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Non-Abelian eikonals

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A functional formulation and partial solution is given of the non-Abelian eikonal problem associated with the exchange of noninteracting, charged or colored bosons between a pair of fermions, in the large s small t limit. A simple, functional ''contiguity'' prescription is devised for extracting those terms which exponentiate, and appear to generate the leading, high-energy behavior of each perturbative order of this simplest non-Abelian eikonal function; the lowest nontrivial order agrees with the corresponding SU(N) perturbative amplitude, while higher-order contributions to this eikonal generate an ''effective Reggeization'' of the exchanged bosons, resembling previous results for the perturbative amplitude. One exact and several approximate examples are given, including an application to self-energy radiative corrections. In particular, for this class of graphs and to all orders in the coupling, we calculate the leading-log eikonal for SU(2). Based on this result, we conjecture the form of the eikonal scattering amplitude for SU(N). [S0556-2821(97)08804-8]

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I. INTRODUCTION

One of the most persistent problems in the application of field theory methods to particle scattering has been the inability to generalize, in a direct functional, nonperturbative way, Abelian eikonal models to their non-Abelian counterparts. Many efforts in this direction have of course been made over the past several decades, using the partial, perturbative summation of an eikonal function [1], or a variety of nonperturbative approximations [2]. In Ref. [2], for example, a ''mean-field'' approximation was made to the relevant functional integrals corresponding to the exchange of neutral vector mesons (NVM's) between scattering nucleons, which include the restrictions of SU(2) isospin; and the result, while ''approximately correct,'' left a certain unease in its wake. A more modern example is the problem of how to include SU(3) color restrictions in four-dimensional QCD, which must be faced if one is to attempt any functional calculation using the recent, exact and approximate Green's functions $G_c(x,y|A)$ of that theory [3]; or, indeed, the new dimensional-transmutation–flux-string expansion of quark-quark scattering amplitudes [4].

We give in this paper a complete, if formal, representation of the simplest non-Abelian eikonal, corresponding to multiple gluon exchange between scattering quarks *without* virtual gluon-gluon interactions; we extract that portion which can be easily isolated, and define a particular, ordered-exponential (OE) representation of the remainder, which can be expanded or approximated in various ways. In particular, we define a simple, functional procedure called ''contiguity,'' which, in an immediate way, isolates at least a subset of those terms that are definitely exponentiated, and can be represented to all orders by a perturbative expansion of the eikonal. These terms correspond to the leading s dependence in the lowest, nontrivial order, and we argue that they correspond to the extraction of the leading s dependence in every perturbative order of the eikonal function. For quark-quark scattering, the result duplicates the essence of well-known, leading-log perturbative estimates previously calculated for amplitudes [1]. The method will be illustrated in two contexts, and its applicability discussed for more general, non-Abelian problems; in particular, for this class of graphs and to all orders in the coupling, we calculate the leading-log eikonal of SU(2).

To our knowledge this is the first time that such estimates have been obtained in a purely functional context, while the contiguity technique opens the way for an attack on other, more complicated, non-Abelian eikonal problems, such as those which involve virtual gluon-gluon interactions (in particular, the so-called ''towers'' and their generalizations), as well as self-energy and vertex effects of non-Abelian, virtual-gluon emission and absorption by a single quark. However, by treating only boson exchange, without self-interactions between the exchanged bosons, we are apparently going to violate requirements of gauge invariance, which for perturbative, Yang-Mills gluons, require the simultaneous computation of all relevant graphs of a given order, and not just the simple eikonal graphs considered here. Surely the same sort of inclusion must eventually be true for any nonperturbative attempt. We ask the reader to suspend judgment on this point until the final discussion presented in the summary of Sec. VI; and to realize that, while gauge invariance must of course be insured in any computation whose results are going to be compared with experiment, we are proposing a functional attack on that part of the problem of immediate concern to the scattering quarks. This is important because a functional treatment contains all powers of the coupling; and it is useful because there exists an additional, computational step by which gauge invariance can be reestablished—including the relevant contributions generated by all gluon-gluon interactions—later on. The main thrust of the present paper is the functional extraction of leading-log, energy dependence of the simplest, non-Abelian eikonal.

We begin at that stage of a quark-quark scattering amplitude where mass-shell amputation (MSA) has already been carried out on the fermion Green's functions $\langle p_{1,2}|G_c[A]|p'_{1,2}\rangle$ approximated in a no-recoil fashion [5], and the essential structure of the eikonal function which describes non-Abelian (NVM) exchange between a pair of fermions [quarks, for SU(3)] has been recognized [6] as

$$e^{i\chi}=\exp\left(-i\int\frac{\delta}{\delta A_{\rm I}}\,Q\,\frac{\delta}{\delta A_{\rm II}}\right)\left[\exp\left(-ig_1\int_{-\infty}^{+\infty}ds p_1^{\mu}A^{a}_{{\rm I}\mu}(z_1-sp_1)\lambda_a^{\rm I}\right)\right]_+ \times\left[\exp\left(-ig_2\int_{-\infty}^{+\infty}dt p_2^{\nu}A^{b}_{{\rm II}\nu}(z_2-tp_2)\lambda_b^{\rm II}\right)\right]_{+|A_{\rm I}=A_{\rm II}=0,g_1=g_2=g}, \tag{1.1}$$

where $z_{1,2}$ and $p_{1,2}$ are the fermions' configuration and momentum coordinates, and $Q^{ab}_{\mu\nu}$ is the appropriate boson propagator. Equation (1.1) defines ''linkages'' between a pair of OE's, and the result will necessarily be a ''doubly ordered exponential.'' How this can be transformed into a pair of single OE's, how the leading logs of the latter may be extracted, leaving but a single OE, and how that OE can, for SU(2), be summed explicitly over all perturbative orders, is the main content of this paper.

More precisely, the preferred method of obtaining the eikonal in the conventional case, where the conventional, no-recoil approximation of $G_c[A]$ destroys coordinate symmetry of this Green's function, is to calculate not $T_{\rm eik}$ but, before MSA:

$$\frac{\partial^2 T_{\rm eik}}{\partial g_1\partial g_2}=\frac{i}{g_1g_2}\,\frac{\delta}{\delta\phi(0)}\,\frac{\delta}{\delta\psi(0)}\,e^{-i\int\delta/\delta A_{\rm I}Q\,\delta/\delta A_{\rm II}}\left[\exp\left(-ig_1\int_{-\infty}^{+\infty}ds\,\phi(s)p_1\cdot A_{\rm I}(z_1-sp_1)\cdot\lambda^{\rm I}\right)\right]_+ \times\left[\exp\left(-ig_2\int_{-\infty}^{+\infty}dt\,\psi(t)p_2\cdot A_{\rm II}(z_2-tp_2)\cdot\lambda^{\rm II}\right)\right]_{+|\phi(s)=\psi(s)=1,A_{\rm I}=A_{\rm II}=0} \tag{1.2}$$

and integrate over $g_{1,2}$ [with the boundary conditions $T_{\rm eik}(g_1,0)=T_{\rm eik}(0,g_2)=0$] after the necessary functional linkages have been performed [6]; it has been assumed that the RHS of Eq. (1.2) is a function of z_1-z_2, and the subsequent $\delta^{(4)}(q_1+q_2)$ statement of four-momentum conservation has been suppressed. For simplicity we consider the quantity of Eq. (1.1) as representative of the correct eikonal—it is exactly the eikonal in the absence of non-Abelian complications—even though it is quite possible to produce, upon integration over the couplings of Eq. (1.2), combinations which are more complicated than that of Eq. (1.1). However, Eq. (1.1) is representative of the full, non-Abelian structure of the problem, and we here restrict attention to this quantity. The noncommuting objects λ_a are taken to be the Gell-Mann matrices of SU(N). We again emphasize that more complicated eikonal graphs, such as the ''tower graphs'' of Cheng and Wu [1], are not included in this analysis, although they can be formally inserted by the functional methods outlined in the last chapters of Refs. [1] and [5].

