Study of quark propagator solutions to the Dyson-Schwinger equation in a confining model

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(Received 26 June 1996)

We solve the Dyson-Schwinger equation for the quark propagator in a model with singular infrared behavior for the gluon propagator. We require that the solutions, easily found in configuration space, be tempered distributions and thus have Fourier transforms. This severely limits the boundary conditions that the solutions may satisify. The sign of the dimensionful parameter that characterizes the model gluon propagator can be either positive or negative. If the sign is negative, we find a unique solution. It is singular at the origin in momentum space, falls off like $1/p^2$ as $p^2 \to +/-\infty$, and is truly nonperturbative, in that it is singular in the limit that the gluon-quark interaction approaches zero. If the sign of the gluon propagator coefficient is positive, we find solutions that are, in a sense that we exhibit, unconstrained linear combinations of advanced and retarded propagators. These solutions are singular at the origin in momentum space, fall off like $1/p^2$ asympotically, exhibit ''resonantlike'' behavior at the position of the bare mass of the quark when the mass is large compared to the dimensionful interaction parameter in the gluon propagator model, and smoothly approach a linear combination of free-quark advanced and retarded two-point functions in the limit that the interaction approaches zero. In this sense, these solutions behave in an increasingly ''particlelike'' manner as the quark becomes heavy. The Feynman propagator and the Wightman function are not tempered distributions and therefore are not acceptable solutions to the Schwinger-Dyson equation in our model. On this basis we advance several arguments to show that the Fourier-transformable solutions we find are consistent with quark confinement, even though they have singularities on the real p^2 axis. [S0556-2821(97)08104-6]

PACS number(s): 12.38.Aw, 12.38.Lg

I. INTRODUCTION

A classic approach to understanding the behavior of confined particles is to model and solve the Dyson-Schwinger (DS) equations for the particles' propagators. Since confinement is generally regarded as an infrared phenomenon, the emphasis is naturally on the infrared region of the kernels of the DS equations. Taking clues from studies of the infrared behavior of propagators in pure Yang-Mills theory, one can adopt a vector-meson propagator model motivated by such studies, insert it in the kernel for the fermionic propagator equation, and study issues such as fermion confinement, chiral symmetry breaking, the interplay between the scales for these two phenomena, and gauge dependence of solutions. There are two extreme views of the infrared behavior of the gluon propagator. One is that the singularity at $q^2=0$ is much stronger than the $1/q^2$ behavior of the perturbative propagator, with variants of $1/q^4$ often proposed, and the other, in complete contrast, is that the propagator vanishes as $q^2 \to 0$. Because of the wide and rather successful application of the former type of behavior to bound state problems, we will adopt a frequently studied model of this type, proposed some time ago by one of us [1,2], for our analysis.

Intuition for the interpretation and application of quantum field theories is built upon an intimate interplay between configuration space and momentum space considerations. The interaction Lagrangian and its symmetry properties are studied in configuration space, and space-time boundary conditions of the Green functions of the theory are crucial to their interpretation. On the other hand, the particle spectrum and the scattering and decay processes contained in the theory are more intuitively assessed in momentum space. The particle content is revealed in the Green functions by their branch cuts and poles in momentum space. The complementarity of the configuration space and momentum space views is especially clear in the interchangeability of the terms short and long distance with ultraviolet and infrared (or hard and soft) to describe the physics of a situation. It is not surprising, therefore, that we assume that Green functions in (Minkowski) configuration space have Fourier transforms to momentum space and vice versa. Indeed, this property is at the foundation of the standard approach to particles and fields.

We return to the study of fermion propagators in the infrared domain to see what insight can be gained by requiring that the solutions to the DS equation be Fourier transformable. The question is not idle, since several solutions proposed in the literature as possible models of confined behavior do not have Fourier transforms [2–4]. This consideration is one of the motivations for the present work. Important collateral questions that will occupy us are those of the propagator behavior in the large mass limit, the asymptotic behavior in both timelike and spacelike directions in momentum space, and the behavior of the propagator in the limit that the infrared ''gluon-fermion'' interaction is turned off. We choose a simple enough model that ''Abelianized'' Ward-Takahashi identities can be enforced at the fermion-gauge boson vertex and still leave us with a model whose propagator we can solve for exactly and whose Fourier transform we can evaluate. Perhaps unique to the present study is that we remain strictly in Minkowski space in setting up and solving our model equations.[1]

[1]With care taken to handle the continuation to Minkowski space properly, an equivalent Euclidean space treatment can be given.

II. DEFINING THE MODEL AND SOLVING FOR THE PROPAGATOR

We begin by developing the model for our study of Fourier-transformable solutions to the fermion Dyson-Schwinger equation. Our crucial ingredient for the DS equation is the infrared gluon propagator model [1,2] used a number of times since [3–5]. As mentioned in the Introduction, a number of studies of the gluon propagator suggest [6] that

$$D(q^2)\to\frac{\mu^2}{q^4}\quad\text{as}\quad q^2\to 0, \tag{1}$$

where, in the Landau gauge,

$$D_{\mu\nu}(q^2)=\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)D(q^2). \tag{2}$$

The Fourier transform of $1/(q^2+i\epsilon)^{2}$[2] does not exist [7], however, and a regularization must be prescribed to define a gluon propagator that has a Fourier transform. Defining $D^{(\lambda)}(q^2)$ as

$$D^{(\lambda)}(q^2)=\frac{64\pi^2}{3}\,\mu^{2(\lambda-1)}\,\frac{(\lambda-2)}{(q^2+i\epsilon)^{\lambda}}, \tag{3}$$

where μ has dimensions of mass, the limit[2]

$$\lim_{\lambda\to 2} D^{(\lambda)}_{\mu\nu}(q^2)=i\mu^2\delta^4(q)(2\pi)^4 g_{\mu\nu}, \tag{4}$$

where μ^2 can be positive or negative, defines the propagator for our infrared DS equation study.[3] The simple form of the confining propagator then is

$$D^{\mu\nu}(x)=i\mu^2 g^{\mu\nu} \tag{5}$$

in configuration space.[4]

