complex plane in the whole interval of allowed τ values. Let us define real and imaginary parts of $\Phi(\tau)$ by

$$\Phi(\tau)=\Phi_1(\tau)+i\Phi_2(\tau). \tag{10}$$

Then Eq. (3) can be transformed into the normal system of two nonlinear differential equations of the first order:

$$\begin{aligned}\frac{d\Phi_1(\tau)}{d\tau}=\frac{1}{\rho\tau}\Bigg\{&-\rho\Phi_1(\tau)+\Bigg[\frac{1}{2}([D^2(s,\tau)+\Phi_1^4(\tau)+\Phi_2^4(\tau)\\&-2D(s,\tau)\Phi_1^2(\tau)+2D(s,\tau)\Phi_2^2(\tau)\\&+2\Phi_1^2(\tau)\Phi_2^2(\tau)]^{1/2}+D(s,\tau)-\Phi_1^2(\tau)\\&+\Phi_2^2(\tau))\Bigg]^{1/2}\Bigg\},\\\frac{d\Phi_2(\tau)}{d\tau}=\frac{1}{\rho\tau}\Bigg\{&-\rho\Phi_2(\tau)+\Bigg[\frac{1}{2}([D^2(s,\tau)+\Phi_1^4(\tau)+\Phi_2^4(\tau)\\&-2D(s,\tau)\Phi_1^2(\tau)+2D(s,\tau)\Phi_2^2(\tau)\\&+2\Phi_1^2(\tau)\Phi_2^2(\tau)]^{1/2}-D(s,\tau)+\Phi_1^2(\tau)\\&-\Phi_2^2(\tau))\Bigg]^{1/2}\Bigg\}.\end{aligned} \tag{11}$$

In contradistinction to the differential equations solved in paper [4] the right-hand sides of both the Eqs. (11) do not contain any singular points in the interval $R=\langle\varepsilon,\tau_{\max}\rangle$; here ε is a finite (arbitrarily small) positive number and $\tau_{\max}$ corresponds to the minimal measured value of t. Both the right-hand sides are represented by continuous functions (satisfying the Lipschitz condition) in variables τ, Φ_1, and Φ_2 provided the elastic hadron differential cross section, generating the function $D(s,\tau)$ [Eq. (4)] is a smooth function of τ. It follows from Peano's and Picard's theorems [18], that the solution of Eqs. (11) [with initial condition (5) or with $\Phi(\varepsilon)=1+\Phi'(0)\varepsilon$] exists and is unique and continuously differentiable.

If the expression under the square root in original differential equation (3) is non-negative then its solution, i.e., the real function Φ should coincide with the solution of the system of differential equations (11); therefore, the imaginary part Φ_2 should be zero in this region. But if this expression starts to be generally complex then Φ_2 becomes different from zero. Equations (11) must be solved numerically, which requires the function $D(s,\tau)$ to be represented by a smooth function. That may be guaranteed by representing the differential cross section with the help of elastic hadron modulus $|F^N(s,\tau)|$ parameterized, e.g., according to our previous paper [13].

It has been found by solving Eqs. (11) under new conditions (with the help of the Runge-Kutta method) that Eqs. (11) possess a real solution only for $\tau\leqslant\tau_1$, i.e., for $|t|\leqslant|t_1|$, where $t_1=-0.172\ \mathrm{GeV}^2$ in the pp elastic scattering and $t_1=-0.133\ \mathrm{GeV}^2$ in the $\overline{pp}$ elastic scattering. The influence of the new more exact approach of separating Coulomb and hadron parts in elastic scattering on the boundary limits is not, therefore, substantial. Evidently, the value of τ, where the solution $\Phi(\tau)$ of Eqs. (3) becomes complex coincides with the first smallest root of Eq. (6). Outside these regions, i.e., for all $\tau>\tau_1$, the function $\Phi(\tau)$ ceases to be real and becomes complex. The resulting τ dependence of the function $\Phi(\tau)$ (i.e., of both the real and imaginary parts of it) for the case of $\overline{pp}$ scattering is shown in Fig. 2. For the case of pp elastic scattering at an energy of 53 GeV the picture is similar.

