S theory

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The representation theory of the maximally extended superalgebra with 32 fermionic and 528 bosonic generators is developed in order to investigate nonperturbative properties of the democratic secret theory behind strings and other p-branes. The presence of Lorentz nonsinglet central extensions is emphasized, their role for understanding up to 13 hidden dimensions and their physical interpretation as boundaries of p-branes are elucidated. The criteria for a new larger set of BPS-like nonperturbative states is given and the methods of investigation are illustrated with several explicit examples. [S0556-2821(97)04702-4]

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I. INTRODUCTION

String and p-brane dualities have led to the notion that there is a mysterious, fascinating, secret supersymmetric theory, which may be called M theory, F theory, S theory, ... It includes all possible closed and open (i.e., with boundaries) p-branes, such as strings, membranes, five-branes, ..., interacting with each other in various flat, curved, or compactified spacetimes. It has been proposed that this theory, which includes up to 12 (or 13, see below) hidden dimensions, may be the mother, father, or sire of all physical theories. Some of the M and F aspects have been discussed by other authors [1–3]. D-branes [4] provide some handle on the theory.

In this paper I will concentrate on some of the S properties of the secret theory, and will emphasize an algebraic approach [5] based on a superalgebra with 32 fermionic and 528 bosonic generators [6,5]. The collection of all bosonic operators form a 32×32 symmetric matrix S given by the anticommutator of the supercharges $\{Q,Q\}\sim S$. The structure and symmetries of S are related to p-branes, dualities, and hidden dimensions. Global properties, certain states, and certain nonperturbative properties of the underlying secret theory may be studied by analyzing the representations of this superalgebra. This line of investigation will be called S theory.

In this paper I will outline the elements of S theory. One of the points to be emphasized is the presence of Lorentz nonsinglet central extensions in S, which so far received little attention. In the usual treatment of nonperturbative properties of M or F theory, the Lorentz singlet central extensions play a major role [e.g., Bogomol'ni-Prasad-Sommerfield (BPS-) like states or black holes]. Here, I will argue that the Lorentz nonsinglets also play a major role and that their presence is required in order to see the full extent of hidden dimensions and duality symmetries. In Sec. II I will give a physical interpretation of these central extensions in terms of boundary variables for p-branes. In Sec. III the symmetry structure of the generalized algebra and the connection to hidden dimensions (up to a total of 13 dimensions) will be explained. In Sec. IV the representations of the superalgebra will be constructed and, as a special case, the criteria for obtaining the quantum numbers of the generalized BPS-type states will be given. As an application of S theory, I will show that there are new nonperturbative BPS states that carry quantum numbers (eigenvalues of central extensions) that are Lorentz nonsinglets, and I will construct some explicit examples. Since these quantum numbers have an interpretation in terms of p-brane boundaries these BPS states are related to open p-branes. When they are included in the spectrum along with more familiar D-brane BPS states they form *larger multiplet structures of symmetries* that exhibit additional hidden dimensions as well as dualities.

Results such as the ones described, which follow only from the properties of the superalgebra, are assumed to be valid nonperturbatively in the full theory. Therefore, they must be useful handles for constructing and analyzing the fundamental underlying theory.

II. LORENTZ NONSINGLET CENTRAL EXTENSIONS

In the generalized superalgebra $\{Q^a_\alpha,Q^b_\beta\}=S^{ab}_{\alpha\beta}$, where $S^{ab}_{\alpha\beta}=\delta^{ab}\ \gamma^\mu_{\alpha\beta}P_\mu+$ ''central extensions,'' there are generally Lorentz nonsinglet central extensions $Z^{ab}_{\mu_1\cdots\mu_p}$ with p Lorentz indices. The structure and properties of these additional operators are explained more fully in the following sections. As explained in [5] a nonzero $Z^{ab}_{\mu_1\cdots\mu_p}$ which is not a Lorentz singlet does not violate the no-go theorem of [7] as long as there are extended objects. The Lorentz singlets Z^{ab}, with $p=0$, are well known to represent the quantum numbers of black holes in M theory. In this section I will clarify the physical interpretation of $Z^{ab}_{\mu_1\cdots\mu_p}$ for $p\geqslant1$ as related to p-branes.

For simplicity, I will assume that all p-branes propagate in flat backgrounds (such as flat spacetime in direct product with tori or their orbifolds in compactified dimensions). Under this assumption all 528 components of S commute with one another as justified below. Similar considerations in curved backgrounds would yield a more complicated algebra that is more difficult to analyze (for example, momenta do not commute in curved backgrounds).

Just like momentum, all possible values of these central extensions must be included in the representation of the superalgebra in order to take into account all possible states in the representation consistent with the Lorentz group. But physical considerations would determine if they are spacelike, lightlike, or timelike, and hence there are classes of representations, just like the Poincaré group.

These central extensions are closely associated with the boundaries of p-branes as well as the topology of the background geometry in which they propagate. In a flat d-dimensional spacetime, as assumed in this paper, the Lorentz nonsinglets are present only if the p-brane has boundaries. This is seen as follows: In the low energy effective theory, the gauge potential with $p+1$ antisymmetric Lorentz indices has a source term in its equation of motion (the notation is explained in [5]):

$$\partial_\lambda \partial^{[\lambda} A_{ab}^{\mu_0\mu_1\cdots\mu_p]}(x) = J_{ab}^{\mu_0\mu_1\cdots\mu_p}(x). \tag{1}$$

The current is nonvanishing when constructed from a p-brane $X_\mu(\tau,\sigma_1,\cdots,\sigma_p)$ and its superpartners

$$J^{ab}_{\mu_0\mu_1\cdots\mu_p}(x) = \int d\tau d\sigma_1 \ldots d\sigma_p \sum_i z_i^{ab} \delta^d(x - X^{(i)} \times(\tau,\sigma_{1,}\ldots\sigma_p))\partial_\tau X^{(i)}_{[\mu_0} \ldots \partial_{\sigma_p} X^{(i)}_{\mu_p]} \times(\tau,\sigma_{1,}\ldots\sigma_p) + \cdots, \tag{2}$$

where the index i is a label for many p-branes and z_i^{ab} is their coupling to the pair of supercharges labeled by a,b. The Lorentz nonsinglet central extension is the integral of the current over a spacelike ''slice'' in spacetime

$$Z^{ab}_{\mu_1\ldots\mu_p} = \int d^{d-1}\Sigma^{\mu_0} J^{ab}_{\mu_0\mu_1\ldots\mu_p}(x), \tag{3}$$

where, e.g., one may use a noncovariant notation by choosing $\mu_0=0$, $d^{d-1}\Sigma^0 = d^{d-1}x$, i.e., the volume of space at fixed time. This is one expression for $Z^{ab}_{\mu_1\ldots\mu_p}$. Another expression is obtained by substituting the left-hand side of Eq. (1) in Eq. (3). Then, the integrand is a total divergence and, therefore, it can be expressed as a surface integral involving the asymptotic values of $A_{ab}^{0\mu_1\cdots\mu_p}$ at infinity of physical space ($r\to\infty$). Therefore, in any classical solution of the effective low energy theory, a nontrivial asymptotic behavior $A_{ab}^{0\mu_1\cdots\mu_p} \sim Z_{ab}^{\mu_1\cdots\mu_p}/r^{d-2}$ would have an interpretation in terms of p-branes through Eq. (2).

