

## Analytic form for the effective Lagrangian of QED and its application to pair production and photon splitting

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(Received 16 May 1996)

We derive an analytic form for the Heisenberg-Euler Lagrangian in the limit where the component of the electric field parallel to the magnetic field is small. We expand these analytic functions to all orders in the field strength ( $F_{\mu\nu}F^{\mu\nu}$ ) in the limits of weak and strong fields, and use these functions to estimate the pair-production rate in arbitrarily strong electric fields and the photon-splitting rate in arbitrarily strong magnetic fields. [S0556-2821(97)05404-0]

PACS number(s): 12.20.Ds, 97.60.Jd, 98.70.Rz

### I. INTRODUCTION: THE ONE-LOOP EFFECTIVE LAGRANGIAN OF QED

When one-loop corrections are included in the Lagrangian of the electromagnetic field, one obtains a nonlinear correction term

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1. \quad (1)$$

Both terms of the Lagrangian can be written in terms of the Lorentz invariants,

$$I = F_{\mu\nu}F^{\mu\nu} = 2(|\vec{B}|^2 - |\vec{E}|^2) \quad (2)$$

and

$$K = -\left(\frac{1}{2}\epsilon^{\lambda\rho\mu\nu}F_{\lambda\rho}F_{\mu\nu}\right)^2 = -(4\vec{E}\cdot\vec{B})^2, \quad (3)$$

following Heisenberg and Euler [1]. We do not expect terms which are odd powers of  $\epsilon^{\lambda\rho\mu\nu}F_{\lambda\rho}F_{\mu\nu}$  to appear in the effective Lagrangian as these terms would yield a Lagrangian which would violate the  $C$  and  $P$  symmetries of the tree-level Lagrangian.

Heisenberg and Euler [1] and Weisskopf [2] independently derived the effective Lagrangian of the electromagnetic field using electron-hole theory. Schwinger [3] later rederived the same result using quantum electrodynamics. In rationalized electromagnetic units, the Lagrangian is given by

$$\mathcal{L}_0 = -\frac{1}{4}I, \quad (4)$$

$$\mathcal{L}_1 = \frac{e^2}{hc} \int_0^\infty e^{-\zeta} \frac{d\zeta}{\zeta^3} \left\{ i\zeta^2 \frac{\sqrt{-K}}{4} \frac{\cos[(\zeta/B_k)\sqrt{-I/2+i(\sqrt{-K}/2)}] + \cos[(\zeta/B_k)\sqrt{-I/2-i(\sqrt{-K}/2)}]}{\cos[(\zeta/B_k)\sqrt{-I/2+i(\sqrt{-K}/2)}] - \cos[(\zeta/B_k)\sqrt{-I/2-i(\sqrt{-K}/2)}]} + |B_k|^2 + \frac{\zeta^2}{6}I \right\}, \quad (5)$$

where  $B_k = E_k = m^2c^3/e\hbar \approx 2.2 \times 10^{15} \text{ Vcm}^{-1} \approx 4.4 \times 10^{13} \text{ G}$ . Both Heisenberg and Euler [1], and Mielniczuk [4] present alternative expressions for these integrals in terms of infinite series.

### II. THE ANALYTIC EXPANSION

For many interesting problems, one needs an expansion of this Lagrangian in the limit where the component of the electric field in the direction of the magnetic field is small (small  $K$ ),

$$\mathcal{L}_1 = \mathcal{L}_1(I,0) + K \left. \frac{\partial \mathcal{L}_1}{\partial K} \right|_{K=0} + \frac{K^2}{2} \left. \frac{\partial^2 \mathcal{L}_1}{\partial K^2} \right|_{K=0} + \dots, \quad (6)$$

