Gauss's law and gauge-invariant operators and states in QCD

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In this work, we prove a previously published conjecture that a prescription we gave for constructing states that implement Gauss's law for ''pure glue'' QCD is correct. We also construct a unitary transformation that extends this prescription so that it produces additional states that implement Gauss's law for QCD with quarks as well as gluons. Furthermore, we use the mathematical apparatus developed in the course of this work to construct gauge-invariant spinor (quark) and gauge (gluon) field operators. We adapt this SU(3) construction for the SU(2) Yang-Mills case, and we consider the dynamical implications of these developments. [S0556-2821(97)08504-4]

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I. INTRODUCTION

The need to implement Gauss's law in QCD and Yang-Mills theory, and the technical problems that complicate the implementation of Gauss's law in non-Abelian theories have been discussed by a number of authors [1–4]. Strategies for implementing Gauss's law have also been developed [5]. In earlier work [6], we constructed states that implement Gauss's law for Yang-Mills theory and QCD — in fact, for any ''pure glue'' gauge theory, in a temporal gauge formulation that has a non-Abelian SU(N) gauge symmetry. In that work, a state vector $\Psi\ |\phi\rangle$ was defined for which

$$\{b_Q^a(\mathbf{k})+J_0^a(\mathbf{k})\}\Psi\ |\phi\rangle=0\ \ , \tag{1.1}$$

where $b_Q^a(\mathbf{k})$ and $J_0^a(\mathbf{k})$ are the Fourier transforms of $\partial_i\Pi_i^a(\mathbf{r})$ [$\Pi_i^a(\mathbf{r})$ is the momentum conjugate to the gauge field] and of the gluon color charge density

$$J_0^a(\mathbf{r})=g\ f^{abc}A_i^b(\mathbf{r})\ \Pi_i^c(\mathbf{r})\ \ , \tag{1.2}$$

respectively. Since the chromoelectric field $E_i^a(\mathbf{r})=-\Pi_i^a(\mathbf{r})$, Eq. (1.1) expresses the momentum space representation of the non-Abelian ''pure glue'' Gauss's law, and $\{b_Q^a(\mathbf{k})+J_0^a(\mathbf{k})\}$ is referred to as the ''Gauss's law operator'' for the ''pure glue'' case; $|\phi\rangle$ is a perturbative state annihilated by $\partial_i\Pi_i^a(\mathbf{r})$. In Ref. [6], we exhibited an explicit form for the operator Ψ: namely,

$$\Psi=\|\ \exp(\mathcal{A})\ \|\ \ , \tag{1.3}$$

where bracketing between double bars denotes a normal order in which all gauge fields and functionals of gauge fields appear to the left of all momenta conjugate to gauge fields. $\mathcal{A}$ was exhibited as an operator-valued series in Ref. [6]. Its form was conjectured to all orders, and verified for the first six orders.

In the work presented here we will extend our previously published results in the following ways: we will prove our earlier conjecture that the state $\Psi|\phi\rangle$ implements the ''pure glue'' form of Gauss's law; we will extend our work from the ''pure glue'' form of the theory to include quarks as well as gluons; we will construct gauge-invariant operator-valued spinor (quark) and gauge (gluon) fields; and we will adapt the QCD formulation to apply to the SU(2) Yang-Mills theory.

II. IMPLEMENTING THE ''PURE GLUE'' FORM OF GAUSS'S LAW

Our construction of Ψ in Ref. [6] was informed by the realization that the operator Ψ had to implement $\{b_Q^a(\mathbf{k})+J_0^a(\mathbf{k})\}\Psi\ |\phi\rangle=\Psi\ b_Q^a(\mathbf{k})\ |\phi\rangle$, or equivalently that

$$[b_Q^a(\mathbf{k}),\ \Psi]=-J_0^a(\mathbf{k})\Psi+B_Q^a(\mathbf{k})\ \ , \tag{2.1}$$

where $B_Q^a(\mathbf{k})$ is an operator product that has $\partial_i\Pi_i^a(\mathbf{r})$ on its extreme right and therefore annihilates the same states as $b_Q^a(\mathbf{k})$, so that $B_Q^a(\mathbf{k})\ |\phi\rangle=0$ as well as $b_Q^a(\mathbf{k})\ |\phi\rangle=0$.

To facilitate the discussion of the structure of Ψ, the following definitions are useful:

$$a_i^\alpha(\mathbf{r})=A_{Ti}^\alpha(\mathbf{r}) \tag{2.2}$$

denotes the transverse part of the gauge field, and

$$x_i^\alpha(\mathbf{r})=A_{Li}^\alpha(\mathbf{r}) \tag{2.3}$$

denotes the longitudinal part, so that $[a_i^\alpha(\mathbf{r})+x_i^\alpha(\mathbf{r})]=A_i^\alpha(\mathbf{r})$. We also will make use of the combinations

$$\mathcal{X}^\alpha(\mathbf{r})=\left[\frac{\partial_i}{\partial^2}A_i^\alpha(\mathbf{r})\right]\ \ , \tag{2.4}$$

and

$$\mathcal{Q}^\beta_{(\eta)i}(\mathbf{r})=\left[a_i^\beta(\mathbf{r})+\frac{\eta}{\eta+1}x_i^\beta(\mathbf{r})\right]\ \ , \tag{2.5}$$

where η is an integer-valued index.

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We will furthermore refer to the composite operators

$$\psi^{\gamma}_{(\eta)i}(\mathbf{r})=(-1)^{\eta-1}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\mathcal{R}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\,\mathcal{Q}^{\beta}_{(\eta)i}(\mathbf{r}) \quad , \tag{2.6}$$

in which $\mathcal{R}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ is given by

$$\mathcal{R}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})=\prod_{m=1}^{\eta}\mathcal{X}^{\alpha[m]}(\mathbf{r}) \quad , \tag{2.7}$$

and $f^{\vec{\alpha}\beta\gamma}_{(\eta)}$ is the chain of SU(3) structure functions

$$f^{\vec{\alpha}\beta\gamma}_{(\eta)}=f^{\alpha[1]\beta b[1]}\;f^{b[1]\alpha[2]b[2]}\;f^{b[2]\alpha[3]b[3]}\cdots \times f^{b[\eta-2]\alpha[\eta-1]b[\eta-1]}f^{b[\eta-1]\alpha[\eta]\gamma} \quad , \tag{2.8}$$

where repeated indices are to be summed. For $\eta=1$, the chain reduces to $f^{\vec{\alpha}\beta\gamma}_{(1)}\equiv f^{\alpha\beta\gamma}$; and for $\eta=0$, $f^{\vec{\alpha}\beta\gamma}_{(0)}\equiv-\delta_{\beta,\gamma}$. Since the only properties of the structure functions that we will use is their antisymmetry and the Jacobi identity, the formalism we develop will be applicable to SU(2) as well as to other models with an SU(N) gauge symmetry.

The composite operators introduced so far can help us to understand qualitatively how Ψ can implement Eq. (1.1). We observe, for example, the product

$$\psi^{\gamma}_{(1)i}(\mathbf{r})=f^{\alpha\beta\gamma}\mathcal{X}^{\alpha}(\mathbf{r})\,\mathcal{Q}^{\beta}_{(1)i}(\mathbf{r}) = f^{\alpha\beta\gamma}\mathcal{X}^{\alpha}(\mathbf{r})\;[a^{\beta}_{i}(\mathbf{r})+\tfrac{1}{2}\,x^{\beta}_{i}(\mathbf{r})] \quad , \tag{2.9}$$

which as part of the expression

$$\mathcal{A}_1=ig\int d\mathbf{r}\;\psi^{\gamma}_{(1)i}(\mathbf{r})\Pi^{\gamma}_{i}(\mathbf{r}) \quad , \tag{2.10}$$

has the property that its commutator with $b^{a}_{Q}(\mathbf{k})$,

$$\left[b^{a}_{Q}(\mathbf{k}),ig\int d\mathbf{r}\;\psi^{\gamma}_{(1)i}(\mathbf{r})\Pi^{\gamma}_{i}(\mathbf{r})\right] = -g\;f^{a\beta\gamma}\int d\mathbf{r}\;\; e^{-i\mathbf{k}\cdot\mathbf{r}}A^{\beta}_{i}(\mathbf{r})\Pi^{\gamma}_{i}(\mathbf{r}) -\frac{g}{2}f^{a\beta\gamma}\int d\mathbf{r}\;e^{-i\mathbf{k}\cdot\mathbf{r}}\mathcal{X}^{\beta}[\partial_i\Pi^{\gamma}_{i}(\mathbf{r})\;], \tag{2.11}$$

generates $-J^{a}_{0}(\mathbf{k})$ when it acts on a state annihilated by $b^{a}_{Q}(\mathbf{k})$. The expression $\exp(\mathcal{A}_1)$ would therefore have been an appropriate choice for Ψ, were it not for the fact that the commutator $[b^{a}_{Q}(\mathbf{k}),\,\mathcal{A}_1]$ fails to commute with $\mathcal{A}_1$. When Eq. (1.1) is applied to a candidate $\Psi_{\rm cand}=\exp(\mathcal{A}_1)$, the commutator $[b^{a}_{Q}(\mathbf{k}),\,\mathcal{A}_1]$ is often produced within a polynomial consisting of $\mathcal{A}_1$ factors — for example $\mathcal{A}_1^{(n-s)}[b^{a}_{Q}(\mathbf{k}),\,\mathcal{A}_1]\;\mathcal{A}_1^{s}$. $[b^{a}_{Q}(\mathbf{k}),\,\mathcal{A}_1]$ does not commute with $\mathcal{A}_1$, and cannot move freely to annihilate the state at the right of $\Psi_{\rm cand}$, thereby excluding $\exp(\mathcal{A}_1)$ as a viable choice for Ψ.

The normal ordering denoted by bracketing between double bars eliminates this problem, but only at the expense of introducing another problem in its place—one that is more benign, but that nevertheless must be addressed. When normal ordering is imposed, the result of commuting $\|\exp(\mathcal{A}_1)\|$ with $b^{a}_{Q}(\mathbf{k})$ is not the formation of $J^{a}_{0}(\mathbf{k})$ to the left of $\Psi_{\rm cand}$, but the formation of only $f^{a\beta\gamma}\int d\mathbf{r}\;e^{-i\mathbf{k}\cdot\mathbf{r}}\,A^{\beta}_{i}(\mathbf{r})$ to the *left* of it, and of $\Pi^{\gamma}_{i}(\mathbf{r})$ to the extreme *right* of all the gauge fields in the series representation of the exponential. Unwanted terms will be generated as $\Pi^{\gamma}_{i}(\mathbf{r})$ is commuted, term by term, from the extreme right of $\Psi_{\rm cand}$ to the extreme left to form the desired $J^{a}_{0}(\mathbf{k})$. To compensate for these further terms, we modify $\Psi_{\rm cand}$ by adding additional expressions to $\mathcal{A}_1$ to eliminate the unwanted commutators generated as $\Pi^{\gamma}_{i}(\mathbf{r})$ is commuted from the right to the left-hand sides of operator-valued polynomials. The question naturally arises whether the process of adding terms to remove the unwanted contributions from earlier ones, comes to closure — whether an operator-valued series $\mathcal{A}$, that leads to a Ψ for which Eq. (1.1) is satisfied, can be specified to all orders. In Ref. [6] we conjectured that this question can be answered affirmatively, by formulating a recursive equation for $\mathcal{A}$, which we verified to sixth order.

In Ref. [6] we represented $\mathcal{A}$ as the series $\mathcal{A}=\Sigma_{n=1}^{\infty}\mathcal{A}_n$; we also showed that the requirement that $\mathcal{A}$ must satisfy to implement Eq. (1.1) can be formulated as

$$\left\|\left[b^{a}_{Q}(\mathbf{k}),\sum_{n=2}^{\infty}\mathcal{A}_n\right]\exp(\mathcal{A})\right\| - \left\|\,g\;f^{a\beta\gamma}\int d\mathbf{r}\;e^{-i\mathbf{k}\cdot\mathbf{r}}A^{\beta}_{i}(\mathbf{r}) \times[\exp(\mathcal{A}),\;\Pi^{\gamma}_{i}(\mathbf{r})]\right\|\approx 0 \quad , \tag{2.12}$$

where $\approx$ indicates a ''soft'' equality, that only holds when the equation acts on a state $|\phi\rangle$ annihilated by $b^{a}_{Q}(\mathbf{k})$. The commutator $[\exp(\mathcal{A}),\;\Pi^{\gamma}_{i}(\mathbf{r})]$ in Eq. (2.12) reflects the fact that when the gluonic ''color'' charge density is assembled to the left of the candidate Ψ, the momentum conjugate to the gauge field must be moved from the extreme right to the extreme left of $\|\exp(\mathcal{A})\|$. Since $\mathcal{A}$ is a complicated multilinear functional of the gauge fields, but has a simple linear dependence on $\Pi^{\gamma}_{i}(\mathbf{r})$, it is useful to represent it as

$$\mathcal{A}=i\int d\mathbf{r}\;\;\overline{\mathcal{A}^{\gamma}_{i}}(\mathbf{r})\Pi^{\gamma}_{i}(\mathbf{r}) \quad , \tag{2.13}$$

where

$$\overline{\mathcal{A}^{\gamma}_{i}}(\mathbf{r})=\sum_{n=1}^{\infty}g^{n}\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r}) \quad , \tag{2.14}$$

and the $\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})$ are elements in a series whose initial term is $\mathcal{A}^{\gamma}_{(1)i}(\mathbf{r})=\psi^{\gamma}_{(1)i}(\mathbf{r})$. All the $\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})$ consist of gauge fields and functionals of gauge fields only; there are no conjugate momenta, $\Pi^{\gamma}_{i}(\mathbf{r})$, in any of the $\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})$. We also showed in Ref. [6] that Eq. (2.12) is equivalent to

$$[b^{a}_{Q}(\mathbf{k}),\;\mathcal{A}_n]\approx g\;f^{a\beta\gamma}\int d\mathbf{r}\;e^{-i\mathbf{k}\cdot\mathbf{r}}\;A^{\beta}_{i}(\mathbf{r})\;[\mathcal{A}_{n-1},\;\Pi^{\gamma}_{i}(\mathbf{r})] \quad , \tag{2.15}$$

for $\mathcal{A}_n$ with $n>1$, where the $\mathcal{A}_n$ form the series $\mathcal{A}=\Sigma_{n=1}^{\infty}\mathcal{A}_n$, and each $\mathcal{A}_n$ can be represented as

$$\mathcal{A}_n = ig^n \int d\mathbf{r} \;\; \mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})\Pi^{\gamma}_i(\mathbf{r}) \quad . \tag{2.16}$$

If $\mathcal{A}_n$ satisfies Eq. (2.15), then the Ψ defined in Eq. (1.3) will also necessarily satisfy Eq. (1.1), and the state $\Psi\,|\phi\rangle$ will implement the non-Abelian "pure glue" Gauss's law.

In Ref. [6] we gave the form of $\mathcal{A}$ as a functional of the auxiliary operator-valued constituents

$$\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r}) = \prod_{m=1}^{\eta} \frac{\partial_j}{\partial^2}\overline{\mathcal{A}^{\alpha[m]}_j}(\mathbf{r}) = \prod_{m=1}^{\eta} \overline{\mathcal{Y}^{\alpha[m]}}(\mathbf{r})$$

$$= \overline{\mathcal{Y}^{\alpha[1]}}(\mathbf{r})\;\overline{\mathcal{Y}^{\alpha[2]}}(\mathbf{r})\cdots\overline{\mathcal{Y}^{\alpha[\eta]}}(\mathbf{r}) \quad , \tag{2.17}$$

and

$$\overline{\mathcal{B}^{\beta}_{(\eta)i}}(\mathbf{r}) = a^{\beta}_i(\mathbf{r}) + \left(\delta_{ij} - \frac{\eta}{\eta+1}\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}^{\beta}_j}(\mathbf{r}) \quad , \tag{2.18}$$

where

$$\overline{\mathcal{Y}^{\alpha}}(\mathbf{r}) = \frac{\partial_j}{\partial^2}\overline{\mathcal{A}^{\alpha}_j}(\mathbf{r}) \quad \text{and} \quad \mathcal{Y}^{\alpha}_{(s)}(\mathbf{r}) = \frac{\partial_j}{\partial^2}\mathcal{A}^{\alpha}_{(s)j}(\mathbf{r}) \quad . \tag{2.19}$$

The defining equation for $\mathcal{A}$ is the recursive

$$\mathcal{A} = \sum_{\eta=1}^{\infty} \frac{ig^{\eta}}{\eta!}\int d\mathbf{r} \;\; \{\psi^{\gamma}_{(\eta)i}(\mathbf{r})$$

$$+ f^{\vec{\alpha}\beta\gamma}_{(\eta)}\,\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\;\overline{\mathcal{B}^{\beta}_{(\eta)i}}(\mathbf{r})\}\Pi^{\gamma}_i(\mathbf{r}) \quad . \tag{2.20}$$

In Ref. [6], we presented this form as a conjecture that we had verified to sixth order only. In this work, we will prove that $\Psi\,|\phi\rangle$ satisfies the "pure glue" Gauss's law by showing that the $\mathcal{A}$ given in Eq. (2.20) satisfies Eq. (2.15).