Now, what property of the p-brane is represented by $Z^{ab}_{\mu_1\ldots\mu_p}$? As an example, let us perform the integral in Eq. (3) for $p=1$, i.e., for a string. The result is[1]

$$Z^{ab}_\mu = \sum_i z_i^{ab} \int d\sigma \partial_\sigma X^{(i)}_\mu(\tau,\sigma) + \cdots, \tag{4}$$

where the dots represent additional pieces in a conserved current. If the remaining integral over σ is for closed strings propagating in flat spacetime, then the closed string condition $X^{(i)}_\mu(\tau,0) = X^{(i)}_\mu(\tau,2\pi)$ gives $(Z^{ab}_\mu)_{\text{closed}}=0$.[2] But if there are open strings, then the result depends only on the end points

$$(Z^{ab}_\mu)_{\text{open}} = \sum_i z_i^{ab}(X^{(i)}_\mu(\tau,\pi) - X^{(i)}_\mu(\tau,0)) + \cdots. \tag{5}$$

Note that this is a Lorentz vector, and (unless identically zero) it is a continuous spacelike or lightlike variable, but is not timelike. It is obviously translationally invariant.

Similarly, for the general p-brane in flat spacetime the central extensions $Z^{ab}_{\mu_1\ldots\mu_p}$ can be shown to be related to boundary variables that commute with one another as well as with the momentum operators. Hence, all these operators are simultaneously diagonalizable and their continuous eigenvalues must label the physical states at an equal footing with the eigenvalues of momentum, since they are not distinguishable under the symmetries of the superalgebra.

III. THE EXTENDED SUPERALGEBRA

The maximum number of supercharges in a physical theory is 32. This constraint comes from four dimensions, which admits at the most eight supercharges, each with four real components, since there can be no supermultiplet of massless particles with spins higher than two. In an arbitrary number of dimensions, we label the 32 supercharges as

$$Q^a_\alpha = \begin{cases} \alpha = \text{spinor in } d \text{ dims.,} \\ a=1,2,\ldots,N, \text{spinor in } c+2 \text{ dims.,} \end{cases} \tag{6}$$

where $N=1$, when $d=11$; $N=2$, when $d=10,\ldots,$ $N=8$ when $d=4$. Here, d is the dimension of spacetime. Let us define c as the number of compactified string dimensions such that $d+c=10$. It was argued that there are two extra hidden dimensions, one spacelike and one timelike, and that N corresponds to the dimension of the spinor in $c+2$ dimensions [5]. Furthermore, N also corresponds to a dimension of an irreducible representation of the group K, which is the maximal compact subgroup of U duality.

The extended superalgebra has the form

$$\{Q^a_\alpha, Q^b_\beta\} = (S)^{ab}_{\alpha\beta},$$

$$(S)^{ab}_{\alpha\beta} = \delta^{ab}\gamma^\mu_{\alpha\beta} P_\mu + \sum_{p=0,1,2,\ldots} \gamma^{\mu_1\cdots\mu_p}_{\alpha\beta} Z^{ab}_{\mu_1\ldots\mu_p}, \tag{7}$$

where the permutation symmetry of $(\alpha\beta)$ must be the same as (ab) in each term on the right-hand side. The structure of $(S)^{ab}_{\alpha\beta}$ is of central interest in this paper. From it we will learn about the symmetries of the underlying theory (many of them hidden from the point of view of conventional string theories) as well as about the representation space that is related to the physical states of the theory.

[1]It is useful, but not necessary to choose the timelike gauge $X^0(\tau,\sigma)=\tau$, use $\tau=x^0$ because of the delta function, and take the spacetime ''surface'' to be the usual integral over all space at constant time.

[2]A similar expression in compactified dimensions is nonzero because of the closed cycles in a nontrivial topology. Similarly, if spacetime is curved instead of flat, there could be nontrivial contributions for closed strings, if the topology is nontrivial.

For $d \geqslant 10$ there are two types of superalgebras A,B that are outlined below. From the construction of γ matrices given in the appendix, one can see that the type A and type B superalgebras originate with different forms S_A, S_B embedded in a higher 64×64 spinor space in 13 dimensions. For $d \leqslant 9$ there is a one-to-one correspondence between type A,B superalgebras, which are then related by T duality. As we will see below certain hidden symmetries in the form (7) become manifest in the covariant forms S_A, S_B given below in higher dimensions. Hence, the T,U duality between A and B versions may find its roots in 13 dimensions (13D).

The Lorentz generators for $\mathrm{SO}(d-1,1)$ do not commute with the supercharges or with the bosonic generators (except for $p=0$ case), since these are spinors or antisymmetric p-tensors under Lorentz transformations, respectively. In addition to the Lorentz transformations that act as isometries, there are other isometries that act such as (i) Lorentz transformations in $c+2$ compact plus hidden dimensions SO $(c+1,1)$ and (ii) discrete $U-$duality transformations that induce continuous transformations through the maximal compact subgroup K (see [5]). The supercharges, bosonic generators, and the physical states of the theory are classified by the isometries $\mathrm{SO}(d-1,1) \otimes\ \mathrm{SO}(c+1,1)$ or by SO $(d-1,1) \otimes K$. Displaying one of these classifications may hide the other one. The isometries and their intersecting subgroup structure was given in [5] as follows:

$$
\left.\begin{array}{l}
\begin{array}{ccc}
\begin{array}{l} c\ \text{compact}+ \\ 2\ \text{hidden dims} \\ \mathrm{SO}(c+1,1) \end{array} & \rightarrow & \begin{array}{c} \text{two intersecting} \\ \text{classifications} \\ \text{of multiplets of} \\ \text{generators and states} \end{array} \\
\downarrow & & \uparrow \\
\left.\begin{array}{l} \mathrm{SO}(c+1)_{1\ \text{hidden dim.}} \\ \mathrm{SO}(c)\ \text{is common subgroup} \\ \mathrm{SO}(c)_L \otimes \mathrm{SO}(c)_R \end{array}\right\} & \rightarrow & \begin{array}{c} K \\ \text{maximal compact in } U \end{array} \\
\uparrow & & \\
T\ \text{duality}\ \mathrm{SO}(c,c) & &
\end{array}
\end{array}\right\} \rightarrow \begin{array}{l} U \\ \\ \text{duality} \end{array}. \tag{8}
$$

In this paper I will assume a flat background and hence all bosonic generators commute among themselves and with the supercharges as explained in Sec. II. In this case it is easy to find all the representations of the superalgebra and analyze the physical states as discussed in Sec. IV.