In the Abelian case, where $A^a_\mu\to A_\mu$, and the λ_a are missing, the functional operation of Eq. (1.1) may be performed immediately, yielding

$$i\chi=ig^2(p_1\cdot p_2)\int\int_{-\infty}^{+\infty}ds\,dt\,\Delta_c(z_1-z_2-sp_1+tp_2) \tag{1.3}$$

with a propagator $Q_{\mu\nu}(x_1,x_2)=\delta_{\mu\nu}\Delta_c(x_1-x_2)$. The proper-time integrals are easily performed when a Fourier representation of Δ_c is inserted into Eq. (1.3), and one finds

$$i\chi=-i\,\frac{g^2}{2\pi}\,\gamma(s)K_0(\mu b), \tag{1.4}$$

where $\gamma(s)=(s-2m^2)/\sqrt{s(s-4m^2)}$ is that factor depending on the spin of the exchanged boson, of mass μ; the fermion mass is denoted by m, and in this equation, s denotes the total c.m. (energy)^2 of the two quarks. In all subsequent expressions, we shall assume the high-energy limit, where $\gamma(s)\to 1$.

We give in the next section a new, functional formulation of the eikonal of Eq. (1.1), and, in an appropriate kinematical situation, display one exact solution. More generally, a perturbative expansion of this eikonal functional may be defined, and certain obvious terms (which are the most elementary generalizations of the Abelian eikonal) are summed to all orders. In Sec. III, we define the statement of ''functional contiguity,'' which isolates those terms of Eq. (1.1) that are definitely exponentiated, and which appears to generate the leading $\ln(E/m)$ dependence of every perturbative term of the non-Abelian eikonal, when the necessary, doubly ordered exponential is defined in a moderately elegant way. In the next section, we discuss the leading-log approximation, and show how the extraction of such terms (from ''nested'' momentum integrals) can reduce the complexity of the computations to operations upon a single OE; for SU(2), these operations are performed and summed to all orders, and suggest a conjecture for the corresponding eikonal scattering amplitude of SU(N). In Sec. V, we apply the analysis to self-energy processes, as well as to eikonal tower graphs and their generalizations, while Sec. VI contains a summary of our present understanding of this eikonal construction.

II. FORMULATION

In order to perform the functional operation of Eq. (1.1), it is useful to introduce for each OE the functional representation:

$$\left[\exp\left(-ig\int_{-\infty}^{+\infty}ds\,p_\mu A_\mu^a(z-sp)\lambda_a\right)\right]_+ = N'\int d[\alpha]\int d[u]\exp\left(i\int_{-\infty}^{+\infty}ds\,\alpha_a(s)[u_a(s) - gp_\mu A_\mu^a(z-sp)]\right)\left[\exp\left(i\int_{-\infty}^{+\infty}ds\,\lambda_a u_a(s)\right)\right]_+ \tag{2.1}$$

or, more simply, rewriting Eq. (2.1) as $I\otimes \exp[-i\int_{-\infty}^{+\infty}ds\, p_\mu A_\mu^a(z-sp)\alpha_a(s)]$, where N' is an appropriate normalization constant. That Eq. (2.1) is trivially true can be seen by breaking up the $-\infty<s<+\infty$ range into small intervals, and integrating over the $\alpha_a(s_t)$ which leads to a δ functional of the $u_a(s)$, whose integration immediately produces the left-hand side (LHS) of Eq. (2.1). The advantage of this procedure is that the functional linkages of Eq. (1.1) are now Abelian, and may be performed immediately, yielding

$$e^{i\chi}=I_1\otimes\cdot I_2\otimes\exp\left[i\int\int_{-\infty}^{+\infty}ds\;dt\,\alpha_a(s)Q_{ab}(s,t)\beta_b(t)\right] \tag{2.2}$$

with $Q_{a,b}=g^2p_1^{\mu}Q_{\mu\nu}^{ab}p_2^{\mu}$, and where the $I_{1,2}$ denote, from Eq. (2.1), simultaneous functional operations to be performed on the $\alpha_a(s)$ and $\beta_b(t)$ variables.

These final operations are what is now needed, and may be delineated by the insertion of relevant source and parameter dependence, followed by a ‘‘Schwingerian search’’ for an appropriate ‘‘differential characterization.’’ With the definition

$$R(s,t|\xi,\eta)=N'\int d[\alpha]\int d[u]\exp\left(i\int_{-\infty}^{+\infty}\alpha\cdot u\right)\left[\exp\left(i\int_{-\infty}^{s}\lambda^{\rm I}\cdot u\right)\right]_+\exp\left(i\int_{-\infty}^{+\infty}u\cdot\xi\right)N'\int d[\beta]\int d[v]$$
$$\times\exp\left(i\int_{-\infty}^{+\infty}\beta\cdot v\right)\left[\exp\left(i\int_{-\infty}^{t}\lambda^{\rm II}\cdot v\right)\right]_+\exp\left(i\int_{-\infty}^{+\infty}v\cdot\eta\right)\exp\left[i\int\int_{-\infty}^{+\infty}ds'\,dt'\,\alpha_a(s')Q_{ab}(s',t')\beta_b(t')\right], \tag{2.3}$$

comparison with Eq. (2.2) shows that the quantity needed is $\ln R(+\infty,+\infty|,0,0)$. One can create a variety of differential equations involving the proper-time parameters s,t and the sources $\xi_a(s),\eta_b(t)$; but for present purposes, it seems to be sufficient to work with only s and η, so that we consider $R(s,+\infty|0,\eta)=R(s|\eta)$.

We next outline the steps which result in the differential equation (2.6), stated below. Calculation of $(\partial/\partial s)R(s|\eta)$ brings down under the integrals the quantity $i\lambda_a^{\rm I}u_a(s)$, standing to the left of its OE, which may be represented as $\lambda_a^{\rm I}(\delta/\delta\alpha_a)(s)$ acting upon $\exp[i\int\alpha\cdot\xi]$; then a functional integration-by-parts moves this $\delta/\delta\alpha_a(s)$ to act upon the last line of Eq. (2.3), which generates under the functional integrals the net quantity $(-i)\int_{-\infty}^{+\infty}dt\lambda_a^{I}Q_{ab}(s,t)\beta_b(t)$. The procedure may now the reversed, representing $(-i)\beta_b(t)$ by the operation $-\delta/\delta v_b(t)$ acting upon $\exp[i\int v\cdot\beta]$; and using another functional integration-by-parts to convert this to the operation

$$\frac{\delta}{\delta v_b(t)}\left[\left(\exp\left(i\int_{-\infty}^{+\infty}\lambda^{\rm II}v\right)\right)_+\exp\left(i\int v\cdot\eta\right)\right]=i\left[\eta_b(t)+\left(\exp\left(i\int_t^{\infty}\lambda^{\rm II}\cdot v\right)\right)_+\lambda_b^{\rm II}\left(\exp\left(-i\int_t^{\infty}\lambda^{\rm II}v\right)\right)_-\right]$$
$$\times\left(\exp\left(i\int_{-\infty}^{+\infty}\lambda^{\rm II}\cdot v\right)\right)_+\exp\left(i\int v\cdot\eta\right) \tag{2.4}$$

written in terms of the antiordered quantity

$$\left[\exp\left(-i\int_t^{\infty}\lambda^{\rm II}\cdot v\right)\right]_-=\left[\left(\exp\left(i\int_t^{\infty}\lambda^{\rm II}\cdot v\right)\right)_+\right]^{\dagger} =\left[\left(\exp\left(i\int_t^{\infty}\lambda^{\rm II}\cdot v\right)\right)_+\right]^{-1}.$$

We introduce the notation

$$\Lambda_b^{\rm II}(t|iv)=\left[\exp\left(i\int_t^{\infty}\lambda^{\rm II}\cdot v\right)\right]_+\lambda_b^{\rm II}\left[\exp\left(-i\int_t^{\infty}\lambda^{\rm II}\cdot v\right)\right]_- \tag{2.5}$$

and observe that Eq. (2.4) may be rewritten as

$$i[\eta_b(t)+\Lambda_b^{\rm II}(t|\delta/\delta\eta)]\left[\exp\left(i\int_{-\infty}^{+\infty}\lambda^{\rm II}\cdot v\right)\right]_+\exp\left(i\int v\cdot\eta\right)$$

so that, finally, one obtains the differential equation

$$\frac{\partial R(s|\eta)}{\partial s} = i\int_{-\infty}^{+\infty} dt\ \lambda_a^{\mathrm{I}} Q_{ab}(s,t) \times\left[\eta_b(t) + \Lambda_b^{\mathrm{II}}\left(t\left|\frac{\delta}{\delta\eta}\right.\right)\right]\cdot R(s|\eta). \quad (2.6)$$