The form of $\mathcal{A}$ suggests that the proposition that it satisfies Eq. (2.15) is well suited to an inductive proof. We observe that two kinds of terms appear on the right-hand side of Eq. (2.20). One is the inhomogeneous term $\psi^{\gamma}_{(\eta)i}(\mathbf{r})$; the other is the product of $\overline{\mathcal{B}^{\beta}_{(\eta)i}}(\mathbf{r})$ and $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$. $\overline{\mathcal{B}^{\beta}_{(\eta)i}}(\mathbf{r})$ is a functional of $\overline{\mathcal{A}^{\beta}_i}(\mathbf{r})$, and $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ is a multilinear functional of $\overline{\mathcal{Y}^{\beta}}(\mathbf{r})$, which is given as a functional of $\overline{\mathcal{A}^{\beta}_i}(\mathbf{r})$ in Eq. (2.19). It is useful to examine the rth order components of $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ and $\overline{\mathcal{B}^{\beta}_{(\eta)i}}(\mathbf{r})$. These are given, respectively, by

$$\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r}) = \Theta(r-\eta)\sum_{r[1],\cdots,r[\eta]} \delta_{r[1]+\cdots+r[\eta]-r}$$

$$\times \prod_{m=1}^{\eta} \mathcal{Y}^{\alpha[m]}_{(r[m])}(\mathbf{r}) \quad , \tag{2.21}$$

and

$$\mathcal{B}^{\beta}_{(\eta,r)i}(\mathbf{r}) = \delta_r\;a^{\beta}_i(\mathbf{r}) + \left(\delta_{ij} - \frac{\eta}{\eta+1}\frac{\partial_i\partial_j}{\partial^2}\right)\mathcal{A}^{\beta}_{(r)j}(\mathbf{r}) \quad , \tag{2.22}$$

where the subscript r is an integer-valued index that labels the order in the expansion of $\overline{\mathcal{A}^{\gamma}_i}(\mathbf{r})$, and δ_r is the Kronecker "δ" that vanishes unless $r=0$. In Eqs. (2.17) and (2.21), η is a "multiplicity index" that defines the multilinearity of $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ in $\overline{\mathcal{Y}^{\beta}}(\mathbf{r})$. Equations (2.20)–(2.22) demonstrate that an $\mathcal{A}_r$ that appears on the left-hand side (LHS) of Eq. (2.20) is given in terms of the rth order inhomogeneous term $\psi^{\gamma}_{(r)j}(\mathbf{r})\Pi^{\gamma}_j(\mathbf{r})$, and $\mathcal{A}^{\beta}_{(r')j}$ terms on the RHS of this equation in which $r'<r$. To emphasize this very crucial observation, we note that in addition to the g^{η} that appears as an overall factor in Eq. (2.20), each $\overline{\mathcal{A}^{\beta}_j}(\mathbf{r})$ in $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ and $\overline{\mathcal{B}^{\beta}_{(\eta)i}}$ carries its own complement of coupling constants — g^r for each order r. The rth order term on the LHS of Eq. (2.20), therefore, depends on RHS contributions from $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ and $\overline{\mathcal{B}^{\beta}_{(\eta)i}}\,(\mathbf{r})$ whose orders do not add up to r, but only to $r-\eta$. Since the summation in Eq. (2.20) begins with $\eta=1$, the highest possible order of $\mathcal{A}^{\gamma}_{(r')j}$ that can appear on the RHS of Eq. (2.20), when $\mathcal{A}_r$ is on the LHS, is $\mathcal{A}^{\gamma}_{(r-1)j}$ — and that must stem from the $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ with the multiplicity index $\eta=1$. Contributions from $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$ with higher multiplicity indices are restricted to $\mathcal{A}^{\gamma}_{(r')j}$ with even lower order r'. This feature of Eq. (2.20) naturally leads us to consider an inductive proof — one in which we assume Eq. (2.20) for all $\mathcal{A}_r$ with $r\leqslant N$, and then use that assumption to prove it for $\mathcal{A}_r$ with $r=N+1$.

The fact that Eq. (2.15) is a "soft" equation, is an impediment to an inductive proof of the proposition that $\mathcal{A}_n$, defined by Eq. (2.20), satisfies it. In order to carry out the needed inductive proof, we must infer correct "hard" generalizations of both these equations, in which $\mathcal{A}$ is replaced by $i\int d\mathbf{r}\;\;\overline{\mathcal{A}^{\gamma}_i}(\mathbf{r})\;\;V^{\gamma}_i(\mathbf{r})$, where $V^{\gamma}_i(\mathbf{r})$ is *any* field that transforms appropriately, and $\partial_i V^{\gamma}_i(\mathbf{r})$ is not required to annihilate any states. The generalization we seek is an exact equality between operator-valued quantities — one that is true in general, and not only when both sides of the equation project on a specified subset of states. Such a generalization would, in particular, allow us to use many different spatial vectors in the role of $V^{\gamma}_i(\mathbf{r})$ in the course of the inductive proof.

We have made the necessary generalization, and have arrived at the defining equation for the nth order term of $i\int d\mathbf{r}\;\;\overline{\mathcal{A}^{\gamma}_i}(\mathbf{r})V^{\gamma}_i(\mathbf{r})$, that generalizes Eq. (2.20):

$$ig^n\int d\mathbf{r}\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})V^{\gamma}_i(\mathbf{r})$$

$$= \frac{ig^n}{n!}\int d\mathbf{r}\;\;\psi^{\gamma}_{(n)i}(\mathbf{r})V^{\gamma}_i(\mathbf{r})$$

$$+ \sum_{\eta=1}\frac{ig^{\eta}}{\eta!}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\sum_{u=0}\sum_{r=\eta}\delta_{r+u+\eta-n}$$

$$\times\int d\mathbf{r}\;\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r})\mathcal{B}^{\beta}_{(\eta,u)i}(\mathbf{r})V^{\gamma}_i(\mathbf{r}) \quad . \tag{2.23}$$

The generalization of Eq. (2.15) — we make use of the configuration-space representation of the Gauss's law operator in this case, instead of its Fourier transform — is

$$
\begin{aligned}
& i\int d\mathbf{r}'[\partial_i\Pi_i^a(\mathbf{r}),\mathcal{A}^\gamma_{(n)j}(\mathbf{r}')]V_j^\gamma(\mathbf{r}')+\delta_{n-1}f^{a\mu\gamma}A_i^\mu(\mathbf{r})V_i^\gamma(\mathbf{r})-\sum_{\eta=1}\sum_{r=\eta}\delta_{r+\eta-(n-1)}\frac{B(\eta)}{\eta!}f^{a\mu c}f^{\vec{\alpha}c\gamma}_{(\eta)}A_i^\mu(\mathbf{r})\\
&\quad\times\frac{\partial_i}{\partial^2}[\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r})\partial_jV_j^\gamma(\mathbf{r})]+\sum_{\eta=0}\sum_{t=1}\sum_{r=\eta}\delta_{r+t+\eta-n}(-1)^{t-1}\frac{B(\eta)}{\eta!(t-1)!(t+1)}f^{\vec{\mu}a\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(\eta)}\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r})\partial_iV_i^\gamma(\mathbf{r})\\
&\quad+f^{a\mu d}A_i^\mu(\mathbf{r})\sum_{\eta=0}\sum_{t=1}\sum_{r=\eta}\delta_{r+t+\eta-(n-1)}(-1)^t\frac{B(\eta)}{\eta!(t+1)!}f^{\vec{\nu}d\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(\eta)}\frac{\partial_i}{\partial^2}[\mathcal{R}^{\vec{\nu}}_{(t)}(\mathbf{r})\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r})\partial_jV_j^\gamma(\mathbf{r})]\\
&=-if^{a\mu\sigma}A_i^\mu(\mathbf{r})\int d\mathbf{r}'[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{A}^\gamma_{(n-1)j}(\mathbf{r}')]V_j^\gamma(\mathbf{r}')\quad,
\end{aligned}\tag{2.24}
$$

where $B(\eta)$ denotes the ηth Bernoulli number. Equation (2.24) relates $\mathcal{A}^\gamma_{(n)j}(\mathbf{r})$ with $n\geqslant 1$, on the LHS of the equation, to $\mathcal{A}^{\gamma'}_{(n-1)j'}(\mathbf{r}')$ on the RHS; $\mathcal{A}^\gamma_{(n)j}(\mathbf{r})$ with $n=0$ is not required for the representation of $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ given in Eq. (2.14), and therefore does not have to be considered. $\mathcal{A}^\gamma_{(n)j}(\mathbf{r})$ with $n=1$ *is* required, but $[\partial_i\Pi_i^a(\mathbf{r}),\mathcal{A}^\gamma_{(1)j}(\mathbf{r}')]$ cannot be described properly by Eq. (2.24), unless $\mathcal{A}^\gamma_{(0)j}(\mathbf{r})$ on the RHS of Eq. (2.24) is given an appropriate definition. The only equation like Eq. (2.24), but with $\int d\mathbf{r}'[\partial_i\Pi_i^a(\mathbf{r})\mathcal{A}^\gamma_{(1)j}(\mathbf{r}')]V_j^\gamma(\mathbf{r}')$ appearing on its LHS, is Eq. (2.11) with $\Pi_i^\gamma(\mathbf{r})$ replaced by $V_i^\gamma(\mathbf{r})$. We have formulated Eq. (2.24) so that it includes the case of $\int d\mathbf{r}'[\partial_i\Pi_i^a(\mathbf{r}),\mathcal{A}^\gamma_{(1)j}(\mathbf{r}')]V_j^\gamma(\mathbf{r}')$ on the LHS, by including the RHS of Eq. (2.11) for the $n=1$ case. To include that case correctly, we define the degenerate $\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r})$ with $\eta=r=0$ as $\mathcal{M}^{\vec{\alpha}}_{(0,0)}(\mathbf{r})=1$, and the degenerate $\mathcal{A}^\gamma_{(0)j}(\mathbf{r})=0$. We will refer to Eq. (2.24) as the ''fundamental theorem'' for this construction of Ψ.

The general plan for the inductive proof of Eq. (2.24) is as follows: We *assume* Eq. (2.24) for all values of $n\leqslant N$. We then observe that, in the $n=N+1$ case to be proven, the RHS of Eq. (2.24) becomes $\mathsf{RHS}_{(N+1)}=-if^{a\mu\sigma}A_i^\mu(\mathbf{r})\int d\mathbf{r}'[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{A}^\gamma_{(N)\,j}(\mathbf{r}')]V_j^\gamma(\mathbf{r}')$. We use Eq. (2.23) to substitute for the $\mathcal{A}^\gamma_{(N)j}\,V_j^\gamma$ in $\mathsf{RHS}_{(N+1)}$, and evaluate the resulting commutators $[\Pi_i^\sigma(\mathbf{r}),\,\psi^\gamma_{(N)j}(\mathbf{r}')]$, $[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r}')]$, and $[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{B}^\beta_{(\eta,u)j}(\mathbf{r}')]$. Since $\psi^\gamma_{(N)j}(\mathbf{r}')$ is a known inhomogeneity in Eq. (2.23), $[\Pi_i^\sigma(\mathbf{r}),\,\psi^\gamma_{(N)j}(\mathbf{r}')]$ can be explicitly evaluated. In expanding the $f^{\vec{\alpha}\beta\gamma}_{(\eta)}[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r}')]$ that result from the substitution of Eq. (2.23) into $\mathsf{RHS}_{(N+1)}$, we make use of the identity

$$
\begin{aligned}
& f^{\vec{\alpha}\delta\gamma}_{(\eta)}\sum_{r=\eta}\delta_{r+\eta+u-N}[\mathsf{Q}(\mathbf{r}),\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r}')]\\
&\quad=-[\mathcal{P}^{(0)}_{(\alpha,\beta[\eta-1])}f^{\alpha\delta e}f^{\vec{\beta}e\gamma}_{(\eta-1)}]\\
&\qquad\times\sum_{p=\eta-1}\sum_{r[\eta]=1}\delta_{p+r[\eta]+u+\eta-N}[\mathsf{Q}(\mathbf{r}),\,\mathcal{Y}^\alpha_{(r[\eta])}(\mathbf{r}')]\\
&\qquad\times\mathcal{M}^{\vec{\beta}}_{(\eta-1,p)}(\mathbf{r}')\quad,
\end{aligned}\tag{2.25}
$$

where $\mathsf{Q}(\mathbf{r})$ is any arbitrary operator; at times, the commutator $[\mathsf{Q}(\mathbf{r})\,,\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r}')]$ will represent a partial derivative $\partial_j\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r}')$. $\mathcal{P}^{(0)}_{(\alpha,\beta[\eta-1])}$ represents a sum over permutations over the indices labeling the $\mathcal{Y}^{\alpha[\eta]}_{(r[\eta])}(\mathbf{r}')$ factors that constitute $\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r}')$, as shown in Eq. (2.17). $\mathcal{P}^{(0)}_{(\alpha,\beta[\eta-1])}$ is defined by

$$
\begin{aligned}
&[\mathcal{P}^{(0)}_{(e,\beta[\eta-1])}f^{e\delta f}f^{\vec{\beta}f\gamma}_{(\eta-1)}]\mathcal{M}^{\vec{\beta}}_{(\eta-1)}\\
&\quad=\sum_{s=0}^{\eta-1}f^{\vec{\beta}\delta u}_{(s)}f^{uev}f^{\vec{\sigma}v\gamma}_{(\eta-s-1)}\mathcal{M}^{\vec{\beta}}_{(s)}\mathcal{M}^{\vec{\sigma}}_{(\eta-s-1)}\quad.
\end{aligned}\tag{2.26}
$$

Equations (2.25) and (2.26) apply not only to those specific cases, but also to all other operators — such as $\mathcal{R}^{\vec{\alpha}}_{(\eta)}(\mathbf{r}')$ — that similarly are products of factors, identical except for their Lie group indices contracted over chains of structure functions.

With the substitution of Eq. (2.23) into $\mathsf{RHS}_{(N+1)}$, and extensive integration by parts, we have replaced the commutator $[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{A}^\gamma_{(N)j}(\mathbf{r}')]$ which appears in $\mathsf{RHS}_{(N+1)}$, with products of chains of $\mathcal{A}^{\beta'}_{(n')j}(\mathbf{r}')$ and one commutator $[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{A}^\gamma_{(l)j}(\mathbf{r}')]$ with $l\leqslant N-1$. Although the $[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{A}^\gamma_{(N)j}(\mathbf{r}')]$ in $\mathsf{RHS}_{(N+1)}$ is *not* covered by the inductive axiom — it is the RHS of the equation for the $n=N+1$ case — the $[\Pi_i^\sigma(\mathbf{r}),\,\mathcal{A}^\gamma_{(l)j}(\mathbf{r}')]$ with $l\leqslant N-1$, which have been substituted into $\mathsf{RHS}_{(N+1)}$, *are* covered by this axiom. We can therefore use the inductive axiom to replace all these latter commutators by their corresponding left-hand side equivalents from Eq. (2.24). After extensive algebraic manipulations, we can demonstrate that $\mathsf{RHS}_{(N+1)}$ has been transformed into the *left-hand side* of Eq. (2.24) for the case in which all n have been replaced by $n=N+1$. This, then, completes the inductive proof of Eq. (2.24). The details of the argument are given in two appendices. Appendix A proves some necessary lemmas; Appendix B proves the fundamental theorem.

Finally, in this section, we will make some general remarks about the state $\Psi\,|\phi\rangle$. It is important to realize that $\Psi\,|\phi\rangle$ implements the non-Abelian Gauss's law, but that it is not an eigenstate of the QCD Hamiltonian. Also, $\Psi\,|\phi\rangle$ does not have a bounded norm. This follows from the fact that

$$[\{b_Q^a(\mathbf{k})+J_0^a(\mathbf{k})\},A_i^b(\mathbf{r})]=[\delta_{ab}k_i-igf^{acb}A_i^c(\mathbf{r})]\times\exp[-i\mathbf{k}\cdot\mathbf{r}] \tag{2.27}$$

and that, because Eq. (1.1) holds, the matrix element

$$\mathsf{M}=\langle\phi|\ \Psi^\star[\{b_Q^a(\mathbf{k})+J_0^a(\mathbf{k})\},A_i^b(\mathbf{r})]\Psi\ |\phi\rangle \tag{2.28}$$

can be represented either as $\mathsf{M}=\langle\phi|\ \Psi^\star(\delta_{ab}k_i-igf^{acb}A_i^c(\mathbf{r}))\Psi\ |\phi\rangle$ or, alternatively, as $\mathsf{M}=[0\cdot\langle\phi|\ \Psi^\star\ \Psi\ |\phi\rangle]$. The apparent incompatibility between these two expressions for M has led some authors to argue that QCD in the temporal gauge is inconsistent [7]. However, as was pointed out by Rossi and Testa [8], the appropriate inference from these two identities is not that QCD in the temporal gauge is inconsistent, but that the gauge-invariant states that implement the non-Abelian Gauss's law do not have bounded norms. The quantity $[0\cdot\langle\phi|\ \Psi^\star\ \Psi\ |\phi\rangle\ ]$ only makes sense when the product of 0 and an infinite norm is carefully defined. This has been done in Ref. [8] in the context of a functional formulation in which a redundancy of gauge-invariant states is eliminated with a constraint that controls the residual gauge invariance that remains after the temporal gauge has been selected. The identical argument — that the norms of the states that implement Gauss's law are unbounded, but that the criticism made in Ref. [7] is unjustified — was also made in the context of an analogy with ordinary quantum mechanics [9]. With proper care, states with unbounded norms can be used in canonical formulations. Even the $|\phi\rangle$ states, whose structure was given in Ref. [6], have unbounded norms and were used in connection with the Fermi formulation of the subsidiary condition for QED in covariant gauges [10].

III. THE INCLUSION OF QUARKS

In Eq. (1.1), we have implemented the ''pure glue'' form of Gauss's law. The complete Gauss's law operator, when the quarks are included as sources for the chromoelectric field, takes the form

$$\hat{\mathcal{G}}^a(\mathbf{r})=\partial_i\Pi_i^a(\mathbf{r})+gf^{abc}A_i^b(\mathbf{r})\Pi_i^c(\mathbf{r})+j_0^a(\mathbf{r}) \quad , \tag{3.1}$$

where

$$j_0^a(\mathbf{r})=g\ \psi^\dagger(\mathbf{r})\frac{\lambda^a}{2}\psi(\mathbf{r}) \quad , \tag{3.2}$$

and where the λ^a represent the Gell-Mann matrices. To implement the ''complete'' Gauss's law—a form that incorporates quark as well as gluon color—we must solve the equation

$$\hat{\mathcal{G}}^a(\mathbf{r})\ \hat{\Psi}\ |\phi\rangle=0 \quad . \tag{3.3}$$

Our approach to this problem will be based on the fact that $\hat{\mathcal{G}}^a(\mathbf{r})$ and $\mathcal{G}^a(\mathbf{r})$ are unitarily equivalent, so that

$$\hat{\mathcal{G}}^a(\mathbf{r})=\mathcal{U}_{\mathcal{C}}\ \mathcal{G}^a(\mathbf{r})\ \mathcal{U}_{\mathcal{C}}^{-1} \quad , \tag{3.4}$$

where $\mathcal{U}_{\mathcal{C}}=e^{\mathcal{C}_0}e^{\overline{\mathcal{C}}}$ and where $\mathcal{C}_0$ and $\overline{\mathcal{C}}$ are given by

$$\mathcal{C}_0=i\int d\mathbf{r}\ \mathcal{X}^\alpha(\mathbf{r})\ j_0^\alpha(\mathbf{r}) \quad \text{and} \quad \overline{\mathcal{C}}=i\int d\mathbf{r}\ \overline{\mathcal{Y}^\alpha}(\mathbf{r})j_0^\alpha(\mathbf{r}) \quad . \tag{3.5}$$

We can demonstrate this unitary equivalence by noting that Eq. (3.4) can be rewritten as

$$e^{-\mathcal{C}_0}\ \hat{\mathcal{G}}^a(\mathbf{r})\ e^{\mathcal{C}_0}=e^{\overline{\mathcal{C}}}\ \mathcal{G}^a(\mathbf{r})\ e^{-\overline{\mathcal{C}}} \quad . \tag{3.6}$$