A. Type A

In 11D there are only three terms [6]

$$
\{Q_\alpha, Q_\beta\} = (S_A)_{\alpha\beta},
$$

$$
(S_A)_{\alpha\beta} = (C\gamma^m)_{\alpha\beta} P_m + (C\gamma^{m_1 m_2})_{\alpha\beta} Z_{m_1 m_2} + (C\gamma^{m_1 \cdots m_5})_{\alpha\beta} X_{m_1 \cdots m_5}, \tag{9}
$$

where $m=0,1,2,\ldots,10$, and the γ matrices are 32×32 (for more details see the Appendix). When reduced to lower dimensions, this 32×32 matrix $(S_A)_{\alpha\beta}$ takes the form of Eq. (7). As is well known, in 10D $(S_A)_{\alpha\beta}$ is distinguishable from the type-B $(S_B)_{\alpha\beta}$ given below, but in nine dimensions or less the reduced $(S_A)_{\alpha\beta}$ and reduced $(S_B)_{\alpha\beta}$ have similar content that is related by T duality.

It is also possible to consider 12 dimensions with signature (10,2) [8] since the Weyl spinor is real and 32 dimensional. Then, the extended superalgebra can be written covariantly in 12D [5]:

$$
\{Q_\alpha, Q_\beta\} = (S_A)_{\alpha\beta},
$$

$$
(S_A)_{\alpha\beta} = (C\gamma^{M_1 M_2})_{\alpha\beta} Z_{M_1 M_2} + (C\gamma^{M_1 \cdots M_6})_{\alpha\beta} Z^{+}_{M_1 \cdots M_6}, \tag{10}
$$

where $M=0', m$, and $m=0,1,2,\ldots,10$, with two timelike dimensions denoted by $M=0'$, 0. The relation between 11- and 12-dimensional γ matrices $\gamma^m_{\alpha\beta}$, $\gamma^M_{\alpha\beta}$, $\Gamma^M_{\alpha\beta}$ is given in the Appendix. Similarly, the bosonic generators are related by

$$
\begin{array}{ll}
Z_{M_1 M_2} \rightarrow P_m \oplus Z_{m_1 m_2} & 66 = 11 + 55, \\
Z^{+}_{M_1 \cdots M_6} \rightarrow X_{m_1 \cdots m_5} & 462 = 462.
\end{array} \tag{11}
$$

The six-index tensor is self-dual in 12D.

Note that there is no 12D translation operator P_M in the (10,2) version (10). Therefore, the extension of the theory from (10,1) to (10,2) is *not the naive extension that would have implied two time coordinates*, since the corresponding canonical conjugate momenta are not present in the theory. There is only one time translation operator P_0, hence there is only one time coordinate that can be recognized after the reduction to 11 or lower dimensions. Nevertheless, there is an obvious SO(10,2) covariance in the form (10). Therefore, there is a hidden SO(10,2) covariance in the forms (9) or in Eq. (7) for $d \leqslant 9$, since they are equivalent to Eq. (10). We emphasize again that the 12D SO(10,2) generators L_{MN} do not appear on the right-hand side; i.e., $Z_{M_1M_2}$ are not the L_{MN}.

B. Type B

The type-A superalgebra in $d=10$ can be written as (see Appendix)

$$\{Q^{\bar{a}}_{\bar{\alpha}},Q^{\bar{b}}_{\bar{\beta}}\}=(S_B)^{\bar{a}\bar{b}}_{\bar{\alpha}\bar{\beta}},$$

$$(S_B)^{\bar{a}\bar{b}}_{\bar{\alpha}\bar{\beta}}=\overline{\gamma}^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}}(c\overline{\tau_i})^{\bar{a}\bar{b}}P^i_{\bar{\mu}}+\overline{\gamma}^{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}_{\bar{\alpha}\bar{\beta}}c^{\bar{a}\bar{b}}Y_{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}+\overline{\gamma}^{\bar{\mu}_1\cdots\bar{\mu}_5}_{\bar{\alpha}\bar{\beta}}(c\overline{\tau_i})^{\bar{a}\bar{b}}X^i_{\bar{\mu}_1\cdots\bar{\mu}_5}, \tag{12}$$

where $\overline{\alpha},\overline{\beta}=1,2,\ldots,16$ and $\overline{a},\overline{b}=1,2$ while $\overline{\mu}=0,1,\ldots,9$ and $i=0',1',2'$. The $X^i_{\mu_1\cdots\mu_5}$ are self-dual in 10D. Here $\overline{\gamma}^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}}$ are 16×16 10D γ matrices obtained from the list of 13D γ matrices in Eq. (A1) by omitting the first two factors in the direct products (i.e., $\gamma_0=1\otimes1_8$, $\gamma_9=\tau_3\otimes1_8$, etc.). The $\overline{\tau}_i^{ab}\equiv(i\tau_2,\tau_3,\tau_1)$ are $2\times2\,\gamma$ matrices in a hidden 3D Minkowski space, with a charge conjugation matrix $c^{ab}=i\tau_2^{ab}=\varepsilon^{ab}$. They are obtained from the 13D γ matrices in the Appendix by keeping only the first factor in $(\Gamma_{0'},\Gamma_{10},\Gamma_A)$ which corresponds to $\overline{\tau_i}$. The reason for the split of 13D into 10D+3D is the type-B chiral projection, as explained in the Appendix. The 13D covariance is lost because of the projection, but a clear identification of the dimensions labeled by $\overline{\mu},i$ survives. This algebra is covariant under $SO(1,9)\otimes SO(1,2)_B$.