The general form of the DS equation is, in momentum space,

$$1=(\not p-m)S(p)-i\int\frac{d^4k}{(2\pi)^4}\,\gamma_\mu D^{\mu\nu}(k)\Lambda_\nu(p+k,p), \tag{6}$$

where Λ_ν is defined in Appendix B. Inserting our model propagator (4), we obtain

$$1=(\not p-m)S(p)+\mu^2\gamma^\mu\cdot\Lambda_\mu(p,p). \tag{7}$$

We assume that $\Lambda_\mu(p+k,p)$ obeys the Ward identity

$$\Lambda_\mu(p,p)=-\frac{\partial}{\partial p^\mu}\,S(p), \tag{8}$$

which is exact in an Abelian gauge theory and true also in non-Abelian gauge theory if the ghost contributions to the Ward-Takahashi identity are of order k and higher. The DS equation (7) then reads

$$1=(\not p-m)S(p)-\mu^2\gamma^\mu\cdot\frac{\partial}{\partial p^\mu}\,S(p), \tag{9}$$

in our model. Taking the Fourier transform of Eq. (9), we find that the x-space DS equation that forms the basis for the present investigation is

$$\delta^4(x)=(i\not\partial-m)S(x)+i\mu^2\gamma\cdot xS(x). \tag{10}$$

Phenomenologically, the dimensionful parameter μ can be expressed in terms of the QCD hadron dynamics scale by fitting the pion decay constant, for example.

The solutions to Eq. (10) have the form [2]

$$S(x)=e^{-\mu^2x^2/2}S_0(x), \tag{11}$$

where $S_0(x)$ is a general solution of the free DS equation. In order to avoid an exponential blow up that invalidates the Fourier transform, we must choose an appropriate $S_0(x)$.[5] For the case $\mu^2>0$, this means that the free propagator choice must be [11]

$$\overline{S_0}(x)=-(i\not\partial+m)[1+C\epsilon(x_0)]\overline{\Delta}(x^2), \tag{12}$$

where C is an arbitrary constant and $\epsilon(x_0)\overline{\Delta}(x^2)$ obeys the free homogeneous DS equation. The choices $C=\mp 1$ yield advanced and retarded Green functions, respectively. Here $\Delta(x^2)$ obeys the inhomogeneous Klein-Gordon equation and has the form [11]

$$\overline{\Delta}(x^2)=\frac{1}{4\pi}\left(\delta(x^2)-\frac{m^2}{2}\,\theta(x^2)\,\frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}}\right), \tag{13}$$

and $\theta(x^2)$ prohibits the $x^2<0$ region where, otherwise, the full propagator (11) would blow up. For the $\mu^2<0$ case in Eq. (4), one needs a solution with $\theta(-x^2)$, which follows from Eq. (13) by adding the appropriate solution to the homogeneous equation: namely,[6]

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[2]The distribution $(q^2+i\epsilon)^{-\lambda}$ has, as a function of λ, a pole at $\lambda=2$ with residue $-i\pi^2\delta^4(q)$. See the discussion in Chap. III, Sec. 2.4, or the summary chapter, in Ref. [7]. The contribution from the second term in Eq. (2) is equal, in the limit $\lambda\to 2$, to 1/4 that of the first term when the gluon propagator is inserted in the DS equation and relativistic invariance and regularity of the vertex function at $q=0$ are taken into account.

[3]Blaha [8] studied PP $[1/(q^2+i\epsilon)^2]$, but in a perturbative context. Pagels studied an alternative prescription, with a quark propagator vanishing as $\lambda\to 2$, to handle the $1/q^4$ singularity. This leads to a different DS equation from ours [9].

[4]The configuration space propagator (5) is consistent with confinement, since it clearly does not satisfy the cluster decomposition property [10].

[5]Momentum space solutions to this model that have been offered in the literature [2–4] are not Fourier transformable because they diverge exponentially on the real four-momentum-squared axis at timelike infinity.

[6]This solution is the unique one in which $\theta(-x^2)$ appears, as required in the $\mu^2<0$ case. That is, for this case, $C=0$.

$$\widetilde{\Delta}(x^2)=\overline{\Delta}(x^2)+\frac{m^2}{4\pi}\frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}}$$

$$=\frac{1}{4\pi}\left(\delta(x^2)+\frac{m^2}{2}\,\theta(-x^2)\,\frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}}\right), \quad (14)$$

with $\widetilde{S}_0(x)=-(i\not\partial+m)\widetilde{\Delta}(x^2)$. Note that Eq. (13) represents a tempered distribution and, therefore, has a Fourier transform, while Eq. (14) does not. We emphasize that a Feynman propagator is *not* an acceptable choice for $S_0(x)$ in Eq. (11), because the corresponding solution $S(x)$ does not have a Fourier transform.

In summary, we have the two cases

$$\overline{S}(x)=\exp\left(-\frac{|\mu^2|}{2}\,x^2\right)\overline{S}_0(x) \quad (15a)$$

and

$$\widetilde{S}(x)=\exp\left(\frac{|\mu^2|}{2}\,x^2\right)\widetilde{S}_0(x), \quad (15b)$$

corresponding to the choices $\mu^2>0$ and $\mu^2<0$, respectively, for the gluon propagator model in Eq. (4). The noninteracting Green functions[7] $S_0(x)$ and $\widetilde{S}_0(x)$ are manifestly regained in the $\mu\to 0$ limit. Avoiding the exponential blowup as $x^2\to-\infty$ in Eq. (15a) and as $x^2\to+\infty$ in Eq. (15b) dictates the choices of $\Delta(x)$ and $\widetilde{\Delta}(x)$ in Eqs. (13) and (14), as necessary conditions to ensure that Fourier transforms to momentum space exist.