In this form, the unitary equivalence can be shown to be a direct consequence of the fundamental theorem—i.e., Eq. (2.24). We observe that the LHS of Eq. (3.6) can be expanded, using the Baker-Hausdorff-Campbell (BHC) theorem, as

$$e^{-\mathcal{C}_0}\ \hat{\mathcal{G}}^a(\mathbf{r})\ e^{\mathcal{C}_0}=\hat{\mathcal{G}}^a(\mathbf{r})+\mathcal{S}_{(1)}^a+\cdots+\mathcal{S}_{(n)}^a+\cdots \quad , \tag{3.7}$$

where $\mathcal{S}_{(1)}^a=-[\mathcal{C}_0,\ \hat{\mathcal{G}}^a(\mathbf{r})]$ and $\mathcal{S}_{(n)}^a=-(1/n)[\mathcal{C}_0,\ \mathcal{S}_{(n-1)}^a]$. We observe that

$$\mathcal{S}_{(1)}^a=-\left[\delta_{a,c}+gf^{abc}\mathcal{X}^b(\mathbf{r})+gf^{abc}A_i^b(\mathbf{r})\frac{\partial_i}{\partial^2}\right]j_0^c(\mathbf{r}) \quad , \tag{3.8}$$

and that

$$\mathcal{S}_{(n)}^a=\frac{(-1)^{n+1}}{n\ !}\Big([g^{n-1}f_{(n-1)}^{\vec{\alpha}a\gamma}\ \mathcal{R}_{(n-1)}^{\vec{\alpha}}(\mathbf{r}) +g^n\ f_{(n)}^{\vec{\alpha}a\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})]j_0^\gamma(\mathbf{r})+g^nf^{abc}\ f_{(n-1)}^{\vec{\alpha}c\gamma}\ A_i^b(\mathbf{r}) \times\frac{\partial_i}{\partial^2}[\mathcal{R}_{(n-1)}^{\vec{\alpha}}(\mathbf{r})\ j_0^\gamma(\mathbf{r})]\Big) \quad . \tag{3.9}$$

Equation (3.9) shows that two $g^nf_{(n)}^{\vec{\alpha}a\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ j_0^\gamma(\mathbf{r})$ terms will appear in this series: one in $\mathcal{S}_{(n)}^a$, and one in $\mathcal{S}_{(n+1)}^a$. The sum of these terms will have the coefficient $[1/n!-1/(n+1)!]=1/(n+1)(n-1)!$. When the BHC series is summed, we find that

$$e^{-\mathcal{C}_0}\ \hat{\mathcal{G}}^a(\mathbf{r})\ e^{\mathcal{C}_0}=\hat{\mathcal{G}}^a(\mathbf{r})-j_0^a(\mathbf{r})-gf^{abc}A_i^b(\mathbf{r})\frac{\partial_i}{\partial^2}j_0^c(\mathbf{r})-\sum_{n=1}(-1)^ng^n\frac{1}{(n-1)!(n+1)}f_{(n)}^{\vec{\alpha}a\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ j_0^\gamma(\mathbf{r})+gf^{abc}A_i^b(\mathbf{r}) \times\sum_{n=1}(-1)^ng^n\frac{1}{(n+1)!}f_{(n)}^{\vec{\alpha}c\gamma}\frac{\partial_i}{\partial^2}[\mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ j_0^\gamma(\mathbf{r})] \quad . \tag{3.10}$$

To prepare for the evaluation of $e^{\overline{\mathcal{C}}}\ \mathcal{G}^a(\mathbf{r})\ e^{-\overline{\mathcal{C}}}$, the RHS of Eq. (3.6), we multiply both sides of Eq. (2.24) for the nth order term, $\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})$, by g^n, and then sum over the integer-valued indices r and n (in that order). The result—a formulation of the fundamental theorem that no longer applies to the individual orders, $\mathcal{A}^{\gamma}_{(n)j}(\mathbf{r})$, but to their sum, $\overline{\mathcal{A}^{\gamma}_{j}}(\mathbf{r})$—is

$$i\int d\mathbf{r}'[\partial_i\Pi^a_i(\mathbf{r}),\overline{\mathcal{A}^{\gamma}_j}(\mathbf{r}')]\,V^{\gamma}_j(\mathbf{r}')+igf^{a\beta d}A^{\beta}_i(\mathbf{r})\int d\mathbf{r}'[\Pi^d_i(\mathbf{r}),\overline{\mathcal{A}^{\gamma}_j}(\mathbf{r}')]\,V^{\gamma}_j(\mathbf{r}')$$

$$\begin{aligned}&=-gf^{a\mu d}\,A^{\mu}_i(\mathbf{r})\,V^d_i(\mathbf{r})+\sum_{\eta=1}\frac{g^{\eta+1}B(\eta)}{\eta!}f^{a\beta c}f^{\vec{\alpha}c\gamma}_{(\eta)}\,A^{\beta}_i(\mathbf{r})\frac{\partial_i}{\partial^2}[\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\,\partial_jV^{\gamma}_j(\mathbf{r})]\\&-\sum_{\eta=0}\sum_{t=1}(-1)^{t-1}g^{t+\eta}\frac{B(\eta)}{\eta!(t-1)!(t+1)}f^{\vec{\mu}a\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(\eta)}\,\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\,\partial_iV^{\gamma}_i(\mathbf{r})\\&-gf^{a\beta d}A^{\beta}_i(\mathbf{r})\sum_{\eta=0}\sum_{t=1}(-1)^tg^{t+\eta}\frac{B(\eta)}{\eta!(t+1)!}f^{\vec{\mu}d\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(\eta)}\frac{\partial_i}{\partial^2}[\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\,\partial_jV^{\gamma}_j(\mathbf{r})]\quad.\end{aligned}\tag{3.11}$$

If we again use the BHC expansion, as in Eq. (3.7), but this time to represent

$$e^{\overline{\mathcal{C}}}\,\mathcal{G}^a(\mathbf{r})\,e^{-\overline{\mathcal{C}}}=\mathcal{G}^a(\mathbf{r})+\overline{\mathcal{S}^a_{(1)}}+\cdots+\overline{\mathcal{S}^a_{(n)}}+\cdots,\tag{3.12}$$

we find that the first order term, $\overline{\mathcal{S}^a_{(1)}}$ can be obtained directly from Eq. (3.11) and is

$$\begin{aligned}\overline{\mathcal{S}^a_{(1)}}&=-gf^{a\mu\gamma}\,A^{\mu}_i(\mathbf{r})\frac{\partial_i}{\partial^2}j^{\gamma}_0(\mathbf{r})+\sum_{s=1}\frac{g^{s+1}B(s)}{s!}f^{a\beta c}f^{\vec{\alpha}c\gamma}_{(s)}\,A^{\beta}_i(\mathbf{r})\frac{\partial_i}{\partial^2}[\mathcal{M}^{\vec{\alpha}}_{(s)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})]\\&-\sum_{s=0}\sum_{t=1}(-1)^{t-1}g^{t+s}\frac{B(s)}{s!(t-1)!(t+1)}f^{\vec{\mu}a\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(s)}\,\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(s)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})\\&-gf^{a\beta d}\,A^{\beta}_i(\mathbf{r})\sum_{s=0}\sum_{t=1}(-1)^tg^{t+s}\frac{B(s)}{s!(t+1)!}f^{\vec{\mu}d\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(s)}\frac{\partial_i}{\partial^2}[\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(s)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})]\quad;\end{aligned}\tag{3.13}$$

the kth order term is

$$\begin{aligned}\overline{\mathcal{S}^a_{(k)}}&=\frac{g^k}{k\,!}f^{a\mu d}f^{\vec{\alpha}d\gamma}_{(k-1)}\,A^{\mu}_i(\mathbf{r})\frac{\partial_i}{\partial^2}[\mathcal{M}^{\vec{\alpha}}_{(k-1)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})]+\sum_{s=1}\frac{g^{s+k}B(s)}{s!\,k!}f^{a\beta c}f^{\vec{\alpha}c\gamma}_{(s+k-1)}A^{\beta}_i(\mathbf{r})\frac{\partial_i}{\partial^2}[\mathcal{M}^{\vec{\alpha}}_{(s+k-1)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})]\\&-\sum_{s=0}\sum_{t=1}(-1)^{t-1}g^{t+s+k-1}\frac{B(s)}{s!k!(t-1)!(t+1)}f^{\vec{\mu}a\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(s+k-1)}\,\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(s+k-1)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})\\&-gf^{a\beta d}\,A^{\beta}_i(\mathbf{r})\sum_{s=0}\sum_{t=1}(-1)^tg^{t+s+k-1}\frac{B(s)}{s!k!(t+1)!}f^{\vec{\mu}d\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(s+k-1)}\frac{\partial_i}{\partial^2}[\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(s+k-1)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})]\quad.\end{aligned}\tag{3.14}$$

When we sum over the entire series, we can change variables in the integer-valued indices to $\eta=k+s-1$, and perform the summation over η and s, with $k=\eta-s+1$. The summation over s then involves nothing but the Bernoulli numbers and fractional coefficients, so that we obtain

$$\begin{aligned}e^{\overline{\mathcal{C}}}\,\mathcal{G}^a(\mathbf{r})\,e^{-\overline{\mathcal{C}}}&=\mathcal{G}^a(\mathbf{r})-gf^{a\beta\gamma}\,A^{\beta}_i(\mathbf{r})\frac{\partial_i}{\partial^2}j^{\gamma}_0(\mathbf{r})+\sum_{\eta=1}g^{\eta+1}D^{\eta}_0(\eta)f^{a\beta c}f^{\vec{\alpha}c\gamma}_{(\eta)}\,A^{\beta}_i(\mathbf{r})\frac{\partial_i}{\partial^2}[\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})]\\&+\sum_{\eta=0}\sum_{t=1}(-1)^tg^{t+\eta}\frac{D^{\eta}_0(\eta)}{(t-1)!(t+1)}f^{\vec{\mu}a\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(\eta)}\,\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\,j^{\gamma}_0(\mathbf{r})\\&-gf^{a\mu d}\,A^{\mu}_i(\mathbf{r})\sum_{\eta=0}\sum_{t=1}(-1)^tg^{t+\eta}\frac{D^{\eta}_0(\eta)}{(t+1)!}f^{\vec{\mu}d\lambda}_{(t)}f^{\vec{\alpha}\lambda\gamma}_{(\eta)}\frac{\partial_i}{\partial^2}[\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\,\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\,j^{\alpha}_0(\mathbf{r})]\quad,\end{aligned}\tag{3.15}$$

where $D_0^\eta(\eta)$ is the sum over Bernoulli numbers defined in Eq. (B18). $D_0^\eta(\eta)$ has the values $D_0^\eta(\eta)=0$ for $\eta\neq 0$, and $D_0^0(0)=1$. Since $f_{(0)}^{\vec{\alpha}\lambda\gamma}=-\delta_{\lambda,\gamma}$, we find that substitution of these values into Eq. (3.15) reduces it identically to Eq. (3.10) and thereby proves Eqs. (3.4) and (3.6), demonstrating the unitary equivalence of $\hat{\mathcal{G}}^a(\mathbf{r})$ and $\mathcal{G}^a(\mathbf{r})$.

The demonstration of unitary equivalence of $\hat{\mathcal{G}}^a(\mathbf{r})$ and $\mathcal{G}^a(\mathbf{r})$ enables us to assign two different roles to $\mathcal{G}^a(\mathbf{r})$. On the one hand, $\mathcal{G}^a(\mathbf{r})$ can be viewed as the Gauss's law operator for ''pure glue'' QCD and $\hat{\mathcal{G}}^a(\mathbf{r})$ as the Gauss's law operator for the theory that includes quarks as well as gluons. But $\mathcal{G}^a(\mathbf{r})$ can also be viewed as the Gauss's law operator for QCD *with* interacting quarks and gluons, in a representation in which all operators and states have been transformed with a similarity transformation that transforms $\hat{\mathcal{G}}^a(\mathbf{r})$ into $\mathcal{G}^a(\mathbf{r})$ and that similarly transforms all other operators and states as well, but that leaves matrix elements unchanged. We will designate the representation in which $\hat{\mathcal{G}}^a(\mathbf{r})$ represents the Gauss's law operator for QCD with quarks as well as gluons, and in which $\mathcal{G}^a(\mathbf{r})$ represents the ''pure glue'' Gauss's law operator, as the ''common'' or $\mathcal{C}$ representation. The unitarily transformed representation, in which $\mathcal{G}^a(\mathbf{r})$ represents the Gauss's law operator for QCD with interacting quarks and gluons, will be designated the $\mathcal{N}$ representation. We can use the relationship between these two representations to construct states that implement the ''complete'' Gauss's law—Eq. (3.3)—from

$$\mathcal{G}^a(\mathbf{r})\ \Psi\ |\phi\rangle=0, \tag{3.16}$$

which is the ''pure glue'' form of Gauss's law in the $\mathcal{C}$ representation. We can simply view Eq. (3.16) as the statement of the complete Gauss's law—the version that includes interacting quarks and gluons—but in the $\mathcal{N}$ representation. In order to transform Eq. (3.16)—now representing Gauss's law with interacting quarks and gluons—from the $\mathcal{N}$ to the $\mathcal{C}$ representation, we make use of the fact that

$$\hat{\mathcal{G}}^a(\mathbf{r})\ \hat{\Psi}\ |\phi\rangle=\mathcal{U}_{\mathcal{C}}\ \mathcal{G}^a(\mathbf{r})\ \mathcal{U}_{\mathcal{C}}^{-1}\ \mathcal{U}_{\mathcal{C}}\ \Psi\ |\phi\rangle=0\ \ , \tag{3.17}$$

identifying $\hat{\Psi}\ |\phi\rangle=\mathcal{U}_{\mathcal{C}}\ \Psi\ |\phi\rangle$ as a state that implements Gauss's law for a theory with quarks and gluons, in the $\mathcal{C}$ representation. In Sec. IV, we will discuss the relation between gauge invariance and the implementation of Gauss's law. As was reiterated in Ref. [4], the Gauss's law operator is the generator of local gauge transformations—which are time-independent in the temporal gauge—so that functional integrals over gauge-invariant states are annihilated by the Gauss's law operator. The apparatus we developed in this and preceding sections for constructing states that implement Gauss's law will therefore be instrumental in finding explicit operator-valued representations of gauge-invariant spinor and gauge fields.

IV. GAUGE-INVARIANT SPINOR AND GAUGE FIELDS

We can apply the unitary equivalence demonstrated in the preceding section to the construction of gauge-invariant spinor and gauge field operators. We observe that Gauss's Law has a central role in generating local gauge transformations, in which the operator-valued gauge and spinor fields in a gauge theory—QCD in this case—are gauge-transformed by an arbitrary c-number field $\omega^a(\mathbf{r})$ consistent with the gauge condition that underlies the canonical theory. In this, the temporal gauge, such gauge transformations are implemented by

$$\mathcal{O}(\mathbf{r})\rightarrow\mathcal{O}'(\mathbf{r})=\exp\left(-\frac{i}{g}\int\hat{\mathcal{G}}^a(\mathbf{r}')\omega^a(\mathbf{r}')\ d\mathbf{r}'\right)\mathcal{O}(\mathbf{r})\times\exp\left(\frac{i}{g}\int\hat{\mathcal{G}}^a(\mathbf{r}')\ \omega^a(\mathbf{r}')\ d\mathbf{r}'\right)\ \ , \tag{4.1}$$

where $\omega^a(\mathbf{r})$ is time-independent, and where $\mathcal{O}(\mathbf{r})$ represents any of the operator-valued fields of the gauge theory and $\mathcal{O}'(\mathbf{r})$ its gauge-transformed form [11]. Equation (4.1) applies to QCD with quarks and gluons, and is expressed in the $\mathcal{C}$ representation. It is obvious that any operator-valued field that commutes with $\hat{\mathcal{G}}^a(\mathbf{r})$ is gauge-invariant.

We can also formulate the same gauge transformations in the $\mathcal{N}$ representation, in which case they take the form

$$\mathcal{O}_{\mathcal{N}}(\mathbf{r})\rightarrow\mathcal{O}'_{\mathcal{N}}(\mathbf{r})=\exp\left(-\frac{i}{g}\int\mathcal{G}^a(\mathbf{r}')\ \omega^a(\mathbf{r}')\ d\mathbf{r}'\right)\mathcal{O}_{\mathcal{N}}(\mathbf{r})\times\exp\left(\frac{i}{g}\int\mathcal{G}^a(\mathbf{r}')\ \omega^a(\mathbf{r}')\ d\mathbf{r}'\right)\ \ , \tag{4.2}$$

where $\mathcal{O}_{\mathcal{N}}(\mathbf{r})$ now represents a spinor or gauge field in the $\mathcal{N}$ representation. Equation (4.2) has the same form as the equation that implements gauge-transformations for ''pure glue'' QCD in the $\mathcal{C}$ representation, but it has a very different meaning. In Eq. (4.2), the operator-valued field $\mathcal{O}_{\mathcal{N}}(\mathbf{r})$, and $\mathcal{G}^a(\mathbf{r})$ which here represents the *entire* Gauss's law — quarks and gluons included — both are in the $\mathcal{N}$ representation.