Again, as in the type-A algebra, there is only one time translation operator. The usual 10D momentum operator $P_{\bar{\mu}}$ corresponds to the $i=0'$ component of $P^i_{\bar{\mu}}$ or, equivalently, to the trace part of $(c\overline{\tau_i})^{ab}P^i_{\bar{\mu}}$. Likewise, the 3D momentum operator P^i corresponds to the $\overline{\mu}=0$ component of $P^i_{\bar{\mu}}$ or to the trace part of $\overline{\gamma}^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}}P^i_{\bar{\mu}}$. The time translation operator for either 10D or 3D is the same one, namely, $P_0^{0'}$ and it corresponds to the trace part of the full 32×32 matrix $(S_B)^{\bar{a}\bar{b}}_{\bar{\alpha}\bar{\beta}}$. This is the same time translation operator that appears as the trace of S_A in the type-A superalgebra, given above in the 12D or 11D covariant form.

C. More about the duality map A⇔B

Once the theory is compactified to $d=9$ (or fewer dimensions), the two types S_A,S_B reduce to two forms that are in one-to-one correspondence to each other, but are not identical. Both of these forms display $SO(8,1)\otimes SO(2,1)$ isometry, one in the form $SO(8,1)\otimes SO(2,1)_B$ coming from $SO(9,1)\otimes SO(2,1)_B$ and the other $SO(8,1)\otimes SO(2,1)_A$ coming from SO(10,2). By comparing the reduced forms of S_A,S_B given below we find the map between the A,B types in nine dimensions:

$$S_B=\overline{\gamma}^{9}_{\bar{\alpha}\bar{\beta}}(c\overline{\tau_i})^{\bar{a}\bar{b}}P^i_9+\overline{\gamma}^{\mu}_{\bar{\alpha}\bar{\beta}}(c\overline{\tau_i})^{\bar{a}\bar{b}}P^i_\mu+(\overline{\gamma^9\gamma^{\mu_1\mu_2}})_{\bar{\alpha}\bar{\beta}}c^{\bar{a}\bar{b}}Y_{9\mu_1\mu_2}+\overline{\gamma}^{\mu_1\mu_2\mu_3}_{\bar{\alpha}\bar{\beta}}c^{\bar{a}\bar{b}}Y_{\mu_1\mu_2\mu_3}+(\overline{\gamma^9\gamma^{\mu_1\cdots\mu_4}})_{\bar{\alpha}\bar{\beta}}(c\overline{\tau_i})^{\bar{a}\bar{b}}X^i_{9\mu_1\cdots\mu_4}$$

$$i=(0',10,A)=(0',1',2'),\quad \mu=0,1,\ldots,8, \tag{13}$$

$$S_A=(C\gamma^{IJ})_{\alpha\beta}Z_{IJ}+(C\gamma^\mu\gamma^I)_{\alpha\beta}Z_{\mu I}+(C\gamma^{\mu_1\mu_2})_{\alpha\beta}Z_{\mu_1\mu_2}+(C\gamma^{\mu_1\mu_2\mu_3}\gamma^{0'}\gamma^9\gamma^{10})_{\alpha\beta}Z^+_{\mu_1\mu_2\mu_3 0'910}$$

$$+(C\gamma^{\mu_1\cdots\mu_4}\gamma^{IJ})_{\alpha\beta}Z^+_{\mu_1\cdots\mu_4 IJ}\quad I=(0',10,9),\quad \mu=0,1,\ldots,8. \tag{14}$$

These expressions are obtained by rewriting the original expressions for S_A,S_B, that were given in terms of the 64×64 γ matrices Γ in the appendix, and specializing the indices $M=(\mu,I)$ or (μ,i), respectively. There is a correspondence term by term: $Z_{IJ}\leftrightarrow\varepsilon_{IJK}P_9^K$, $Z_{\mu I}\leftrightarrow P^i_\mu$, etc. However, the $32\times32\gamma$ matrices of type A that multiply these coefficients are not of the direct product form $\overline{\tau}\otimes\overline{\gamma}$ of type B. Furthermore, the type A,B indices I,i, respectively, label different sets of compactified dimensions embedded in 13D ($I=0',10,9$, versus $i=0',10,A$, where A labels the 13th dimension). Hence, the T duality that exists between types A and B is closely related to the map provided by the above expressions, and it involves a *"duality" transformation that corresponds to relabeling some of the 13 dimensions.*

IV. REPRESENTATIONS, BPS STATES

The superalgebra in d dimensions (7) has two types of isometries: spacetimelike isometries $SO(c+1,1)$ and duality isometries K ($\subset U$) in addition to the Lorentz isometry $SO(d-1,1)$. The classification of the various generators has been tabulated in various dimensions (d,c) elsewhere [5]. The physical states of the theory must be classified as the representation spaces of the superalgebra. Therefore, it is expected that the physical states form supermultiplets consistent with these isometries and that they reveal the structures of the hidden dimensions and dualities displayed in Eq. (8). Some work in this direction has been reported before [9,5]. Here, we describe a more systematic approach and provide examples of new BPS states that belong in larger multiplets along with previously known BPS states. The new element is the inclusion of quantum numbers that carry Lorentz indices.

In the case of Abelian bosonic generators, as assumed in this paper, all representations are found by analogy to representations of standard supersymmetry. The main new ingredient is that instead of the standard momentum operators, we now have 528 Abelian operators in S that are simultaneosly diagonalizable. In previous work only the Lorentz singlet central extensions were included in addition to the standard momentum in seeking representations, but here we include all bosonic operators. Recall that those that carry p Lorentz indices are physically relevant for the description of boundaries of p-branes in ordinary spacetime (not just in compactified spacetime). These 528 operators are at an equal footing since they are mixed with the momenta and with one another by the $SO(c+1,1)$ and K isometries.

The representation space is constructed as follows. A reference state is chosen such that it is labeled by

$$|S,R_c\rangle \quad \text{or} \quad |S,R_K\rangle, \tag{15}$$

where S represents the eigenvalues of all commuting 528 bosonic generators, and $R_{c,K}$ is a representation of the isometry $SO(d-1,1)\otimes SO(c+1,1)$ or a representation of $SO(d-1,1)\otimes K$, respectively (as discussed in [5] R_c must form a collection of irreducible representations that can be expanded in terms of R_K and vice versa). Then all possible powers of the fermionic generators are applied to the reference state in order to obtain the full supermultiplet.

For ''long'' multiplets there are $2^{32/2}$ combinations of linearly independent powers of fermionic operators applied on the reference state [fermionic plus bosonic spinor representations of SO(32)]. Since the reference state has dimension $\dim(R_{c,K})$, then the dimension of the full supermultiplet is $2^{16}\times \dim(R_{c,K})$. Furthermore, since each supergenerator is classified under $SO(d-1,1)\otimes SO(c+1,1)$ or $SO(d-1,1)\otimes K$, it is straightforward to obtain the representation content of each state under these groups. We will argue in the next section that these supermultiplets hide even bigger structures associated with 12D or 13D.