The only solution to the homogeneous form of Eq. (10) that reduces to the free Wightman function when $\mu^2=0$ is

$$S_W(x)=e^{-(\mu^2/2)x^2}(i\not\partial+m)W_0(x), \quad (16)$$

where $W_0(x)$ is the free-field scalar Wightman function [11]

$$W_0(x)=\frac{m}{4\pi\sqrt{x^2}}\,K_1(\sqrt{x^2}m), \quad (17)$$

in terms of a standard Hankel function. If Wightman functions are tempered distributions satisfying certain regularity conditions, one can prove that free fermion asymptotic states exist [12]. The Schwinger model, which is solvable, illustrates a connection between pathologies of the Wightman functions and confinement. The fermion Wightman functions in Coulomb gauge blow up exponentially in configuration space and fermion states, at the same time, do not appear in the spectrum [13]. Since $S_W(x)$ as defined above is not a tempered distribution, this suggests that our model is consistent with fermion confinement.

Let us now take up the evaluation of the Fourier transforms of Eqs. (15a) and (15b) and examine their behavior in momentum space. The essential calculations that must be performed are the Fourier transforms of $e^{-|\mu^2|x^2}\overline{\Delta}(x^2)$ and $e^{+|\mu^2|x^2}\widetilde{\Delta}(x^2)$; namely, choosing $e^{-|\mu^2|x^2}\overline{\Delta}(x^2)$ for discussion, we have, adopting the convention $\mu^2>0$,

$$\overline{B}(p^2)=-m\int d^4x\; e^{ip\cdot x}e^{-(\mu^2/2)x^2}\overline{\Delta}(x^2)[1+C\epsilon(x_0)] \quad (18)$$

and

$$p^2\overline{A}(p^2)=-i\int d^4x\; e^{ip\cdot x}e^{-(\mu^2/2)x^2}\not p\not p\{\overline{\Delta}(x^2)[1+C\epsilon(x_0)]\}$$

$$=(p^2+\mu^2 p\cdot\partial_p)\overline{B}(p^2)/m. \quad (19)$$

We have defined $\overline{A}(p^2)$ and $\overline{B}(p)$ in terms of $\overline{S}(p)$ to be

$$\overline{S}(p)=\not p\overline{A}(p^2)+\overline{B}(p^2)=\int d^4x\; e^{ip\cdot x}\overline{S}(x). \quad (20)$$

Details of our evaluation of the Fourier transform (18) are given in the Appendix A, where we present a procedure that can be applied to any function $F(x^2)$ which has a one-dimensional (in x^2) Fourier transform and whose four-dimensional Fourier transform is a function of p^2. We also show there how to choose contours that give improved convergence for the numerical evaluation of the Fourier transform for timelike and spacelike values of the momentum space argument. Writing the $C=0$ result for $B(p^2)$ derived in Appendix A in the form

$$\frac{\overline{B}(p^2)}{m}=\frac{i}{2}\int_{-\infty}^{\infty}d\nu\;\epsilon(\nu)\exp\left(-ip^2\nu+i\,\frac{m^2\nu}{1-2i\mu^2\nu}\right), \quad (21)$$

we factor out $e^{-m^2/2\mu^2}$ and introduce a variable λ^2 as follows:

$$\exp\left(i\,\frac{m^2\nu}{1-2i\mu^2\nu}\right)=\exp\left(-\frac{m^2}{2\mu^2}\right)\int_{-\infty}^{\infty}d\tau\;\delta(\nu-\tau)$$

$$\times\exp\left(\frac{m^2}{2\mu^2}\,\frac{1}{1-2i\mu^2\tau}\right)$$

$$=\exp\left(-\frac{m^2}{2\mu^2}\right)\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\int_{-\infty}^{\infty}d\lambda^2$$

$$\times\exp[i\lambda^2(\nu-\tau)]$$

$$\times\exp\left(\frac{m^2}{2\mu^2}\,\frac{1}{1-2i\mu^2\tau}\right). \quad (22)$$

Substituting Eq. (22) into Eq. (21), exchanging the order of λ^2 and τ integration, and evaluating the integral over ν produces

$$\frac{\overline{B}}{m}(p^2)=\mathrm{PP}\int_{-\infty}^{\infty}d\lambda^2\,\frac{\sigma(\lambda^2)}{p^2-\lambda^2}, \quad (23)$$

where

[7] $\widetilde{S}_0(x)$ does not have a Fourier transform. Thus Eq. (15b) is a truly nonperturbative solution. As we will see below, although $\widetilde{S}_0(x^2)$ is the $\mu\to 0$ limit of Eq. (15b), the Fourier transform of Eq. (15b) is singular as $\mu\to 0$.

$$\sigma(\lambda^2)=\exp\left(-\frac{m^2}{2\mu^2}\right)\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\exp\left(-i\lambda^2\tau+\frac{m^2}{2\mu^2}\frac{1}{1-2i\mu^2\tau}\right)$$
$$=\exp\left(\frac{-m^2}{2\mu^2}\right)\left[\delta(\lambda^2)+\theta(\lambda^2)\frac{m}{2\mu^2\lambda}\times\exp\left(-\frac{\lambda^2}{2\mu^2}\right)I_1\left(\frac{\lambda m}{\mu^2}\right)\right]$$
$$\equiv\exp\left(\frac{-m^2}{2\mu^2}\right)\delta(\lambda^2)+\overline{\sigma}(\lambda^2)\theta(\lambda^2). \tag{24}$$

Thus our representation for $C=0$ is

$$\frac{\overline{B}(p^2)}{m}=\mathrm{PP}\left[\frac{e^{-m^2/2\mu^2}}{p^2}+\int_0^{\infty}d\lambda^2\,\frac{\overline{\sigma}(\lambda^2)}{p^2-\lambda^2}\right]. \tag{25a}$$

Equation (23) represents $\overline{B}(p^2)$ as a superposition of free propagators of mass λ. The Fourier transform of $\epsilon(x_0)\Delta(x^2,\lambda)$ is obtained from that of $\Delta(x^2,\lambda)$ by the substitution $(p^2-\lambda^2)^{-1}\to -i\pi\epsilon(p_0)\delta(p^2-\lambda^2)$. Therefore, the Fourier transform of the second term in Eq. (18) is

$$C\left[i\pi\epsilon(p_0)e^{-m^2/2\mu^2}\left[\delta(p^2)+\theta(p^2)\frac{m}{2\mu^2}\frac{1}{\sqrt{p^2}}I_1\left(\frac{m\sqrt{p^2}}{\mu^2}\right)\right]e^{-p^2/2\mu^2}\right]. \tag{25b}$$

Equations (25a) and (25b) show several key features of the momentum space behavior of the propagator, which is the solution to the DS equation in our model, and we turn to a discussion of these points in the next section.

