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Manifestly gauge covariant treatment of lattice chiral fermions

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We propose a lattice formulation of the chiral fermion which maximally respects the gauge symmetry and simultaneously is free of the unwanted species doublers. The formulation is based on the lattice fermion propagator and composite operators, rather than on the lattice fermion action. The fermionic determinant is defined as a functional integral of an expectation value of the gauge current operator with respect to the background gauge field: The gauge anomaly is characterized as the nonintegrability. We perform some perturbative tests to confirm the gauge covariance and an absence of the doublers. The formulation can be applied rather straightforwardly to numerical simulations in the quenched approximation. [S0556-2821(97)03405-X]

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The chiral fermion on the lattice has refused its manifestly gauge-invariant treatment [1]. There even exists the no-go theorem [2] for such an endeavor. In the continuum counterpart, the chiral fermion develops a curious phenomenon, called the quantum anomaly [3] or more definitely the gauge anomaly [4]. It can be argued that the difficulty in the lattice chiral gauge theory is a natural consequence of the gauge anomaly.

Suppose that we start with a well-regularized fermionic partition function defined by a manifestly gauge-invariant lattice fermion action. In the continuum limit, the gauge anomaly is a gauge variation of the partition function. Therefore, if the fermion content is not free of the gauge anomaly, the partition function should not be gauge invariant—this contradicts the very gauge invariance of the formulation. There are two possible resolutions: One is an appearance of the species doublers which cancel the gauge anomaly [5]. Another is a pathology in the continuum limit such as the non-Lorentz covariance [6]. The "trouble" with the lattice regularization is that it always regularizes ultraviolet divergences, even when a gauge-invariant regularization should be impossible due to the gauge anomaly.

The above reasoning suggests that the appearance of the unwanted doublers is quite natural. However, the problem in the conventional approach is, of course, that the doublers appear even in the anomaly-free cases. Presumably, the ideal lattice formulation of the chiral fermion will be the one which distinguishes the anomaly-free gauge representations from the anomalous ones. That unknown gauge-invariant lattice action should have a structure that can be written down, for example, for the spinor representation of so(4) but not for the fundamental representation of su(3), because the latter is anomalous. Such an ideal formulation seems to require a further deeper understanding on the origin of the quantum anomaly.

In this work, we simply abandon a direct gauge-invariant definition of the fermionic partition function. We take an indirect route. Nevertheless, we attempt to respect the gauge symmetry as much as possible within a range consistent with the gauge anomaly.

Instead of directly defining the fermion action and the partition function, we start with the propagator and the gauge current operator on the lattice. This formulation may be regarded as a first quantization approach, compared to the conventional ones. The important fact for us is that although the partition function cannot be regularized gauge invariantly in general, the gauge current can always be regularized gauge *covariantly* even if the gauge representation is anomalous. This type of regularization scheme in the continuum theory is known as the covariant regularization [7].

In the covariant regularization, fermion loop diagrams are defined as an expectation value of the gauge current $J^{\mu a}(x)$, in the presence of the background gauge field. The ultraviolet divergence of the diagram is then regularized by inserting a gauge-invariant dumping factor into the fermion propagator. In this way, the gauge invariance associated with all the gauge vertices *except* that of $J^{\mu a}(x)$, is preserved. The basic idea is that a possible breaking of the gauge symmetry due to the anomaly is forced on the $J^{\mu a}(x)$ vertex as much as possible. The gauge anomaly $D_\mu\langle J^{\mu a}(x)\rangle$ thus defined has the covariant form [8,9] because of the gauge invariance at external vertices. On the other hand, a gauge singlet-operator such as the fermion number current is always regularized gauge invariantly. The scheme thus spoils the Bose symmetry in general but it is restored when the theory is free of the gauge anomaly. The scheme is very powerful and applicable to any chiral gauge theories including the Yukawa couplings.