Some of the irreducible supermultiplets are shorter than the naive counting would indicate. This happens whenever there is a linear combination of fermionic generators that vanishes on the reference state

$$\epsilon^{\alpha}_{a(k)} Q^a_{\alpha}|S,R_{c,K}> = 0, \quad k=0,1,\ldots,2n. \tag{16}$$

The supermultiplets associated with such reference states are the BPS-type states. This gives the analogue of shorter multiplets of ordinary supersymmetry with central extensions, but here the possibilities are much richer since there are many more central extensions.

It is important to emphasize that since the momentum P_μ (and, in particular, the mass) is mixed with all other 528 quantum numbers under $SO(c+1,1)$, *these multiplets can contain states of different masses*. We see then that the multiplets have plenty of information about the hidden dimensions or duality symmetries of the theory. The more familiar string states at various excitation levels are part of the multiplet; the additional states needed to complete the multiplet become the prediction of S theory. In previous work some simple examples in this direction were provided [9,5].

Now, we give the covariant criteria, consistent with all the isometries, for the presence of BPS-like supermultiplets. Since Eq. (16) must hold, then it implies that the 32×32 matrix S must have zero eigenvalues with multiplicity n

$$\epsilon^{\alpha}_{a(k)}\{Q^a_{\alpha},Q^b_{\beta}\}|S,R_{c,K}\rangle = 0 \rightarrow \epsilon^{\alpha}_{a(k)} S^{ab}_{\alpha\beta}=0. \tag{17}$$

Therefore, the determinant vanishes

$$\det(S)=0. \tag{18}$$

By writing out the secular equation $\det(S-\lambda)=0$, the multiplicity of the zero eigenvalue (i.e., $2n$) can be determined. In our notation the energy term in $S$is proportional to the identity $S\sim P_0+\cdots$ as described at the end of Sec. III B. Hence, adding the λ is superfluous; we can instead count the multiplicity of the energy eigenvalue at which $\det(S)=0$. This condition is consistent with all the isometries and, therefore, the collection of all BPS-like states that satisfy it must form a shorter supermultiplet of the superalgebra and of the isometries. It is worth emphasizing that all the information about the multiplet is contained in the reference state (15).

V. HIGHER DIMENSIONS AND BPS STATES

As discussed in Sec. III, the form of S in (d,c) dimensions is a rewriting of the original $S_{A,B}$ embedded in 12 or 13 dimensions. Furthermore, the criteria for the BPS-like states (as well as for longer multiplets) are consistent with the higher symmetries that are displayed by the original $S_{A,B}$. Therefore, the long or shorter supermultiplets that are identified in any dimension must also be consistent with the symmetry structure of the hidden 12 or 13 dimensions, i.e., $SO(10,2)_A$ or $SO(9,1)\otimes SO(2,1)_B$. Some examples are provided below.

A. From type IIA superstring to 12D supergravity

As is well known by now the black holes of type IIA superstring can be thought of as Kaluza-Klein states of 11D supergravity compactified to 10D. In our language these BPS states correspond to a reference state

$$|p_\mu, p_{10}, \quad R_{c,K}=\mathbf{1}\rangle, \tag{19}$$

where, in addition to momentum, the only nonzero central extension in 10D is the quantized 11th momentum $p_{10}=n/R$ where R is the radius of compactification (related to the coupling constant as argued by Witten [1]). The remaining bosonic 517 $(=528-11)$ central extensions are set equal to zero. Then Eqs. (9) and (10) and the BPS conditions simplify to

$$S_A = p_\mu(C\gamma^\mu)+p_{10}(C\gamma^{10}),$$

$$\det(S_A)\sim(p_0^2-\vec{p}^{\,2}-p_{10}^2)^{16}=(M_{10}-p_{10})^{16}(M_{10}+p_{10})^{16}, \tag{20}$$

$$\det S_A = 0 \leftrightarrow M_{10}=|p_{10}|, \quad \text{multiplicity}=16,$$

where M_{10} is the mass in 10D, $M_{10}^2=p_0^2-\vec{p}^{\,2}$. Sixteen supersymmetries vanish and $32-16=16$ act nontrivially. Therefore, the dimension of this short supermultiplet is $2^{16/2}=256$, consisting of 2^7 bosons plus 2^7 fermions. This has the same content as the degrees of freedom of 11D supergravity. As is well known, the presence of the 11th momentum p_{10}, as a central extension, indicates the presence of the 11th dimension. In addition, 11D manifests itself in the 256-dimensional supermultiplet of states, since this multiplet is constructed from the direct products of the 32 supercharges that form a spinor basis for 11D.

Now, we discuss the hidden 12D structure. The first hint is that the 32 supercharges form a basis for the chiral spinor representation for 12D. Therefore, the multiplicity 256 is consistent with 12D Lorentz transformations SO(10,2). However, there is no 12th momentum. Indeed, as discussed in Sec. III, a 12th momentum should not be expected, since it does not appear in the superalgebra of type A (9,10). Instead, one should seek the central extensions $Z_{M_1M_2}$, $Z^+_{M_1\ldots M_6}$. Consider a reference state labeled by

$$|Z_{M_1M_2}\neq 0,\quad Z^+_{M_1\ldots M_6}=0\,,\quad R_{c,K}=\mathbf{1}\rangle, \tag{21}$$

with the further special condition $(Z^3)_{M_1M_2}=0$ which is still SO(10,2) covariant [contractions of indices require the (10,2) metric]. A form of $Z_{M_1M_2}$ that satisfies this requirement is

$$Z_{M_1M_2}=\Lambda_{M_1}^{N_1}Z^0_{N_1N_2}\Lambda_{M_2}^{N_2},$$
$$Z^0_{M_1M_2}=\begin{pmatrix}0 & p_{10} & p_\mu\\ -p_{10} & 0 & 0\\ -p_\mu & 0 & 0\end{pmatrix}, \tag{22}$$

where Λ is a SO(10,2) boost and $Z^0_{M_1M_2}$ is the solution in Eq. (20). This form is equivalent to the cross product of two 12D vectors that are orthogonal and one of them is null in 12D:

$$Z_{M_1M_2}=\tfrac{1}{2}(\tilde P_{M_1}\tilde P'_{M_2}-\tilde P_{M_2}\tilde P'_{M_1}),$$
$$\tilde P\cdot\tilde P'=\tilde P\cdot\tilde P=0. \tag{23}$$

The tildes are used to emphasize that these are 12D vectors. S_A and its determinant have the form

$$S_A=C\gamma^{M_1M_2}\tilde P_{M_1}\tilde P'_{M_2},$$
$$\det S_A=[\tilde P^2\tilde P'^2-(\tilde P\cdot\tilde P')^2]^{16}. \tag{24}$$

Therefore, the zero eigenvalue is 16-fold degenerate. Written in this form the reference state is covariant under SO(10,2), but yet it is equivalent to the 11D reference state in Eq. (20) up to a SO(10,2) boost. Combining this reference state with the fact that the 32 supercharges form a spinor representation of SO(10,2), the resulting supermultiplet with a 256 degeneracy (just as in 11D supergravity) must also be consistent with SO(10,2).