where the terms are given by the integrals

$$\begin{aligned} \mathcal{L}_1(I,0) &= \frac{e^2}{hc} \frac{I}{2} X_0\left(\frac{1}{\xi}\right) \\ &= \frac{e^2}{hc} \frac{I}{2} \int_0^\infty e^{-u/\xi} \frac{du}{u^3} \left(-u \coth u + 1 + \frac{u^2}{3}\right), \quad (7) \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial \mathcal{L}_1}{\partial K} \right|_{K=0} &= \frac{e^2}{hc} \frac{1}{16I} X_1\left(\frac{1}{\xi}\right) \\ &= \frac{e^2}{hc} \frac{1}{16I} \int_0^\infty e^{-u/\xi} \frac{du}{u^2} \\ &\quad \times \left(\coth u - \frac{2}{3}u^2 \coth u - u \operatorname{csch}^2 u\right), \quad (8) \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial^2 \mathcal{L}_1}{\partial K^2} \right|_{K=0} &= \frac{e^2}{hc} \frac{1}{384I^3} X_2 \left( \frac{1}{\xi} \right) \\ &= \frac{e^2}{hc} \frac{1}{384I^3} \int_0^\infty e^{-u/\xi} \frac{du}{u^2} \left( -9ucsch^2u - 4u^3csch^2u \right. \\ &\quad \left. + 2u^2cothu + 15cothu \right. \\ &\quad \left. + \frac{8}{15}u^4cothu - 6u^2coth^3u \right), \end{aligned} \tag{9}$$

and we have defined

$$\xi = \frac{1}{B_k} \sqrt{\frac{I}{2}} \quad \text{and} \quad u = \frac{\xi}{B_k} \sqrt{\frac{I}{2}}. \tag{10}$$

Note that  $\xi$  is a dimensionless measure of the strength of the field.

The auxiliary functions  $X_i$  may be calculated analytically:

$$\begin{aligned} X_0(x) &= 4 \int_0^{x/2-1} \ln[\Gamma(v+1)]dv + \frac{1}{3} \ln\left(\frac{1}{x}\right) + 2 \ln 4 \pi - 4 \ln A \\ &\quad - \frac{5}{3} \ln 2 - \left[ \ln 4 \pi + 1 + \ln\left(\frac{1}{x}\right) \right] x + \left[ \frac{3}{4} + \frac{1}{2} \ln\left(\frac{2}{x}\right) \right] x^2, \end{aligned} \tag{11}$$

$$X_1(x) = -2X_0(x) + xX_0^{(1)}(x) + \frac{2}{3}X_0^{(2)}(x) - \frac{2}{9} \frac{1}{x^2}, \tag{12}$$

$$\begin{aligned} X_2(x) &= -24X_0(x) + 9xX_0^{(1)}(x) + (8 + 3x^2)X_0^{(2)}(x) \\ &\quad + 4xX_0^{(3)}(x) - \frac{8}{15}X_0^{(4)}(x) + \frac{8}{15} \frac{1}{x^2} + \frac{16}{15} \frac{1}{x^4}, \end{aligned} \tag{13}$$

where

$$X_0^{(n)}(x) = \frac{d^n X_0(x)}{dx^n}. \tag{14}$$

Because of the near symmetry between  $I$  and  $K$  in Eq. (6), higher derivatives with respect to  $K$  may be calculated, in principle, and represented by a sum of derivatives of  $\mathcal{L}(I,0)$  with respect to  $I$ . The constant  $A$  is defined as

$$\ln A = \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n i \ln i \right) - \left[ \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n - \frac{n^2}{4} \right] \tag{15}$$

$$= \frac{1}{12} - \zeta^{(1)}(-1) = 0.248\,754\,477, \tag{16}$$

in analogy to the Euler-Mascheroni constant ( $\gamma$ ) [5]. Here  $\zeta^{(1)}(x)$  denotes the first derivative of the Riemann  $\zeta$  function.

Barnes [5] evaluates the definite integral of  $\ln\Gamma(x)$  in terms of the  $G$  function:

$$\begin{aligned} \int_0^{x/2-1} \ln[\Gamma(v+1)]dv &= \left(\frac{x}{2}-1\right) \ln\Gamma\left(\frac{x}{2}\right) - \ln G\left(\frac{x}{2}\right) - \frac{x^2}{8} \\ &\quad + \frac{x}{4}(1 + \ln 2 \pi) - \frac{1}{2} \ln 2 \pi, \end{aligned} \tag{17}$$

where

$$\begin{aligned} G(z) &= (2\pi)^{(z-1)/2} \exp\left(-\frac{z(z-1)}{2}\right) \\ &\quad \times \exp\left(-\gamma \frac{(z-1)^2}{2}\right) \prod_{k=1}^\infty \left[ \left(1 + \frac{z-1}{k}\right)^k \right. \\ &\quad \left. \times \exp\left(1 - z + \frac{(z-1)^2}{2k}\right)\right]. \end{aligned} \tag{18}$$