III. PROPERTIES OF THE FERMION PROPAGATOR

A. $\mu^2>0$ case

The most obvious features of Eqs. (25a) and (25b) are the singularities at $p^2=0$. As we emphasize in Sec. IV, these are not the singularities of a Feynman propagator. Next, we note that, if $\mu^2\neq 0$, the λ^2 integral clearly converges since

$$I_1(x)\to\frac{1}{\sqrt{2\pi x}}e^x \quad \text{as} \quad x\to\infty,$$

and

$$\frac{1}{\lambda}e^{-\lambda^2/2\mu^2}I\left(\frac{\lambda m}{2\mu^2}\right)\to\sqrt{\frac{\mu}{\pi m}}\frac{1}{\lambda^{3/2}}e^{-\lambda^2/2\mu^2+\lambda m/\mu^2},$$

which is strongly convergent. Therefore the asympototic behavior of $B(p^2)$ is

$$\overline{B}(p^2)\to\frac{m}{p^2} \quad \text{as} \quad p^2\to\pm\infty, \tag{26}$$

and the free-propagator ultraviolet behavior is reproduced.[8]

Next we consider the $\mu^2/m^2\to 0$ limit of the expressions (25a) and (25b) for $B(p^2)$. The singularities at the origin vanish exponentially in this limit. What happens to the principal part integral? The asymptotic expansion for $I_1(x)$ shown above allows one to write the limit in the form

$$\overline{\sigma}(\lambda^2)\to\frac{m}{2\mu^2\lambda}\frac{1}{\sqrt{2\pi}}\frac{\mu}{\sqrt{m\lambda}}e^{-(\lambda-m)^2/2\mu^2}\to\frac{\sqrt{m}}{\lambda^{3/2}}\delta(\lambda-m),$$

which yields

$$\frac{\overline{B}(p^2)}{m}\to\mathrm{PP}\left(\frac{1}{p^2-m^2}\right)+i\pi C\epsilon(p_0)\delta(p^2-m^2) \quad \text{as} \quad \frac{\mu^2}{m^2}\to 0, \tag{27}$$

which is the Fourier transform of the free Green function $[1+C\epsilon(x_0)]\Delta(x^2)$ in Eqs. (12) and (13). This result establishes that there is a smooth limit where the free momentum space Green function is the Fourier transform of the free configuration space Green function. This smooth limit does not obtain in our $\mu^2<0$ case of Eq. (4) (see Sec. III B below) or in the solutions reported in the literature [2–4]. $B(p^2)m$ is graphed for several values of μ^2/m^2 in Fig. 1. The sharpening of the resonancelike behavior at $p^2\cong m^2$ and the disappearance of the pole at $p^2=0$ as $\mu^2/m^2\to 0$ are clearly shown. Thus in the limit as the mass of the fermion becomes large compared to the scale associated with the infrared behavior of the gluon propagator, the fermion propagator becomes more and more particlelike, in the sense that it behaves like $1/(p^2-m^2)$ everywhere. The pole at $p^2=0$ is *not* an actual particle pole with the $i\epsilon$ prescription corresponding to a time-ordered product that insures unitarity in the perturbative expansion. This is true for any value of C in Eq. (12).

A blowup of the region near $p^2=0$ for the $2\mu^2/m^2=0.2$ case is shown in Fig. 2 to indicate just how sharp the pole is in this case where its weighting factor is $e^{-m^2/2\mu^2}=e^{-5}$.

Figure 3 shows the value of $[B(p^2)/m](p^2-m^2)$ as a function of p^2/m^2 for the case $2\mu^2/m^2=0.2$. The rapid approach to the free Green function behavior for large p^2/m^2 is readily apparent.

B. Case $\mu^2<0$: An example of a singular $\mu^2\to 0$ limit

The free Green function $\tilde{\Delta}(x^2)$, Eq. (14), and the corresponding $\tilde{S}_0(x)$ are not tempered distributions and do not have Fourier transforms. Nonetheless, the solution (15b) to the DS equation with the vertex (8) and gluon infrared propagator (4) *does* have a Fourier transform because the exponential factor $e^{|\mu^2|x^2}$ controls the $\exp(m\sqrt{|x^2|})$ divergence of $J_1(m\sqrt{x^2})$ as $x^2\to-\infty$.

Following the same steps as before, one arrives at Eq. (21), but with the opposite sign in front of μ^2. The representation corresponding to Eq. (25a) is

[8]In fact, expanding Eq. (21) in powers of $(1/p^2)$ shows that the asympototic behavior is given by $B(p^2)/m\to 1/(p^2-m^2)+O(m^4/p^6)$, as $p^2\to\infty$.