Therefore, I conjecture that there should be a reformulation (or generalization, perhaps, by including auxiliary fields) of 11D supergravity that is consistent with more hidden dimensions, is SO(10,2) covariant, and contains the same 256 physical components of fields that are present in 11D supergravity. However, for covariance, the fields should be allowed to depend on more than 11 dimensions, and be consistent with the covariant central extensions given above in the form of equations of motion for the fields

$$\tilde P^2\phi+\cdots=0,\quad \tilde P\cdot\tilde P'\phi+\cdots=0. \tag{25}$$

Such a reformulation would provide one of the simplest low energy effective field theories that describe a sector of the fundamental theory consistently with SO(10,2). In view of the present remarks it may be useful to revive some old attempts in such a direction [10]. Toward this goal, perhaps, a first step should be generalizing the superparticle action with additional degrees of freedom in 12D, such as $\tilde P,\tilde P'$ and superpartners. The canonical quantization of such a generalized superparticle should yield the specialized form of the superalgebra in Eq. (10):

$$\{Q_\alpha,Q_\beta\}=(C\gamma^{M_1M_2})_{\alpha\beta}\tilde P_{M_1}\tilde P'_{M_2}. \tag{26}$$

The form (24) can satisfy $\det S_A=0$ with a slightly less constrained SO(10,2) vectors $\tilde P,\tilde P'$. This corresponds to a larger class of solutions that are not connected to Eq. (20) by SO(10,2) boosts.

1. More 11D↔12D solutions

It is possible to display some special solutions with more central extensions consistent with 11D, and covariantized to 12D by boosts. For example, using 11D γ matrices as in Eq. (10), one may take

$$S_A=C\not P+C\not P'\not P+C\not X_1\not X_2\not X_3\not X_4\not P, \tag{27}$$

where $\not P\equiv P\cdot\gamma,\not P'\equiv P'\cdot\gamma,\not X_i\equiv X_i\cdot\gamma$ are 11D vectors dotted with γ matrices. To ensure the antisymmetry of $Z_{\mu\nu},X_{\mu_1\ldots\mu_5}$, the vectors are taken orthogonal to one another. Furthermore, taking P_μ lightlike in 11D [i.e., $M_{10}=|p_{10}|$ as in Eq. (20)] guarantees that the BPS condition is satisfied

$$\det S_A=\det(C+C\not Z+C\not X_1\not X_2\not X_3\not X_4)\det(\not P)=0. \tag{28}$$

Since

$$\det(\not P)=(M_{10}+p_{10})^{16}(M_{10}+p_{10})^{16}, \tag{29}$$

the multiplicity of the zero eigenvalue is again 16. In this case the reference state has more nonzero central extensions describing more complicated p-branes. These are probably related to one another by various dualities.

Any 11D solution can be boosted to a 12D covariant form by applying an overall SO(10,2) transformation and then identifying the $Z_{M_1M_2}$, $Z^+_{M_1\ldots M_6}$. In the present solution both of these are nonzero, albeit of special forms rather than being the most general. It is because of their special form that the degeneracy of the zero eigenvalue is still 16. With more general forms the degeneracy (and hence the size of the shorter supermultiplets) would be different.

2. Excited levels

The excited states of type IIA (perturbative) string give only a subset of the states of the full secret theory. We have conjectured in the past that the correct set should correspond to supermultiplets in 11D and found some evidence for this [9,5]. We now modify this conjecture because we expect the full theory to be consistent with representations of the superalgebra as described in Sec. III. Then, the excited levels should be classified according to reference states with nontrivial $R_{c,K}$, but the same form of S_A as the base. The well-known (perturbative) excited string states should fill part of these multiplets. The remainder of the multiplet is a prediction about the properties of the underlying secret theory, as seen in examples in our previous work.

B. Example of a vector central extension

Consider superstring theory of type II compactified to 9D. The base state is labeled with a 9D momentum p_μ, a quantized Kaluza-Klein momentum k_9, and a winding number w_9. If these are the only nontrivial central extensions then they are embedded in the 12D Z_{MN} as follows (where the order of the indices is taken as $M,N=9,10,0',0,1,2,\dots,8$)

$$Z_{MN}=\begin{pmatrix} 0 & -w_9 & -k_9 & 0 & 0 & \cdots & 0\\ w_9 & 0 & 0 & 0 & 0 & \cdots & 0\\ k_9 & 0 & 0 & p_0 & p_1 & \cdots & p_8\\ 0 & 0 & -p_0 & 0 & 0 & \cdots & 0\\ 0 & 0 & -p_1 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & -p_8 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{30}$$

while $Z^+_{M_1\dots M_2}=0$. This form is covariant under Lorentz transformations SO(8,1) but noncovariant under the isometries of the superalgebra given in Eq. (8). The perturbative string states are well known at all excited levels. The nonperturbative black hole states satisfy the well-known BPS condition $M_9=|k_9\pm w_9|$, where $M_9^2=p_0^2-\vec{p}^2$. This corresponds to requiring the vanishing of the determinant

$$\det S=(M_9-w_9-k_9)^8(M_9-w_9+k_9)^8(M_9+w_9+k_9)^8 \times(M_9+w_9-k_9)^8, \tag{31}$$

which has an eightfold degeneracy for the zero eigenvalue. This means that eight supergenerators vanish and $32-8=24$ of them act nontrivially on the reference state, giving a well-known shorter supermultiplet of dimension $2^{24/2}=2^{11}_{\text{bosons}}+2^{11}_{\text{fermions}}$.