The integral of  $\ln\Gamma(x)$  may also be expressed in terms of the generalized Riemann  $\zeta$  function [6]:

$$\begin{aligned} \int_0^{x/2-1} \ln[\Gamma(v+1)]dv &= \zeta^{(1)}\left(-1, \frac{x}{2}-1\right) - \zeta^{(1)}(-1) \\ &\quad + \frac{\ln 2 \pi}{2} \left(\frac{x}{2}-1\right) - \frac{x}{4} \left(\frac{x}{2}-1\right) \\ &\quad + \left(\frac{x}{2}-1\right) \ln\left(\frac{x}{2}-1\right). \end{aligned} \tag{19}$$

Our expression for  $X_0$  was also found by Dittrich *et al.* [6]. Ivanov [7] derived a similar expression as well, but his result differs from ours and that of Dittrich *et al.* in the constant term. Unlike Ivanov's expression, ours approaches zero as  $\xi$  goes to 0 which from examination of Eq. (7) is the correct limiting behavior. In addition, the above form for  $X_0$  reproduces the asymptotic strong-field limit given by Heisenberg and Euler [1].

These functions can be expanded in both the weak-field and strong-field limits. In the weak-field limit ( $\xi < 0.5$ ), we obtain

$$X_0\left(\frac{1}{\xi}\right) = - \sum_{j=1}^\infty \frac{2^{2j} B_{2(j+1)}}{j(j+1)(2j+1)} \xi^{2j} \tag{20}$$

$$X_1\left(\frac{1}{\xi}\right) = -\frac{14}{45} \xi^2 + \frac{1}{3} \sum_{j=2}^\infty \frac{2^{2j} [6B_{2(j+1)} - (2j+1)B_{2j}]}{j(2j+1)} \xi^{2j} \tag{21}$$

$$\begin{aligned} X_2\left(\frac{1}{\xi}\right) &= \frac{1}{15} \sum_{j=3}^\infty \frac{2^{2j}}{j} \left[ 2j(2j-1)B_{2(j-1)} + 60(j-1)B_{2j} \right. \\ &\quad \left. - 180 \frac{j-2}{2j+1} B_{2(j+1)} \right] \xi^{2j}, \end{aligned} \tag{22}$$

where  $B_j$  denotes the  $j$ th Bernoulli number. In the strong-field limit ( $\xi > 0.5$ ), we obtain

$$\begin{aligned}
X_0\left(\frac{1}{\xi}\right) &= \left(\frac{1}{3}\ln 2 - 4\ln A + \frac{1}{3}\ln \xi\right) + (1 - \ln \pi + \ln \xi) \xi^{-1} \\
&+ \left(\frac{1}{2}(\ln 2 - \gamma) + \frac{3}{4} + \frac{1}{2}\ln \xi\right) \xi^{-2} \\
&+ \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{2^{j-2}} \frac{1}{j} \frac{1}{j-1} \zeta(j-1) \xi^{-j}, \quad (23)
\end{aligned}$$

$$\begin{aligned}
X_1\left(\frac{1}{\xi}\right) &= -\frac{2}{3}\xi + \left(8\ln A - \frac{1}{3} - \frac{2}{3}\gamma\right) \\
&+ \left(\ln \pi + \frac{1}{18}\pi^2 - 2 - \ln \xi\right) \xi^{-1} \\
&+ \left(-\frac{1}{2} - \frac{1}{6}\zeta(3)\right) \xi^{-2} + \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{2^{j-2}} \\
&\times \left(\frac{j-2}{j(j-1)}\zeta(j-1) + \frac{1}{6}\zeta(j+1)\right) \xi^{-j}, \quad (24)
\end{aligned}$$

$$\begin{aligned}
X_2\left(\frac{1}{\xi}\right) &= \frac{16}{15}\xi^3 - 4\xi + \left(-6 - 8\gamma + \frac{4}{15}\zeta(3) + 96\ln A\right) \\
&+ \left(-27 + \pi^2 - 15\ln \pi - \frac{1}{225}\pi^4 - 15\ln \xi\right) \xi^{-1} \\
&+ \left(\frac{2}{5}\zeta(5) - 9 - 4\zeta(3)\right) \xi^{-2} \\
&+ \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{2^{j-2}} \left[\frac{3(j+4)(j-2)}{j(j-1)}\zeta(j-1)\right. \\
&\left. + (j+2)\left(\zeta(j+1) - \frac{j+1}{30}\zeta(j+3)\right)\right] \xi^{-j}. \quad (25)
\end{aligned}$$