With the boundary condition $R(-\infty|\eta)=1$, easily seen as appropriate from the definition of $R(s|\eta)$, the solution to Eq. (2.6) may be written as an OE

$$R(s,t) = \left(\exp\left[i\int_{-\infty}^{s} ds' \int_{-\infty}^{+\infty} dt' \lambda_a^{\mathrm{I}} Q_{ab}(s',t')\left[\eta_b(t') + \Lambda_b^{\mathrm{II}}\left(t'\left|\frac{\delta}{\delta\eta}\right.\right)\right]\right]\right)_{+s'} \quad (2.7)$$

with the ordering indicated for the s' variables only. With $s\to+\infty$ and $\eta\to 0$, we then have a representation of Eq. (2.3) which apparently involves a single OE; however, it should be noted that the second ordering will be found in the definition of Λ_b^{II}, Eq. (2.5), so that there do exist two sets of "orderings," although they can now be addressed separately. In fact, the "t orderings" can be defined from the integral solution to the differential equation satisfied by $\Lambda_a^{\mathrm{II}}(t|\delta/\delta\eta)$; the latter may immediately be obtained from its definition (2.5):

$$\frac{\partial}{\partial t}\Lambda_a^{\mathrm{II}}\left(t\left|\frac{\delta}{\delta\eta}\right.\right) = 2if_{acd}\frac{\delta}{\delta\eta_c(t)}\Lambda_d^{\mathrm{II}}\left(t\left|\frac{\delta}{\delta\eta}\right.\right)$$

which, together with the boundary condition at $t=\infty$, generates

$$\Lambda_a^{\mathrm{II}}\left(t\left|\frac{\delta}{\delta\eta}\right.\right) = \lambda_a^{\mathrm{II}} - 2if_{acd}\int_t^{\infty} dt' \frac{\delta}{\delta\eta_c(t')}\Lambda_a^{\mathrm{II}}\left(t'\left|\frac{\delta}{\delta\eta}\right.\right) \quad (2.8)$$

whose repeated iterations contain all the t orderings of the problem, and where the f_{abc} are the structure constants of the SU(N) algebra.

Conventional eikonal models replace $Q_{a,b}$ by $\delta_{a,b}Q(s,t)$, and in the absence of any other isospin-color vector, we may expect that the result will generate the products $\lambda^{\mathrm{I}}\cdot\lambda^{\mathrm{II}}$. The latter may then be replaced by eigenvalues appropriate to the scattering problem; for example, in the SU(2) isospin scattering of two nucleons, those eigenvalues are given by $I(I+1)/2-3/4$, for singlet ($I=0$) or triplet ($I=1$) total isospin; for SU(3), the situation is somewhat more complicated, as one tries to extract the overall, contribution of the eikonal to the singlet scattering amplitude [1].

While Eq. (2.7) is a formal solution of the problem, certain terms of its expansion can be summed without difficulty. To see this, consider the expansion of Eq. (2.7) up to quadratic Q dependence:

$$\begin{aligned} R|_{s\to\infty} \simeq{}& 1 + i\int\int_{-\infty}^{+\infty} ds'dt'\lambda_a^{\mathrm{I}}Q_{ab}(s',t')\left[\eta_b(t') + \Lambda_b^{\mathrm{II}}\left(t'\left|\frac{\delta}{\delta\eta}\right.\right)\right] \\ &+ i^2\int\int_{-\infty}^{+\infty} ds_1dt_1\lambda_a^{\mathrm{I}}Q_{a_1b_1}(s_1,t_1)\int_{-\infty}^{s_1}ds_2\int_{-\infty}^{+\infty}dt_2\lambda_{a_2}^{\mathrm{I}}Q_{a_2,b_2}(s_2,t_2) \\ &\times\left[\eta_{b_1}(t_1) + \Lambda_{b_1}^{\mathrm{II}}\left(t_1\left|\frac{\delta}{\delta\eta}\right.\right)\right]\cdot\left[\eta_{b_2}(t_2) + \Lambda_{b_2}^{\mathrm{II}}\left(t_2\left|\frac{\delta}{\delta\eta}\right.\right)\right]_{n\to 0} + \cdots. \end{aligned} \quad (2.9)$$

With the definition of $\Lambda_b^{\mathrm{II}}(t|\delta/\delta\eta)$, it is clear that the only contribution of the linear Q terms is

$$i\int\int_{-\infty}^{+\infty} ds dt \lambda_a^{\mathrm{I}} Q_{ab}(s,t)\lambda_b^{\mathrm{II}}, \quad (2.10)$$

while the $\delta/\delta\eta$-independent part of the quadratic Q terms of Eq. (2.9) yields

$$i^2\int\int_{-\infty}^{+\infty} ds_1dt_1\lambda_{a_1}^{\mathrm{I}}Q_{a_1b_1}(s_1,t_1)\lambda_{b_1}^{\mathrm{II}}\int_{-\infty}^{s_1}ds_2 \times\int_{-\infty}^{+\infty}dt_2\lambda_{a_2}^{\mathrm{I}}Q_{a_2b_2}(s_2,t_2)\lambda_{b_2}^{\mathrm{II}}. \quad (2.11)$$

This structure, obtained from the first term, λ_b^{II}, in the iterative expansion of Λ_b^{II}, Eq. (2.8):

$$\Lambda_b^{\mathrm{II}}\left(t\left|\frac{\delta}{\delta\eta}\right.\right) \simeq \lambda_b^{\mathrm{II}} - 2if_{bcd}\lambda_d^{\mathrm{II}}\int_t^{\infty}dt'\frac{\delta}{\delta\eta_c(t')} + \cdots \quad (2.12)$$

will appear in every term of the complete expansion of R, and generates the OE:

$$\left(\exp\left[i\int\int_{-\infty}^{+\infty} ds\ dt\lambda_a^{\mathrm{I}}Q_{ab}(s,t)\lambda_b^{\mathrm{II}}\right]\right)_{+(s)}. \quad (2.13)$$

If, as typical, $Q_{ab}=\delta_{ab}Q(s,t)$, all the $\lambda_a^{\mathrm{I}}\cdot\lambda_b^{\mathrm{II}}$ terms in the expansion of Eq. (2.13) combine to form the products $\lambda^{\mathrm{I}}\cdot\lambda^{\mathrm{II}}$, at which point the OE becomes an ordinary exponential (oe):

$$\exp\left[i(\lambda^{\mathrm{I}}\cdot\lambda^{\mathrm{II}})\int\int_{-\infty}^{+\infty} ds\ dtQ(s,t)\right], \quad (2.14)$$

where the combination $\lambda^{\rm I}\cdot\lambda^{\rm II}$ may be replaced by its appropriate eigenvalue. The value of the integrals of Eq. (2.14) may be read off from Eqs. (1.3) and (1.4).

It is instructive to continue with the example of Eq. (2.9) and calculate the first commutator term, as in Eq. (2.12), to this quadratic Q dependence; it is

$$2if_{b_1b_2d}\int\int_{-\infty}^{+\infty}ds_1dt_1\lambda^{\rm I}_{a_1}Q_{a_1b_1}(s_1,t_1)\int_{-\infty}^{s_1}ds_2 \times\int_{t_1}^{\infty}dt_2\lambda^{\rm I}_{a_2}Q_{a_2b_2}(s_2,t_2)\lambda^{\rm II}_d. \tag{2.15}$$

If, again $Q_{a,b}=\delta_{a,b}Q(s,t)$, the antisymmetry of Eq. (2.15) under b_1,b_2 exchange is converted to a like antisymmetry under a_1,a_2 exchange, so that the pair $\lambda^{\rm I}_{a_1}\lambda^{\rm I}_{a_2}$ may be replaced by $if_{a_1a_2c}\lambda^{\rm I}_c$. One then finds the double summation $\Sigma_{a_1a_2}f_{a_1a_2c}f_{a_1a_2d}=C_2\delta_{cd}$, where $C_2(N)=N$ denotes the value of the quadratic Casimir invariant of the adjoint representation; and Eq. (2.15) becomes

$$-2C_2(\lambda^{\rm I}\cdot\lambda^{\rm II})\int\int_{-\infty}^{+\infty}ds\ dtQ(s,t) \times\int_{-\infty}^{s}ds_1\int_t^{\infty}dt_1Q(s_1,t_1). \tag{2.16}$$

In a typical eikonal situation corresponding to NVM exchange, $Q(s,t)=g^2(p_1\cdot p_2)\Delta_c(z_1-z_2-sp_1+tp_2)$, and the integrals of Eq. (2.16) may be evaluated to yield the leading $\ln(E/m)$ dependence:

$$i\,\frac{C_2}{2\pi}\left(\frac{g^2}{\pi}\right)^2(\lambda^{\rm I}\cdot\lambda^{\rm II})\ln(E/m)K_0^2(\mu b), \tag{2.17}$$

where $4E^2$ denotes the total c.m. (energy)^2 of the scattering quarks. The form of Eq. (2.17) is worth noting, for it contains the new feature of a $\ln(E/m)$ dependence multiplying reasonable, impact-parameter dependence; as explained in great detail in Ref. [1], it is the first appearance of an effective Reggeization of the exchanged gluon, and it appears directly in the eikonal function.