First, I generalize this by including the central extension P_{10} which is a Lorentz singlet, and whose presence is required by U duality.[3] Then

$$Z_{MN}=\begin{pmatrix} 0 & -w_9 & -k_9 & 0 & 0 & \cdots & 0\\ w_9 & 0 & -k_{10} & 0 & 0 & \cdots & 0\\ k_9 & k_{10} & 0 & p_0 & p_1 & \cdots & p_8\\ 0 & 0 & -p_0 & 0 & 0 & \cdots & 0\\ 0 & 0 & -p_1 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & -p_8 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{32}$$

gives

$$\det S=[(M_9-w_9)^2-k_9^2-k_{10}^2]^8[(M_9+w_9)^2-k_9^2-k_{10}^2]^8, \tag{33}$$

which has manifest $K\otimes SO(8,1)$ symmetry. The degeneracy of the zero eigenvalue is still eight, hence the size of the supermultiplet is $2^{11}_{\text{bosons}}+2^{11}_{\text{fermions}}$ but the base has one more quantum number, and it displays the explicit K isometry. The mass formula is K invariant.

$$M_9=|w_9\pm\sqrt{k_9^2+k_{10}^2}|. \tag{34}$$

So far, this is insufficient to also display the SO $(c+1,1)=SO(2,1)$ isometry given in Eq. (8). As explained there, the SO(2) subgroup is the same as the one appearing in K. The SO(2,1) transformations mix the indices $M=0',9,10$. When these are applied to the Z_{MN} above they require the more general covariant form[4]

$$Z_{MN}=\begin{pmatrix} 0 & -w_9 & -k_9 & z_{90} & z_{91} & \cdots & z_{98}\\ w_9 & 0 & -k_{10} & z_{100} & z_{101} & \cdots & z_{108}\\ k_9 & k_{10} & 0 & p_0 & p_1 & \cdots & p_8\\ -z_{90} & -z_{100} & -p_0 & 0 & 0 & \cdots & 0\\ -z_{91} & -z_{101} & -p_1 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ -z_{98} & -z_{108} & -p_8 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

The additional $z_{9\mu},z_{10\mu}$ are Lorentz vectors that are interpreted as positions of end points of strings in 9D as explained in Sec. II. Therefore, they must be spacelike vectors. Together with the timelike $p_\mu\equiv z_{0'\mu}$, they form a triplet $z_{i\mu}$ of SO(2,1). When these are included in the reference state, we obtain a supermultiplet consistent with SO $(2,1)\otimes SO(8,1)$. The determinant of $S\sim C\gamma^{MN}Z_{MN}$ is

$$\det S=\left[\mathrm{Tr}Z^4-\left(\mathrm{Tr}\frac{Z^2}{2}\right)^2\right]^8. \tag{36}$$

[3] In this case $U=SL(2)\times SO(1,1)$ which has the maximal compact subgroup $K=SO(2)\times Z_2$. Under K the Lorentz singlet central extensions form a doublet (k_9,k_{10}) plus a singlet w_9.

[4] $Z^+_{M_1\dots M_2}=0$ is still consistent with the isometries so far, but without turning on this central extension as well, the full 12D covariance remains hidden.

The BPS determinant condition is evidently invariant under $SO(2,1)\otimes SO(8,1)$, the degeneracy of the zero eigenvalue is eight, and the size of the supermultiplet is $2^{11}_{\rm bosons}+2^{11}_{\rm fermions}$. This supermultiplet is consistent with the isometries $SO(2,1)\otimes SO(8,1)$ (as well as with K) since all supercharges and all central extensions are complete multiplets of the isometries (for their classification in every dimension see [5]).

In this way we have extended the perturbative superstring multiplets to the larger supermultiplets containing nonperturbative states of the secret theory consistent with 12D or 13D.

VI. CONCLUSIONS

In the examples of Sec. IV only some of the central extensions were turned on. This was sufficient to illustrate the method of investigation as well as the relevance and properties of the Lorentz nonsinglet central extensions. Even with this limited set of examples it is clear that the spectrum and multiplet structure of the secret theory is much richer than previously thought. More work is needed to find all the solutions of $\det S_{A,B}=0$ and identify all the distinct shorter multiplets. The number of supersymmetries on BPS states in the examples of this paper were 8,16, but all numbers from 0 to 32 are possible in more general examples. The algebraic approach outlined in this paper seems to be sufficiently powerful to elucidate some of the nonperturbative global properties of the secret theory. For example, by starting with known perturbative string states at any excitation level and using the multiplets of S theory one can, in principle, make predictions on the spectrum of the secret theory. Some preliminary examples of this type were given before [9,5]. Similar considerations should also apply to scattering amplitudes, etc. In S theory one finds that there are up to 13 hidden dimensions, some of which remain hidden from the point of view of perturbative approaches involving p-branes. One of the appealing aspects of the S theory approach is to treat all 528 bosonic generators on an equal footing, thus elucidating the duality and hidden dimensions as simple consequences of the isometries of the maximally extended superalgebra.

There should be many variations of S theory by taking some of the 528 bosonic generators to be non-Abelian. Some of them may also have nontrivial commutation relations with the supergenerators [11]. Such variations of the superalgebra must be related to the geometrical properties of the background in which the p-branes propagate, as opposed to the flat background assumed in the present paper. Evidently, the representation theory of the corresponding superalgebra will be more difficult but more interesting. In this way it should be possible to find relations between the results of M, F, and S theories.

APPENDIX

The γ matrices in 12D with signature (10,2), or in 13D with signature (11,2), may be given explicitly in the following 64×64 purely real (Majorana) representation, using direct products of Pauli matrices,

$$\begin{aligned}
&\Gamma_{0'}=i\tau_2\otimes 1\otimes 1\otimes 1_8, && \Gamma_9=\tau_1\otimes\sigma_1\otimes\tau_3\otimes 1_8,\\
&\Gamma_0=\tau_1\otimes i\sigma_2\otimes 1\otimes 1_8, && \Gamma_8=\tau_1\otimes\sigma_1\otimes\tau_1\otimes 1_8,\\
&\Gamma_{10}=\tau_1\otimes\sigma_3\otimes 1\otimes 1_8, && \Gamma_i=\tau_1\otimes\sigma_1\otimes\tau_2\otimes g_i,\\
&\Gamma_A=\tau_3\otimes 1\otimes 1\otimes 1_8, && C=1\otimes i\sigma_2\otimes 1\otimes 1_8,\\
&\Gamma_B=1\otimes\sigma_3\otimes 1\otimes 1_8, &&
\end{aligned} \tag{A1}$$

where the g_i are purely imaginary 8×8 antisymmetric γ matrices for the remaining seven dimensions. C is the charge conjugation matrix, it has the property that $C\Gamma_M$ is symmetric for the 12D γ matrices, $M=0',0,1,\ldots,10$, or $C\Gamma_M C^{-1}=-(\Gamma_M)^T$.