The Lagrangian may also be expanded in terms of the invariants themselves or the electric and magnetic fields. In the weak-field limit, we obtain

$$\mathcal{L}_1(I, 0) = \frac{e^2}{hc} \left( \frac{1}{180} \frac{I^2}{B_k^2} - \frac{1}{630} \frac{I^3}{B_k^4} + \frac{1}{630} \frac{I^4}{B_k^6} + \dots \right), \quad (26)$$

$$\left. \frac{\partial \mathcal{L}_1}{\partial K} \right|_{K=0} = \frac{e^2}{hc} \left( -\frac{7}{720} \frac{1}{B_k^2} + \frac{13}{5040} \frac{I}{B_k^4} - \frac{11}{3780} \frac{I}{B_k^6} + \dots \right), \quad (27)$$

$$\begin{aligned}
\left. \frac{\partial^2 \mathcal{L}_1}{\partial K^2} \right|_{K=0} &= \frac{e^2}{hc} \left( \frac{19}{15120} \frac{1}{B_k^6} - \frac{127}{23760} \frac{I}{B_k^8} \right. \\
&\left. + \frac{5527}{180180} \frac{I^2}{B_k^{10}} + \dots \right). \quad (28)
\end{aligned}$$

This weak-field expansion agrees with the Heisenberg and Euler [1] result.

In the strong-field limit, for direct comparison with Heisenberg and Euler [1], we define  $a = E/E_k$  and  $b = B/B_k$  and take the limit  $b \gg 1$  and  $a \ll 1$ . We take,  $\xi^2 = b^2 - a^2$ ,  $\xi \approx b - a^2/2b$  and  $K = -16B_k^4(ab)^2$ . We obtain

$$\begin{aligned}
\mathcal{L}_1(a, b) &= 4 \frac{e^2}{hc} B_k^2 \left[ b^2 \left( \frac{\ln b}{12} - \ln A + \frac{\ln 2}{12} \right) + \frac{b}{4} (\ln b + 1 - \ln \pi) \right. \\
&+ \frac{\ln b}{8} + \frac{3}{16} + \frac{\ln 2 - \gamma}{8} - \frac{a^2}{12} (\ln b + \ln 2 - \gamma) \\
&\left. + b \left( \frac{a^2}{12} + \frac{a^4}{90} + \dots \right) + \dots \right], \quad (29)
\end{aligned}$$

which agrees numerically with the corresponding expansion in Heisenberg and Euler [1].

### III. PAIR PRODUCTION IN AN ARBITRARILY STRONG ELECTRIC FIELD

In a strong electric field with no magnetic field, the value of the first invariant is negative,  $I = -2|\vec{E}|^2$  and  $K = 0$ . The analytic expressions for the Lagrangian are valid for values of  $\xi$  throughout the complex plane, with a branch cut along the negative real axis. Using an imaginary value of

$$\xi = i \left( \frac{E}{E_k} \right) = iy, \quad y > 0 \quad (30)$$

and taking  $w = 2(4\pi\hbar)^{-1}$ ,  $\text{Im}\mathcal{L}$  gives the pair production rate per unit volume [8]. From examination of Eq. (20), for  $\xi < 0.5$  the pair-production rate is apparently zero. However, since Eq. (20) is a power series in  $\xi/2$ , the imaginary part of  $X_0$  may be exponentially small. Berestetskii *et al.* [8] derive, for a weak field,

$$w = 2(4\pi\hbar)^{-1} \text{Im}\mathcal{L} \sim \frac{1}{4\pi^3} \left( \frac{\hbar}{mc} \right)^{-3} \left( \frac{\hbar}{mc^2} \right)^{-1} y^2 \exp\left(-\frac{\pi}{y}\right). \quad (31)$$

To simplify the numerics, we use an alternate definition of  $X_0(x)$  obtained by means of a change of variables

$$\begin{aligned}
X_0(x) &= 4 \left( \frac{x}{2} - 1 \right) \int_0^1 \ln \left[ \Gamma \left( u \left( \frac{x}{2} - 1 \right) + 1 \right) \right] du + \frac{1}{3} \ln \left( \frac{1}{x} \right) \\
&+ 2 \ln 4 \pi - \left( 4 \ln A + \frac{5}{3} \ln 2 \right) - \left[ \ln 4 \pi + 1 + \ln \left( \frac{1}{x} \right) \right] x \\
&+ \left[ \frac{3}{4} + \frac{1}{2} \ln \left( \frac{2}{x} \right) \right] x^2. \quad (32)
\end{aligned}$$