Once the expectation value of the gauge current is obtained in the covariant regularization, the fermionic determinant may be defined as a functional integral of $\langle J^{\mu a}(x)\rangle$ with respect to the background gauge field. However, it is obvious that the integration is possible only when there exists a Bose symmetry among all the gauge vertices. In other words, the gauge anomaly should satisfy the Wess-Zumino condition [10] which is a consequence of the integrability. Since the covariant anomaly breaks the Bose symmetry and the Wess-Zumino condition [8,9], we cannot define the fermionic determinant from the integration of $\langle J^{\mu a}(x)\rangle$, provided that it

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has the covariant gauge anomaly.[1] Therefore, following the covariant regularization, the gauge anomaly is characterized as the nonintegrability of this integration process. This is a consistent picture because the fermionic determinant cannot be gauge invariant when the gauge anomaly is present. Our proposal in this work in spirit can be regarded as a lattice version of the covariant regularization. We notice that the covariant regularization itself does not require the action level realization [11].

We proceed as follows: In the continuum theory, the propagator of a massless *Dirac* fermion is expressed as

$$\langle T\psi(x)\overline{\psi}(y)\rangle=\frac{-1}{i\not{D}}\,\delta(x-y)$$
$$=i\not{D}\,\frac{1}{g^{\mu\nu}D_\mu D_\nu+i[\gamma^\mu,\gamma^\nu]F_{\mu\nu}/4}\,\delta(x-y), \tag{1}$$

where $\not{D}\equiv\gamma^\mu(\partial_\mu+iA_\mu)$ is the vector type, i.e., nonchiral, covariant derivative, and $F_{\mu\nu}$ is the field strength. In the second line, the denominator has been rewritten as a *second* derivative, that may allow a lattice propagator free of the doubler's massless pole. As the propagator on the lattice, therefore, we take

$$\langle T\psi(x)\overline{\psi}(y)\rangle=G(x,y)$$
$$=i\not{D}(x)\,\frac{1}{\Box(x)+[\gamma^\mu,\gamma^\nu][U_{\mu\nu}(x)-1]/(4a^2)}$$
$$\times\delta(x,y), \tag{2}$$

where $\delta(x,y)\equiv\delta_{x,y}/a^4$ and $\not{D}(x)$ is the standard lattice covariant derivative

$$\not{D}(x)=\sum_\mu\,\gamma^\mu\,\frac{1}{2a}\,[U_\mu(x)e^{a\partial_\mu}-e^{-a\partial_\mu}U_\mu^\dagger(x)] \tag{3}$$

[$U_\mu(x)$ is the link variable [12] and a is the lattice spacing]. To avoid the unwanted massless pole, we define the covariant lattice d'Alembertian by

$$\Box(x)\equiv-\sum_\mu\,\frac{1}{a^2}\,[U_\mu(x)e^{a\partial_\mu}+e^{-a\partial_\mu}U_\mu^\dagger(x)-2]. \tag{4}$$

For the free theory, this is $2\Sigma_\mu(1-\cos ak_\mu)/a^2$ in the momentum space and does not have the doubler's zero at $k_\mu=\pi/a$. Equation (4) is nothing but the Wilson term [13] apart from one extra $1/a$. In Eq. (2), $U_{\mu\nu}(x)$ is the standard plaquette variable [12] $U_{\mu\nu}(x)\equiv U_\mu(x)U_\nu(x+a^\mu)U_\mu^\dagger(x+a^\nu)U_\nu^\dagger(x)$. With the parametrization $U_\mu(x)=\exp[iaA_\mu(x)]$, the lattice propagator (2) obviously reduces to the continuum one (1) in the naive continuum limit. The choice (2) is by no means unique and another definition would work as well.