Γ_A is a 13th γ matrix that is the product of the 12D Γ_M and it anticommutes with them (i.e., analogue of γ_5 in 4D). Γ_A commutes with C and Γ_B. The chiral projector $\frac{1}{2}(1+\Gamma_A)$ serves to project to the 32×32 subspace that is of interest for the type A sector of the theory. Since the chirally projected sector distinguishes the 13th γ matrix, the maximal covariance is broken down from 13D to 12D in the type A sector.

When the antisymmetric products of p γ matrices $\Gamma_{M_1M_2\ldots M_p}$ are multiplied by the projector $1+\Gamma_A$, only the 12D covariant $66\rightarrow\frac{1}{2}(1+\Gamma_A)C\Gamma_{M_1M_2}$ and 462 $\rightarrow\frac{1}{2}(1+\Gamma_A)C\Gamma_{M_1M_2\ldots M_6}$ are symmetric matrices. Therefore, only the two-index and self-dual (in 12D) six-index tensors can appear in the 12D type-A superalgebra. Thus, S_A is a linear combination of these as in Eq. (9). In this 32×32 subspace we may replace each Γ_M by $\frac{1}{2}(1+\Gamma_A)\Gamma_M\frac{1}{2}(1-\Gamma_A)\rightarrow\gamma_M$, where we denote the $32\times 32\gamma$ matrices γ_M by $\gamma_M=(1,\gamma_m)$, with $\gamma_{0'}=1$ and γ_m, $m=0,1,\ldots,10$ given by omitting the first τ_i factors in the expressions of the other 11 Γ_m given above, i.e., $\gamma_0=i\sigma_2\otimes 1\otimes 1_8=C$, $\gamma_{10}=\sigma_3\otimes 1\otimes 1_8$, etc. This form may be used in Eq. (10) to simplify it to the 11D notation of Eq. (9).

$\frac{1}{2}(1+\Gamma_B)$ is the projector to the 32-dimensional subspace relevant for the type-B sector of the theory. Γ_B is the product of the usual 10D $\Gamma_\mu\gamma$ matrices $\Gamma_B\sim\Gamma_0\Gamma_1\ldots\Gamma_9$. One can also write

$$\Gamma_B\sim\Gamma_{0'}\Gamma_{10}\Gamma_A\sim\Gamma_0\Gamma_1\ldots\Gamma_9. \tag{A2}$$

Γ_B commutes with each $\Gamma_i\equiv(\Gamma_{0'},\Gamma_{10},\Gamma_A)$ and anticommutes with C and each $\Gamma_\mu=(\Gamma_0,\Gamma_1,\ldots,\Gamma_9)$

$$[\Gamma_B,\Gamma_i]=0,\quad \{\Gamma_B,\Gamma_\mu\}=\{\Gamma_B,C\}=0. \tag{A3}$$

Therefore, this projection breaks the symmetry from 13D to $10D\otimes 3D$ since it treats the 3D differently than the 10D. The three Γ_i may be regarded as the γ matrices for a 3D hidden Minkowski space just as the ten Γ_μ are the γ matrices for the 10D Minkowski space. This extra space is evidently related to the geometrical origin of the $SL(2,R)$ symmetry of the type B sector.

S_B is constructed from 528 linearly independent symmetric 32×32 matrices of type B. In 64×64 notation these are given by

$$\frac{1}{2}(1+\Gamma_B)C\Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}(\Gamma_A\Gamma_i) \quad : \quad \frac{1}{2}\,\frac{10\times 9\times 8\times 7\times 6}{1\times 2\times 3\times 4\times 5}\times 3=378,$$

$$\frac{1}{2}(1+\Gamma_B)C\Gamma_{\mu_1\mu_2\mu_3}(\Gamma_A) \quad : \quad \frac{10\times 9\times 8}{1\times 2\times 3}=120,$$

$$\frac{1}{2}(1+\Gamma_B)C\Gamma_{\mu_1}(\Gamma_A\Gamma_i) \quad : \quad 10\times 3=30, \tag{A4}$$

where the chirally projected five-index γ matrices are self-dual in 10D. In the chirally projected B sector, these matrices reduce to 32×32 blocks which may be conveniently written in the form of direct products of 2×2 times 16×16 matrices $\overline{\tau}\otimes\overline{\gamma}$ as in Eq. (12), where the 2×2 part comes directly from the first factor and the 16×16 part comes from the last three factors in the γ matrix expressions Γ in Eqs. (A1) and (A4).

One may consider SO(1,2)=SL(2,R) rotations of S_B in the extra 3D subspace, leaving unaffected the usual ten dimensions. To make the connection to 13D, we give it in the form of rotations in the 64×64 spinor space

$$\delta S_B=[\epsilon^{ij}\Gamma_{ij},S_B]. \tag{A5}$$

One can show that

$$\epsilon^{ij}\Gamma_{ij}\left(\frac{1}{2}(1+\Gamma_B)C\Gamma_{\mu_1\cdots\mu_p}\right) =\left(\frac{1}{2}(1+\Gamma_B)C\Gamma_{\mu_1\cdots\mu_p}\right)\Gamma_A\epsilon^{ij}\Gamma_{ij}\Gamma_A \tag{A6}$$

when $p=$odd. Using this identity we see easily that the following commutators simplify

$$\left[\epsilon^{ij}\Gamma_{ij},\left(\frac{1}{2}(1+\Gamma_B)C\Gamma_{\mu_1\cdots\mu_p}(\Gamma_A\Gamma_{k\ldots})\right)\right] =\frac{1}{2}(1+\Gamma_B)C\Gamma_{\mu_1\cdots\mu_p}\Gamma_A[\epsilon^{ij}\Gamma_{ij},(\Gamma_{k\ldots})]. \tag{A7}$$

The last commutator is just the rule for performing rotations in the 3D subspace. This shows that only the 3D indices rotate under these SO(1,2) rotations embedded in 13D rotations and, hence, verifies that the construction of the 528 matrices (A4) is the right one. Furthermore, this result is consistent with using the direct product notation of Eq. (12). The construction of Eq. (A4) is useful because it exhibits the precise embedding of the type-B space in the spinor space of 13D.

From the constructions given above, we see that the type A and type B are different projections within the same 64×64 spinor space of 13D. Hence, the duality of the type-A and type-B sectors of the theory has its origins in the spinor space for 13D.

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