Before discussing how such contributions may be extracted and summed in this functional context, it may be appropriate to note that there is at least one kinematical context in which Eq. (2.13) is the exact result. This is the special case where $Q_{a,b(s,t)}=Q_{a,b}(s)\delta(s-t)$, when the functional derivatives of Eq. (2.12) can never appear (due to a misordering of subsequent, proper-time variables).

Another example where differences may be expected from the usual eikonal forms results from the appearance of a $Q_{a,b}=f_{abc}\xi_cQ(s,t)$, where ξ_c is a color vector in the flux-string model of Ref. [4]. Because this Q is proportional to a delta function of the square of the x_1-x_2 variables of Eq. (1.3), it produces an OE with only s dependence, and the kinematical forms which appear are quite different from the examples noted above.

Other formulations of the solution to Eq. (2.6) are possible, such as the representation of $R(s|\eta)$ by a Fourier functional transform, and the subsequent conversion of Eq. (2.6) to a differential equation linear in parametric and functional derivatives. However, because of the noncommutation of the λ_a, this route does not seem to lead to any real simplification.

III. CONTIGUITY

A representation for the general structure of all such terms may be obtained by the following argument. Return to the differential equation (2.6) for $R(s|\eta)$ and make the ansatz: $R=R_0U_0$, where we shall assume in all that follows that $Q_{a,b}=\delta_{a,b}Q(s,t)$. The quantity $R_0(s)$ is defined by

$$R_0(s)\equiv\left(\exp\left[i\int_{-\infty}^{s}ds'\int_{-\infty}^{+\infty}dt'\lambda^{\rm I}_aQ(s',t') \times[\eta_a(t')+\lambda^{\rm II}_a]\right]\right)_{+(s')} \tag{3.1}$$

and substitution of Eq. (3.1) into Eq. (2.6) then yields

$$\frac{\partial U_0}{\partial s}=i\int_{-\infty}^{+\infty}dt\ R_0^{-1}(s)\lambda^{\rm I}_aQ(s,t) \times\Delta\Lambda^{\rm II}_a\left(t\left|\ \frac{\delta}{\delta\eta}\right.\right)R_0(s)\cdot U_0(s),$$

$$\Delta\Lambda^{\rm II}_a=\Lambda^{\rm II}_a-\lambda^{\rm II}_a\ ,$$

with the solution

$$U_0(s)=\left(\exp\left[i\int_{-\infty}^{s}ds'\int_{-\infty}^{+\infty}dt'R_0^{-1}(s')\lambda^{\rm I}_aQ(s',t') \times\Delta\Lambda^{\rm II}_a\left(t'\left|\ \frac{\delta}{\delta\eta}\right.\right)R_0(s')\right]\right)_{+(s')} \tag{3.2}$$

from which we require the limits $s\to\infty$, $\eta\to 0$. To quadratic order in Q, one finds that the expansion of U_0 generates Eq. (2.16), as it must; but because of the R_0 factors inside the OE of Eq. (3.2), higher-order terms will, at least in part, involve commutators of the λ dependence of R_0 with neighboring $\lambda^{\rm I},\lambda^{\rm II}$ dependence of Eq. (3.2); those terms will be different from the simple exponentiation of Eq. (2.16), but they will always be of higher perturbative order than that of Eq. (2.16), and are not the leading terms of their own perturbative order. Note that the combination $\Delta\Lambda^{\rm II}_a$ of Eq. (3.2) contains all the multiple commutators, indicated in Eq. (2.12), whose functional derivatives act upon the η dependence of R_0.

To find that term in the eikonal of order $g^{2(n+1)}$ which is the leading term of that order, let us now write

$$\Delta\Lambda^{\rm II}_a\left(t\left|\ \frac{\delta}{\delta\eta}\right.\right)\equiv\sum_{n=1}^{\infty}\Delta_n\Lambda^{\rm II}_a\left(t\left|\ \frac{\delta}{\delta\eta}\right.\right), \tag{3.3}$$

$$\Delta_n\Lambda_a^{\rm II}\left(t\left|\frac{\delta}{\delta\eta}\right.\right)=(-2i)^n f_{ac_1d_1}f_{d_1c_2d_2}\cdots f_{d_{n-1},c_nd_n}\lambda_{d_n}^{\rm II}\times\int_t^\infty dt_1\int_{t_1}^\infty dt_2\cdots\int_{t_{n-1}}^\infty \times dt_n\,\frac{\delta}{\delta\eta_{c_1}(t_1)}\cdots\frac{\delta}{\delta\eta_{c_n}(t_n)}$$

and set $U_0=R_1U_1$, where we define

$$R_1(s)=\left(\exp\left[i\int_{-\infty}^s ds'\int_{-\infty}^{+\infty}dt' R_0^{-1}(s')\lambda_a^{\rm I}Q(s',t')\times\Delta_1\Lambda_a^{\rm II}\left(t'\left|\frac{\delta}{\delta\eta}\right.\right)R_0(s')\right]\right)_{+(s')}.$$

Then, by again solving the appropriate differential equation, we find

$$U_1(s)=\left(\exp\left[i\int_{-\infty}^s ds'\int_{-\infty}^{+\infty}dt'[R_0(s')R_1(s')]^{-1}\times\lambda_a^{\rm I}Q(s',t')\sum_{n=2}^\infty\Delta_n\Lambda_a^{\rm II}\left(t'\left|\frac{\delta}{\delta\eta}\right.\right)\times[R_0(s')R_1(s')]\right]\right)_{+(s')}.$$

Performing this operation sequentially, it is clear that the general structure of the result may be written as

$$R(s|\eta)=R_0(s)\cdot R_1(s)\cdots R_n(s)\cdot U_n(s)\equiv[s_n]U_n(s),$$

where

$$R_n(s)=\left(\exp\left[i\int_{-\infty}^s ds'\int_{-\infty}^{+\infty}dt'[s']_{n-1}^{-1}\lambda_a^{\rm I}Q(s',t')\cdot\Delta_n\Lambda_a^{\rm II}\left(t'\left|\frac{\delta}{\delta\eta}\right.\right)[s']_{n-1}\right]\right)_{+(s')}\tag{3.4}$$

and

$$U_n(s)=\left(\exp\left[i\int_{-\infty}^s ds'\int_{-\infty}^{+\infty}dt'[s']_n^{-1}\lambda_a^{\rm I}Q(s',t')\cdot\sum_{\ell=n+1}^\infty\Delta_\ell\Lambda_a^{\rm II}\left(t'\left|\frac{\delta}{\delta\eta}\right.\right)[s']_n\right]\right)_{+(s')}.\tag{3.5}$$

Because each functional derivative $\delta/\delta\eta$ will generate a term [when operating on $R_0(s)$] proportional to $Q\sim g^2\Delta_c$, the log of R_n contains all powers of g^{2m}, with $m\geqslant n+1$. The lowest order term, with $m=n+1$, will contain the largest power of $\ln^n(E/m)$, while higher-order terms constructed from the same R_n will have no higher-order log; rather, the terms containing $\ln^m(E/m)$, $m>n+1$, will come from the corresponding, lowest-order terms of R_m.

