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Brans wormholes

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It is shown that three of the four Brans solutions of classes I–IV admit wormhole geometry. Two-way traversable wormholes in the Brans-Dicke theory are allowed not only for the negative values of the coupling parameter ω ($\omega < -2$), as concluded earlier, but also for arbitrary positive values of ω ($\omega < \infty$). It also follows that the scalar field ϕ plays the role of exotic matter violating the weak energy condition. [S0556-2821(97)06804-5]

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Researches on wormhole physics by Morris, Thorne, and Yurtsever [1,2] have opened up, in recent years, a new frontier in theoretical physics. There already exist a number of investigations exploring the possible existence of wormhole geometries in different physical situations [3–6]. The occurrence of exotic matter having negative energy density [weak energy condition (WEC) violation] offers an intriguing possibility as to whether wormholes might act as effective gravitational lenses in astrophysical scenarios. Such a possibility has been conjectured by Cramer *et al.* [6] who also recommend an analysis of massive compact halo objects (MACHO's) search data for the detection of such lens effects. However, all the above analyses were carried out only within the framework of Einstein's general relativity theory (GRT). On the other hand, it is known that the GRT can be recovered in the limiting case $\omega \to \infty$ of the Brans-Dicke theory (BDT). In addition to the well-known utility of the BDT in local and cosmological problems, it is often invoked in the interpretation of physical phenomena on a galactic scale as well. For example, there are attempts aimed at explaining the observed flat rotation curves in the vast domain of dark galactic haloes [7,8]. It, therefore, seems only natural that in the context of wormhole physics, too, one looks for wormhole solutions of BDT. The case of dynamic wormholes has been dealt with by Accetta *et al.* [9] while the search for static wormhole geometry in BDT has been initiated only recently by Agnese and La Camera [10]. They show that a static spherically symmetric Brans-Dicke (BD) solution, obtained in a certain gauge by Krori and Bhattacharjee [11], does indeed support a two-way traversable wormhole for $\omega < -2$ and one way for $\omega > -3/2$.

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In the present paper, we wish to examine how many of the Brans I–IV classes of solutions [12], which also include the case considered in [10], support wormhole geometry. It is demonstrated that, of the four classes, as many as three represent wormhole solutions provided the range of parameters are chosen appropriately. The range, obtained by Agnese and La Camera [10], of the coupling parameter for wormhole solutions, viz., $\omega < -2$, seems unduly restrictive. Our analysis reveals that ω may take on arbitrary positive values as well. It will also be apparent that the presence of the BD scalar field ϕ cannot prevent WEC violation showing that the latter is not a consequence of the GRT alone.

The next four sections will deal with four classes of Brans solutions, respectively. The final section concludes the results obtained in the paper.

The BD field equations are

$$\Box^2 \phi = \frac{8\pi}{3+2\omega} T^{\mu}_{M\mu}, \tag{1}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{\phi} T_{M\mu\nu} - \frac{\omega}{\phi^2}\left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\right] - \frac{1}{\phi}[\phi_{;\mu;\nu} - g_{\mu\nu}\Box^2\phi], \tag{2}$$

where $\Box^2 \equiv (\phi^{;\rho})_{;\rho}$ and $T_{M\mu\nu}$ is the matter energy-momentum tensor excluding the ϕ field, ω is a dimensionless coupling parameter. Brans [12] presented four classes of solutions to BDT. The general metric, in isotropic coordinates (r,θ,φ,t), is given by $(G=c=1)$

$$d\tau^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\nu(r)}r^2[d\theta^2 + \sin^2\theta d\varphi^2]. \tag{3}$$

Brans solutions correspond to the gauge $\beta - \nu = 0$. Class I solutions are given by

$$e^{\alpha(r)}=e^{\alpha_0}\left[\frac{1-B/r}{1+B/r}\right]^{1/\lambda}, \quad (4)$$

$$e^{\beta(r)}=e^{\beta_0}\left(1+\frac{B}{r}\right)^2\left[\frac{1-B/r}{1+B/r}\right]^{(\lambda-C-1)/\lambda}, \quad (5)$$

$$\phi(r)=\phi_0\left[\frac{1-B/r}{1+B/r}\right]^{C/\lambda}, \quad (6)$$

$$\lambda^2\equiv(C+1)^2-C\left(1-\frac{\omega C}{2}\right)>0, \quad (7)$$

α_0, β_0, B, C, and ϕ_0 are constants. The constants α_0 and β_0 are determined by an asymptotic flatness condition as $\alpha_0=\beta_0=0$, while B is determined by the requirement of having Schwarzschild geometry in the weak field limit such that $B=\lambda M/2$, $M>0$ is the central mass of the configuration. Clearly B and λ must have the same sign.