With this definition and the property of the  $\Gamma$  function,  $\ln \Gamma(\bar{x}) = \overline{\ln \Gamma(x)}$ , we see that  $X_0(\bar{x}) = \overline{X_0(x)}$ , so

$$\begin{aligned}
w &= 2(4\pi\hbar)^{-1} \text{Im}\mathcal{L}|_{I=-2y^2E_k^2, K=0} \\
&= i \frac{e^2}{8\pi^2 \hbar^2 c} E_k^2 y^2 \left[ X_0\left(-\frac{i}{y}\right) - X_0\left(\frac{i}{y}\right) \right] \\
&= \frac{1}{8\pi^2} \left( \frac{\hbar}{mc} \right)^{-3} \left( \frac{\hbar}{mc^2} \right)^{-1} \left[ -\frac{1}{3} \pi y^2 - 8y^2 \text{Im}Q(y) \right. \\
&\left. - 2y(\ln y + \ln 4\pi + 1) + 4y \text{Re}Q(y) + \frac{1}{2} \pi \right], \quad (33)
\end{aligned}$$

where

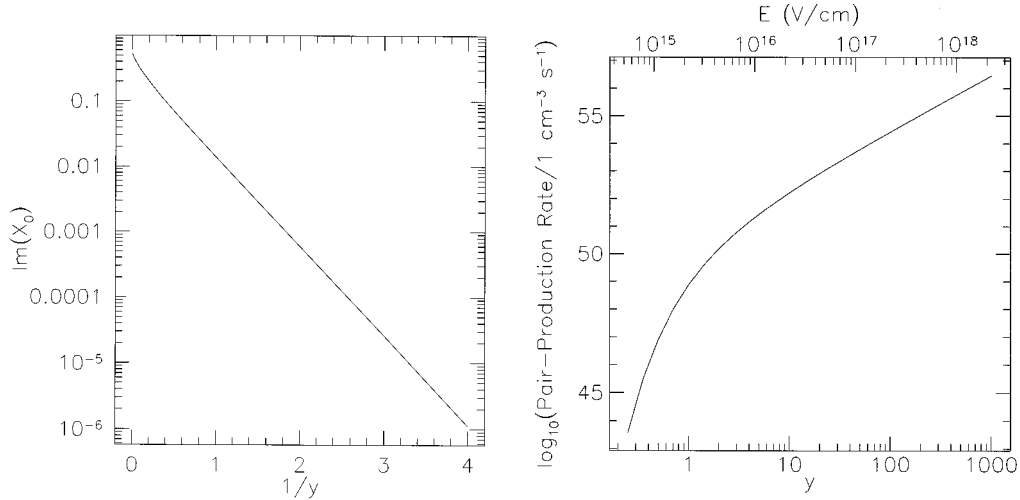


FIG. 1. The left panel depicts the imaginary part of  $X_0$  as a function of  $1/y$ . For weak fields the imaginary component is approximately  $\pi^{-1}\exp(-\pi/y)$ . The right panel depicts the pair-production rate for near and supercritical fields.

$$Q(y) = \int_0^1 \ln \left[ \Gamma \left( u \left( \frac{i}{2y} - 1 \right) + 1 \right) \right] du \quad (34)$$

and the scaling constant

$$\frac{1}{8\pi^2} \left( \frac{\hbar}{mc} \right)^{-3} \left( \frac{\hbar}{mc^2} \right)^{-1} = 1.7 \times 10^{51} \text{ cm}^{-3} \text{ s}^{-1}. \quad (35)$$

This expression for  $w$ , the pair-production rate, agrees with the Itzykson-Zuber result [9] for an arbitrarily strong electric field. If we return to Eq. (11) and use the properties of the  $\Gamma$  function along the imaginary axis, we find that the imaginary part of  $X_0$  for imaginary arguments is given by

$$\text{Im}X_0 \left( \frac{1}{iy} \right) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\pi n/y}}{n^2}, \quad (36)$$

which yields a pair-production rate in agreement with the earlier result [9].