In order to define ''contiguity,'' imagine that R_n is expanded in powers of g^2, by expanding its OE:

$$R_n|_{s\to\infty}\simeq 1+i\int\int_{-\infty}^{+\infty}ds\;dt[s]_{n-1}^{-1}\lambda_a^{\rm I}Q_{ab}(s,t)\Delta_n\Lambda_a^{\rm II}\left(t\left|\frac{\delta}{\delta\eta}\right.\right)[s]_{n-1}+i^2\int\int_{-\infty}^{+\infty}ds\;dt[s]_{n-1}^{-1}\lambda_a^{\rm I}Q(s,t)\Delta_n\Lambda_a^{\rm II}\left(t\left|\frac{\delta}{\delta\eta}\right.\right)[s]_{n-1}\times\int_{-\infty}^s ds_1\int_{-\infty}^{+\infty}dt_1[s_1]_{n-1}^{-1}\lambda_{a_1}^{\rm I}Q(s_1,t_1)\Delta_n\Delta_{a_1}^{\rm II}\left(t_1\left|\frac{\delta}{\delta\eta}\right.\right)[s_1]_{n-1}+\cdots,\tag{3.6}$$

where $[s]_{n-1}=R_0(s)\ldots R_{n-1}(s)$. ''Contiguity'' suggests that the leading dependence of $\ln(R_n)$ will be obtained if each $\Delta_n\Lambda_{a_j}^{\rm II}(t_j|\delta/\delta\eta)$ operates directly upon the $[S_j]_{n-1}$ factor contiguous to it, that is, immediately to its right. This can be seen in the simplest, nontrivial terms of order g^4 and g^6, and, we subsequently argue, is true for all terms; however, what is clear from this definition is that terms contributing to each order of the contiguity operation can be summed and calculated directly from the OE form of R_n, writing

$$R_n|_{s\to\infty}=\left(\exp\left[i\int\int_{-\infty}^{+\infty}dsdt\;[s]_{n-1}^{-1}\lambda_a^IQ(s,t)\underbrace{\Delta_n\Lambda_a^{II}\left(t\left|\frac{\delta}{\delta\eta}\right.\right)[s]_{n-1}}\right]\right)_{+(s)}\tag{3.7}$$

where the factor-pairing notation is meant to express the subset of terms extracted by contiguity.

The entire g^{2n} dependence of the eikonal, that is, of $\ln(R)$, can be obtained by considering the following sequence of ascending powers of g^2, in the limit of $s\to\infty$, $\eta\to 0$: All g^2 dependence is given by R_0, $\ln(R_0)=i(\lambda^{\rm I}\cdot\lambda^{\rm II})\int\int_{-\infty}^{+\infty}ds\;dtQ(s,t)$; all $(g^2)^2$ dependence is given by the contiguity calculation of R^1, which generates our previous result,

$$\ln(R_1)=-2C_2(\lambda^I\cdot\lambda^{II})\int\int_{-\infty}^{+\infty}dsdtQ(s,t)\times\int_{-\infty}^s ds_1\int_t^\infty dt_1Q(s_1,t_1);$$

all $(g^2)^3$ dependence is given by the contiguity calculation of R_2, and by the g^2 expansion of the $[s]_0^{-1}$ and $[s]_0$ factors of R_1; all $(g^2)^4$ dependence is given by the contiguity calculation of R_3, by the g^2 expansion of the $[s]_1^{-1}$ and $[s]_1$ factors of R_2, and by the g^4 expansion of the $[s]_0^{-1}$ and $[s]_0$ factors of R_0; etc.

In this way, one constructs the complete $g^{2(n+1)}$ dependence of $\ln(R)=\ln(R_0\ldots R_n)$. Those exponential, eikonal terms obtained directly from contiguity will contain one or more terms proportional to a single factor of $\lambda^{\mathrm{I}}\cdot\lambda^{\mathrm{II}}$, while the g^{2p} expansions of the $[s_j]$ and $[s_i]^{-1}$ appear to generate more complicated group factors, similar to those found in the perturbative calculations of the amplitude [1]. We argue in the next section that the leading $\ln(E/m)$ dependence to the eikonal of order $g^{2(n+1)}$ comes only from the contiguity calculations of R_n, when the functional differentiation is performed only on the $R_0(s)$ factor of $[s]_n$. Using simple functional techniques, the sum of these leading contributions over all orders n is constructed for the eikonal of SU(2).

IV. LEADING LOGS

We here give a qualitative discussion of the leading $\ln(E/m)$ dependence of this class of non-Abelian eikonals (where, we again remind the reader, interacting gluons are not included). For this, consider first those terms of order $g^{2(n+1)}$ in the expression for $\ln(R_n)$ arising from the contiguity operation of $\Delta_n\Lambda_a^{\mathrm{II}}$ upon the factor $[s]_{n-1}$ standing to its immediate right, as in Eq. (3.7). In particular, the leading terms of that order will come from the $\Delta_n\Lambda_a^{\mathrm{II}}$ operation upon the $R_0(s)$ functional in $[s]_{n-1}$ [rather than the same-g^2-order contribution to the eikonal from $\ln(R_{n-1})$, with $\Delta_{n_1}\Lambda_a^{\mathrm{II}}$ acting upon $R_1(s)$ in $[s]_{n-2}$, etc].

For clarity, we carry the discussion through for $n=2$, and then generalize to arbitrary n; for the moment, we suppress the f_{abc} factors arising in the t-dependent iterations of $\Lambda_a^{\mathrm{II}}(t|\delta/\delta\eta)$, but we explicitly write the s-dependent permutations that are generated by the functional differentiation of $\Delta_2\Lambda_a^{\mathrm{II}}(t|\delta/\delta\eta)$ upon $R_0(s)$, which are proportional to

$$\int_t^\infty dt_1\,\frac{\delta}{\delta\eta_{c_1}(t_1)}\int_{t_1}^\infty dt_2\,\frac{\delta}{\delta\eta_{c_2}(t_2)}\times\left(\exp\left[i\int_{-\infty}^{s}ds'\int_{-\infty}^{+\infty}dt'\times Q(s',t')\lambda_a^{\mathrm{I}}\eta_a(t')\right]\right)_{+(s')|\eta\to 0}. \tag{4.1}$$

We have neglected in this $R_0(s)$ its exponential $i(\lambda^{\mathrm{I}}\cdot\lambda^{\mathrm{II}})\int_{-\infty}^{s}ds'\int_{-\infty}^{+\infty}dt'Q(s',t')$ dependence because, as explained below, it can only contribute to orders g^{2p}, $p>n+1$, and carries no additional $\ln(E/M)$ factors. Suppressing the superscript I for each λ_c^{I}, the functional operations of Eq. (4.1) yield

$$i^2\int_t^\infty dt_1\int_{t_1}^\infty dt_2\int_{-\infty}^{s}ds_1\int_{-\infty}^{s}ds_2 Q(s_1,t_1)Q(s_2,t_2)\times[\lambda_{c_1}\lambda_{c_2}\theta(s_1-s_2)+\lambda_{c_2}\lambda_{c_1}\theta(s_2-s_1)] \tag{4.2}$$

and suggest the obvious generalization for $n>2$ as

$$i^n\int_t^\infty dt_1\cdots\int_{t_{n-1}}^\infty dt_n\sum_{\mathrm{perm}}\int_{-\infty}^{s}ds_1\cdots\times\int_{-\infty}^{s_{n-1}}ds_n\cdot\lambda_{c_1}\cdots\lambda_{c_n}Q(s_1,t_1)\cdots Q(s_n,t_n), \tag{4.3}$$

in which the n c_1 indices are permuted, with a corresponding permutation of the s_i, in $n!$ different ways.

For our estimates of the $\ln(E/m)$ dependence, we use the standard Fourier propagator representation, $\Delta_c(x)=(2\pi)^{-4}\int d^4k(k^2+\mu^2-i\epsilon)^{-1}e^{ik-x}$, and (improperly) take the kinematic limits for each (mass-shell) quark: $E-p=0$, rather than the more accurate $E-p\simeq m^2/2E$. Any integral that we find containing an UV log divergence is really proportional to a corresponding factor of $\ln(E/m)$, which dependence appears when proper (but much more complicated) kinematics are used.