Using the lattice propagator (2), we define a fermion bilinear operator as[2]

$$\langle\overline{\psi}(x)\mathcal{M}\psi(x)\rangle\equiv-\operatorname{tr}\mathcal{M}G(x,y)|_{x=y}, \tag{5}$$

where the minus sign is due to the Fermi statistics. The gauge current of a right-handed *chiral* fermion is simply *defined* by taking $\mathcal{M}=T^a\gamma^\mu P_R$, where $P_R\equiv(1+\gamma_5)/2$ is the chirality projection operator. This is possible because nothing flips the chirality along the fermion line. When the Yukawa coupling is involved, this simple recipe using the Dirac propagator does not work and we will comment on later the generalization.

An important property of definition (5) is the manifest gauge covariance. Namely, under the gauge transformation on the link variable $U_\mu(x)\to V(x)U_\mu(x)V^\dagger(x+a^\mu)$, the propagator (2) is transformed as $G(x,y)\to V(x)G(x,y)\times V^\dagger(y)$. Consequently, the bilinear operator (5) transforms

$$\langle\overline{\psi}(x)\mathcal{M}\psi(x)\rangle\to\langle\overline{\psi}(x)V^\dagger(x)\mathcal{M}V(x)\psi(x)\rangle, \tag{6}$$

which means that the composite operator has a definite transformation property under the gauge transformation on the external gauge field. Note that the covariance holds for a finite lattice spacing as well as the continuum limit $a\to0$. In particular, a gauge *singlet* operator, for which $\mathcal{M}$ commutes with the gauge generator, is regularized *gauge invariantly*. In the continuum limit, the gauge anomaly should have the gauge covariant form provided that the limit is not pathological.

Since we are not assuming the underlying fermion action in the present formulation, various symmetric properties are unfortunately not manifest. Nevertheless the gauge covariance (6) is powerful enough to derive the Ward identity associated with external gauge vertices. The vertex function, being a gauge current type operator $\mathcal{M}=T^a\gamma^\mu\mathcal{N}$ inserted, is defined by

$$\langle\overline{\psi}(x)T^a\gamma^\mu\mathcal{N}\psi(x)\rangle\equiv\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\left[a^4\sum_{x_j,\mu_j,a_j}A_{\mu_j}^{a_j}(x_j)\right.$$
$$\left.\times\int_{-\pi/a}^{\pi/a}\frac{d^4p_j}{(2\pi)^4}\,e^{ip_j(x-x_j)}e^{iap_{j_{\mu_j}}/2}\right]$$
$$\times\Gamma_{\mathcal{N}}^{\mu\mu_1\cdots\mu_n aa_1\cdots a_n}(p_1,p_2,\ldots,p_n), \tag{7}$$

where the term independent of the gauge field ($n=0$) identically vanishes. For example, when the constant matrix $\mathcal{N}$ in Eq. (7) commutes with the gauge generator, we find

[1]When the gauge group is Abelian, the Wess-Zumino condition is trivial and gives no constraint. The existence of the covariant anomaly implies the nonintegrability also in this case because, $0\neq\delta\partial_\mu^x\langle J^\mu(x)\rangle/(\delta A_\nu(y))\neq\partial_\mu^x\delta\langle J^\nu(y)\rangle/(\delta A_\mu(x))=0$, where the left-hand side is the U(1) gauge anomaly and the right-hand side means the current is covariantly regularized.

[2]It is equally easy to define, say, the two-point function of baryon-type composite operators. Note that the present formulation is also applicable to the vector gauge theory such as QCD.

$$p_\nu \lim_{a\to 0}\Gamma_{\mathcal{N}}^{\mu\nu ab}(p)=0,$$

$$-ip_\nu \lim_{a\to 0}\Gamma_{\mathcal{N}}^{\mu\nu\rho abc}(p,q)-f^{bad}\lim_{a\to 0}\Gamma_{\mathcal{N}}^{\mu\rho dc}(p)=0, \tag{8}$$

and higher point identities, by examining a variation of both sides of Eq. (7) under an infinitesimal gauge transformation. The second relation is nothing but the covariant convergence of the gauge current at one of external vertices in the three-point function. Note, however, that the gauge covariance (6) itself does *not* imply

$$i(p_\mu+q_\mu)\lim_{a\to 0}\Gamma_{\mathcal{N}}^{\mu\nu\rho abc}(p,q)-f^{abd}\lim_{a\to 0}\Gamma_{\mathcal{N}}^{\nu\rho dc}(p)=0. \tag{9}$$