It is easy to see that the spinor field $\psi(\mathbf{r})$ is a gauge-invariant spinor in the $\mathcal{N}$ representation, because $\psi(\mathbf{r})$ and $\mathcal{G}^a(\mathbf{r}')$ trivially commute. To produce $\psi_{\mathsf{GI}}(\mathbf{r})$, this gauge-invariant spinor transposed into the $\mathcal{C}$ representation, we make use of

$$\psi_{\mathsf{GI}}(\mathbf{r})=\mathcal{U}_{\mathcal{C}}\ \psi(\mathbf{r})\ \mathcal{U}_{\mathcal{C}}^{-1}\quad\text{and}\quad\psi^\dagger_{\mathsf{GI}}(\mathbf{r})=\mathcal{U}_{\mathcal{C}}\ \psi^\dagger(\mathbf{r})\ \mathcal{U}_{\mathcal{C}}^{-1}\ \ . \tag{4.3}$$

We can easily carry out the unitary transformations in Eq. (4.3) to give

$$\psi_{\mathsf{GI}}(\mathbf{r})=V_{\mathcal{C}}(\mathbf{r})\ \psi(\mathbf{r})\quad\text{and}\quad\psi^\dagger_{\mathsf{GI}}(\mathbf{r})=\psi^\dagger(\mathbf{r})\ V_{\mathcal{C}}^{-1}(\mathbf{r})\ \ , \tag{4.4}$$

where

$$V_{\mathcal{C}}(\mathbf{r})=\exp\left(-ig\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\exp\left(-ig\mathcal{X}^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\ \ , \tag{4.5}$$

and

$$V_{\mathcal{C}}^{-1}(\mathbf{r})=\exp\left(ig\mathcal{X}^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\exp\left(ig\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\ \ . \tag{4.6}$$

Because we have given an explicit expression for $\overline{\mathcal{Y}^\alpha}(\mathbf{r})$ in Eqs. (2.19) and (2.20), Eq. (4.4) represents complete, nonperturbative expressions for gauge-invariant spinors in the $\mathcal{C}$ representation. We can, if we choose, expand Eqs. (4.4) to arbitrary order. We then find that to $O(g^3)$, we agree with Refs. [12,13] in which a perturbative construction of a gauge-invariant spinor is carried out to $O(g^3)$. Furthermore, in the $\mathcal{C}$ representation, $\psi(\mathbf{r})$ gauge-transforms as

$$\psi(\mathbf{r})\rightarrow\psi'(\mathbf{r})=\exp\left(i\omega^{\alpha}(\mathbf{r})\frac{\lambda^{\alpha}}{2}\right)\psi(\mathbf{r}) \quad . \tag{4.7}$$

Since $\psi_{\mathrm{GI}}(\mathbf{r})$ has been shown to be gauge-invariant, it immediately follows that $V_{\mathcal{C}}(\mathbf{r})$ gauge-transforms as

$$V_{\mathcal{C}}(\mathbf{r})\rightarrow V_{\mathcal{C}}(\mathbf{r})\exp\left(-i\omega^{\alpha}(\mathbf{r})\frac{\lambda^{\alpha}}{2}\right) \quad \text{and}$$

$$V_{\mathcal{C}}^{-1}(\mathbf{r})\rightarrow\exp\left(i\omega^{\alpha}(\mathbf{r})\frac{\lambda^{\alpha}}{2}\right)V_{\mathcal{C}}^{-1}(\mathbf{r}) \quad . \tag{4.8}$$

The procedure we have used to construct gauge-invariant spinors is not applicable to the construction of gauge-invariant gauge fields, because we do not have ready access to a form of the gauge field that is trivially gauge invariant in either the $\mathcal{C}$ or the $\mathcal{N}$ representation. We will, however, discuss two methods for constructing gauge-invariant gauge fields. One method is based on the observation that the states $|\phi\rangle$ for which

$$\hat{\mathcal{G}}^{a}\hat{\Psi}|\phi\rangle=\hat{\mathcal{G}}^{a}\mathcal{U}_{\mathcal{C}}\Psi|\phi\rangle=0 \tag{4.9}$$

include any state $|\phi_{A^b_{T\,i}(\mathbf{r})}\rangle$ in which the transverse gauge field $A^b_{T\,i}(\mathbf{r})$ acts on another $|\phi\rangle$ state. This is an immediate consequence of the fact that $\hat{\mathcal{G}}^a\ \hat{\Psi}=\hat{\Psi}\,b^a_Q(\mathbf{k})+B^a_Q(\mathbf{k})$, and that $A^b_{T\,i}(\mathbf{r})$ trivially commutes with $\partial_i\Pi^a_i(\mathbf{r}')$. We use the commutator algebra for the operator-valued fields to maneuver the transverse gauge field, along with all further gauge field functionals generated in this process, to the left of $\mathcal{U}_{\mathcal{C}}\Psi$ in $\mathcal{U}_{\mathcal{C}}\Psi A^b_{T\,i}(\mathbf{r})\,|\phi\rangle$. We then obtain the result that

$$\hat{\Psi}A^b_{T\,i}(\mathbf{r})|\phi\rangle=A^b_{\mathrm{GI}\,i}(\mathbf{r})\hat{\Psi}|\phi\rangle, \tag{4.10}$$

where $A^b_{\mathrm{GI}\,i}(\mathbf{r})$ is a gauge-invariant gauge field created in the process of commuting $A^b_{T\,i}(\mathbf{r})$ past the Ψ to its left. The gauge-invariance of $A^b_{\mathrm{GI}\,i}(\mathbf{r})$ follows from the fact that the Gauss's law operator $\hat{\mathcal{G}}^a$ annihilates both sides of Eq. (4.10). Equations (4.9) and (4.10) require that the commutator $[\hat{\mathcal{G}}^a,\ A^b_{\mathrm{GI}\,i}(\mathbf{r})\ ]=0$, and it then follows directly from Eq. (4.1) that $A^b_{\mathrm{GI}\,i}(\mathbf{r})$ is gauge-invariant. It only remains for us to find an explicit expression for $A^b_{\mathrm{GI}\,i}(\mathbf{r})$. We first observe from Eqs. (3.4) and (3.5) that the gauge field and all functionals of gauge fields commute with $\mathcal{U}_{\mathcal{C}}$. We further see that

$$A^b_{\mathrm{GI}\,i}(\mathbf{r})\Psi=[\Psi,\ A^b_{T\,i}(\mathbf{r})]+A^b_{T\,i}(\mathbf{r})\Psi. \tag{4.11}$$

When we expand Ψ as

$$\begin{aligned}\Psi=\|\exp(\mathcal{A})\|=&\left\|\exp\left(i\int d\mathbf{r}\ \overline{\mathcal{A}^{\gamma}_k}(\mathbf{r})\ \Pi^{\gamma}_k(\mathbf{r})\right)\right\|=1+i\int d\mathbf{r}_1\overline{\mathcal{A}^{\gamma}_k}(\mathbf{r}_1)\ \Pi^{\gamma}_k(\mathbf{r}_1)\\&+\frac{(i)^2}{2}\int d\mathbf{r}_1\ d\mathbf{r}_2\ \overline{\mathcal{A}^{\gamma_1}_{k_1}}(\mathbf{r}_1)\ \overline{\mathcal{A}^{\gamma_2}_{k_2}}(\mathbf{r}_2)\ \Pi^{\gamma_1}_{k_1}(\mathbf{r}_1)\ \Pi^{\gamma_2}_{k_2}(\mathbf{r}_2)+\cdots+\frac{(i)^n}{n\ !}\int d\mathbf{r}_1\ d\mathbf{r}_2\ \cdots\ d\mathbf{r}_n\ \overline{\mathcal{A}^{\gamma_1}_{k_1}}(\mathbf{r}_1)\ \overline{\mathcal{A}^{\gamma_2}_{k_2}}(\mathbf{r}_2)\ \cdots\\&\times\ \overline{\mathcal{A}^{\gamma_n}_{k_n}}(\mathbf{r}_n)\ \Pi^{\gamma_1}_{k_1}(\mathbf{r}_1)\ \Pi^{\gamma_2}_{k_2}(\mathbf{r}_2)\ \cdots\ \Pi^{\gamma_n}_{k_n}(\mathbf{r}_n)+\cdots\end{aligned} \tag{4.12}$$

it becomes evident that

$$\begin{aligned}[\Psi,\ A^b_{T\,i}(\mathbf{r})]=&\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}^b_j}(\mathbf{r})\ +\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}^b_j}(\mathbf{r})\ i\int d\mathbf{r}_1\ \overline{\mathcal{A}^{\gamma}_k}(\mathbf{r}_1)\ \Pi^{\gamma}_k(\mathbf{r}_1)+\cdots\\&+\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}^b_j}(\mathbf{r})\ \frac{(i)^{n-1}}{(n-1)\ !}\int d\mathbf{r}_1\ d\mathbf{r}_2\ \cdots\ d\mathbf{r}_{n-1}\ \overline{\mathcal{A}^{\gamma_1}_{k_1}}(\mathbf{r}_1)\ \overline{\mathcal{A}^{\gamma_2}_{k_2}}(\mathbf{r}_2)\ \cdots\ \overline{\mathcal{A}^{\gamma_{n-1}}_{k_{n-1}}}(\mathbf{r}_{n-1})\\&\times\ \Pi^{\gamma_1}_{k_1}(\mathbf{r}_1)\Pi^{\gamma_2}_{k_2}(\mathbf{r}_2)\ \cdots\ \Pi^{\gamma_{n-1}}_{k_{n-1}}(\mathbf{r}_{n-1})+\cdots\\&=\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}^b_j}(\mathbf{r})\ \Psi \quad ,\end{aligned} \tag{4.13}$$

and therefore that the gauge-invariant gauge field is

$$A^b_{\mathrm{GI}\ i}(\mathbf{r}) = A^b_{T\ i}(\mathbf{r}) + \left[\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right]\overline{\mathcal{A}^b_j}(\mathbf{r})$$
$$= a^b_i(\mathbf{r}) + \overline{\mathcal{A}^b_i}(\mathbf{r}) - \partial_i\overline{\mathcal{Y}^b}(\mathbf{r}) \quad . \tag{4.14}$$

Confirmation of this result can be obtained from the fact that $A^b_{\mathrm{GI}\ i}(\mathbf{r})$ commutes with $\mathcal{G}^a$ — and therefore also with $\hat{\mathcal{G}}^a$. We observe that

$$[\mathcal{G}^a(\mathbf{r}),\, A^b_{\mathrm{GI}\ i}(\mathbf{r}')] = \left[\mathcal{G}^a(\mathbf{r}),\, \left(A_i{}^b{}_T(\mathbf{r}') + \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}^b_j}(\mathbf{r}')\right)\right]$$
$$= \int d\mathbf{y}\, \{[\mathcal{G}^a(\mathbf{r}),\, A^b_j(\mathbf{y})] + [\mathcal{G}^a(\mathbf{r}),\, \overline{\mathcal{A}^b_j}(\mathbf{y})]\} V_{ij}(\mathbf{y}-\mathbf{r}') = 0 \quad , \tag{4.15}$$

where

$$V_{ij}(\mathbf{y}-\mathbf{r}') = \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\delta(\mathbf{y}-\mathbf{r}') \quad . \tag{4.16}$$

Equation (4.15) follows directly from Eq. (3.11); $\int d\mathbf{y}\, [\mathcal{G}^a(\mathbf{r}),\, \overline{\mathcal{A}^b_j}(\mathbf{y})]\, V_{ij}(\mathbf{y}-\mathbf{r}')$ can be identified as the first line of that equation, when the integration over $\mathbf{y}$ in Eq. (4.15) is identified with the integration over $\mathbf{r}'$ in Eq. (3.11), and when the tensor element $V_{ij}(\mathbf{y}-\mathbf{r}')$, with $\mathbf{r}'$ and i fixed, is substituted for the vector component V^γ_j in Eq. (3.11). Similarly, $\int d\mathbf{y}[\mathcal{G}^a(\mathbf{r}),\, A^b_i(\mathbf{y})]\, V_{ij}(\mathbf{y}-\mathbf{r}')$ can be identified as the second line of Eq. (3.11). The remaining three lines of Eq. (3.11) vanish because $\partial_j V_{ij}(\mathbf{y}-\mathbf{r}') = 0$ is an identity. In this way, Eq. (3.11) accounts for the gauge-invariance of $A^b_{\mathrm{GI}\ i}(\mathbf{r})$.

Another method for constructing a gauge-invariant gauge field is based on the observation that $V_{\mathcal{C}}(\mathbf{r})$ can be written as an exponential function. We can make use of the BHC theorem that $e^{\mathsf{u}}e^{\mathsf{v}} = e^{\mathsf{w}}$, where w is a series whose initial term is $\mathsf{u}+\mathsf{v}$, and whose higher order terms are multiples of successive commutators of u and v with earlier terms in that series. Since the commutator algebra of the Gell-Mann matrices λ^α is closed, $V_{\mathcal{C}}(\mathbf{r})$ must be of the form $\exp[-ig\mathcal{Z}^\alpha(\lambda^\alpha/2)]$, where

$$\exp\left[-ig\mathcal{Z}^\alpha\frac{\lambda^\alpha}{2}\right] = \exp\left[-ig\overline{\mathcal{Y}^\alpha}\frac{\lambda^\alpha}{2}\right]\exp\left[-ig\mathcal{X}^\alpha\frac{\lambda^\alpha}{2}\right] \tag{4.17}$$

and $\mathcal{Z}^\alpha$ is a functional of gauge fields (but not of their canonical momenta). $V_{\mathcal{C}}(\mathbf{r})$ therefore can be viewed as a particular case of the operator $\exp[i\omega^\alpha(\mathbf{r})(\lambda^\alpha/2)]$ that gauge-transforms the spinor field $\psi(\mathbf{r})$; ω^α in this case is $\mathcal{Z}^\alpha$ and therefore a functional of gauge fields that commutes with all other functionals of gauge and spinor fields. Moreover, we can refer to the Euler-Lagrange equation (in the $A_0 = 0$ gauge) for the spinor field $\psi(\mathbf{r})$:

$$\left[im + \gamma_j\left(\partial_j - ig\, A^\alpha_j(\mathbf{r})\frac{\lambda^\alpha}{2}\right) + \gamma_0\partial_0\right]\psi(\mathbf{r}) = 0, \tag{4.18}$$

where we have used the same noncovariant notation for the gauge fields as in Ref. [3] [i.e., $A^\alpha_j(\mathbf{r})$ designates contravariant and ∂_j covariant quantities], and where $\gamma_0 = \beta$ and $\gamma_j = \beta\alpha_j$. Although the gauge fields are operator-valued, they commute with all other operators in Eq. (4.18)—with the exception of the derivatives ∂_j —so that, when only time-independent gauge-transformations are considered, $V_{\mathcal{C}}(\mathbf{r})$, acting as an operator that gauge-transforms ψ, behaves as though $\mathcal{Z}^\alpha$ were a c number. The gauge-transformed gauge field, that corresponds to the gauge-transformed spinor $\psi_{\mathrm{GI}}(\mathbf{r}) = V_{\mathcal{C}}(\mathbf{r})\ \psi(\mathbf{r})$, therefore also is gauge-invariant; it is given by

$$\left[A^b_{\mathrm{GI}\ i}(\mathbf{r})\,\frac{\lambda^b}{2}\right] = V_{\mathcal{C}}(\mathbf{r})\left[A^b_i(\mathbf{r})\frac{\lambda^b}{2}\right]V_{\mathcal{C}}^{-1}(\mathbf{r}) + \frac{i}{g}V_{\mathcal{C}}(\mathbf{r})\ \partial_i V_{\mathcal{C}}^{-1}(\mathbf{r}) \quad . \tag{4.19}$$

Since *further* gauge transformations must be carried out simultaneously on $\psi(\mathbf{r})$ and $V_{\mathcal{C}}(\mathbf{r})$, and must leave $\psi_{\mathrm{GI}}(\mathbf{r})$ untransformed, $A^b_{\mathrm{GI}\ i}(\mathbf{r})$ must also therefore remain untransformed by further gauge transformations. $A^b_{\mathrm{GI}\ i}(\mathbf{r})$ thus is identified as a gauge-invariant gauge field.

To find an explicit form for $[A^b_{\mathrm{GI}\ i}(\mathbf{r})(\lambda^b/2)]$ from the RHS of Eq. (4.19), we use Eq. (2.23), with $V^\gamma_j(\mathbf{r}) = \delta_{ij}(\lambda^\gamma/2)$, to obtain

$$\left[a^\gamma_i(\mathbf{r}) + \overline{\mathcal{A}^\gamma_i}(\mathbf{r}) - \sum_{\eta=1}^{\infty}\frac{g^\eta}{\eta!}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\ \mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\ \overline{\mathcal{B}^\beta_{(\eta)i}}(\mathbf{r})\right]\frac{\lambda^\gamma}{2}$$
$$= \left[a^\gamma_i(\mathbf{r}) + \sum_{\eta=1}^{\infty}\frac{g^\eta}{\eta!}\psi^\gamma_{(\eta)i}(\mathbf{r})\right]\frac{\lambda^\gamma}{2}. \tag{4.20}$$

It is straightforward but tedious to show that

$$\left[a^\gamma_i(\mathbf{r}) + \sum_{\eta=1}^{\infty}\frac{g^\eta}{\eta!}\psi^\gamma_{(\eta)i}(\mathbf{r})\right]\frac{\lambda^\gamma}{2}$$
$$= \exp\left(-ig\ \mathcal{X}^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\left[A^\gamma_i(\mathbf{r})\frac{\lambda^\gamma}{2} + \frac{i}{g}\partial_i\right]\exp\left(ig\ \mathcal{X}^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right) , \tag{4.21}$$

$$\left[a^\gamma_i(\mathbf{r}) - \sum_{\eta=1}^{\infty}\frac{g^\eta}{\eta!}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\ \mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\ a^\beta_i(\mathbf{r})\right]\frac{\lambda^\gamma}{2}$$
$$= \exp\left(ig\ \overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\left[a^\gamma_i(\mathbf{r})\frac{\lambda^\gamma}{2}\right]\exp\left(-ig\ \overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right) , \tag{4.22}$$

$$\left[\partial_i\overline{\mathcal{Y}^\gamma}(\mathbf{r})-\sum_{\eta=1}^{\infty}\frac{g^\eta}{\eta!}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\,\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\;\partial_i\overline{\mathcal{Y}^\beta}(\mathbf{r})\right]\frac{\lambda^\gamma}{2}$$

$$=\exp\left(ig\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\left[\partial_i\overline{\mathcal{Y}^\gamma}(\mathbf{r})\frac{\lambda^\gamma}{2}\right]\exp\left(-ig\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right),\tag{4.23}$$

and

$$\left[\overline{\mathcal{A}_i^\gamma}+\partial_i\overline{\mathcal{Y}^\gamma}(\mathbf{r})-\sum_{\eta=1}^{\infty}\frac{g^\eta}{\eta!}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\,\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\times\left(\overline{\mathcal{A}_i^\beta}(\mathbf{r})+\frac{1}{\eta+1}\partial_i\overline{\mathcal{Y}^\beta}(\mathbf{r})\right)\right]\frac{\lambda^\gamma}{2}$$

$$=\exp\left(ig\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\left[\overline{\mathcal{A}_i^\gamma}(\mathbf{r})\frac{\lambda^\gamma}{2}+\frac{i}{g}\partial_i\right]\times\exp\left(-ig\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right).\tag{4.24}$$

Equations (4.20)–(4.24) lead to

$$V_{\mathcal{C}}(\mathbf{r})\left[A_i^b(\mathbf{r})\;\frac{\lambda^b}{2}\right]V_{\mathcal{C}}^{-1}(\mathbf{r})+\frac{i}{g}V_{\mathcal{C}}(\mathbf{r})\;\partial_iV_{\mathcal{C}}^{-1}(\mathbf{r})$$

$$=A^b_{T\,i}(\mathbf{r})\frac{\lambda^b}{2}+\left[\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right]\overline{\mathcal{A}_j^b}(\mathbf{r})\frac{\lambda^b}{2},\tag{4.25}$$

so that the identical gauge-invariant gauge field is given in Eqs. (4.14) and (4.19). In the gauge-invariant gauge field, as in the earlier case of the gauge-invariant spinor, we find that when we expand Eq. (4.14)—this time to $O(g^2)$—we agree with Refs. [12,13] in which a perturbative construction of a gauge-invariant gauge field is carried out to that order.