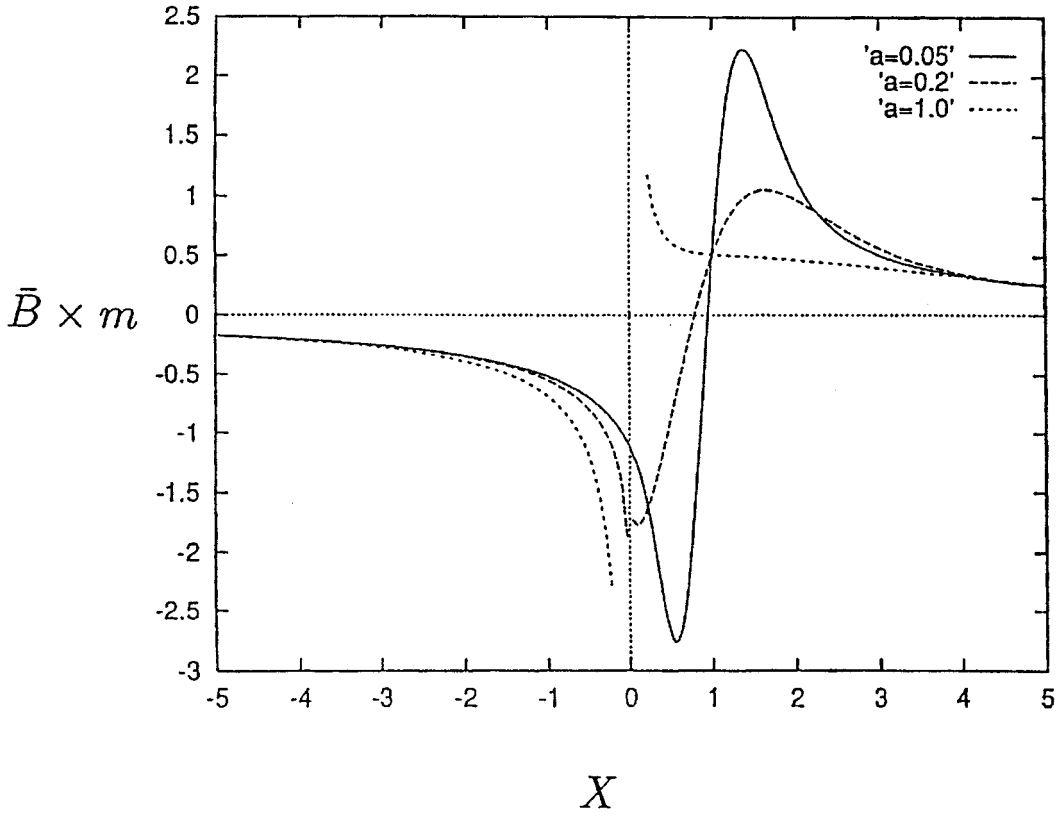

FIG. 1. Plot of $m\overline{B}(p^2)$ vs $x=p^2/m^2$ for the case $C=0$, Eq. (21). Curves for the values $a=2\mu^2/m^2=0.05$ (solid line), $a=0.2$ (long-dashed line), and $a=1.0$ (short-dashed line) are shown. The onset of the pole is visible for the $a=0.2$ case, and it is the dominant feature in the $a=1.0$ case, where the ''resonant'' behavior at $x=1(p^2=m^2)$ is gone. The region around $x=0$ is excluded from all three plots so that all three cases fit on the figure (the jump across $x=0$ does not show up on this scale for the $a=0.05$ case).

$$\frac{\widetilde{B}}{m}(p^2)=\mathrm{PP}\left[e^{m^2/2\mu^2}\frac{1}{p^2}+e^{m^2/2\mu^2}\frac{m}{2\mu^2}\int_0^\infty \frac{d\lambda^2}{\lambda}\frac{1}{p^2+\lambda^2}e^{-\lambda^2/2\mu^2}J_1\left(\frac{m\lambda}{\mu^2}\right)\right] \tag{28}$$

In the limit $p^2\to\pm\infty$, $\widetilde{B}(p^2)/m\to 1/p^2$ as in the previous case. In the limit $\mu^2/m^2\to 0$, there is no $\delta(\lambda-m)$ behavior, and there is no pole at $p^2=m^2$. The whole expression *diverges* as $e^{m^2/2\mu^2}$ in the (singular) $\mu^2/m^2\to 0$ limit. The original free Green function $\widetilde{\Delta}(x^2)$ is not Fourier transformable, and so the singular nature of the $\mu^2/m^2\to 0$ limit merely reflects that fact.

IV. DISCUSSION OF RESULTS AND CONCLUSIONS

We have reexamined a model for the infrared gluon propagator and quark-gluon vertex previously discussed in the literature [1–5]. We found those solutions to the quark propagator DS equation that admit Fourier transforms; we work directly in Minkowski space. Such solutions lend themselves to the study of timelike and spacelike behavior of the propagator without appeal to transformation to Euclidean space and continuation of the solutions found to the timelike region. The first solution ($\mu^2>0$ case) presented has a smooth limit to a combination of advanced and retarded Green functions of the free Dirac equation when the interaction is turned off. In momentum space, the real part of the free Green function is simply $\mathrm{PP}[1/(p^2-m^2)]$, which is also the behavior of the full solution to the interacting model in the $\pm p^2\to\infty$ limit, as indicated in Fig. 3. The full propagator shows an interesting particlelike behavior as the mass parameter grows large compared to the infrared scale that characterizes the gluon propagator. This behavior is shown in Fig. 1, and it gives an explicit picture of the increasingly ''free-particlelike'' behavior expected as quarks become heavy. This is particularly true of the top quark, of course. The solution has a pole at $p^2=0$ that dominates when the quark mass parameter is of the order of or less than the infrared scale in the gluon propagator, as shown in Fig. 1. The singularity at the origin is suppressed by the $\exp(-m^2/2\mu^2)$ factor displayed in Eqs. (25a) and (25b) in the large quark mass limit. Though the model is not realistic, since we do not include the ultraviolet contribution from the gluon propagator, it does have the interesting feature that, while the region near $p^2=0$ dominates when the quark mass is small, the region near $p^2=m^2$ dominates as the quark mass gets large. Regarding confinement, we note that the gluon propagator model, which is constant in configuration space, clearly violates the cluster decomposition property, which has been considered to be a sufficient condition for confinement. For the quark propagator, the absence of a singularity on the real p^2 axis is often taken to be a sufficient condition for confinement. In contrast, the quark propagator solution that we find