If this relation would hold, it implies the covariant convergence of arbitrary current operators and contradicts with possible anomalies. The crucial point in this formulation is that vertices associated with the external gauge field and the vertex of the composite current operator are treated differently. Thus, in general, $\lim_{a\to 0}\Gamma_{\mathcal{N}}^{\mu\nu\rho abc}$ is not symmetric under an exchange $\mu\leftrightarrow\nu$ and $a\leftrightarrow b$. It is this breaking of the Bose symmetry which allows the manifest gauge covariance of the formulation. However, as was already noted, the Bose symmetry will be restored in the continuum limit when the theory is free of the gauge anomaly.

To verify the above properties and that the unwanted doublers are really absent in the present formulation, we explicitly evaluated some of the vertex functions in the continuum limit. After a somewhat lengthly calculation using the technique in [14], we find that the two-point function is given by

$$\lim_{a\to 0}\Gamma_{\mathcal{N}}^{\mu\nu ab}(p)=-\frac{1}{48\pi^2}\,\mathrm{tr}\;T^aT^b\gamma^\mu\mathcal{N}(\not{p}p^\nu-\gamma^\nu p^2)\times\left[\ln\frac{4\pi}{-a^2p^2}-\gamma+\frac{5}{3}+4\pi^2\left(J-\frac{5}{24}K\right)\right], \tag{10}$$

where $J=0.0465\ldots$ and $K=0.309\ldots$ are numerical constants [14]. The Lorentz covariance is restored and there is no nonlocal divergence in Eq. (10). Also the quadratically divergent terms, which are proportional to $g^{\mu\nu}/a^2$, are canceled out, as the Ward identity (8) and the hypercubic symmetry indicate. By taking $\mathcal{N}=P_R$, Eq. (10) gives the vacuum polarization tensor of a right-handed chiral fermion:

$$\Pi^{\mu\nu ab}(p)=-\frac{1}{24\pi^2}\,\mathrm{tr}\;T^aT^b(p^\mu p^\nu-g^{\mu\nu}p^2)\times\left[\ln\frac{4\pi}{-a^2p^2}-\gamma+\frac{5}{3}+4\pi^2\left(J-\frac{5}{24}K\right)\right]. \tag{11}$$

It is transverse, as is constrained by the Ward identity (8) *and* the logarithmic divergence has the correct coefficient as a *single* chiral fermion. Thus we see that the formulation, in fact, respects the gauge covariance and simultaneously is free of the species doublers at least in the perturbative treatment.

For the three-point function, we computed the divergence of the vector and the axial gauge currents:

$$i(p_\mu+q_\mu)\lim_{a\to 0}\Gamma_{1}^{\mu\nu\rho abc}(p,q)=0,$$

$$i(p_\mu+q_\mu)\lim_{a\to 0}\Gamma_{\gamma_5}^{\mu\nu\rho abc}(p,q)=\frac{i}{4\pi^2}\,\mathrm{tr}\;T^a\{T^b,T^c\}\varepsilon^{\nu\rho\alpha\beta}p_\alpha q_\beta. \tag{12}$$

Both relations are consistent with the Ward identity and actually the first of Eq. (12) may be derived solely from Eq. (8) and a general argument. The second relation should be interpreted as the covariant anomaly because of the underlying gauge covariance: It has the unique covariantized form

$$D_\mu \lim_{a\to 0}\langle\overline{\psi}(x)T^a\gamma^\mu\psi(x)\rangle=0,$$

$$D_\mu \lim_{a\to 0}\langle\overline{\psi}(x)T^a\gamma^\mu\gamma_5\psi(x)\rangle=\frac{i}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\;T^aF_{\mu\nu}F_{\rho\sigma}. \tag{13}$$

Therefore, the gauge anomaly of a chiral fermion (note that P_R is inserted in the gauge current) has the covariant form with the correct coefficient. We also find the correct fermion number anomaly [15] of a single chiral fermion by substituting $T^a\to 1$ in Eq. (13).