The class I solution above is exactly the one considered in [10]. It can be easily verified that Eq. (6) of [10] is just our Eq. (7) above. The important point is that the exponents in Eqs. (4)–(6) depend on two parameters ω and C satisfying the inequality (7). This implies that the range of ω is dictated by the range of C, which, in turn, is to be dictated by the requirements of wormhole geometry as we shall see soon.

In their analysis, Agnese and La Camera [10] use post-Newtonian values to parametrize their two exponents A and B (equivalently, our ω and C) by a single parameter $\gamma\,[=(1+\omega)/(2+\omega)]$. This procedure leads, after suitable readjustment of notations, to the equality that $C=\gamma-1=-1/(\omega+2)$, which certainly constitutes a stronger condition than the inequality (7). As a further consequence, we find $\lambda^2=(\omega+1.5)/(\omega+2)>0$ which implies that the range $-2<\omega<-1.5$ must be excluded *a priori* as it corresponds to imaginary λ. Therefore, it seems more logical to use the inequality (7) *per se* for the analysis.

In order to investigate whether a given solution represents a wormwhole geometry, it is convenient to cast the metric into Morris-Thorne canonical form:

$$d\tau^2=-e^{2\Phi(R)}dt^2+\left[1-\frac{b(R)}{R}\right]^{-1}dR^2 +R^2[d\theta^2+\sin^2\,\theta d\varphi^2], \quad (8)$$

where $\Phi(R)$ and $b(R)$ are called the redshift and shape functions, respectively. These functions are required to satisfy some constraints, enumerated in [1], in order that they represent a wormhole. It is, however, important to stress that the choice of coordinates (Morris-Thorne) is purely a matter of convenience and not a physical necessity. For instance, one could equally well work directly with isotropic coordinates using the analyses of Visser [3] but the final conclusions would be the same. Redefining the radial coordinate $r\to R$ as

$$R=re^{\beta_0}\left(1+\frac{B}{r}\right)^2\left[\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right]^{\Omega}, \quad \Omega=1-\frac{C+1}{\lambda}, \quad (9)$$

we obtain the functions $\Phi(R)$ and $b(R)$ as

$$\Phi(R)=\alpha_0+\frac{1}{\lambda}\left[\ln\left\{1-\frac{B}{r(R)}\right\}-\ln\left\{1+\frac{B}{r(R)}\right\}\right], \quad (10)$$

$$b(R)=R\left[1-\left\{\frac{\lambda\{r^2(R)+B^2\}-2r(R)B(C+1)}{\lambda\{r^2(R)-B^2\}}\right\}^2\right]. \quad (11)$$

The throat of the wormhole occurs at $R=R_0$ such that $b(R_0)=R_0$. This gives minimum allowed r-coordinate radii $r_0^{\pm}$ as

$$r_0^{\pm}=B[(1-\Omega)\pm\sqrt{\Omega(\Omega-2)}]. \quad (12)$$

The values $R_0^{\pm}$ can be obtained from Eq. (9) using this $r_0^{\pm}$. Noting that $R\to\infty$ as $r\to\infty$, we find that $b(R)/R\to 0$ as $R\to\infty$. Also $b(R)/R\leqslant 1$ for all $R\geqslant R_0$. The redshift function $\Phi(R)$ has a singularity at $r=r_s=B$. In order that a wormhole be two-way traversable, the minimum allowed values $r_0^{\pm}$ must exceed $r_s=B$. The extent to which this requirement is satisfied depends on specific values of Ω. Several cases are possible.

(i) $-\infty<\Omega<0$ [$\Rightarrow\lambda<C+1$]. We see that $r_0^{+}>B$ while $r_0^{-}<B$. Hence a real, positive throat radius R_0^{+} exists only when $r=r_0^{+}$. The function $\Phi(R)$ is also nonsingular for $R\geqslant R_0^{+}>0$ and it is finite everywhere. We therefore have a two-way traversable wormhole. On the other hand, if $r=r_0^{-}<B$, the corresponding value R_0^{-} is imaginary and hence does not represent a wormhole.