In the strong-field limit we use Eq. (23) and take the imaginary part:

$$w = \frac{1}{8\pi^2} \left( \frac{\hbar}{mc} \right)^{-3} \left( \frac{\hbar}{mc^2} \right)^{-1} \left[ -\frac{\pi}{3} y^2 + 2(1 - \ln \pi + \ln y) y + \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{3k} \zeta(2k)}{2^{2k-1} k(2k+1)} y^{-(2k-1)} \right]. \quad (37)$$

Figure 1 depicts the imaginary component of  $X_0$  for  $I = -2y^2 E_k^2$  and the pair-production rate per unit volume. From the left panel, we verify that the imaginary component of  $X_0$  is approximately  $\pi^{-1}\exp(-\pi/y)$  for weak fields. The right panel shows the pair-production rate which increases as  $y^2$  for strong fields and is damped exponentially in weak fields.

#### IV. AUXILIARY FUNCTIONS FOR PHOTON SPLITTING

To calculate the photon-splitting rates we follow the technique by Adler [10] for the low-frequency limit. In this limit,

Adler expresses the opacity for photon splitting by means of auxiliary functions which are simply derivatives of the Lagrangian and, therefore, of the functions  $X_i$  above:

$$\kappa[\parallel \rightarrow \parallel + \parallel] = \frac{\alpha^6}{2\pi^2} \frac{\hbar^{13}}{m^{16} c^{24}} B^6 \sin^6 \theta \frac{\omega^5}{30} C_1(\xi)^2 \quad (38)$$

$$= \frac{\alpha^3}{2\pi^2} \left( \frac{B \sin \theta}{B_k} \right)^6 \left( \frac{\hbar \omega}{mc^2} \right)^5 \frac{C_1(\xi)^2}{30} \left( \frac{mc}{\hbar} \right) \quad (39)$$

$$= 17.0 \text{ cm}^{-1} \left( \frac{B \sin \theta}{B_k} \right)^6 \left( \frac{\hbar \omega}{mc^2} \right)^5 C_1(\xi)^2, \quad (40)$$

$$\kappa[\parallel \rightarrow \perp + \perp] = \frac{\alpha^6}{2\pi^2} \frac{\hbar^{13}}{m^{16} c^{24}} B^6 \sin^6 \theta \frac{\omega^5}{30} C_2(\xi)^2, \quad (41)$$

$$\kappa[\perp \rightarrow \parallel + \perp] = 2 \frac{\alpha^6}{2\pi^2} \frac{\hbar^{13}}{m^{16} c^{24}} B^6 \sin^6 \theta \frac{\omega^5}{30} C_2(\xi)^2. \quad (42)$$

The conversion of  $\perp$  to  $\parallel + \perp$  proceeds through two channels, hence the twofold increase in the opacity for this process.  $C_1$  and  $C_2$  are defined by

$$\frac{\partial^3 \mathcal{L}_{\text{Adler}}}{\partial \mathcal{F}^3} \Big|_{\mathcal{G}=0, \mathcal{F}=(1/2)B^2} = \frac{64}{4\pi} \cdot \frac{\partial^3 \mathcal{L}_1}{\partial I^3} \Big|_{K=0, I=2B^2} \quad (43)$$

$$= -\frac{\alpha^3 \hbar^6}{2\pi^2 m^8 c^{10}} C_1(\xi)$$

$$= -\frac{\alpha}{2\pi^2 B_k^4} C_1(\xi), \quad (44)$$

$$\frac{\partial^3 \mathcal{L}_{\text{Adler}}}{\partial \mathcal{F} \partial \mathcal{G}^2} \Big|_{\mathcal{G}=0, \mathcal{F}=(1/2)B^2} = -\frac{128}{4\pi} \cdot \frac{\partial^2 \mathcal{L}_1}{\partial I \partial K} \Big|_{K=0, I=2B^2} \quad (45)$$

$$= -\frac{\alpha}{2\pi^2 B_k^4} C_2(\xi), \quad (46)$$

and  $\theta$  is the angle between the direction of propagation of the photon and the external magnetic field. The factors of 128 and 64 result from the definitions of Adler's  $\mathcal{F}$  and  $\mathcal{G}$  in terms of  $I$  and  $K$ :

$$\mathcal{F} = \frac{1}{2}(|\vec{B}|^2 - |\vec{E}|^2) = \frac{I}{4}, \quad (47)$$

$$\mathcal{G} = \vec{B} \cdot \vec{E} = \frac{1}{4}\sqrt{-K}. \quad (48)$$

An additional factor of  $4\pi$  appears because we are using rationalized Gaussian units, while Adler employs unrationalized units.