V. THE CASE OF YANG-MILLS THEORY

Because of the simplicity of the SU(2) structure constants, it is instructive to examine $\overline{\mathcal{A}_j^a}(\mathbf{r})$—its defining equation and its role in the ''fundamental theorem''—for the case of Yang-Mills theory. For that purpose, we substitute ϵ^{abc}—the structure constants of SU(2)—for the f^{abc} required for SU(3), in the equations that pertain to $\overline{\mathcal{A}_j^a}(\mathbf{r})$. $\epsilon^{\vec{\alpha}\beta\gamma}_{(\eta)}$, the SU(2) equivalent of the $f^{\vec{\alpha}\beta\gamma}_{(\eta)}$ that are important in the definition of $\overline{\mathcal{A}_j^a}(\mathbf{r})$, is given by

$$\epsilon^{\vec{\alpha}\beta\gamma}_{(\eta)}=(-1)^{\eta/2-1}\delta_{\alpha[1]\alpha[2]}\delta_{\alpha[3]\alpha[4]}\cdots\times\delta_{\alpha[\eta-3]\alpha[\eta-2]}\;\epsilon^{\alpha[\eta-1]\beta b}\;\epsilon^{b\alpha[\eta]\gamma}\tag{5.1}$$

and

$$\epsilon^{\vec{\alpha}\beta\gamma}_{(\eta)}=(-1)^{(\eta-1)/2}\delta_{\alpha[1]\alpha[2]}\delta_{\alpha[3]\alpha[4]}\cdots\times\delta_{\alpha[\eta-2]\alpha[\eta-1]}\;\epsilon^{\alpha[\eta]\beta\gamma}\tag{5.2}$$

for even and odd η, respectively. We can use Eqs. (5.1) and (5.2) to write the SU(2) version of Eq. (2.20) for $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$, which appears (implicitly) as the coefficient of the $\Pi_i^\gamma(\mathbf{r})$ on the LHS of that equation. In doing so, we separate $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ into two parts

$$\overline{\mathcal{A}_i^\gamma}(\mathbf{r})=\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\mathcal{X}}+\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\overline{\mathcal{Y}}},\tag{5.3}$$

where $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\mathcal{X}}$ represents the part of $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ that depends only on ''known'' quantities that stem from the $\psi^\gamma_{(n)i}(\mathbf{r})$ and are functionals of gauge fields; $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\overline{\mathcal{Y}}}$ represents the part that implicitly contains the $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ itself. In Sec. II, we showed how the perturbative expansion of $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ proceeds with the construction of the nth order term, $\mathcal{A}^\gamma_{(n)i}(\mathbf{r})$, from the $\psi^\gamma_{(n)i}(\mathbf{r})$ of the same order, and from $\mathcal{A}^\gamma_{(n')i}(\mathbf{r})$ of lower orders—in the SU(2) case, the latter originating from $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\overline{\mathcal{Y}}}$. The explicit forms of $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\mathcal{X}}$ and $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\overline{\mathcal{Y}}}$ are

$$\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\mathcal{X}}=g\epsilon^{\alpha\beta\gamma}\mathcal{X}^\alpha(\mathbf{r})\;A_i^\beta(\mathbf{r})\frac{\sin(\mathcal{N})}{\mathcal{N}}-g\epsilon^{\alpha\beta\gamma}\mathcal{X}^\alpha(\mathbf{r})\partial_i\mathcal{X}^\beta(\mathbf{r})\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}-g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\mathcal{X}^\mu(\mathbf{r})\mathcal{X}^\alpha(\mathbf{r})A_i^\beta(\mathbf{r})\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}$$
$$+g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\mathcal{X}^\mu(\mathbf{r})\mathcal{X}^\alpha(\mathbf{r})\partial_i\mathcal{X}^\beta(\mathbf{r})\left[\frac{1}{\mathcal{N}^2}-\frac{\sin(\mathcal{N})}{\mathcal{N}^3}\right]\tag{5.4}$$

and

$$\overline{\mathcal{A}_i^\gamma}(\mathbf{r})_{\overline{\mathcal{Y}}}=g\epsilon^{\alpha\beta\gamma}\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})\left(A^\beta_{T\,i}(\mathbf{r})+\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}_j^\beta}(\mathbf{r})\right)\frac{\sin(\overline{\mathcal{N}})}{\overline{\mathcal{N}}}+g\epsilon^{\alpha\beta\gamma}\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})\partial_i\overline{\mathcal{Y}^\beta}(\mathbf{r})\frac{1-\cos(\overline{\mathcal{N}})}{\overline{\mathcal{N}}^2}+g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\;\overline{\mathcal{Y}^\mu}(\mathbf{r})\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})$$
$$\times\left(A^\beta_{T\,i}(\mathbf{r})+\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}_j^\beta}(\mathbf{r})\right)\frac{1-\cos(\overline{\mathcal{N}})}{\overline{\mathcal{N}}^2}+g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\;\overline{\mathcal{Y}^\mu}(\mathbf{r})\;\overline{\mathcal{Y}^\alpha}(\mathbf{r})\partial_i\overline{\mathcal{Y}^\beta}(\mathbf{r})\left[\frac{1}{\overline{\mathcal{N}}^2}-\frac{\sin(\overline{\mathcal{N}})}{\overline{\mathcal{N}}^3}\right],\tag{5.5}$$

where

$$\mathcal{N}(\mathbf{r})\equiv\mathcal{N}=[g^2\mathcal{X}^{\delta}(\mathbf{r})\mathcal{X}^{\delta}(\mathbf{r})]^{1/2}\quad , \tag{5.6}$$

and

$$\overline{\mathcal{N}}(\mathbf{r})\equiv\overline{\mathcal{N}}=[g^2\,\overline{\mathcal{Y}^{\delta}}(\mathbf{r})\,\overline{\mathcal{Y}^{\delta}}(\mathbf{r})]^{1/2}\quad . \tag{5.7}$$

There is a striking resemblance in the structure of Eqs. (5.4) and (5.5) on the one hand, and $(A^{\gamma})_i'$, the gauge-transformed gauge field A_i^{γ}, where the gauge transformation is by a finite gauge function ω^{γ}. $(A^{\gamma})_i'$ is given by

$$\begin{aligned}(A^{\gamma})_i' &= \left(A_i^{\gamma}+\frac{1}{g}\partial_i\omega^{\gamma}\right)-\epsilon^{\alpha\beta\gamma}\left(\omega^{\alpha}\,A_i^{\beta}\frac{\sin(|\omega|)}{|\omega|}+\frac{1}{g}\omega^{\alpha}\,\partial_i\omega^{\beta}\frac{1-\cos(|\omega|)}{|\omega|^2}\right)\\ &\quad-\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\left(\omega^{\mu}\omega^{\alpha}\,A_i^{\beta}\frac{1-\cos(|\omega|)}{|\omega|^2}+\frac{\omega^{\mu}\omega^{\alpha}\partial_i\omega^{\beta}}{g}\left(\frac{1}{|\omega|^2}-\frac{\sin(|\omega|)}{|\omega|^3}\right)\right)\quad .\end{aligned} \tag{5.8}$$

The SU(2) version of Eq. (3.11)—our so-called ''fundamental theorem''—can similarly be given. In that case, the summations over order and multiplicity indices can be absorbed into trigonometric functions, and we obtain the much simpler equation

$$\begin{aligned}
&i\int d\mathbf{r}'[\partial_i\Pi_i^a(\mathbf{r}),\,\overline{\mathcal{A}_j^{\gamma}}(\mathbf{r}')]V_j^{\gamma}(\mathbf{r}')+ig\epsilon^{a\beta d}A_i^{\beta}(\mathbf{r})\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\,\overline{\mathcal{A}_j^{\gamma}}(\mathbf{r}')]V_j^{\gamma}(\mathbf{r}')\\
&\quad=-g\epsilon^{a\mu d}\,A_i^{\mu}(\mathbf{r})\,V_i^d(\mathbf{r})-\frac{g^2}{2}\epsilon^{a\beta c}\epsilon^{\alpha c\gamma}\,A_i^{\beta}(\mathbf{r})\frac{\partial_i}{\partial^2}(\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\,\partial_jV_j^{\gamma}(\mathbf{r}))\\
&\qquad-g^3\epsilon^{a\beta c}\epsilon_{(2)}^{\vec{\alpha}c\gamma}\,A_i^{\beta}(\mathbf{r})\frac{\partial_i}{\partial^2}\left(\mathcal{M}_{(2)}^{\vec{\alpha}}(\mathbf{r})\left[\frac{1}{2\overline{\mathcal{N}}}\cot\left(\frac{\overline{\mathcal{N}}}{2}\right)-\frac{1}{\overline{\mathcal{N}}^2}\right]\partial_jV_j^{\gamma}(\mathbf{r})\right)\\
&\qquad+g\epsilon^{\mu a\gamma}\mathcal{X}^{\mu}(\mathbf{r})\left[\frac{\sin(\mathcal{N})}{\mathcal{N}}-\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}\right]\partial_iV_i^{\gamma}(\mathbf{r})+g^2\,\epsilon_{(2)}^{\vec{\mu}a\gamma}\,\mathcal{R}_{(2)}^{\vec{\mu}}(\mathbf{r})\left[\frac{\cos(\mathcal{N})}{\mathcal{N}^2}-\frac{\sin(\mathcal{N})}{\mathcal{N}^3}\right]\partial_iV_i^{\gamma}(\mathbf{r})\\
&\qquad+\frac{g^2}{2}\epsilon^{\mu a\lambda}\epsilon^{\alpha\lambda\gamma}\mathcal{X}^{\mu}(\mathbf{r})\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\left[\frac{\sin(\mathcal{N})}{\mathcal{N}}-\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}\right]\partial_iV_i^{\gamma}(\mathbf{r})\\
&\qquad+\frac{g^3}{2}\epsilon_{(2)}^{\vec{\mu}a\gamma}\,\epsilon^{\alpha\lambda\gamma}\mathcal{R}_{(2)}^{\vec{\mu}}(\mathbf{r})\,\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\left[\frac{\cos(\mathcal{N})}{\mathcal{N}^2}-\frac{\sin(\mathcal{N})}{\mathcal{N}^3}\right]\partial_iV_i^{\gamma}(\mathbf{r})+g^3\epsilon^{\mu a\lambda}\,\epsilon_{(2)}^{\vec{\alpha}\lambda\gamma}\mathcal{X}^{\mu}(\mathbf{r})\mathcal{M}_{(2)}^{\vec{\alpha}}(\mathbf{r})\left[\frac{\sin(\mathcal{N})}{\mathcal{N}}-\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}\right]\\
&\qquad\times\left(\frac{1}{2\overline{\mathcal{N}}}\cot\left(\frac{\overline{\mathcal{N}}}{2}\right)-\frac{1}{\overline{\mathcal{N}}^2}\right)\partial_iV_i^{\gamma}(\mathbf{r})+g^4\,\epsilon_{(2)}^{\vec{\mu}a\lambda}\,\epsilon_{(2)}^{\vec{\alpha}\lambda\gamma}\mathcal{R}_{(2)}^{\vec{\mu}}(\mathbf{r})\,\mathcal{M}_{(2)}^{\vec{\alpha}}(\mathbf{r})\left[\frac{\cos(\mathcal{N})}{\mathcal{N}^2}-\frac{\sin(\mathcal{N})}{\mathcal{N}^3}\right]\left(\frac{1}{2\overline{\mathcal{N}}}\cot\left(\frac{\overline{\mathcal{N}}}{2}\right)-\frac{1}{\overline{\mathcal{N}}^2}\right)\partial_iV_i^{\gamma}(\mathbf{r})\\
&\qquad-g^2\epsilon^{a\beta d}\epsilon^{\mu d\gamma}A_i^{\beta}(\mathbf{r})\,\frac{\partial_i}{\partial^2}\left(\mathcal{X}^{\mu}(\mathbf{r})\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}\partial_jV_j^{\gamma}(\mathbf{r})\right)-g^3\epsilon^{a\beta d}\epsilon_{(2)}^{\vec{\mu}d\gamma}A_i^{\beta}(\mathbf{r})\,\frac{\partial_i}{\partial^2}\left(\mathcal{R}_{(2)}^{\vec{\mu}}(\mathbf{r})\frac{\sin(\mathcal{N})-\mathcal{N}}{\mathcal{N}^3}\partial_jV_j^{\gamma}(\mathbf{r})\right)\\
&\qquad-\frac{g^3}{2}\epsilon^{a\beta d}\epsilon^{\mu d\lambda}\epsilon^{\alpha\lambda\gamma}A_i^{\beta}(\mathbf{r})\frac{\partial_i}{\partial^2}\left(\mathcal{X}^{\mu}(\mathbf{r})\,\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}\partial_jV_j^{\gamma}(\mathbf{r})\right)\\
&\qquad-\frac{g^4}{2}\epsilon^{a\beta d}\epsilon_{(2)}^{\vec{\mu}d\lambda}\epsilon^{\alpha\lambda\gamma}A_i^{\beta}(\mathbf{r})\frac{\partial_i}{\partial^2}\left(\mathcal{R}_{(2)}^{\vec{\mu}}(\mathbf{r})\,\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\frac{\sin(\mathcal{N})-\mathcal{N}}{\mathcal{N}^3}\partial_jV_j^{\gamma}(\mathbf{r})\right)\\
&\qquad-g^4\epsilon^{a\beta d}\epsilon^{\mu d\lambda}\epsilon_{(2)}^{\vec{\alpha}\lambda\gamma}A_i^{\beta}(\mathbf{r})\frac{\partial_i}{\partial^2}\left(\mathcal{X}^{\mu}(\mathbf{r})\,\mathcal{M}_{(2)}^{\vec{\alpha}}(\mathbf{r})\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2}\left[\frac{1}{2\overline{\mathcal{N}}}\cot\left(\frac{\overline{\mathcal{N}}}{2}\right)-\frac{1}{\overline{\mathcal{N}}^2}\right]\partial_jV_j^{\gamma}(\mathbf{r})\right)\\
&\qquad-g^5\epsilon^{a\beta d}\epsilon_{(2)}^{\vec{\mu}d\lambda}\epsilon_{(2)}^{\vec{\alpha}\lambda\gamma}A_i^{\beta}(\mathbf{r})\frac{\partial_i}{\partial^2}\left(\mathcal{R}_{(2)}^{\vec{\mu}}(\mathbf{r})\,\mathcal{M}_{(2)}^{\vec{\alpha}}(\mathbf{r})\frac{\sin(\mathcal{N})-\mathcal{N}}{\mathcal{N}^3}\left[\frac{1}{2\overline{\mathcal{N}}}\cot\left(\frac{\overline{\mathcal{N}}}{2}\right)-\frac{1}{\overline{\mathcal{N}}^2}\right]\partial_jV_j^{\gamma}(\mathbf{r})\right)\quad .
\end{aligned} \tag{5.9}$$

To account for the general structure of Eqs. (5.4) and (5.5), we observe from Eqs. (4.17) and (4.18) that the unitary transformation that transforms the spinor field to its gauge-invariant form *is itself a gauge transformation*. $V_{\mathcal{C}}(\mathbf{r})$ therefore is an operator that gauge-transforms the spinor $\psi(\mathbf{r})$ to a form *that is then invariant to any further gauge transformations*. And $A^b_{\mathrm{GI}\ i}(\mathbf{r})$, which is the corresponding gauge transform of the gauge field $A^b_i(\mathbf{r})$, is similarly invariant to any further gauge transformations. Equation (4.14) identifies $\overline{\mathcal{A}^b_i}(\mathbf{r})$ as an essential constituent of $A^b_{\mathrm{GI}\ i}(\mathbf{r})$, and Eqs. (5.4) and (5.5) specialize $\overline{\mathcal{A}^b_i}(\mathbf{r})$ to its SU(2) structure. It is therefore not surprising to find that the relation between $\overline{\mathcal{A}^b_i}(\mathbf{r})$ and $A^b_i(\mathbf{r})$ anticipates the relation between $A^b_{\mathrm{GI}\ i}(\mathbf{r})$ and $A^b_i(\mathbf{r})$—i.e., that $A^b_{\mathrm{GI}\ i}(\mathbf{r})$ is the gauge-transform of $A^b_i(\mathbf{r})$ by the finite gauge function $\mathcal{Z}^b(\mathbf{r})$, defined in Eq. (4.17).

VI. DISCUSSION

This paper has addressed four main topics: The first has been a proof of a previously published conjecture that states, constructed in an earlier work [6] and given in Eqs. (1.1), (1.3), and (2.20), implement the ''pure glue'' form of Gauss's law for QCD. Another has been the construction of a unitary transformation that extends these states so that they implement Gauss's law for QCD with quarks as well as gluons. The third topic is the construction of gauge-invariant spinor and gauge field operators. And the last topic is the application of the formalism to the SU(2) Yang-Mills case.

Implementation of Gauss's law is always required in a gauge theory, but in earlier work it was shown that in QED and other Abelian gauge theories, the failure to implement Gauss's law does not affect the theory's physical consequences [14,15]. And, in fact, it is known that the renormalized S matrix in perturbative QED is correct, in spite of the fact that incident and scattered charged particles are detached from all fields, including the ones required to implement Gauss's law. In contrast, the validity of perturbative QCD is more limited. It is not applicable to low energy phenomena. And, it is likely that all perturbative results in QCD are obscured, in some measure, by long-range effects, so that the implications of QCD for even high-energy phenomenology are still not fully known. In particular, color confinement is not well understood. One possible avenue for exploring QCD dynamics beyond the perturbative regime is the use of gauge-invariant operators and states in formulating QCD dynamics. Although dynamical equations for gauge-invariant operator-valued fields have not yet been developed, we believe that the mathematical apparatus we have constructed in this paper can serve as a basis for reaching such an objective.

We also note a feature of this work that is most clearly evident in the SU(2) example. The recursive equation for $\overline{\mathcal{A}^b_i}(\mathbf{r})$ — Eq. (2.20) in the SU(3) case, with an arbitrary $V^\gamma_i(\mathbf{r})$ replacing the $\Pi^\gamma_i(\mathbf{r})$, and Eqs. (5.3)–(5.5) in the SU(2) Yang-Mills theory — have many of the features that we associate with finite gauge transformations applied to a gauge field. This is particularly conspicuous for the parts of $\overline{\mathcal{A}^\gamma_i}(\mathbf{r})_{\mathcal{X}}$ and $\overline{\mathcal{A}^\gamma_i}(\mathbf{r})_{\overline{\mathcal{Y}}}$ that correspond to the ''pure gauge'' components of $(A^\gamma)'_i$ displayed in Eq. (5.8). These ''pure gauge'' parts are $\overline{\mathcal{A}^\gamma_i}(\mathbf{r})^{(\mathrm{pg})}_{\mathcal{X}}$ and $\overline{\mathcal{A}^\gamma_i}(\mathbf{r})^{(\mathrm{pg})}_{\overline{\mathcal{Y}}}$, respectively, and are given by

$$\overline{\mathcal{A}^\gamma_i}(\mathbf{r})^{(\mathrm{pg})}_{\mathcal{X}} = -g\,\epsilon^{\alpha\beta\gamma}\,\mathcal{X}^\alpha(\mathbf{r})\,\partial_i\mathcal{X}^\beta(\mathbf{r})\,\frac{1-\cos(\mathcal{N})}{\mathcal{N}^2} + g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\mathcal{X}^\mu(\mathbf{r})\mathcal{X}^\alpha(\mathbf{r})\partial_i\mathcal{X}^\beta(\mathbf{r}) \times\left[\frac{1}{\mathcal{N}^2}-\frac{\sin(\mathcal{N})}{\mathcal{N}^3}\right] \tag{6.1}$$

and

$$\overline{\mathcal{A}^\gamma_i}(\mathbf{r})^{(\mathrm{pg})}_{\overline{\mathcal{Y}}} = g\,\epsilon^{\alpha\beta\gamma}\,\overline{\mathcal{Y}^\alpha}(\mathbf{r})\,\partial_i\overline{\mathcal{Y}^\beta}(\mathbf{r})\frac{1-\cos(\overline{\mathcal{N}})}{\overline{\mathcal{N}}^2} + g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\,\overline{\mathcal{Y}^\mu}(\mathbf{r})\,\overline{\mathcal{Y}^\alpha}(\mathbf{r})\,\partial_i\overline{\mathcal{Y}^\beta}(\mathbf{r}) \times\left[\frac{1}{\overline{\mathcal{N}}^2}-\frac{\sin(\overline{\mathcal{N}})}{\overline{\mathcal{N}}^3}\right]. \tag{6.2}$$

The ''pure gauge'' parts of $\overline{\mathcal{A}^\gamma_i}(\mathbf{r})_{\mathcal{X}}$ and $\overline{\mathcal{A}^\gamma_i}(\mathbf{r})_{\overline{\mathcal{Y}}}$ correspond to the pure gauge part of $(A^\gamma)'_i$, with $-g\mathcal{X}^\gamma(\mathbf{r})$ and $g\overline{\mathcal{Y}^\gamma}(\mathbf{r})$ corresponding to the gauge function $\omega^\gamma(\mathbf{r})$, and $\mathcal{N}$ and $\overline{\mathcal{N}}$ corresponding to $|\omega|$, respectively. This correspondence suggests that, in addition to the iterative solution of Eq. (2.20), which we have discussed extensively in this work, there may be nonperturbative solutions that cannot be represented as an iterated series and that are related to the nontrivial topological sectors of non-Abelian gauge fields [16].