(ii) $\Omega=0$ [$\Rightarrow\lambda=C+1$]. This gives a minimum allowed radius $r_0^{\pm}=B$ and the function $\Phi(R)$ is singular at the corresponding radius $R_0^{\pm}=4B$. Thus we obtain a non-Schwarzschild one-way wormhole since $C\neq 0$ and the scalar field ϕ is present. The choice $C=0$ indicates the absence of the ϕ field and we have what is known as the one-way Schwarzschild wormhole.

(iii) $0<\Omega<2$ [$\Rightarrow\lambda>C+1$]. In this case, $r_0^{\pm}$ and hence $R_0^{\pm}$ are imaginary. Hence, no wormhole can be constructed.

(iv) $2\leqslant\Omega<\infty$. If λ assumes a positive sign and so does B, then $r_0^{\pm}$ and $R_0^{\pm}$ both become negative and hence wormholes are not possible. Let λ assume a negative sign so that $B=-B'$, $B'>0$. Then, from Eq. (12), we get $r_0^{-}>B'$, $r_0^{+}<B'$. The function Φ has no horizon at $r=r_0^{-}$ and is finite for $r\geqslant r_0^{-}$ and we have a two-way wormhole with a corresponding throat radius $R=R_0^{-}$. But if $r=r_0^{+}$, then $\Phi(R)$ is undefined, and we cannot have a wormhole. The case $\Omega=2$ corresponds to case (ii) above.

Summing up, we see that two-way wormhole solutions are allowed only in the ranges $-\infty<\Omega<0$ and $2<\Omega<\infty$ (with λ negative, $\lambda=-\lambda'$, $\lambda'>0$). Let us write out Ω in terms of ω and C explicitly:

$$\Omega=1-\frac{C+1}{\lambda}=1-\frac{C+1}{\pm[(C+1)^2-C(1-\omega C/2)]^{1/2}}. \quad (13)$$

It is evident that $(C+1)$ and λ must have the same sign for $\Omega<0$. Suppose both have minus signs. Then, $C+1=-t$, $t>0$, say. The following inequality must hold:

$$t>[t^2+(1+t)\{1+(\omega/2)(1+t)\}]^{1/2}\Rightarrow(1+t)\omega<-2.$$

It is possible to choose t in such a way that ω may take on any arbitrary value in the open interval $(-2,0)$. Suppose again that both $(C+1)$ and λ have plus signs. Then, $C+1=s$, $s>0$, say. The following must hold:

$$s>[s^2-(s-1)\{1-(\omega/2)(s-1)\}]^{1/2}$$
$$\Rightarrow -(s-1)\{1-(\omega/2)(s-1)\}<0.$$

Now, two cases are possible: (a) If $0<s<1$, take $s-1=a$, then $a<0$. We then have $a\omega<-2\Rightarrow -\infty<\omega<\infty$. (b) If $1<s<\infty$, take $s-1=b>0$. Then, $b\omega<2$. In the limit $b\rightarrow 0+$, we have $\omega<+\infty$. In other words, ω can take on arbitrary positive values if a and b are appropriately chosen. For $2<\Omega<\infty$, we must have $(C+1)>\lambda'$ and we find $\omega<\infty$ from the same analysis as above.

The combined energy density of the gravitational (second-order derivatives of $g_{\mu\nu}$) + scalar (ϕ) field $(T_g+T_\phi)_{00}$ is obtained by computing the Einstein tensor G_{00} such that

$$G_{00}=\frac{1}{8\phi}(T_g+T_\phi)_{00}=\frac{1}{R^2}\frac{db}{dR}. \tag{14}$$

From Eq. (11), we obtain

$$\frac{db}{dR}=\frac{4r^2B^2}{(r^2-B^2)^2}[\Omega(2-\Omega)]. \tag{15}$$

If $\Omega<0$ or $\Omega>2$, then $db/dR<0$. This implies that, with ϕ everywhere non-negative, $G_{00}<0$. This shows that the scalar field ϕ plays the role of exotic matter at the wormhole throat. The same conclusion was reached also in [10].

The axially symmetric embedded surface $z=z(R)$ shaping the wormhole's spatial geometry is obtained from

$$\frac{dz}{dR}=\pm\left[\frac{R}{b(R)}-1\right]^{-1/2}. \tag{16}$$

For a coordinate-independent description of wormhole physics, one may use proper length l instead of R such that

$$l=\pm\int_{R_0^+}^{R}\frac{dR}{[1-b(R)/R]^{1/2}}. \tag{17}$$

In the present case,

$$l=\pm\int_{r_0^+}^{r}e^{\beta(r)}dr. \tag{18}$$

This integral is not integrable in a closed form. Nonetheless, it can be seen that $l\rightarrow\pm\infty$ as $r\rightarrow\pm\infty$.