We have observed that, besides the manifest gauge covariance (6), the present formulation possesses many desired features at least in the perturbative treatment. At this point the reader might be wondering if the present formulation is equivalent to a nonlocal fermion action

$$S=a^4\sum_x \overline{\psi}(x)\{\Box(x)+[\gamma^\mu,\gamma^\nu]\times[U_{\mu\nu}(x)-1]/(4a^2)\}\frac{-1}{i\not{D}(x)}P_R\psi(x), \tag{14}$$

because it obviously corresponds to the propagator (2); hence, the nonlocality leads to some pathology. This interpretation is *not* correct. If our formulation is simply based on the action (14), the gauge current would be defined by $\langle\psi(x)T^a\gamma^\mu\psi(x)\rangle_{\mathrm{BS}}\equiv-\delta\ln\int\Pi_y d\psi(y)d\psi(y)e^S/[\delta A_\mu^a(x)]$ and the definition obviously respects the Bose symmetry among all the gauge vertices. As a consequence, we have the consistent form of gauge anomaly in the continuum limit which contradicts with the manifest gauge invariance of Eq. (14). As was already argued, we then expect the doublers or a pathology such as a breaking of the Lorentz covariance. Our formulation based on the prescription (5) with $\mathcal{M}=T^a\gamma^\mu P_R$, on the other hand, explicitly spoils the Bose symmetry but instead respects the maximal background gauge covariance. The possible gauge anomaly has the covariant form. Therefore, two approaches are completely different *even in the continuum limit*.

For simplicity of the presentation, we have neglected the possible Yukawa couplings up to now, which is important in realistic chiral gauge theories. The generalization of Eqs. (2)

and (5) is however straightforward. In the continuum theory, the covariant derivative is generalized as $\mathcal{D}\equiv\partial\!\!\!/_R P_R + \partial\!\!\!/_L P_L - iG\phi_R P_R - iG\phi_L P_L$, with obvious notations. Expression (1) is therefore replaced by $-1/(i\mathcal{D}) = i\mathcal{D}^\dagger/(\mathcal{D}\mathcal{D}^\dagger)$ and the following steps are almost identical. The lattice d'Alembertian (4) may be used for the right-handed and the left-handed components, respectively.

From the above analyses, the present proposal seems to provide a gauge covariant (or invariant for a gauge singlet operator) definition of composite operators without the unwanted doublers. We then have to integrate the gauge current expectation value to construct the fermionic determinant. A cancellation of the gauge anomaly is the integrability condition in the continuum limit, as was already noted. However, this fact is not so useful practically because the analytical integration is a formidable task and, the continuum limit is never reached in numerical simulations. Clearly we ought to study the integrability with a *finite* lattice spacing [and associated modifications of Eqs. (2) and (5), if necessary] for setting up a nonperturbative framework. This analysis is in progress and will be reported elsewhere. Here, we simply note that what is needed in the Metropolis simulation is not the fermionic determinant itself but the *difference* of the determinant between two gauge field configurations. This is the lattice analogue of the gauge current expectation value.

However, the integration is not necessary at all if one is contented with the *quenched* approximation. The application to numerical simulations is straightforward once having the lattice fermion propagator such as Eq. (2). We therefore believe that our proposal, even in the present form, has a range of practical application at least within the quenched approximation.

Note added. After this paper was accepted for publication, the author was aware that a similar proposal had already been made [16]. However, the point that we do not assume the underlying nonlocal action is the crucial difference. Our consideration also explains why the correct axial anomaly evaluated from a composite current operator definition [16] and the correct vacuum polarization tensor are not reproduced in the corresponding nonlocal action calculations [17].

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