For $n=2$, let us examine both permutations, and include the $i\lambda_a^{\mathrm{I}}\int\int ds\ dtQ(s,t)$ factor of Eq. (3.7), whose $[s]_1^{-1}$ has been replaced by unity [because it can only contribute to higher orders with no corresponding increase in the number of $\ln(E/m)$ factors]. Each factor of Q carries with it $p_1\cdot p_2\sim E^2$ dependence, which is removed by the explicit E factors associated with the s and t integrations, in standard eikonal fashion; and we suppress all such canceling E dependence. With $Q(s,t)=g^2(p_1\cdot p_2)\Delta_c(z-sp_1+tp_2)$, where $z=z_1-z_2=(b,z_3,z_0)$ is the difference of configuration coordinates of the scattering quarks, the first of the two permutations of Eq. (4.1) will lead to

$$\int d^4\overline{k^{(+)}}e^{i\overline{k}z}\delta(\overline{k^{(+)}})\delta(\overline{k^{(-)}})\int d^2k_1\int d^2k_2\int dk_1^{(+)}\int dk_1^{(-)}\int dk_2^{(+)}\int dk_2^{(-)}$$
$$\times[\omega^2(\overline{k}-k_1-k_2)+(k_1^{(+)}+k_2^{(+)})(k_1^{(-)}+k_2^{(-)})-i\epsilon]^{-1}[\omega_1^2+k_1^{(+)}k_1^{(-)}-i\epsilon]^{-1}$$
$$\times[\omega_2^2+k_2^{(+)}k_2^{(-)}-i\epsilon]^{-1}(k_2^{(-)}+i\epsilon)^{-1}(k_1^{(-)}+k_2^{(-)}+i\epsilon)^{-1}(k_2^{(+)}-i\epsilon)^{-1}(k_1^{(+)}+k_2^{(+)}-i\epsilon)^{-1},$$

where $\overline{k}=k+k_1+k_2$, $k^{(\pm)}=k_3\pm k_0$, $\omega^2=\mu^2+k_\perp^2$, and $\omega^2(k-k_1-k_2)^2=\mu^2+(k-k_1-h_2)_\perp^2$, with $\perp$ components referring to the transverse 1,2 directions (the impact parameter vector b) in the c.m. of the scattering quarks; all momentum integrals run from $-\infty$ to $+\infty$. The $i\epsilon$ factors are important, and—aside from the $-i\epsilon$ of the standard Feynman propagator denominators—arise upon calculating $\int_{-\infty}^{s}ds_1\int_{-\infty}^{s_1}ds_2$ and $\int_t^\infty dt_1\int_{t_1}^\infty dt_2$, when one insists upon the proper definition of the integrand at the $\pm\infty$ limits of integration. The second permutation of Eq. (4.2) leads to the same form with the interchange of $k_1^{(-)}$ and $k_2^{(-)}$ and, it will become clear immediately, to the same leading-log dependence.

It is best to begin by performing the $\int dk_{1,2}^{(-)}$, integrations, which by simple contour evaluation require $k_1^{(+)}>0$ and $k_1^{(+)}+k_2^{(+)}>0$, and generate

$$(-2\pi i)^2\int \frac{d^2k}{\omega^2}\,e^{ib\cdot k_\perp}\int \frac{d^2k_1}{\omega_1^2}\,e^{ib\cdot k_1}\int \frac{d^2k_2}{\omega_2^2}\,e^{ib\cdot k_2} \times\int_\epsilon^K \frac{dk_2^{(+)}}{k_2^{(+)}}\int_\epsilon^K \frac{dk_1^{(+)}}{k_2^{(+)}+k_1^{(+)}}\left\{1+\frac{k_2^{(+)}\omega_1^2}{k_2^{(+)}\omega_1^2+k_1^{(+)}\omega_2^2}\right\}, \tag{4.4}$$

where we have inserted upper (K) and lower (ϵ) cutoffs for the $k_{1,2}^{(+)}$ integrations, and have replaced the transverse $k_\perp$ variable by $(k+k_1+k_2)_\perp$. Each of the three factors $\int d^2k\ \omega^{-2}e^{ik\cdot b}$ generates a term $(2\pi)K_0(\mu b)$, and the ''1'' of the curly bracket of Eq. (4.4) produces a ''nested'' contribution for the $k_{1,2}^{(+)}$ integrals of amount $(1/2)\ln_2(K/\epsilon)\to(1/2)\ln^2(E^2/m^2)$, when the replacement $E-p_3\simeq m^2/2E$ is used. In contrast, the second term of the curly bracket of Eq. (4.4) generates a contribution proportional to $\ln(E/m)$, and can be dropped as subleading. Quite generally, a ''nesting'' of the $k_i^{(+)}$ momenta follows directly from the ordered t limits of the iterates of $\Delta_n\Lambda_a^{\rm II}$, while the sum over all permutations of the $\lambda_{c_1}\cdots\lambda_{c_n}$ follows from the ordered s limits of the terms obtained upon functional differentiation of $R_0(s)$ by $\Delta_n\Lambda^{\rm II}$. The leading-log result for each $\lambda_{c_1}\cdots\lambda_{c_n}$ permutation is proportional to $(1/n!)\ln^n(E^2/m^2)$.

One can easily see that any expansion of the $i(\lambda^{\rm I}\cdot\lambda^{\rm II})\int_{-\infty}^{+\infty}ds\int_{-\infty}^{+\infty}dt\ Q(s,t)$ portion of the exponent of $R_0(s)$, in conjunction with the above forms, must always produce subleading, dependence, because at least one of the nested $k^{(+)}$ denominators needed for leading-log behavior will be missing. Further, one can also see the reason for the importance of the contiguity prescription, for—when the OE's defining each R_n are each expanded in powers of g^2—all the noncontiguous $\Delta_{\not\ell}\Lambda^{\rm II}(t|\delta/\delta\eta)$ operations will display ''improper,'' or out-of-sequence, limits for the s integrals, which will generate a similar sort of subleading behavior. For this standard choice of propagator, contiguity generates a first subdivision of terms containing the desired, leading-log dependence; and the latter are then isolated by the retention of only $R_0(s)$ in each factor of $[s]_{n-1}$, and the neglect of every $[s]_{n-1}^{-1}$, in each $R_n(s)$.

Perturbative eikonal analyses quite similar to the above have appeared long ago, in connection with multiperipheral processes of scalar ''tower'' exchange. There also one expects $k_i^{(-)}\sim 0$ and large, nested, $k_i^{(+)}$ momenta. What is different here (aside from trivial, complex, multiplicative factors) is that one must also include the sums over all $\lambda_{c_1}\cdots\lambda_{c_1}$ permutations, and the general form of such a sum is not clear for SU(N).

For SU(2), however, this computation can be carried through, and we now sketch that calculation. Its essence is to replace the leading-log dependence by another method of extraction which does not arise from the nested $k_i^{(+)}$ integrations, but yields, term-for-term and order-by-order, the same results. This method is defined by retaining the same f_{abc} factors obtained from the $\Delta_n\Lambda_a^{\rm II}$ interactions, and multiplying those that contribute to order n by the terms

$$\frac{i^n}{n!}\sum_{\rm perms}\int_{-\infty}^{s}ds_1\cdots\int_{-\infty}^{s_{n-1}}ds_n\int_t^\infty dt_1\cdots \times\int_t^\infty dt_nQ(s_1,t_1)\cdots Q(s_n,t_n)\lambda_{c_1}\cdots\lambda_{c_n} \tag{4.5}$$

and then summing over all n. It is easy to see that the leading-log contributions of Eq. (4.5) are identical to those of Eq. (4.4); the only difference is that the $\int_e^K dk(+)/k(+)$ contributions of Eq. (4.5) are not nested, and that the $(n!)^{-1}$ which follows from Eq. (4.4) because of nesting is, in Eq. (4.5), inserted by hand. This replacement can be made for arbitrary SU(N); but the next step, summing over all permutations of the s ordering, seems to be straightforward only for SU(2).

Because the t integrals of Eq. (4.5) are not ordered, we introduce the symbol $A_c(t)=\int_t^\infty dt'[\delta/\delta\eta_c(t')]$, and $\Sigma_a^{\rm II}[A]$ as the sum of all functional operations, which when performed on $R_0(s)$, generate the correct sequence of ϵ_{abc} coefficients multiplying Eq. (4.5). $\Delta\Sigma^{\rm II}[A]$ corresponds to the set of all the iterations of the SU(2) version of $\Delta\Lambda^{\rm II}(t|\delta/\delta\eta)$, where a factor of $(n!)^{-1}$ is inserted for each nth order, and the operators $A_{c_1}\cdots A_{c_n}$ replace the t-ordered $\int_t^\infty dt_1\int_{t_1}^\infty dt_2\cdots\int_{t_{n-1}}^\infty dt_n[\delta/\delta n_{c_1}(t_1)]\cdots[\delta/\delta\eta_{c_n}(t_n)]$ of the expansion of Eq. (2.8). We work directly with the contiguity approximation to U_0 (rather than to the R_n separately), in which R_0^{-1} is replaced by unity, and the leading-log simplification of R_0 is used, as in Eq. (4.5); everywhere, the $f_{abc}\to\epsilon_{abc}$, and $\lambda_c\to\sigma_c$. One may now examine the first four terms of this expansion, and it then becomes clear, by inspection, that the full sum over all such A dependence may be written as

$$\Delta\sum_a[A]=[A^2\delta_{ab}-A_aA_b]\sigma_b\,\frac{1}{A^2}\{\cosh(A)-1\} -i\epsilon_{acd}\sigma_d\cdot A_c\,\frac{\sinh(A)}{A}, \tag{4.6}$$

where $A^2=\Sigma_cA_c^2$, and $A=[A^2]^{1/2}$. To obtain Eq. (4.6), one repeatedly uses the SU(2) property $\Sigma_c\epsilon_{abc}\epsilon_{cde}=\delta_{ad}\delta_{be}-\delta_{ae}\delta_{bd}$.