Class II solutions are given by

$$\alpha(r)=\alpha_0+\frac{2}{\Lambda}\arctan\left(\frac{r}{B}\right), \tag{19}$$

$$\beta(r)=\beta-\frac{2(C+1)}{\Lambda}\arctan\left(\frac{r}{B}\right)-\ln\left(\frac{r^2}{r^2+B^2}\right), \tag{20}$$

$$\phi(r)=\phi_0 e^{(2C/\Lambda)\arctan(r/B)}, \tag{21}$$

$$\Lambda^2\equiv C\left(1-\frac{\omega C}{2}\right)-(C+1)^2>0. \tag{22}$$

The constants α_0 and β_0 are determined by using an asymptotic flatness condition and the constant B is determined by the weak field condition as follows:

$$\alpha_0=-\frac{\pi}{\Lambda},\quad \beta_0=\frac{\pi(C+1)}{\Lambda},\quad B=\frac{\Lambda M}{2}, \tag{23}$$

where $M>0$ is the central mass of the configuration. The inequality (22) fixes the range of ω:$C\geqslant -1\Rightarrow\omega<-2$, or, $C<-1\Rightarrow -2<\omega<-3/2$. The sign of Λ is left undetermined. Under the radial coordinate transformation $r\rightarrow R$

$$R=r\left(1+\frac{B^2}{r^2}\right)\exp\left[1-\frac{2}{\pi}\arctan\left(\frac{r}{B}\right)\right]\beta_0, \tag{24}$$

class II solutions yield

$$\Phi(R)=-\frac{\pi}{\Lambda}+\frac{2}{\Lambda}\arctan\left(\frac{r(R)}{B}\right), \tag{25}$$

$$b(R)=R\left[1-\left\{1+\frac{2B}{r^2(R)+B^2}\left(\frac{r(R)(C+1)}{\Lambda}-B\right)\right\}^2\right]. \tag{26}$$

Once again, $R\rightarrow\infty$ as $r\rightarrow\infty$ and all the conditions for a two-way wormhole are satisfied by the above $\Phi(R)$ and $b(R)$. The function $\Phi(R)$ has no horizon, is finite everywhere, and $\Phi(R)\rightarrow 0$ as $R\rightarrow\infty$. The r radii of the throat are given by

$${r_0}^{\pm}=\frac{B\beta_0}{\pi}[-1\pm(1+\beta_0^2/\pi^2)^{1/2}]. \tag{27}$$

As usual, putting these values in Eq. (24), we can find ${R_0}^{\pm}$. Notice that finite positive values of r (except $r=0$) correspond to finite positive values of R. Thus we require that ${r_0}^{\pm}>0$ so that we can have ${R_0}^{\pm}>0$. Rewriting Eq. (27) as ${r_0}^{+}=pM(1+C)$, where $p>0$ is any arbitrary real number, we find that the range $C>-1$ allows two-way wormhole solutions since it ensures ${r_0}^{+}>0$. In the same way, ${r_0}^{-}=-qM(1+C)$ where $q>0$ is any arbitrary real number and $C<-1$ implies a finite positive ${R_0}^{-}$ for the wormhole throat radius in the range $-2<\omega<-3/2$.

It can be verified that

$$\left.\frac{db}{dR}\right|_{R=R^{\pm}_0}=-1 \tag{28}$$

and hence there occurs a WEC violation. The flaring-out condition $d^2z/dR^2>0$ is also satisfied, since it can be verified that

$$\left.\frac{d^2z}{dR^2}\right|_{R=R^{\pm}_0}=\frac{1}{R_0^{\pm}}>0. \tag{29}$$

The proper length l is given by

$$l=\pm e^{\beta_0}\int_{r_0^{\pm}}^{r}e^{\beta(r)}dr=\pm e^{\beta_0}[(r-{r_0}^{\pm})+\cdots]. \tag{30}$$

Again, $R\rightarrow\pm\infty\Leftrightarrow l\rightarrow\pm\infty$ as $r\rightarrow\pm\infty$.