Given the analytic forms for  $X_0$  and  $X_1$ , we obtain

$$C_1(\xi) = \frac{1}{4\xi^7} \left[ X_0^{(3)} \left( \frac{1}{\xi} \right) + 3X_0^{(2)} \left( \frac{1}{\xi} \right) \xi - 3X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^2 \right], \quad (49)$$

$$C_2(\xi) = -\frac{1}{4\xi^5} \left[ X_1^{(1)} \left( \frac{1}{\xi} \right) + 2X_1 \left( \frac{1}{\xi} \right) \xi \right]. \quad (50)$$

The functions  $C_1$  and  $C_2$  have the appropriate limits as  $\xi \rightarrow 0$  which correspond to the lowest order hexagon diagrams for the splitting process:

$$C_1(I) = 16 \sum_{j=2}^{\infty} \frac{2^{j-1}(j-1)B_{2(j+1)}}{2j+1} \left( \frac{I}{B_k^2} \right)^{j-2} \quad (51)$$

$$= 6 \times \frac{8}{315} - \frac{64}{105} \frac{I}{B_k^2} + \frac{320}{99} \left( \frac{I}{B_k^2} \right)^2 \dots, \quad (52)$$

$$C_2(I) = \frac{1}{3} \sum_{j=2}^{\infty} \frac{2^{j+1}(j-1)(6B_{2(j+1)} - (2j+1)B_{2j})}{j(2j+1)} \left( \frac{I}{B_k^2} \right)^{j-2} \quad (53)$$

$$= 6 \times \frac{13}{945} - \frac{176}{945} \frac{I}{B_k^2} + \frac{332}{495} \left( \frac{I}{B_k^2} \right)^2 \dots \quad (54)$$

In the strong-field limit, we obtain

$$C_1(\xi) = \frac{1}{3} \xi^{-4} - \frac{3}{4} \xi^{-5} \left( \ln \xi - \ln \pi + \frac{2}{3} \right) - \xi^{-6} - \sum_{j=3}^{\infty} \frac{(-1)^j j^2 - 4}{2^j j - 1} \zeta(j-1) \xi^{-j-4}, \quad (55)$$

$$C_2(\xi) = \frac{1}{6} \xi^{-3} + \xi^{-4} \left( \frac{1}{6} + \frac{1}{3} \gamma - 4 \ln A \right) + \frac{3}{4} \xi^{-5} \left( \ln \xi - \ln \pi + \frac{5}{3} - \frac{1}{18} \pi^2 \right) + \frac{1}{2} \xi^{-6} \left( 1 + \frac{1}{3} \zeta(3) \right) + \sum_{j=3}^{\infty} \frac{(-1)^j (j+2)}{2^j} \times \left( \frac{j-2}{j(j-1)} \zeta(j-1) + \frac{\zeta(j+1)}{6} \right) \xi^{-j-4}. \quad (56)$$

These expressions for the photon-splitting rate are valid only in the low-frequency limit since the Heisenberg-Euler Lagrangian neglects the gradients of the field. When these gradients are neglected, the results from Schwinger's proper-

time integration [3] used by Adler [10] reduce to these results obtained from the Heisenberg-Euler Lagrangian. Baier, Milstein, and Shaisultanov [11] have also obtained similar results for arbitrary field strengths (and photon frequencies) using an operator diagram technique.

Baier *et al.* and Adler's methods differ in spirit, but yield the same results. Baier, Katkov, and Strakhovenko developed the operator diagram technique [12]. In this formalism, the photon-splitting matrix element is evaluated with Feynman diagrams [13] using electron propagators in an external field. On the other hand, Adler [10] calculates the expectation value of the current [ $\langle j_\mu(x) \rangle$ ] order by order in the external photon fields using Schwinger's formalism [3] and relates this expectation value to the photon-splitting matrix element.