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APPENDIX A: SOME NECESSARY LEMMAS

In this appendix we will prove a number of lemmas required for the inductive proof of Eq. (2.24) — the fundamental identity that enables us to construct states that implement the non-Abelian Gauss's law. The first group of lemmas pertains to the sums over permutations of structure constants that arise when $\Pi^\sigma_i(\mathbf{r})$ and $\partial_i\Pi^\alpha_i(\mathbf{r})$ are commuted with $\overline{\mathcal{A}^\gamma_j}(\mathbf{r}')$. The first of these identities is

$$[\mathcal{P}^{(0)}_{(e,\beta[m-1])}f^{e\delta f}f^{\vec{\beta}f\gamma}_{(m-1)}]\mathcal{U}^{\vec{\beta}}_{(m-1)} = \sum_{s=0}^{m-1}\frac{m!}{(m-s-1)!(s+1)!}f^{\vec{\beta}\delta g}_{(m-s-1)} \times f^{gf\gamma}f^{\vec{\sigma}ef}_{(s)}\,\mathcal{U}^{\vec{\beta}}_{(m-s-1)}\,\mathcal{U}^{\vec{\sigma}}_{(s)}\ . \tag{A1}$$

$\mathcal{U}^{\vec{\alpha}}_{(\eta)}=\Pi^{\eta}_{m=1}\mathcal{U}^{\alpha[m]}(\mathbf{r})$ is a product of operator-valued functions $\mathcal{U}^{\alpha[m]}(\mathbf{r})$ that differ only in the index $\alpha[m]$ that refers to the adjoint representation of the Lie group to which the gauge fields belong, and for which $[\mathcal{U}^\gamma(\mathbf{r}),\mathcal{U}^\lambda(\mathbf{r}')]=0$. To prove Eq. (A1) we generalize Eq. (2.26) by defining the more general permutation operator

$$[\mathcal{P}^{(j)}_{(e,\beta[m-j-1])} f^{e\delta f} f^{\vec{\beta} f\gamma}_{(m-j-1)}]\mathcal{U}^{\vec{\beta}}_{(m-j-1)} = \sum_{s=0}^{m-j-1} \frac{(s+j)!}{s!j!} f^{\vec{\beta}\delta u}_{(s)} f^{uev} f^{\vec{\sigma} v\gamma}_{(m-s-j-1)} \mathcal{U}^{\vec{\beta}}_{(s)} \mathcal{U}^{\vec{\sigma}}_{(m-s-j-1)} \quad . \tag{A2}$$

We can designate the individual permutations that appear in Eq. (A2) as

$$\mathsf{p}_{e,\beta[m-j-1]}(s) = f^{\vec{\beta}\delta u}_{(s)} f^{uev} f^{\vec{\tau} v\gamma}_{(m-j-s-1)} \mathcal{U}^{\vec{\beta}}_{(s)} \mathcal{U}^{\vec{\tau}}_{(m-j-s-1)} \tag{A3}$$

for $s=0,1,2,\ldots,m-j-1$, so that Eq. (A2) can be expressed as

$$[\mathcal{P}^{(j)}_{(e,\beta[m-j-1])} f^{e\delta f} f^{\vec{\beta} f\gamma}_{(m-j-1)}]\mathcal{U}^{\vec{\beta}}_{(m-j-1)} = \sum_{s=0}^{m-j-1} \frac{(s+j)!}{s!j!} \mathsf{p}_{e,\beta[m-j-1]}(s) \quad . \tag{A4}$$

We can transform $\mathsf{p}_{e,\beta[m-j-1]}(s)$ by using the Jacobi identity

$$f^{ceb[1]} f^{b[1]\beta b[2]} = f^{cb[2]b[1]} f^{\beta eb[1]} + f^{c\beta b[1]} f^{eb[2]b[1]} \quad . \tag{A5}$$

As we use the Jacobi identity to transform each permutation in Eq. (A4), turn by turn, each such transformation augments the coefficient of the immediately following permutation by the accumulated sum of all preceding permutations [i.e., applying the Jacobi identity to $\mathsf{p}_{e,\beta[m-j-1]}(s)$ contributes an additional $\mathsf{p}_{e,\beta[m-j-1]}(s+1)$ term]. Since $\sum_{n=0}^{s}(n+j)!/n!j! = (s+j+1)!/s!(j+1)!$, we find, after the Jacobi identity has been applied to the last possible set of permutations on the RHS of Eq. (A4), that we obtain

$$\begin{aligned}[\mathcal{P}^{(j)}_{(e,\beta[m-j-1])} f^{e\delta f} f^{\vec{\beta} f\gamma}_{(m-j-1)}] \mathcal{U}^{\vec{\beta}}_{(m-j-1)} &= [\mathcal{P}^{(j+1)}_{(f,\beta[m-j-2])} f^{f\delta g} f^{\vec{\beta} g\gamma}_{(m-j-2)}] f^{e\sigma f} \mathcal{U}^{\vec{\beta}}_{(m-j-2)} \mathcal{U}^{\sigma} \\ &\quad + \frac{m!}{(m-j-1)!(j+1)!} f^{\vec{\beta}\delta f}_{(m-j-1)} f^{ef\gamma} \mathcal{U}^{\vec{\beta}}_{(m-j-1)} \quad .\end{aligned} \tag{A6}$$

The last term on the RHS of Eq. (A6) is the last permutation in Eq. (A4), whose coefficient has now been increased to $(s+j+1)!/s!(j+1)!$ with $s=m-j-1$ by the application of the Jacobi identity to all the earlier permutations. In this permutation, the structure constant that contains e is already on the extreme right of all other structure constants, so that the Jacobi identity can no longer be applied to its product with the structure constant on its right. For that reason, the sum over permutations on the RHS of Eq. (A6) contains one fewer elements than the sum over permutations on the LHS of that equation.

Applying Eq. (A6) sequentially to

$$[\mathcal{P}^{(s)}_{(e,\beta[m-s-1])} f^{e\delta u} f^{\vec{\beta} u\gamma}_{(m-s-1)}] \mathcal{U}^{\vec{\beta}}_{(m-s-1)} \quad , \tag{A7}$$

for $s=j,j+1,j+2,\ldots,m-2$, thus decreasing the number of terms in the sum over permutations by one with each operation until the last permutation has vanished, leads to

$$\begin{aligned}[\mathcal{P}^{(j)}_{(e,\beta[m-j-1])} f^{e\delta u} f^{\vec{\beta} u\gamma}_{(m-j-1)}]\mathcal{U}^{\vec{\beta}}_{(m-j-1)} &= \sum_{s=j}^{m-1} \frac{m!}{(m-s-1)!(s+1)!} f^{\vec{\beta}\delta v}_{(m-s-1)} f^{vu\gamma} f^{\vec{\sigma} eu}_{(s-j)} \\ &\quad \times \mathcal{U}^{\vec{\beta}}_{(m-s-1)} \mathcal{U}^{\vec{\sigma}}_{(s-j)} \quad ,\end{aligned} \tag{A8}$$

one of the important lemmas established in this appendix. For $j=0$, Eq. (A8) becomes Eq. (A1). Equation (A8) with different values for j can be combined to obtain other useful identities. By combining the $j=0$ and $j=1$ versions of Eq. (A8), we obtain

$$\begin{aligned}[\mathcal{P}^{(0)}_{(e,\beta[m-1])} f^{e\delta u} f^{\vec{\beta} u\gamma}_{(m-1)}]\mathcal{U}^{\vec{\beta}}_{(m-1)} &= m f^{e\delta u} f^{\vec{\beta} u\gamma}_{(m-1)} \mathcal{U}^{\vec{\beta}}_{(m-1)} - \sum_{t=0}^{m-2} \frac{m!(m-t-1)}{(m-t)!t!} \\ &\quad \times f^{\vec{\beta}\delta v}_{(t)} f^{vu\gamma} f^{\vec{\sigma} eu}_{(m-t-1)} \mathcal{U}^{\vec{\beta}}_{(t)} \mathcal{U}^{\vec{\sigma}}_{(m-t-1)} \quad ,\end{aligned} \tag{A9}$$

and

$$\begin{aligned}[\mathcal{P}^{(0)}_{(e,\beta[m-1])} f^{e\delta u} f^{\vec{\beta} u\gamma}_{(m-1)}]\mathcal{U}^{\vec{\beta}}_{(m-1)} &= f^{e\delta u} f^{\vec{\beta} u\gamma}_{(m-1)} \mathcal{U}^{\vec{\beta}}_{(m-1)} + \sum_{s=0}^{m-2} \frac{(m-1)!}{(s+1)!(m-s-2)!} \\ &\quad \times f^{\vec{\beta}\delta v}_{(m-s-1)} f^{vu\gamma} f^{\vec{\sigma} eu}_{(s)} \mathcal{U}^{\vec{\beta}}_{(m-s-1)} \mathcal{U}^{\vec{\sigma}}_{(s)} \quad .\end{aligned} \tag{A10}$$

Our next objective is to evaluate the contribution to Eq. (2.24) from $(i/n!)\int d\mathbf{r}'\,\psi^{\gamma}_{(n)j}(\mathbf{r}')V^{\gamma}_j(\mathbf{r}')$, the inhomogeneous term in the recursive equation for $i\int d\mathbf{r}'\,\overline{\mathcal{A}^{\gamma}_j}(\mathbf{r}')V^{\gamma}_j(\mathbf{r}')$. From Eq. (2.6) we observe that

$$\begin{aligned}\frac{i}{n!}\int d\mathbf{r}'[\Pi^d_i(\mathbf{r}),\,\psi^{\gamma}_{(n)j}(\mathbf{r}')]V^{\gamma}_j(\mathbf{r}') &= \frac{i}{n!}(-1)^{n-1} f^{\vec{\alpha}\beta\gamma}_{(n)} \int d\mathbf{r}'\,\mathcal{R}^{\vec{\alpha}}_{(n)}(\mathbf{r}')\,[\Pi^d_i(\mathbf{r}),\,\mathcal{Q}^{\beta}_{(n)j}(\mathbf{r}')]V^{\gamma}_j(\mathbf{r}') \\ &\quad + \frac{i}{n!}(-1)^{n-1} f^{\vec{\alpha}\beta\gamma}_{(n)} \int d\mathbf{r}'[\Pi^d_i(\mathbf{r}),\,\mathcal{R}^{\vec{\alpha}}_{(n)}(\mathbf{r}')]\mathcal{Q}^{\beta}_{(n)j}(\mathbf{r}')V^{\gamma}_j(\mathbf{r}') \quad .\end{aligned} \tag{A11}$$

We use integration by parts and the identity

$$f^{\vec{\alpha}\delta\gamma}_{(m)}\,[\mathsf{Q},\,\mathcal{U}^{\vec{\alpha}}_{(m)}] = -[\mathcal{P}^{(0)}_{(\alpha,\beta[m-1])} f^{\alpha\delta e} f^{\vec{\beta} e\gamma}_{(m-1)}][\,\mathsf{Q},\,\mathcal{U}^{\alpha}(\mathbf{r})]\,\mathcal{U}^{\vec{\beta}}_{(m-1)} \quad , \tag{A12}$$

where Q represents an arbitrary operator for which $[\mathsf{Q}, \mathcal{U}^{\gamma}(\mathbf{r})]$ commutes with $\mathcal{U}^{\lambda}(\mathbf{r}')$, to obtain

$$\frac{i}{n!}\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \psi_{(n)j}^{\gamma}(\mathbf{r}')]V_j^{\gamma}(\mathbf{r}')=\frac{1}{n!}(-1)^{n-1}f_{(n)}^{\vec{\alpha}d\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ V_i^{\gamma}(\mathbf{r})+\frac{1}{(n+1)!}(-1)^n f_{(n)}^{\vec{\alpha}d\gamma}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ \partial_j V_j^{\gamma}(\mathbf{r}))$$
$$+\frac{1}{(n+1)!}(-1)^{n+1}[\mathcal{P}^{(0)}_{(\alpha,\sigma[n-1])}f^{\alpha de}f_{(n-1)}^{\vec{\sigma}e\gamma}]\frac{\partial_i}{\partial^2}(x_j^{\alpha}(\mathbf{r})\ \mathcal{R}_{(n-1)}^{\vec{\sigma}}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}))$$
$$+\frac{1}{n!}(-1)^{n-1}[\mathcal{P}^{(0)}_{(d,\sigma[n-1])}f^{d\beta e}f_{(n-1)}^{\vec{\sigma}e\gamma}]\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n-1)}^{\vec{\sigma}}(\mathbf{r})\ \mathcal{Q}_{(n)j}^{\beta}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}))\ . \qquad \text{(A13)}$$

Equation (A1) enables us to rewrite Eq. (A13) as

$$\frac{i}{n!}\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \psi_{(n)j}^{\gamma}(\mathbf{r}')]\ V_j^{\gamma}(\mathbf{r}')=\frac{1}{n!}(-1)^{n-1}f_{(n)}^{\vec{\alpha}d\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ V_i^{\gamma}(\mathbf{r})+\frac{1}{(n+1)!}(-1)^n f_{(n)}^{\vec{\alpha}d\gamma}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ \partial_j V_j^{\gamma}(\mathbf{r}))$$
$$-\sum_{s=0}^{n-1}\frac{1}{(n+1)(s+1)!(n-s-1)!}f_{(s)}^{\vec{\tau}\beta e}f^{eu\gamma}f_{(n-s-1)}^{\vec{\sigma}du}(-1)^{n+1}$$
$$\times\frac{\partial_i}{\partial^2}(\mathcal{R}_{(s)}^{\vec{\tau}}(\mathbf{r})\ \mathcal{R}_{(n-s-1)}^{\vec{\sigma}}(\mathbf{r})\ x_j^{\beta}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}))+\sum_{s=0}^{n-1}\frac{1}{s!(n-s)!}f_{(n-s-1)}^{\vec{\sigma}du}f^{eu\gamma}f_{(s)}^{\vec{\tau}\beta e}(-1)^{n+1}$$
$$\times\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n-s-1)}^{\vec{\sigma}}(\mathbf{r})\ \mathcal{R}_{(s)}^{\vec{\tau}}(\mathbf{r})\ \mathcal{Q}_{(n)j}^{\beta}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}))\ , \qquad \text{(A14)}$$

and the identity

$$\frac{1}{s!(n-s)!}\mathcal{Q}_{(s)j}^{\alpha}(\mathbf{r})=\frac{1}{s!(n-s-1)!}\left[\frac{1}{(n-s)}\mathcal{Q}_{(n)j}^{\alpha}(\mathbf{r})-\frac{1}{(n+1)(s+1)}x_j^{\alpha}(\mathbf{r})\right]\ , \qquad \text{(A15)}$$

finally leads to

$$\frac{i}{n!}\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \psi_{(n)j}^{\gamma}(\mathbf{r}')]V_j^{\gamma}(\mathbf{r}')=\frac{1}{n!}(-1)^{n-1}f_{(n)}^{\vec{\alpha}d\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ V_i^{\gamma}(\mathbf{r})+\frac{1}{(n+1)!}(-1)^n f_{(n)}^{\vec{\alpha}d\gamma}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ \partial_j V_j^{\gamma}(\mathbf{r}))$$
$$+\frac{1}{n!}f_{(n-1)}^{\vec{\sigma}du}f^{eu\gamma}(-1)^n\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n-1)}^{\vec{\sigma}}(\mathbf{r})\ a_j^e(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}))$$
$$+\sum_{s=1}^{n-1}\frac{1}{s!(n-s)!}f_{(n-s-1)}^{\vec{\sigma}du}f^{eu\gamma}(-1)^{n-s}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n-s-1)}^{\vec{\sigma}}(\mathbf{r})\ \psi_{(s)j}^{e}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}))\ . \qquad \text{(A16)}$$

Similarly, the same algebraic identities used to obtain Eq. (A16) can be used to transform

$$\frac{i}{n!}\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\ \psi_{(n)j}^{\gamma}(\mathbf{r}')]V_j^{\gamma}(\mathbf{r}')=\frac{1}{n!}(-1)^{n-1}f_{(n)}^{\vec{\alpha}\beta\gamma}\int d\mathbf{r}'\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r}')[\partial_i\Pi_i^b(\mathbf{r}),\ \mathcal{Q}_{(n)j}^{\beta}(\mathbf{r}')]\ V_j^{\gamma}(\mathbf{r}')$$
$$+\frac{i}{n!}(-1)^{n-1}f_{(n)}^{\vec{\alpha}\beta\gamma}\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r}')]\mathcal{Q}_{(n)j}^{\beta}(\mathbf{r}')V_j^{\gamma}(\mathbf{r}') \qquad \text{(A17)}$$

to

$$\frac{i}{n!}\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\ \psi_{(n)j}^{\gamma}(\mathbf{r}')]V_j^{\gamma}(\mathbf{r}')=\frac{1}{(n-1)!(n+1)}(-1)^{n-1}f_{(n)}^{\vec{\alpha}b\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ \partial_jV_j^{\gamma}(\mathbf{r})$$
$$+\frac{1}{(n-1)!}(-1)^{n-1}f^{b\beta e}f_{(n-1)}^{\vec{\sigma}e\gamma}\ A_j^{\beta}(\mathbf{r})\ \mathcal{R}_{(n-1)}^{\vec{\sigma}}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r})$$
$$+\frac{n-1}{n!}(-1)^{n-1}f^{ed\gamma}f_{(n-1)}^{\vec{\tau}bd}\ \mathcal{R}_{(n-1)}^{\vec{\tau}}(\mathbf{r})\ a_j^e(\mathbf{r})\ V_j^{\gamma}(\mathbf{r})$$
$$+\sum_{s=1}^{n-2}\frac{n-s-1}{s!(n-s)!}(-1)^{n-s-1}f^{ed\gamma}f_{(n-s-1)}^{\vec{\tau}bd}\ \mathcal{R}_{(n-s-1)}^{\vec{\tau}}(\mathbf{r})\ \psi_{(s)j}^{e}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}) \quad . \tag{A18}$$