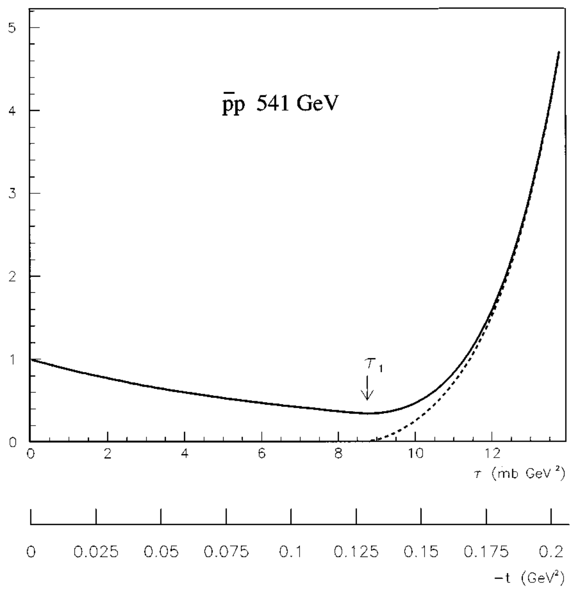

FIG. 2. The τ dependence of both real and imaginary parts of function $\Phi(\tau)$ for $\overline{pp}$ peripheral elastic scattering at energy of 541(546) GeV; Re $\Phi(\tau)$-full line, Im$\Phi(\tau)$-dotted line.

The reality of the function $\Phi(\tau)$ is a basic condition for Martin's equations to be mathematically consistent. Outside the reality region Martin's equation lose any physical sense. Consequently, they can never be used at higher values of τ especially in the region of diffractive minimum. Any physical sense cannot be attributed to attempts [10,11,19], either, in which some corrections are added to get a numerical fit with experimental data. The mentioned inconsistency increases when in some papers [10,11] the standard West and Yennie relative phase [12] is made use of, combining the basic assumption of the exponential differential cross section with strong experimental nonexponentiality (for details see papers [13,14]). Consequently, one must conclude that the quoted papers [6–10] contain a significant mathematical inconsistency using Martin's equations in the regions of τ, where $\Phi(\tau)$ must be complex.

There is only one mathematically consistent way of enabling Martin's equations to be used at the values $\tau > \tau_1$ (at finite energies); namely, to use higher derivatives of function $\Phi(\tau)$ in equations exhibiting geometrical scaling [3,4]. In such a case we should obtain, however, more complicated equations for both the real and imaginary parts of elastic hadron amplitude, and the simplicity and the beauty of the Martin's equations would be lost.

Note added. During the preparation of this work a paper by Kawasaki, Maehara, and Yonezawa [20] has been submitted for publication in which they criticize results from our previous paper [4]. Their paper contains practically two basic statements which both contradict our conclusions. First, they argue that the equation

$$\frac{d\sigma^N(s,\tau)}{d\tau} = \frac{d\sigma^N(s,0)}{d\tau}\left[\Phi^2(\tau) + \rho^2\left(\frac{d}{d\tau}[\tau\Phi(\tau)]\right)^2\right]\frac{1}{1+\rho^2}, \tag{12}$$

derived with the help of the Martin's equations [1] is valid in the whole range of measured t values. And second, they state that the same holds for Eq. (5) of Ref. [20] [or Eq. (3.3) of Ref. [4]] being under some limiting conditions equivalent to Eq. (12). We will show now that any of these statements may be hardly regarded as true.

(i) The Martin's equations are based on the assumption that the function $\Phi(\tau)$ is real. We have shown in Ref. [4] and demonstrated to a greater detail in this paper that $\Phi(\tau)$ may be real only in the interval $(0,\tau_1)$. $\Phi(\tau)$ becomes complex for any $\tau > \tau_1$.

(ii) One can, of course, postulate the validity of Eq. (5) (of Ref. [20]) for the whole range of all measured τ values. Then we must look for the solution of this first order differential equation with one initial condition

$$\alpha(0) = \arctan(\rho). \tag{13}$$

When the given equation cannot be solved analytically we have shown in Ref. [4] that there is a sudden change in $\alpha(\tau)$ behavior in a close neighborhood of τ_1, which is in full agreement with the characteristics of Eq. (12). The authors of Ref. [20] have performed, however, a valuable analysis when they showed that Eq. (5) (of Ref. [20]) has a nodal point in τ_1. The individual solutions in this point are then distinguished by the values of integration constant C_1 (see, e.g., Eq. (15) of Ref. [20]) in a corresponding linear approximation [21]; this constant C_1 may possess any real value in principle. It is practically impossible to establish its actual value corresponding to the initial condition (13) as evaluation from the corresponding numerical solution exhibits changes by many tenths of order when τ goes near τ_1. Consequently, the resulting solution presented in Ref. [20] represents a combination of three components belonging to different initial conditions, given by Eq. (13), specified by $C_1 = 0$ and $\alpha = \pi/2$ in a close neighborhood of diffractive minimum.

Realizing the existence of the nodal point enables us to understand why Dias de Deus and Kroll [3] could already have proposed a similar solution many years ago starting from similar initial conditions in $\tau = 0$ and in the diffraction minimum, since both the different branches had to meet in the nodal point.

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