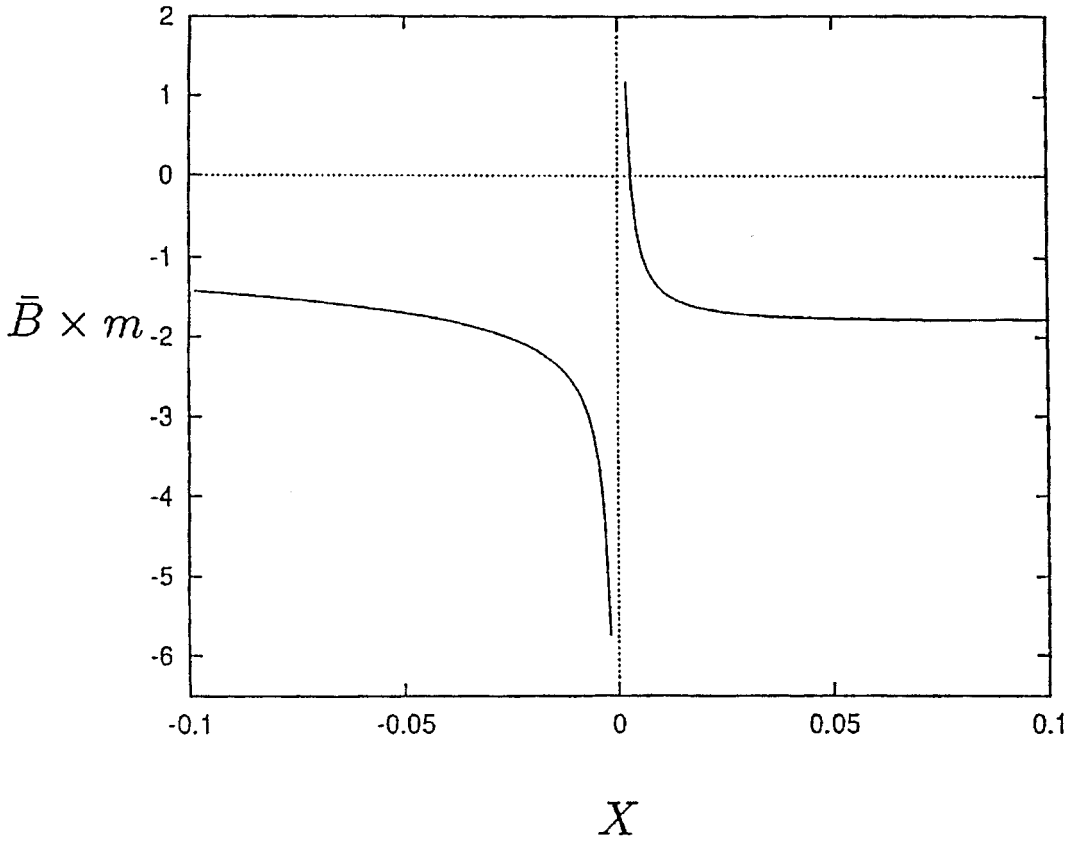

FIG. 2. Blow up of the $p^2=0$ region for the $a=0.2$ plot to display the onset of the singularity.

is singular at the origin in momentum space. This singular behavior appears to be consistent with confinement, since the $i\epsilon$ prescription necessary to define Feynman diagrams that build in the connection with unitary S-matrix elements between free outgoing and incoming colored quark states is not permitted by the DS equation [(9) or (10)]. Further reinforcing our point about confinement is the fact that the function $S_W(x)$, Eq. (16), which reduces to the free two-point Wightman function in our model, is not a tempered distribution, one of the conditions assumed for the Wightman functions when they are used for proving the existence of free-fermion asymptotic states. If an ultraviolet term were added to the gluon propagator to make it more realistic, what effect on our solution to the quark DS equation might one find? Certainly the $1/p^2$ behavior at large, spacelike values of p^2 should be not affected, since this is consistent with asymptotic freedom. At large timelike values of p^2, the $1/p^2$ behavior that we find may be modified, but the asymptotic behavior should still admit a Fourier transform for the solution. The specific singularity structure on the finite real axis may be changed, but the model for the infrared region is compatible with the cluster decomposition property, so one may expect that there will be no poles or cuts with the Feynman $i\epsilon$ prescription that would indicate loss of confinement. Finally, as noted in [2], our model of the gluon propagator does not produce spontaneous chiral symmetry breaking by itself, but the ultraviolet term may drive the breakdown.

The second Fourier transformable solution ($\mu^2<0$ case) that we presented has the same ''free-quark'' asymptotic behavior in momentum space as the first, but the particlelike resonant behavior at $p^2=m^2$ in the large mass limit is missing. Furthermore, the large mass (or small interaction strength) limit is singular, in keeping with the fact that, while this solution to the full, interacting DS equation has a Fourier transform, the solution to the free equation does not. When taken in momentum space, the limit to the free Green function simply does not exist. The latter feature is shared by the solutions presented in [2], [3], and [4], where the (Euclidean) momentum space propagators, extended to timelike values of the argument, are not tempered distributions and have no Fourier transform.

In conclusion, we have presented the tempered-distribution solutions to the DS equation for a simple model in four-dimensional Minkowski space with a confining gluon propagator and a nontrivial quark gluon vertex. We offer the results as an interesting, instructive, and useful addition to the literature on the dynamics of confined quarks.

ACKNOWLEDGMENTS

We thank Pankaj Jain and V. A. Miranski for discussions. The computational facilities of the Kansas Institute for Theoretical and Computational Science were used for portions of this work. This research has been supported in part by Department of Energy Grant No. DE-FG02-85ER40214.