We can combine Eqs. (A16) and (A18) to obtain

$$-f^{b\mu d}\ A_i^{\mu}(\mathbf{r})\frac{i}{(n-1)!}\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \psi_{(n-1)j}^{\gamma}(\mathbf{r}')]V_j^{\gamma}(\mathbf{r}')$$
$$=\frac{i}{n!}\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\ \psi_{(n)j}^{\gamma}(\mathbf{r}')]\ V_j^{\gamma}(\mathbf{r}')-\frac{1}{(n-1)!(n+1)}(-1)^{n-1}f_{(n)}^{\vec{\alpha}b\gamma}\ \mathcal{R}_{(n)}^{\vec{\alpha}}(\mathbf{r})\ \partial_jV_j^{\gamma}(\mathbf{r})$$
$$-\frac{n-1}{n!}(-1)^{n-1}f^{ed\gamma}f_{(n-1)}^{\vec{\tau}bd}\ \mathcal{R}_{(n-1)}^{\vec{\tau}}(\mathbf{r})\ a_j^e(\mathbf{r})\ V_j^{\gamma}(\mathbf{r})-\sum_{s=1}^{n-2}\frac{n-s-1}{s!(n-s)!}(-1)^{n-s-1}f^{ed\gamma}f_{(n-s-1)}^{\vec{\tau}bd}$$
$$\times\mathcal{R}_{(n-s-1)}^{\vec{\tau}}(\mathbf{r})\ \psi_{(s)j}^{e}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r})-f^{b\mu d}A_i^{\mu}(\mathbf{r})\frac{1}{n!}(-1)^{n-1}f_{(n-1)}^{\vec{\alpha}d\gamma}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n-1)}^{\vec{\alpha}}(\mathbf{r})\ \partial_jV_j^{\gamma}(\mathbf{r}))$$
$$-f^{b\mu d}A_i^{\mu}(\mathbf{r})\frac{1}{(n-1)!}f_{(n-2)}^{\vec{\sigma}dh}f^{eh\gamma}(-1)^{n-1}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n-2)}^{\vec{\sigma}}(\mathbf{r})\ a_j^e(\mathbf{r})\ V_j^{\gamma}(\mathbf{r}))-f^{b\mu d}A_i^{\mu}(\mathbf{r})$$
$$\times\sum_{s=1}^{n-2}\frac{1}{s!(n-s-1)!}f_{(n-s-2)}^{\vec{\sigma}dh}f^{eh\gamma}(-1)^{n-s-1}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(n-s-2)}^{\vec{\sigma}}(\mathbf{r})\ \psi_{(s)j}^{e}(\mathbf{r})\ V_j^{\gamma}(\mathbf{r})) \quad . \tag{A19}$$

APPENDIX B: PROOF OF FUNDAMENTAL THEOREM

In this section we will prove Eq. (2.24) by an inductive argument that assumes that Eq. (2.24) holds for all $n\leqslant N$, and then demonstrates that it must also hold for $n=(N+1)$. The theorem is trivial for $n=2$ and $n=1$, in the latter case with the previously established convention that $\mathcal{A}_{(0)j}^{\alpha}(\mathbf{r})=0$. The structure of Eq. (2.23), which defines $\mathcal{A}_{(n)j}^{\gamma}(\mathbf{r})V_j^{\gamma}(\mathbf{r})$ recursively in terms of the inhomogeneous term $(ig^n/n!)\int d\mathbf{r}\psi_{(n)j}^{\gamma}(\mathbf{r})V_j^{\gamma}(\mathbf{r})$ as well as other $\mathcal{A}_{(n')j'}^{\gamma}(\mathbf{r})V_{j'}^{\gamma}(\mathbf{r})$ with $n'<n$, is ideally suited to an inductive argument.

We will transform $-igf^{b\mu d}A_i^{\mu}(\mathbf{r})\ \int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \mathcal{A}_{(N)j}^{\alpha}(\mathbf{r}')]V_j^{\alpha}(\mathbf{r}')$—the RHS of Eq. (2.24) for $n=N+1$—into the corresponding LHS of that equation, using Eq. (2.24) as an inductive axiom only for those $\mathcal{A}_{(n)j}^{\alpha}(\mathbf{r}')$ that have $n<N$. We set

$$-igf^{b\mu d}A_i^{\mu}(\mathbf{r})\ \int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \mathcal{A}_{(N)j}^{\alpha}(\mathbf{r}')]\ V_j^{\alpha}(\mathbf{r}')=\mathsf{A}+\mathsf{B}+\mathsf{C} \quad , \tag{B1}$$

where

$$\mathsf{A}=-gf^{b\mu d}\ A_i^{\mu}(\mathbf{r})\ \frac{ig^N}{N!}\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \psi_{(N)j}^{\gamma}(\mathbf{r}')]\ V_j^{\gamma}(\mathbf{r}') \quad , \tag{B2}$$

$$\mathsf{B}=-gf^{b\mu d}\ A_i^{\mu}(\mathbf{r})\ \sum_{m=1}\frac{ig^N}{m!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{u=0}\sum_{r=m}\ \delta_{r+u+m-N}\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\ \mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r}')]\ \mathcal{B}_{(m,u)j}^{\delta}(\mathbf{r}')\ V_j^{\gamma}(\mathbf{r}') \quad , \tag{B3}$$

and

$$\mathsf{C}=-gf^{b\mu d}\ A_i^{\mu}(\mathbf{r})\ \sum_{m=1}\frac{ig^N}{m!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{u=0}\sum_{r=m}\ \delta_{r+m+u-N}\int d\mathbf{r}'\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r}')[\Pi_i^d(\mathbf{r}),\ \mathcal{B}_{(m,u)j}^{\delta}(\mathbf{r}')]\ V_j^{\gamma}(\mathbf{r}') \quad . \tag{B4}$$

We will represent A, B, and C by dividing each of them into parts as shown by

$$\mathsf{A}=\mathsf{A}_{(1)}+\mathsf{A}_{(2)}+\mathsf{A}_{(3)} \quad , \tag{B5}$$

so that the subscript (1) designates those terms that contain commutators with $\partial_i\Pi_i^b(\mathbf{r})$, the subscript (3) designates terms that contain $\partial_i V_i^\alpha(\mathbf{r})$, and the subscript (2) labels residues, most of which cancel as the proof proceeds to its conclusion. The representation of A as a sum of its properly subscripted constituents is easily obtained from Eq. (A19). We use Eq. (A12) to represent B as

$$\mathsf{B}=-gf^{b\mu d}\,A_i^\mu(\mathbf{r})\sum_{m=1}\frac{ig^N}{m!}[\mathcal{P}^{(0)}_{(\alpha,\beta[m-1])}\,f^{\alpha\delta e}f^{\vec{\beta}e\gamma}_{(m-1)}\,]\sum_{u=0}\sum_{p=m-1}\sum_{r[m]=1}\delta_{p+r[m]+u+m-N}$$
$$\times\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\,\mathcal{A}^\alpha_{(r[m])j}(\mathbf{r}')]\frac{\partial_j}{\partial^2}(\mathcal{M}^{\vec{\beta}}_{(m-1,p)}(\mathbf{r}')\,\mathcal{B}^\delta_{(m,u)k}(\mathbf{r}')\,V_k^\gamma(\mathbf{r}')) \quad , \tag{B6}$$

and invoke the inductive axiom to represent $gf^{b\mu d}A_i^\mu(\mathbf{r})\;i\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\,\mathcal{A}^\alpha_{(r[m])j}(\mathbf{r}')]\;V_j^\alpha(\mathbf{r}')$ in terms of the LHS of Eq. (2.24) for all values of $r[m]<N$. When we equate the operator-valued vector quantity $(g^N/m!)[\mathcal{P}^{(0)}_{(\alpha,\beta[m-1])}f^{\alpha\delta e}f^{\vec{\beta}e\gamma}_{(m-1)}](\partial_j/\partial^2)(\mathcal{M}^{\vec{\beta}}_{(m-1,p)}(\mathbf{r}')\;\mathcal{B}^\delta_{(m,u)k}(\mathbf{r}')\;V_k^\gamma(\mathbf{r}'))$ in Eq. (B6) to the arbitrary vector field $V_j^\alpha(\mathbf{r}')$ in Eq. (2.24), we obtain

$$\mathsf{B}_{(1)}=\sum_{m=1}\frac{ig^{N+1}}{m!}[\mathcal{P}^{(0)}_{(\alpha,\beta[m-1])}f^{\alpha\delta e}f^{\vec{\beta}e\gamma}_{(m-1)}]\sum_{u=0}\sum_{p=m-1}\sum_{r[m]=0}\delta_{p+r[m]+m+u-N}$$
$$\times\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\,\mathcal{A}^\alpha_{(r[m]+1)j}(\mathbf{r}')]\frac{\partial_j}{\partial^2}(\mathcal{M}^{\vec{\beta}}_{(m-1,p)}(\mathbf{r}')\;\mathcal{B}^\delta_{(m,u)k}(\mathbf{r}')\;V_k^\gamma(\mathbf{r}')) \quad , \tag{B7}$$

and, after summing over $r[m]$ and representing the sums over permutations by using the lemma given in Eq. (A1),

$$\mathsf{B}_{(2)}=f^{b\mu\alpha}A_i^\mu(\mathbf{r})\sum_{m=1}\sum_{u=0}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=v}\delta_{r+q+m+u-N}\frac{g^{N+1}}{(v+1)!(m-v-1)!}f^{\vec{\beta}\delta g}_{(m-v-1)}f^{gf\gamma}f^{\vec{\sigma}\alpha f}_{(v)}$$
$$\times\frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\mathcal{M}^{\vec{\sigma}}_{(v,q)}(\mathbf{r})\mathcal{B}^\delta_{(m,u)j}(\mathbf{r})V_j^\gamma(\mathbf{r}))-\sum_{m=1}\sum_{u=0}\sum_{n=1}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=n+v}\delta_{r+q+m+n+u-N}$$
$$\times f^{\vec{\beta}\delta h}_{(m-v-1)}f^{\alpha h\gamma}f^{\vec{\sigma}c\alpha}_{(n+v)}f^{b\mu c}\frac{g^{N+1}B(n)}{n!(v+1)!(m-v-1)!}A_i^\mu(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\sigma}}_{(n+v,q)}(\mathbf{r})\;\mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\;\mathcal{B}^\delta_{(m,u)j}(\mathbf{r})\;V_j^\gamma(\mathbf{r}))$$
$$+\sum_{m=1}\sum_{u=0}\sum_{n=0}\sum_{t=1}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=n+v}\delta_{r+q+n+t+u+m-(N+1)}(-1)^{t-1}\frac{g^{N+1}B(n)}{n!(t-1)!(t+1)(v+1)!(m-v-1)!}$$
$$\times f^{\vec{\beta}\delta h}_{(m-v-1)}f^{\alpha h\gamma}f^{\vec{\sigma}\lambda\alpha}_{(n+v)}f^{\vec{\mu}b\lambda}_{(t)}\;\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\;\mathcal{M}^{\vec{\sigma}}_{(n+v,q)}(\mathbf{r})\mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\;\mathcal{B}^\delta_{(m,u)i}(\mathbf{r})\;V_i^\gamma(\mathbf{r})$$
$$+\sum_{m=1}\sum_{u=0}\sum_{n=0}\sum_{t=1}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=n+v}\delta_{r+q+n+t+m+u-N}(-1)^t\frac{g^{N+1}B(n)}{n!(t+1)!(v+1)!(m-v-1)!}$$
$$\times f^{\vec{\beta}\delta h}_{(m-v-1)}f^{\alpha h\gamma}f^{\vec{\sigma}\lambda\alpha}_{(n+v)}f^{b\mu d}f^{\vec{\nu}d\lambda}_{(t)}\;A_i^\mu(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{R}^{\vec{\nu}}_{(t)}(\mathbf{r})\mathcal{M}^{\vec{\sigma}}_{(n+v,q)}(\mathbf{r})\;\mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\;\mathcal{B}^\delta_{(m,u)j}(\mathbf{r})\;V_j^\gamma(\mathbf{r})) \quad . \tag{B8}$$

We will represent C as

$$\mathsf{C}=\mathsf{C}(a)+\mathsf{C}(\mathcal{A}) \quad , \tag{B9}$$

where $\mathsf{C}(a)$ includes the commutator of $\Pi_i^\alpha(\mathbf{r})$ with the $a_j^\delta(\mathbf{r}')$ part of $\mathcal{B}^\delta_{(m,u)j}(\mathbf{r}')$ and $\mathsf{C}(\mathcal{A})$ includes the commutator of $\Pi_i^\alpha(\mathbf{r})$ with the $\mathcal{A}^\delta_{(u)j}(\mathbf{r}')$ part of $\mathcal{B}^\delta_{(m,u)j}(\mathbf{r}')$. $\mathsf{C}(a)$ is given by

$$\mathsf{C}(a)=-gf^{b\mu\delta}\,A_i^\mu(\mathbf{r})\sum_{m=1}\sum_{r=m}\delta_{r+m-N}\frac{g^N}{m!}f^{\vec{\alpha}\delta\gamma}_{(m)}\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)(\mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r})\;V_j^\gamma(\mathbf{r})) \tag{B10}$$

and $\mathsf{C}(\mathcal{A})$ is given by

$$\mathsf{C}(\mathcal{A})=-gf^{b\mu d}\,A_i^{\mu}(\mathbf{r})\sum_{m=1}\frac{ig^N}{m!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{u=1}\sum_{r=m}\delta_{r+m+u-N}\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\,\mathcal{A}_{(u)j}^{\delta}(\mathbf{r}')]\left(\delta_{jk}-\frac{m}{m+1}\frac{\partial_j\partial_k}{\partial^2}\right)(\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r}')\;V_k^{\gamma}(\mathbf{r}')). \tag{B11}$$

We again invoke the inductive axiom to transform $\mathsf{C}(\mathcal{A})$ by representing $ig\;f^{b\mu d}\;A_i^{\mu}(\mathbf{r})\int d\mathbf{r}'[\Pi_i^d(\mathbf{r}),\,\mathcal{A}_{(u)j}^{\delta}(\mathbf{r}')]\;V_j^{\delta}(\mathbf{r}')$ in terms of the LHS of Eq. (2.24) for values of $u<N$, and identify $(g^N/m!)f_{(m)}^{\vec{\alpha}\delta\gamma}[\delta_{jk}-m/(m+1)\times(\partial_j\partial_k/\partial^2)](\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r}')V_k^{\gamma}(\mathbf{r}'))$ as the vector field $V_j^{\alpha}(\mathbf{r}')$ in Eq. (2.24); after summing over the common integer-valued variable in the two Kronecker δ functions in the resulting expression, we obtain

$$\mathsf{C}(\mathcal{A})=\mathsf{C}(\mathcal{A})_{(1)}+\mathsf{C}(\mathcal{A})_{(2)}+\mathsf{C}(\mathcal{A})_{(3)}\quad, \tag{B12}$$

with

$$\mathsf{C}(\mathcal{A})_{(1)}=\sum_{m=1}\frac{ig^{N+1}}{m!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{u=0}\sum_{r=m}\delta_{r+m+u-N}\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\,\mathcal{A}_{(u+1)j}^{\delta}(\mathbf{r}')]\left(\delta_{jk}-\frac{m}{m+1}\frac{\partial_j\partial_k}{\partial^2}\right)(\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r}')\;V_k^{\gamma}(\mathbf{r}'))\quad, \tag{B13}$$

$$\begin{aligned}\mathsf{C}(\mathcal{A})_{(2)}=&f^{b\mu\delta}\,A_i^{\mu}(\mathbf{r})\sum_{m=1}\frac{g^{N+1}}{m!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{r=m}\delta_{r+m-N}\left(\delta_{ij}-\frac{m}{m+1}\frac{\partial_i\partial_j}{\partial^2}\right)(\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})V_j^{\gamma}(\mathbf{r}))\\&-\sum_{m=1}\sum_{r=m}\frac{g^{N+1}}{(m+1)!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{n=1}\sum_{s=n}\delta_{s+n+r+m-N}\frac{B(n)}{n!}f^{b\mu c}f_{(n)}^{\vec{\sigma}c\delta}\,A_i^{\mu}(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{M}_{(n,s)}^{\vec{\sigma}}(\mathbf{r})\;\partial_j\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})\;V_j^{\gamma}(\mathbf{r}))\\&+\sum_{m=1}\sum_{r=m}\frac{g^{N+1}}{(m+1)!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{n=0}\sum_{t=1}\sum_{s=n}\delta_{r+m+s+n+t-(N+1)}\,(-1)^{t-1}\frac{B(n)}{n!(t-1)!(t+1)}f_{(t)}^{\vec{\mu}b\lambda}f_{(n)}^{\vec{\sigma}\lambda\delta}\mathcal{R}_{(t)}^{\vec{\mu}}(\mathbf{r})\\&\times\mathcal{M}_{(n,s)}^{\vec{\sigma}}(\mathbf{r})\;\partial_i\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})\;V_i^{\gamma}(\mathbf{r})+\sum_{m=1}\sum_{r=m}\frac{g^{N+1}}{(m+1)!}f_{(m)}^{\vec{\alpha}\delta\gamma}f^{b\mu d}\,A_i^{\mu}(\mathbf{r})\\&\times\sum_{n=0}\sum_{t=1}\sum_{s=n}\delta_{r+m+s+n+t-N}(-1)^t\frac{B(n)}{n!(t+1)!}f_{(t)}^{\vec{\nu}d\lambda}f_{(n)}^{\vec{\sigma}\lambda\delta}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(t)}^{\vec{\nu}}(\mathbf{r})\;\mathcal{M}_{(n,s)}^{\vec{\sigma}}(\mathbf{r})\;\partial_j\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})\;V_j^{\gamma}(\mathbf{r}))\quad,\end{aligned} \tag{B14}$$

and

$$\begin{aligned}\mathsf{C}(\mathcal{A})_{(3)}=&-\sum_{m=1}\sum_{r=m}\frac{g^{N+1}}{(m+1)!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{n=1}\sum_{s=n}\delta_{r+m+s+n-N}\frac{B(n)}{n!}f^{b\mu c}f_{(n)}^{\vec{\sigma}c\delta}\,A_i^{\mu}(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{M}_{(n,s)}^{\vec{\sigma}}(\mathbf{r})\;\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})\;\partial_jV_j^{\gamma}(\mathbf{r}))\\&+\sum_{m=1}\sum_{r=m}\frac{g^{N+1}}{(m+1)!}f_{(m)}^{\vec{\alpha}\delta\gamma}\sum_{n=0}\sum_{t=1}\sum_{s=n}\delta_{r+m+s+n+t-(N+1)}(-1)^{t-1}\frac{B(n)}{n!(t-1)!(t+1)}\\&\times f_{(t)}^{\vec{\mu}b\lambda}f_{(n)}^{\vec{\sigma}\lambda\delta}\mathcal{R}_{(t)}^{\vec{\mu}}(\mathbf{r})\mathcal{M}_{(n,s)}^{\vec{\sigma}}(\mathbf{r})\;\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})\partial_iV_i^{\gamma}(\mathbf{r})+\sum_{m=1}\sum_{r=m}\frac{g^{N+1}}{(m+1)!}f_{(m)}^{\vec{\alpha}\delta\gamma}f^{b\mu d}A_i^{\mu}(\mathbf{r})\\&\times\sum_{n=0}\sum_{t=1}\sum_{s=n}\delta_{r+m+s+n+t-N}(-1)^t\frac{B(n)}{n!(t+1)!}f_{(t)}^{\vec{\nu}d\lambda}f_{(n)}^{\vec{\sigma}\lambda\delta}\frac{\partial_i}{\partial^2}(\mathcal{R}_{(t)}^{\vec{\nu}}(\mathbf{r})\mathcal{M}_{(n,s)}^{\vec{\sigma}}(\mathbf{r})\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})\partial_jV_j^{\gamma}(\mathbf{r}))\quad.\end{aligned} \tag{B15}$$