The functional expression of contiguity, of $\Delta\Sigma_a^{\rm II}[A]$ operating on $R_0(s)$, can be performed by first introducing the representations

$$A_a\,\frac{\sinh(A)}{A}=\frac{1}{2\pi}\int d^3u\;\delta(\vec{u}^2-1)\;\frac{\partial}{\partial u_a}\,e^{\vec{u}\cdot\vec{A}} \qquad (4.7)$$

and

$$[A^2\delta_{ab}-A_aA_b]\,\frac{1}{A^2}\,\{\cosh(A)-1\}$$
$$=\frac{1}{2\pi}\int_0^1\frac{d\lambda}{\lambda}\int d^3u\;\delta(\vec{u}^2-1)$$
$$\times\left[\delta_{ab}\left(\frac{\partial}{\partial\vec{u}}\right)^2-\frac{\partial}{\partial u_a}\,\frac{\partial}{\partial u_b}\right]e^{\lambda\vec{u}\cdot\vec{A}}, \qquad (4.8)$$

where (λ, $\mathbf{u}$) are dummy integration variables. The quantity $e^{\lambda\vec{u}\cdot\vec{A}}R_0(s)|_{\eta\to0}$ is then the OE

$$\left(\exp\left[i\int_{-\infty}^{s}ds'\int_{t}^{\infty}dt'Q(s',t')\lambda(\vec{\sigma}^{\rm I}\cdot\vec{u})\right]\right)_{+(s')} \qquad (4.9)$$

and may be replaced by the oe

$$\exp[i\lambda(\vec{\sigma}^{\rm I}\cdot\vec{u})K(s,t)], \qquad (4.10)$$

where $K(s,t)=\int_{-\infty}^{s}ds'\int_{t}^{+\infty}dt'Q(s',t')$. In effect, the lack of A ordering for these leading-log terms has transformed their operation upon $R_0(s)$ into an ordinary exponential with weightings to be determined by the integrations of Eqs. (4.7) and (4.8). These last steps are now easily performed, by the replacement of Eq. (4.10) by $\cos(\lambda uK)+i(\vec{\sigma}^I\cdot\vec{u})[\sin(\lambda uK)/u]$, and its substitution into Eqs. (4.7) and (4.8), whose evaluations yield

$$A_a\,\frac{\sinh(A)}{A}\cdot R_0(s)|_{\eta\to0}=\frac{i}{3}\,\sigma_a^{\rm I}(K\,\cos K+2\,\sin K) \qquad (4.11)$$

and

$$[A^2\delta_{ab}-A_aA_b]\,\frac{1}{A^2}\,\{\cosh(A)-1\}R_0(s)|_{\eta\to0}$$
$$=\frac{4}{3}\,\delta_{ab}\left[\cos(K)-1-\frac{K}{2}\,\sin(K)\right]. \qquad (4.12)$$

From Eqs. (4.6) and (4.10), (4.11), and (4.12), one obtains

$$\Delta\sum_a^{\rm II}[A]R_0(s)|_{\eta\to0}=\frac{1}{3}\,\epsilon_{acd}\sigma_d^{\rm II}\sigma_c^{\rm I}[K\,\cos K+2\,\sin K]$$
$$+\frac{4}{3}\,\sigma_a^{\rm II}\left[\cos K-1-\frac{K}{2}\,\sin K\right]. \qquad (4.13)$$

Multiplying Eq. (4.13) on the left by $\sigma_a^{\rm I}$, antisymmetrizing where appropriate (together with the Casimir relation $\Sigma_{ac}\epsilon_{acd}\epsilon_{ace}=2\,\delta_{de}$), and including the R_0 contribution of the product $R=R_0U_0$, one finds the eikonal given by

$$\chi=(\sigma^{\rm I}\cdot\sigma^{\rm II})\int\int_{-\infty}^{+\infty}ds\;dt\;Q(s,t)$$
$$\times\left\{1-\frac{4}{3}\left[1-\cos K+\frac{K}{2}\,\sin K\right]\right.$$
$$\left.+\frac{2}{3}\,i[K\,\cos K+2\,\sin K]\right\}. \qquad (4.14)$$

Finally, if one imagines expanding Eq. (4.14) in powers of $K(s,t)$, and combines each $K^n(s,t)$ with the remaining integrand of U_0, one may use the easily verified property, correct for the leading-log dependence of each order:

$$\int\int_{-\infty}^{+\infty}ds\;dtQ(s,t)K^n(s,t)\simeq\left[-i\,\frac{g^2}{\pi^2}\,\ln(E/m)K_0(\mu b)\right]^n$$
$$\equiv[-iL]^n \qquad (4.15)$$

so that, upon resumming these terms into the equivalent of Eq. (4.14), in effect the quantity $K(s,t)$ may be replaced by $-iL$ of Eq. (4.15), yielding

$$\chi=-\frac{g^2}{2\pi}\,(\sigma^{\rm I}\cdot\sigma^{\rm II})K_0(\mu b)\left\{1-\frac{4}{3}\,[1-e^L]+\frac{2}{3}\,Le^L\right\} \qquad (4.16)$$

as the complete eikonal in leading-log approximation for the SU(2) problem (e.g., of nucleon-nucleon scattering by the exchange of neutral and charged vector mesons, with conserved isospin).

Perhaps the most obvious feature of Eq. (4.16) is its proportionality to $\sigma^{\rm I}\cdot\sigma^{\rm II}$, which quantity takes on isoscalar or isovector eigenvalues depending on the nature of the initial scattering states. A second interesting property is that, by expressing the exp$[L]$ factors of Eq. (4.16) in terms of

$$e^L=(s/m^2)^{(g^2/2\pi^2)K_0(\mu b)}$$

one finds an ''effective Reggeization'' of the eikonal, where s here again denotes total c.m. (energy)^2. For $\mu\neq0$, there is little contribution to the scattering amplitude for small b; and hence if $K_0(\mu b)$ is approximated as $\sim\exp[-\mu b]$, one obtains forms similar to those found in the Regge-eikonal approximation of multiperipheral scattering, except that this eikonal is real. In fact, the amplitude, constructed in the generic form (and suppressing all inessential factors)

$$T\sim is\int_0^\infty b\;db\;J_0(qb)[1-e^{i\chi(s,b)}] \qquad (4.17)$$

exhibits a variant of a ''hard disk'' scattering solution, in that there are two regions of impact parameter, $b\lessgtr b_0$, which produce different contributions to the amplitude. This can be seen by defining b_0 as that impact parameter where $L(b_0)=1$, $b_0=\mu^{-1}\ln[(g^2/2\pi^2)Y]>\mu^{-1}$, $Y=2\,\ln(E/m)$, and writing the contributions to the amplitude of Eq. (4.17) in terms of integrations over these two regions of b. Since $L(b)=\exp[\mu(b_0-b)]$, and we assume that Y is large, when

$b<b_0$, L is large, as is the eikonal of Eq. (4.16), and the only significant contribution to the amplitude comes from the ''1'' of the first term of Eq. (4.17). When $b>b_0$, L is small, and the only significant contribution to the eikonal comes from the ''1'' of the bracket of Eq. (4.16), which we shall call χ_0; this is the contribution coming from the original R_0 term of Eq. (2.14). This argument leads to the representation of the amplitude of Eq. (4.17) as the sum of two parts:

$$T\sim is\int_0^{b_0} b\ db\ J_0(qb)+is\int_{b_0}^{\infty} b\ db\ J_0(qb)[1-e^{i\chi_0}] \tag{4.18}$$

or as

$$T\sim is\int_0^{b_0} b\ db\ J_0(qb)e^{i\chi_0}+is\int_0^{\infty} b\ db\ J_0(qb)[1-e^{i\chi_0}] \tag{4.19}$$

in which the amplitude is characterized by its simplest eikonal approximation, χ_0, and by the range parameter $b_0(E/m)$ which defines that impact parameter beyond which leading-log corrections force the eikonal to become extremely large and oscillatory, thereby removing its contribution from the amplitude.