APPENDIX A

In this appendix we present details of our evaluation of the Minkowski space Fourier transforms. Our approach is to define the auxiliary, one-dimensional transform

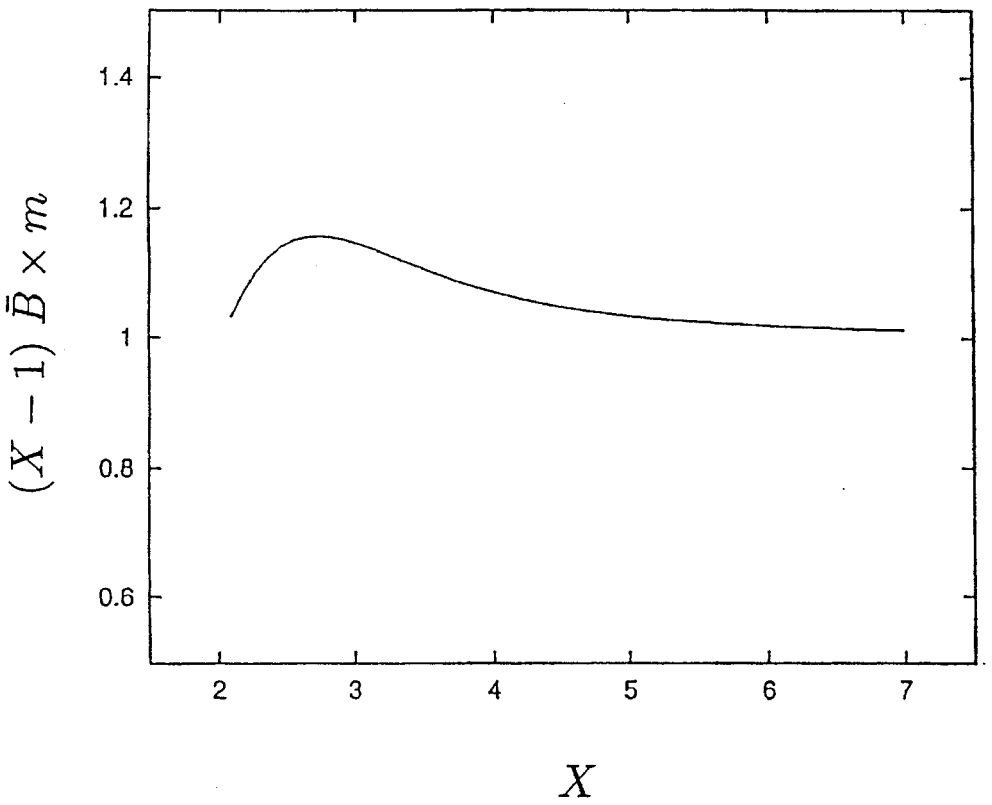

FIG. 3. Plot of $(x-1)\overline{B}(p^2)m$ for the case $a=0.2$. The rapid approach to 1 as $x=p^2/m^2$ grows shows the $1/(x-1)$ asymptotic behavior. As seen in Fig. 1, this behavior is shared by all of the different a-value solutions. The same asymptotic behavior is obeyed by $\tilde{B}(p^2)$ (see text).

$$e^{-(\mu^2/2)x^2}\overline{\Delta}(x^2)=\int_{-\infty}^{\infty}\frac{e^{i\omega x^2}}{2\pi}\,G(\omega)dw \tag{A1}$$

and its inverse

$$G(\omega)=\int_{-\infty}^{\infty}e^{-i\omega x^2}e^{-(\mu^2/2)x^2}\overline{\Delta}(x^2)dx^2. \tag{A2}$$

It is understood that $\mu^2>0$ here. This procedure can be applied to any function of x^2 which has a one-dimensional (in x^2) Fourier transform and whose four-dimensional transform is a function of p^2. Substituting Eq. (13) into Eq. (A2) and evaluating, we find

$$\begin{aligned}G(\omega)&=\int_{-\infty}^{\infty}e^{-x^2[i\omega+(\mu^2/2)]}\left[\frac{1}{4\pi}\left(\delta(x^2)\right.\right.\\&\quad\left.\left.-\frac{m^2}{2}\,\theta(x^2)\,\frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}}\right)\right]dx^2\\&=\frac{1}{4\pi}-\frac{m^2}{8\pi}\int_0^{\infty}e^{-x^2[i\omega+(\mu^2/2)]}\,\frac{J_1(m\sqrt{x^2})}{m\sqrt{x^2}}\,dx^2\\&=\frac{1}{4\pi}\,e^{-m^2/4(i\omega+\mu^2/2)}.\end{aligned} \tag{A3}$$

So the evaluation of the first term in the expression for $B(p^2)$, Eq. (18), now involves the integral

$$\overline{B}(p^2)=\frac{-m}{8\pi^2}\int_{-\infty}^{\infty}d\omega\;e^{-m^2/4[i\omega+(\mu^2/2)]}\int d^4x\;e^{i\omega x^2+ip\cdot x}, \tag{A4}$$

where we have exchanged the order of integration in writing this version of Eq. (18). The x integration is a product of Fresnel integrals that yields

$$\int d^4x\;e^{i\omega x^2+ip\cdot x}=-i\epsilon(\omega)\,\frac{\pi^2}{\omega^2}\,e^{-i(p^2/4\omega)}. \tag{A5}$$

Defining $\nu=4/\omega$, the Fourier transform $\overline{B}(p^2)$ can be written

$$\begin{aligned}\overline{B}(p^2)/m&=\mathrm{Im}\int_0^{\infty}d\nu\;\exp\left(ip^2\nu-i\nu m^2\,\frac{(1-2i\mu^2\nu)}{(1+4\mu^4\nu^2)}\right)\\&=\frac{1}{m^2}\int_0^{\infty}dz\;\exp[-z^2a/(1+a^2z^2)]\\&\quad\times\sin[tz-z/(1+a^2z^2)],\end{aligned} \tag{A6}$$

where $a\equiv 2\mu^2/m^2$, $t\equiv p^2/m^2$, and $z\equiv\nu m^2$. One obtains the Fourier transform of $\tilde{B}(p^2)$ by simply changing the sign of μ^2 in Eq. (A6).