Class III solutions are given by

$$\alpha(r)=\alpha_0-\frac{r}{B}, \tag{31}$$

$$\beta(r)=\beta_0-\ln\left(\frac{r}{B}\right)^2+(C+1)\left(\frac{r}{B}\right), \tag{32}$$

$$\phi(r)=\phi_0 e^{-(Cr/B)}, \tag{33}$$

$$C=\frac{-1\pm\sqrt{-2\omega-3}}{\omega+2}. \tag{34}$$

The redshift and shape functions are

$$\Phi(R)=\alpha_0 - r(R)/B, \tag{35}$$

$$b(R)=R\left[1-\left\{\frac{C+1}{B}r(R)-1\right\}^2\right], \tag{36}$$

where

$$R=r^{-1}B^2\exp\left(\beta_0+\frac{C+1}{B}r\right). \tag{37}$$

Here, too, $R\to\infty$ as $r\to\infty$ but $b(R)/R \nrightarrow 0$ as $R\to\infty$. Also $\Phi(R)\to\infty$ as $R\to\infty$. Asymptotic flatness condition is also not satisfied by this solution. Therefore, there is no question of any wormhole geometry in this case.

Class IV solutions are

$$\alpha(r)=\alpha_0 - 1/Br, \tag{38}$$

$$\beta(r)=\beta_0+(C+1)/Br, \tag{39}$$

$$\phi=\phi_0 e^{-(Cr/B)}, \tag{40}$$

$$C=\frac{-1\pm\sqrt{-2\omega-3}}{\omega+2}. \tag{41}$$

Usual asymptotic flatness and weak field conditions fix α_0, β_0, and B as

$$\alpha_0=\beta_0=0, \quad B=1/M>0. \tag{42}$$

The functions are

$$\Phi(R)=-\alpha_0 - 1/Br(R), \tag{43}$$

$$b(R)=R\left[1-\left\{1-\frac{C+1}{Br(R)}\right\}^2\right], \tag{44}$$

$$R=r\exp[(C+1)/Br]. \tag{45}$$

The wormhole throat occurs at

$$r=r_0=(C+1)/B \Rightarrow R=R_0[(C+1)/B]e. \tag{46}$$

It can be verified from Eq. (41) that $(C+1)>0$ only if $\omega<-2$. No wormhole is possible if $-2<\omega\leqslant-3/2$ or $\omega>-3/2$, since $(C+1)$ is either negative or imaginary.

The proper length is given by

$$l=\pm\int_{r_0}^{r}\exp\left(\frac{C+1}{Br}\right)dr. \tag{47}$$

One can see that if $r\to\pm\infty$, then $R\to\pm\infty$ and $l\to\pm\infty$. It can be verified that all the conditions of a two-way wormhole including the flaring-out condition are satisfied. The peculiarity of this solution is that

$$\frac{db}{dR}=-[(C+1)/Br]^2<0, \tag{48}$$

and hence $G_{00}<0$ for all finite nonzero values of r (and, of course, R). This implies that the entire wormhole, and not only the throat, is made up of exotic material.

The special case $C=-1$ is not of interest as it corresponds to a flat spatial section.

It was shown in the foregoing that three out of the four types of Brans solutions give rise to a two-way traversable wormhole geometry provided the constants are chosen appropriately. The restriction $\omega<-2$ need no longer be strictly maintained, for, as we have seen, ω can also take on positive values in the context of two-way wormholes. This result extends the scope for the feasibility of wormhole scenarios even to the regime of ordinary observations. For example, laser-ranging probes and observations on binary systems put a lower limit of $\omega\geqslant 500$–600 [13–15]. However, there occurs a violation of the WEC at the wormhole throat even for $\omega<+\infty$ (class I solutions), but, unlike in [10], the range of ω (or γ) alone does not cause it. The positive, real values of the throat radii $r_0^{\pm}$, (or $R_0^{\pm}$) containing both ω and C are actually responsible for the WEC violation, as we have just seen. Only in class IV solutions do we see that WEC is violated for all values of r.

A search for wormhole geometry in BDT amounts to an investigation of the extent to which the scalar field ϕ does play the role of exotic matter required for WEC violation. Researches into the existence of matter having negative energy density (or, negative mass) are not new. It was Bondi [16] who initiated the work and, in recent years, we have a number of investigations into the question of negative energy [17–20]. Interestingly, Pollard and Dunning-Davies [20] show that no contradictions arise if negative mass is introduced into Newton's laws of motion.

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