Could the same mechanism be operative for the general case of SU(N)? Even though we cannot perform the closed sum over all orders of leading-log contributions for $N>2$, one can anticipate that for a similar $b_0(E/m)$ the eikonal becomes very large, contributing a rapidly oscillating and negligible contribution to the amplitude, which may be written in the form of Eqs. (4.18) or (4.19), with the $\sigma^{\mathrm{I}}\cdot\sigma^{\mathrm{II}}$ invariant of χ_0 replaced by $\lambda^{\mathrm{I}}\cdot\lambda^{\mathrm{II}}$. We think it a reasonable conjecture that this simple form is the actual result of the complete SU(N) calculation. Of course, this point is somewhat academic, since when energies are large enough to take leading-logs seriously, other processes which have here been neglected (e.g., multiperipheral production) are going to appear. Nevertheless, it is of some theoretical interest to examine an amplitude constructed from the eikonal of Eq. (4.16), under the assumption that $\ln(E/m)\gg 1$; and it will be most interesting to see if similar structures and simplifying approximations are going to appear in the study of other eikonal processes which reflect the growth of inelastic particle production with increasing energies.

V. OTHER PROCESSES

An important variation of the non-Abelian eikonal scattering problem is found when self-energy processes (as in radiative corrections to other QCD n-point functions) are attempted. Here, one may make use of the new, exact, and approximate representations for the needed Green's functions of Ref. [3] in which dependence on the source fields, A_μ and $F_{\mu\nu}$ is that of an OE of linear form; for the simplest example, we omit the $F_{\mu\nu}$ terms, and work in a quenched approximation, so that the sum of all radiative corrections to the fermion propagator will require evaluation of the quantity

$$R(s|\xi)=N'\int d[u]\int d[\alpha]\exp\left(i\int \alpha\cdot u\right)\times\left[\exp\left(i\int_{-\infty}^{s}\lambda\cdot u\right)\right]_+\exp\left(\frac{i}{2}\int\int \alpha_a Q_{ab}\alpha_b\right)\times\exp\left(i\int u\cdot\xi\right) \tag{5.1}$$

in the limit of $s\to\infty$ and $\zeta_a(s')\to 0$. Here, $Q_{a,b}(s,t)$ is considerably more complicated than the corresponding function of an eikonal scattering amplitude (although the resemblance becomes closer if an improper, no-recoil approximation is adopted), but must satisfy $Q_{a,b}(s,t)=Q_{ba}(t,s)$.

Using techniques modeled after those sketched above, it is easy to see that a representation of Eq. (5.1) is given by the formal OE

$$R(s|\xi)=\left(\exp\left[i\int_{-\infty}^{s}ds'\int_{-\infty}^{+\infty}dt'\lambda_a Q_{ab}(s',t')\left[\xi_b(t')+\theta(s'-t')\Lambda_b\left(s',t'\left|\frac{\delta}{\delta\xi}\right.\right)\right]\right]\right)_{+(s')}, \tag{5.2}$$

where $\Lambda_b(s,t|iu)=[\exp(i\int_t^s\lambda\cdot u)]_+\lambda_b[\exp(-i\int_t^s\lambda\cdot u)]_-$. The same, formal expansion corresponding to Eqs. (3.4) and (3.5) may be defined, except that R_0 is now multiplied by the exponential factor $\exp[(i/2)\int\xi Q\xi]$, which has the effect of inserting polynomial ξ dependence into all the exponents of subsequent R_n, and the power-counting arguments given above must be appropriately modified.

Perhaps the most interesting generalization of the forms of Sec. III should appear in eikonal quark-scattering models when gluon-gluon interactions (e.g., the ''tower graphs'' and their generalizations) are taken into account. Before a functional treatment can be attempted, even in the relatively simple models described in the last chapters of Refs. [1] and [5], it is necessary to have a decent representation—as a functional of an equivalent gluon source used to represent internal, ''s-channel'' gluon exchanges—for the Green's function corresponding to the t-channel gluons exchanged between quarks. For the eikonal situation where different spin-one bosonic fields are used to describe distinct t- and s-channel exchanges, respectively, such a representation now exists [7], and can be written down without undue complications; for the single gluonic field of real QCD, the situation is similar but not as straightforward.

If these calculations can be carried through for the tower graphs (corresponding to two-gluon, t-channel exchange between scattering quarks) in a functional context, using contiguity as appropriate, there should then be an immediate functional generalization which includes multiple, t-channel gluon exchanges. Such estimates of the QCD eikonal would be most relevant to high-energy particle scattering experiments.

VI. SUMMARY

In this paper we have shown how the formidable, non-Abelian eikonal combination (1.1) may be written as the OE

$R(s|\eta)$ in the limit as $s\to\infty$, and $\eta\to 0$; and have, by contiguity, isolated a subset of terms which exponentiate and contribute directly to the eikonal function, and which contain appropriate $\ln(E/m)$ dependence associated with the leading-log behavior of every perturbative order. For SU(2), these terms may be summed to all orders, generating an eikonal dependent on the total isospin of the scattering channel, which displays a form of Reggeization peculiar to this set of graphs summed.

Contiguity may also be phrased in terms of the original ansatz, $R(s|\eta)=R_0U_0$, by replacing the exact U_0 of Eq. (3.2) by its contiguity approximation, as used for the SU(2) calculation. However, at least for the specifically perturbative estimates of $\ln(U)$, it appears to be simpler to adopt contiguity in the context of the R_n. As explained in Sec. III, contiguity together with the elimination of obviously subleading terms provides a straightforward method for the estimation of the eikonal's leading-log terms in every perturbative order. We have found an elementary method for summing all such terms in SU(2), and conjecture the form of a simplified eikonal amplitude for all N.

In summary, we cannot here claim to have given the complete solution to the problem of non-Abelian field-theory structure; but, rather, a new and complete functional formulation (for eikonals and related self-energy graphs), and a ''contiguity'' method of extracting those terms which are certainly going to be exponentiated, and which seem to correspond to the identification of leading $\ln(E/m)$ dependence appearing in the construction of specifically non-Abelian eikonals. It is hoped that these new techniques will be useful for other processes, as discussed in the previous sections.

In particular, it is now appropriate to explain to the patient reader how this procedure—which lacks manifest gauge invariance in a Yang-Mills context—can be incorporated within a large scheme, in order to obtain strictly gauge-invariant results for physical scattering amplitudes. There are three separate issues involved. In any eikonal calculation, one is searching for the proper separation of longitudinal/timelike momenta from transverse momenta—this is the problem attempted from first principles by Verlinde and Verlinde [8]—while at the same time, one is trying to sum over the contributions of all perturbative orders for the classes of graphs considered; and, simultaneously, one must insist on the restrictions of gauge invariance.

The eikonal calculation of the present paper, with its ability to extract leading-$\ln(s)$ dependence, is intended to be used as an initial step in a complete functional expression for the scattering of a pair of quarks, which includes all gluonic self-interactions as part of a ''gluonic sector'' described by the methods of Halpern [9], or its slight generalization by Fried [10]. The A_μ dependence of these formulations takes the form of an exponential of linear and quadratic forms, so that the $Q(s,t)$ propagator of Eq. (1.1) is now dependent upon auxiliary fields, and is linked to subsequent functional integrals which describe the gluon self-interactions; extra functional integrations maintain gauge restrictions. The insertion of the forms of this paper then leads, as an intermediate step, to a rather complicated set of functional integrals; but in the integrands of these functional integrals, one has already extracted the leading $\ln(s)$ behavior of the simple eikonal where s is essentially given by quark kinematics. For large s, by a rescaling of the auxiliary functional integrands, one can now try to approximate and to extract relevant gluon self-interaction structure, in this large s, small t limit, and in a gauge invariant way. These calculations are presently underway, and whether they will succeed is not yet known, but this is the reason why a functional evaluation of the leading-log behavior of the simple eikonal form of Eq. (1.1) can be relevant to quarks and gluons.

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