To evaluate $B(p^2)$ for $p^2>0$, Eq. (A6), convergence can be improved by choosing a contour in the first quadrant in the complex z plane that is equivalent to the one shown in Eq. (A6) by Cauchy's theorem. Writing $z=\rho e^{i\theta}$, $0\leq\theta<\pi/2$, one can recast the representation of $B(p^2)$ in the form

$$m\overline{B}(p^2)=\int_0^\infty d\rho\left[\sin\left(x\rho\,\cos\theta+\theta-\frac{\rho\,\cos\theta}{D}\right)\right.$$
$$\left.\times\exp\left(-x\rho\,\sin\theta+\rho\,\frac{\sin\theta}{D}-\frac{a\rho^2}{D}\right)\right], \quad \text{(A7)}$$

where $x=p^2/m^2$, $a=2\mu^2/m^2$, and $D=(1-a\rho\,\sin\theta)^2+a^2\rho^2\cos^2\theta$. The choice $\theta=\pi/4$ is convenient for evaluating Eq. (A7), which converges significantly faster than Eq. (A6) for $p^2>0$ and thereby speeds up the numerical integration. A similar formula holds for $p^2<0$, where $-(\pi/2)<\theta<0$.

APPENDIX B

The configuration space solution of the DS equation has the factorizable form $S(x)=e^{if(x^2)}S_0(x)$ for the vertex outlined in this appendix, for any gluon propagator for which the integral (B15) exists. The $\delta^4(q)$ propagator is a special example where the Ward identity (8) defines the DS equation vertex and leads to the factorized solution (11). Let us see how this result can be generalized to other propagators.

A model for the irreducible vertex Γ_μ that has good symmetry properties [14] and satisfies the Abelian Ward-Takahashi identity can be written in configuration space as

$$\Gamma_\mu(z;x,y)=F_\mu(y-z,x-z)S^{-1}(x,y). \quad \text{(B1)}$$

In Eq. (B1), $S(x,y)$ is the full fermion propagator and

$$F_\mu(y-z,x-z)=\int(e^{-iq\cdot(y-z)}-e^{-iq\cdot(x-z)})\,\frac{(x-y)_\mu}{q\cdot(x-y)}\,\frac{d^4q}{(2\pi)^4}+F^T_\mu, \quad \text{(B2)}$$

where $\partial F^T_\mu/\partial z_\mu=0$. One can represent the Fourier transform of Γ_μ in terms of the fermion propagator as

$$\Gamma_\mu(p+k,p)=\frac{\partial}{\partial p^\mu}\int_0^1 S^{-1}(p+\alpha k)d\alpha+\Gamma^T_\mu(p+k,p). \quad \text{(B3)}$$

The first term on the right-hand side of Eq. (B3) is constructed to satisfy

$$k^\mu\Gamma_\mu(p+k,p)=S^{-1}(p+k)-S^{-1}(p), \quad \text{(B4)}$$

where $k^\mu\Gamma^T_\mu(p+k,p)=0$.

While the vertex Γ_μ is convenient for a discussion of the symmetry properties of a theory, the study of the fermion DS equation is generally more convenient when use is made of the function Λ_μ, defined as

$$\Lambda_\mu(p+k,p)\equiv S(p+k)\Gamma_\mu(p+k,p)S(p). \quad \text{(B5)}$$

In terms of $\Lambda_\mu(p+k,k)$, our model looks like

$$\Lambda_\mu(p+k,p)=\frac{\partial}{\partial p^\mu}\int_0^1 S(p+\alpha k)d\alpha+\Lambda^T_\mu(p+k,p), \quad \text{(B6)}$$

with

$$k^\mu\Lambda_\mu(p+k,p)=S(p)-S(p+k) \quad \text{(B7)}$$

and $k^\mu\Lambda^T_\mu=0$. Equation (B6) must satisfy the Ward identity (8); therefore, Λ^T_μ must vanish at $k=0$. With this in mind, we make the approximation that the first term in Eq. (B6) dominates in the infrared region of interest here, and so we drop Λ^T_μ in what follows.[9]

The general form of the DS equation is then, in momentum space,

$$1=(\not p-m)S(p)-i\int\frac{d^4k}{(2\pi)^4}\,\gamma_\mu D^{\mu\nu}(k)\Lambda_\nu(p+k,p), \quad \text{(B8)}$$

where $D^{\mu\nu}(k)$ is the gauge-boson propagator.

With $\Lambda_\mu(p+k,k)$ modeled by the first term in Eq. (B6), the Fourier transform of Eq. (B8) is a pure differential equation, which we write as

$$\delta^4(x)=(i\not\partial-m)S(x)+[\not\partial f(x^2)]S(x). \quad \text{(B9)}$$

In Eq. (B9) we have

$$\frac{\partial}{\partial x^\nu}f(x^2)=x_\nu\int_0^1[d_1(\alpha^2x^2)+\alpha^2x^2d_2(\alpha^2x^2)]d\alpha$$
$$=2x_\nu\dot f(x^2), \quad \text{(B10)}$$

with

$$\dot f(x^2)\equiv\frac{d}{dx^2}f(x^2)=\frac{1}{2}\int_0^1 d\alpha[d_1(\alpha^2x^2)+\alpha^2x^2d_2(\alpha^2x^2)]. \quad \text{(B11)}$$

In writing Eqs. (B8)–(B10), we have expressed the covariant-gauge, gauge-boson propagator in the general form

$$D_{\mu\nu}(x)=d_1(x^2)g_{\mu\nu}+d_2(x^2)x_\nu x_\nu. \quad \text{(B12)}$$

The general solution to the DS equation (B9) has the remarkably simple factorized form

$$S(x)=e^{if(x^2)}S_0(x). \quad \text{(B13)}$$

The Green function $S_0(x)$ satisfies the free equation

$$(i\not\partial-m)S_0(x)=\delta^4(x)e^{-if(0)}, \quad \text{(B14)}$$

where it is to be understood that arbitrary solutions S_H to the homogeneous equation $(i\not\partial-m)S_H(x)=0$ can always be added to solutions of Eq. (B14). We require that $S_0(x)$ have the standard normalization, and so $f(0)=0$, and the solution to Eq. (B10) can be displayed as

$$f(x^2)=\frac{1}{2}\int_0^{x^2}dx'^2\int_0^1 d\alpha[d_1(\alpha^2x'^2)+\alpha^2x'^2d_2(\alpha^2x'^2)], \quad \text{(B15)}$$

in those circumstances where the integral converges.

[9]For the model of Eq. (4), Λ^T_μ does not contribute anyway, and so the first term of Eq. (B6) is the complete contribution.

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