## V. PHOTON-SPLITTING OPACITIES AND APPLICATION TO NEUTRON STARS

Adler [10] argues that because of dispersive effects, the process  $\parallel \rightarrow \perp + \perp$  dominates the opacity of photons traveling through a strong field. Therefore, we are interested in the function  $C_2(\xi)$  which determines the splitting rate for all magnetic field strengths at photon energies small compared to the mass of the electron. We see immediately from the expansions of  $C_2$  that the opacity has the following behavior for weak and strong fields

$$\kappa[\parallel \rightarrow \perp + \perp] = \begin{cases} 0.116 \text{ cm}^{-1} \left( \frac{B \sin \theta}{B_k} \right)^6 \left( \frac{\hbar \omega}{mc^2} \right)^5, & B \ll B_k, \\ 0.472 \text{ cm}^{-1} \sin^6 \theta \left( \frac{\hbar \omega}{mc^2} \right)^5, & B \gg B_k. \end{cases} \quad (57)$$

We find, in agreement with the recent result of Baier *et al.* [11] and as well as with earlier results [14,15] that the photon-splitting opacity approaches a constant value in the limit of strong fields.

The left panel of Fig. 2 depicts the opacity for photons with  $E = mc^2$  as a function of  $\xi$ . Our formulas are not valid for these high-energy photons, but for low energies the opacity scales as this quantity times the photon energy to the fifth power. The right panel applies these opacities to neutron stars. Neutron stars are observed to have magnetic fields  $\sim 10^{12}$  G (e.g., [16]) and a subset of these objects known as magnetars are suspected to have much stronger fields  $\sim 10^{16}$  G or larger [17]. The figure illustrates the energy of photons with a mean-free path of 10 km. All parallel-polarized photons with this energy or larger would tranverse an optical depth of one or larger, while escaping from the neutron star.

Because of the asymptotic behavior of the function  $C_2(\xi)$ , even in immensely large fields, photons with energies less than 37 keV have opacities less than  $(10 \text{ km})^{-1}$ . This energy corresponds to a temperature of  $4 \times 10^8$  K, so we must conclude that unless the strong magnetic field of the neutron star extends over a distance much greater than 10 km, photon splitting affects the thermal radiation of only the youngest neutron stars.

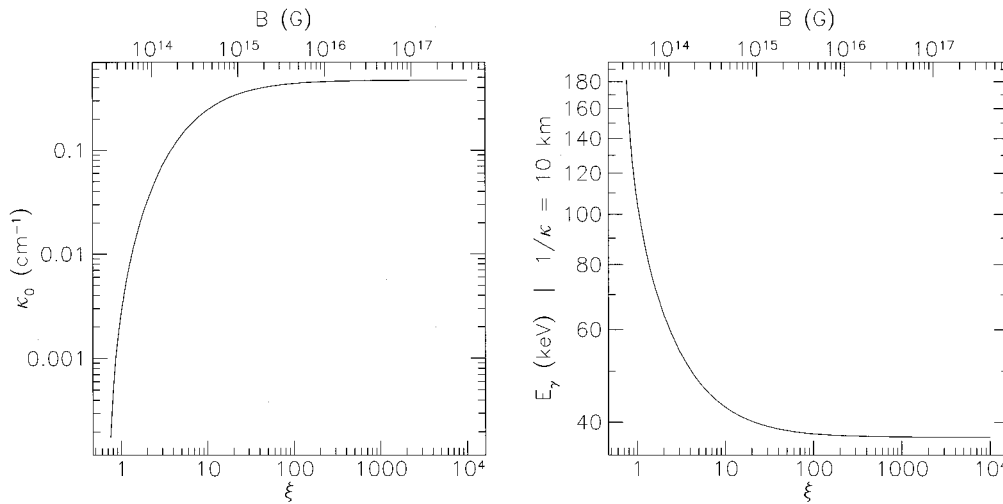


FIG. 2. The left panel depicts the photon-splitting opacity for photons with  $E=mc^2$  as a function  $\xi$  and  $B$  (upper axis). The right panel shows the energy of photons with a mean-free path for splitting of 10 km as a function  $\xi$  and  $B$ .

## VI. CONCLUSION

We have derived a closed form expression for the Heisenberg-Euler effective Lagrangian for quantum electrodynamics as a function of the gauge and Lorentz invariant quantities  $I$  and  $K$  in the limit of small  $K$ . We have calculated from this analytic expression the photon-splitting and pair-production rates in the intense field and found them to agree with previous work. Furthermore, the expressions for the dielectric and permeability tensors in an external field derived from our analytic expression also agree with previous results [18]. We expect that these expressions may be

applied to a wide variety of problems in strong electromagnetic fields, including Compton scattering, photon-photon scattering, and bremsstrahlung.

## ACKNOWLEDGMENTS

This material is based upon work supported by the National Science Foundation Graduate program. L.H. thanks the National Science Foundation for support under the Presidential Faculty Fellows Program.

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