When we transform $\partial_i\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})$ and $\partial_j\mathcal{M}_{(m,r)}^{\vec{\alpha}}(\mathbf{r})$ in $\mathsf{C}(\mathcal{A})_{(2)}$ by using Eq. (A12) and transform the resulting expression by applying Eq. (A1), we obtain an equation that so resembles Eq. (B8) in structure, that it becomes very natural to add $\mathsf{B}_{(2)}$ and $\mathsf{C}(\mathcal{A})_{(2)}$. In carrying out this addition, we note that

$$\mathcal{B}_{(m,u)j}^{\delta}(\mathbf{r})+\frac{v+1}{(m+1)(m-v)}\partial_j\mathcal{Y}_{(u)}^{\delta}(\mathbf{r})=\mathcal{B}_{(m-v-1,u)j}^{\delta}(\mathbf{r})\quad, \tag{B16}$$

and obtain

$$\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}=f^{b\mu\alpha}A_i^\mu(\mathbf{r})\sum_{m=1}\sum_{u=0}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=v}\delta_{r+q+m+u-N}\frac{g^{N+1}}{(v+1)!(m-v-1)!}$$
$$\times f^{\vec{\beta}\delta g}_{(m-v-1)}f^{gf\gamma}f^{\vec{\sigma}\alpha f}_{(v)}\frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\ \mathcal{M}^{\vec{\sigma}}_{(v,q)}(\mathbf{r})\ \mathcal{B}^{\delta}_{(m,u)j}(\mathbf{r})\ V_j^\gamma(\mathbf{r}))+f^{b\mu\delta}\ A_i^\mu(\mathbf{r})$$
$$\times\sum_{m=1}\frac{g^{N+1}}{m!}f^{\vec{\alpha}\delta\gamma}_{(m)}\sum_{r=m}\delta_{r+m-N}\left(\delta_{ij}-\frac{m}{m+1}\frac{\partial_i\partial_j}{\partial^2}\right)(\mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r})\ V_j^\gamma(\mathbf{r}))$$
$$-\sum_{m=1}\sum_{u=0}\sum_{n=1}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=n+v}\delta_{r+q+m+n+u-N}f^{\vec{\beta}\delta h}_{(m-v-1)}f^{\alpha h\gamma}f^{\vec{\sigma}c\alpha}_{(n+v)}f^{b\mu c}$$
$$\times\frac{g^{N+1}B(n)}{n!(v+1)!(m-v-1)!}A_i^\mu(\mathbf{r})\ \frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\sigma}}_{(n+v,q)}(\mathbf{r})\ \mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})$$
$$\times\mathcal{B}^{\delta}_{(m-v-1,u)j}(\mathbf{r})\ V_j^\gamma(\mathbf{r}))+\sum_{m=1}\sum_{u=0}\sum_{n=0}\sum_{t=1}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=n+v}\delta_{r+q+m+n+u+t-(N+1)}(-1)^{t-1}$$
$$\times\frac{g^{N+1}B(n)}{n!(t-1)!(t+1)(v+1)!(m-v-1)!}f^{\vec{\beta}\delta h}_{(m-v-1)}f^{\alpha h\gamma}f^{\vec{\sigma}\lambda\alpha}_{(n+v)}f^{\vec{\mu}b\lambda}_{(t)}\ \mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\mathcal{M}^{\vec{\sigma}}_{(n+v,q)}(\mathbf{r})$$
$$\times\mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\ \mathcal{B}^{\delta}_{(m-v-1,u)i}(\mathbf{r})\ V_i^\gamma(\mathbf{r})+\sum_{m=1}\sum_{u=0}\sum_{n=0}\sum_{t=1}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=n+v}\delta_{r+q+m+n+u+t-N}(-1)^t$$
$$\times\frac{g^{N+1}B(n)}{n!(t+1)!(v+1)!(m-v-1)!}f^{\vec{\beta}\delta h}_{(m-v-1)}f^{\alpha h\gamma}f^{\vec{\sigma}\lambda\alpha}_{(n+v)}f^{b\mu d}f^{\vec{\nu}d\lambda}_{(t)}A_i^\mu(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{R}^{\vec{\nu}}_{(t)}(\mathbf{r})$$
$$\times\mathcal{M}^{\vec{\sigma}}_{(n+v,q)}(\mathbf{r})\ \mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\ \mathcal{B}^{\delta}_{(m-v-1,u)j}(\mathbf{r})V_j^\gamma(\mathbf{r}))\ . \tag{B17}$$

We change the integer-valued variables in the summations of the third, fourth, and fifth terms in Eq. (B17) to $k=m+n$ and $\ell=v+n$, and carry out the summation over k, ℓ, and n. We then observe that combining $\mathsf{B}_{(2)}$ and $\mathsf{C}(\mathcal{A})_{(2)}$ and applying Eq. (B16) has left us with an expression in which the only n dependence is in the Bernoulli numbers, and in fractional coefficients. The indices p and q in the operator-valued functions $\mathcal{M}^{\vec{\sigma}}_{(p,q)}(\mathbf{r})$, and $\mathcal{B}^{\delta}_{(p,q)i}(\mathbf{r})$ all are either k or ℓ, and have no further dependence on the integer-valued summation index n. We therefore can make use of the identity

$$D_s^\ell(\ell)=0\quad\text{for}\quad s=0\quad\text{and}\quad\ell>0\ ,\quad\text{where}\quad D_s^\kappa(\ell)=\sum_{n=s}^{\kappa}\frac{B(n)}{n!(\ell-n+1)!}\ , \tag{B18}$$

and observe that the only surviving contributions to Eq. (B17) from sums over Bernoulli numbers are $D_1^\ell(\ell)=-[1/(\ell+1)!]$ and $D_0^0(0)=1$. We represent $\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}$ as $[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(a)}+[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(b)}+[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(c)}$, where

$$[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}\ ]_{(a)}=f^{b\mu\delta}A_i^\mu(\mathbf{r})\sum_{m=1}\frac{g^{N+1}}{m!}f^{\vec{\alpha}\delta\gamma}_{(m)}\sum_{r=m}\delta_{r+m-N}\left(\delta_{ij}-\frac{m}{m+1}\frac{\partial_i\partial_j}{\partial^2}\right)(\mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r})\ V_j^\gamma(\mathbf{r}))\ , \tag{B19}$$

$$[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(b)}=f^{b\mu\alpha}A_i^\mu(\mathbf{r})\sum_{m=1}\sum_{u=0}\sum_{v=0}^{m-1}\sum_{r=m-v-1}\sum_{q=v}\delta_{r+q+m+u-N}\frac{g^{N+1}}{(v+1)!(m-v-1)!}$$
$$\times f^{\vec{\beta}\delta g}_{(m-v-1)}f^{gf\gamma}f^{\vec{\sigma}\alpha f}_{(v)}\frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\beta}}_{(m-v-1,r)}(\mathbf{r})\ \mathcal{M}^{\vec{\sigma}}_{(v,q)}(\mathbf{r})\ \mathcal{B}^{\delta}_{(m,u)j}(\mathbf{r})\ V_j^\gamma(\mathbf{r}))$$
$$+\sum_{k=2}\sum_{\ell=1}^{k-1}\sum_{u=0}\sum_{r=k-\ell-1}\sum_{q=\ell}\delta_{r+q+k+u-N}f^{\vec{\beta}\delta h}_{(k-\ell-1)}f^{\alpha h\gamma}f^{\vec{\sigma}\nu\alpha}_{(\ell)}f^{b\mu\nu}A_i^\mu(\mathbf{r})\frac{g^{N+1}}{(\ell+1)!(k-\ell-1)!}$$
$$\times\frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\sigma}}_{(\ell,q)}(\mathbf{r})\mathcal{M}^{\vec{\beta}}_{(k-\ell-1,r)}(\mathbf{r})\ \mathcal{B}^{\delta}_{(k-\ell-1,u)j}(\mathbf{r})\ V_j^\gamma(\mathbf{r}))\ , \tag{B20}$$

and

$$[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(c)}=\sum_{k=1}\sum_{u=0}\sum_{r=k-1}\sum_{t=1}\delta_{r+k+u+t-(N+1)}(-1)^{t-1}\frac{g^{N+1}}{(t-1)!(t+1)(k-1)!}$$
$$\times f^{\vec{\beta}\delta h}_{(k-1)}f^{h\lambda\gamma}f^{\vec{\mu}b\lambda}_{(t)}\mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\mathcal{M}^{\vec{\beta}}_{(k-1,r)}(\mathbf{r})\ \mathcal{B}^{\delta}_{(k-1,u)i}(\mathbf{r})\ V^{\gamma}_i(\mathbf{r})$$
$$+\sum_{k=1}\sum_{u=0}\sum_{r=k-1}\sum_{t=1}\delta_{r+k+u+t-N}(-1)^{t}\frac{g^{N+1}}{(t+1)!(k-1)!}f^{\vec{\beta}\delta h}_{(k-1)}f^{h\lambda\gamma}f^{b\mu d}f^{\vec{\nu}d\lambda}_{(t)}\ A^{\mu}_i(\mathbf{r})$$
$$\times\frac{\partial_i}{\partial^2}(\mathcal{R}^{\vec{\nu}}_{(t)}(\mathbf{r})\ \mathcal{M}^{\vec{\beta}}_{(k-1,r)}(\mathbf{r})\ \mathcal{B}^{\delta}_{(k-1,u)j}(\mathbf{r})\ V^{\gamma}_j(\mathbf{r}))\ . \tag{B21}$$

We then note that the second term on the RHS of Eq. (B20) is an expression that has the form $\Sigma_{k=2}\Sigma_{\ell=1}^{k-1}\varphi(k,\ell)$, and that this sum can be expressed as $\Sigma_{k=2}\Sigma_{\ell=1}^{k-1}\varphi(k,\ell)=\Sigma_{k=1}\Sigma_{\ell=0}^{k-1}\varphi(k,\ell)-\Sigma_{k=2}\varphi(k,0)-\varphi(1,0)$; we further observe that a number of the summations in the parts of Eq. (B20) that we have included in $\Sigma_{k=2}\varphi(k,0)$ and $\varphi(1,0)$ can be eliminated because they become degenerate, enabling us to make use of Eq. (2.23) to transform them. We use Eq. (B16) to combine the $\Sigma_{k=1}\Sigma_{\ell=0}^{k-1}\varphi(k,\ell)$ part of this second term on the RHS of Eq. (B20) with the first term in that equation, so that the two $\mathcal{B}^{\delta}_{(\eta,u)j}(\mathbf{r})$ terms are combined into a multiple of $\partial_j\mathcal{Y}^{\delta}_{(u)}(\mathbf{r})$. Finally, we use Eq. (2.25) with $[\mathsf{Q},\ ]=\partial_j$, to obtain

$$[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(b)}=-\sum_{m=1}\sum_{r=m}\delta_{r+m-N}\frac{g^{N+1}}{(m+1)!}f^{\vec{\alpha}\delta\gamma}_{(m)}f^{b\mu\delta}\ A^{\mu}_i(\mathbf{r})\ \frac{\partial_i}{\partial^2}(\partial_j\mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r})\ V^{\gamma}_j(\mathbf{r}))$$
$$-\frac{g^{N+1}}{(N-1)!}f^{\delta h\gamma}f^{b\mu\delta}\ A^{\mu}_i(\mathbf{r})\ \frac{\partial_i}{\partial^2}(\psi^{h}_{(N-1)j}(\mathbf{r})\ V^{\gamma}_j(\mathbf{r}))\ . \tag{B22}$$

We also note that the RHS of Eq. (B21) contains an expression of the form $\Sigma_{k=1}\vartheta(k)$, which can be expressed as $\Sigma_{k=1}\vartheta(k)=\Sigma_{k=2}\vartheta(k)+\vartheta(1)$. As in the case of $\varphi(0)$ above, a number of the summations in $\vartheta(1)$ can be eliminated; we make use of Eq. (2.23) to transform $\Sigma_{k=2}\vartheta(k)$, and then obtain

$$[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(c)}=(-1)^{N-1}\frac{g^{N+1}}{(N-1)!(N+1)}f^{\lambda\delta\gamma}f^{\vec{\mu}b\lambda}_{(N)}\ \mathcal{R}^{\vec{\mu}}_{(N)}(\mathbf{r})\ a^{\delta}_i(\mathbf{r})\ V^{\gamma}_i(\mathbf{r})$$
$$+\sum_{t=1}^{N-1}(-1)^{t-1}\frac{g^{N+1}}{(t-1)!(t+1)(N-t)!}f^{\lambda\delta\gamma}f^{\vec{\mu}b\lambda}_{(t)}\ \mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\ \psi^{\delta}_{(N-t)i}(\mathbf{r})\ V^{\gamma}_i(\mathbf{r})$$
$$+(-1)^{N-1}\frac{g^{N+1}}{N!}f^{\lambda\delta\gamma}f^{b\mu d}f^{\vec{\nu}d\lambda}_{(N-1)}\ A^{\mu}_i(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{R}^{\vec{\nu}}_{(N-1)}(\mathbf{r})\ a^{\delta}_j(\mathbf{r})\ V^{\gamma}_j(\mathbf{r}))$$
$$+\sum_{t=1}^{N-2}(-1)^{t}\frac{g^{N+1}}{(t+1)!(N-t-1)!}f^{\lambda\delta\gamma}f^{b\mu d}f^{\vec{\nu}d\lambda}_{(t)}\ A^{\mu}_i(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{R}^{\vec{\nu}}_{(t)}(\mathbf{r})\ \psi^{\delta}_{(N-t-1)j}(\mathbf{r})\ V^{\gamma}_j(\mathbf{r}))\ . \tag{B23}$$

Making use of Eq. (A19), we observe that $[\mathsf{B}_{(2)}+\mathsf{C}(\mathcal{A})_{(2)}]_{(c)}$ in Eq. (B23) has the same form as $\mathsf{A}_{(2)}$; and since Eq. (B19) has the same structure as $\mathsf{C}(a)$ in Eq. (B10), it is natural to combine these terms to obtain

$$\mathsf{A}_{(2)}+\mathsf{B}_{(2)}+\mathsf{C}(a)+\mathsf{C}(\mathcal{A})_{(2)}=\sum_{m=1}\sum_{r=m}\delta_{r+m-N}\frac{g^{N+1}}{(m+1)!}f^{b\mu c}f^{\vec{\sigma}c\gamma}_{(m)}\ A^{\mu}_i(\mathbf{r})\ \frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\sigma}}_{(m,r)}(\mathbf{r})\ \partial_jV^{\gamma}_j(\mathbf{r}))\ . \tag{B24}$$

We combine all the terms with subscript (1), use Eq. (A12) to eliminate permutations of structure constants, and note that $\partial_i\Pi^b_i(\mathbf{r})$ commutes with $a^{\alpha}_j(\mathbf{r}')$. We then observe that

$$\mathsf{A}_{(1)}+\mathsf{B}_{(1)}+\mathsf{C}(\mathcal{A})_{(1)}=\frac{ig^{N+1}}{(N+1)!}\int d\mathbf{r}'[\partial_i\Pi^b_i(\mathbf{r}),\ \psi^{\gamma}_{(N+1)j}(\mathbf{r}')]V^{\gamma}_j(\mathbf{r}')$$
$$+\sum_{m=1}\frac{ig^{N+1}}{m!}f^{\vec{\alpha}\delta\gamma}_{(m)}\sum_{u=0}\sum_{r=m}\delta_{r+m+u-(N+1)}\int d\mathbf{r}'[\partial_i\Pi^b_i(\mathbf{r}),\ \mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r}')]\ \mathcal{B}^{\delta}_{(m,u)j}(\mathbf{r}')\ V^{\gamma}_j(\mathbf{r}')$$
$$+\sum_{m=1}\frac{ig^{N+1}}{m!}f^{\vec{\alpha}\delta\gamma}_{(m)}\sum_{u=0}\sum_{r=m}\delta_{r+m+u-(N+1)}\int d\mathbf{r}'[\partial_i\Pi^b_i(\mathbf{r}),\ \mathcal{B}^{\delta}_{(m,u)j}(\mathbf{r}')]\ \mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r}')\ V^{\gamma}_j(\mathbf{r}')\ . \tag{B25}$$

If we then use Eq. (2.23) to transform $\mathcal{A}^{\gamma}_{(N+1)j}$, Eq. (B25) can be written as

$$\mathsf{A}_{(1)}+\mathsf{B}_{(1)}+\mathsf{C}\ (\mathcal{A})_{(1)}=i\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\ \mathcal{A}^{\alpha}_{(N+1)j}(\mathbf{r}')]\ V_j^{\alpha}(\mathbf{r}')\quad . \tag{B26}$$

We change the integer-valued variables in the summation in Eq. (B15) to $\ell=m+n$, and carry out the summation over ℓ and n; we then obtain an expression in which the only n dependence is in the Bernoulli numbers and in fractional coefficients. We make use of Eq. (B18) and observe that the only surviving contributions to Eq. (B15) from sums over Bernoulli numbers are $D_0^{\ell-1}(\ell)=-[B(\ell)/\ell!]$ and $D_1^{\ell-1}(\ell)=-[B(\ell)/\ell!]-[1/(\ell+1)!]$; we then obtain

$$\begin{aligned}\mathsf{C}(\mathcal{A})_{(3)}=&-\sum_{\ell=2}\sum_{p=\ell}\delta_{p+\ell-N}\ g^{N+1}\left(\frac{B(\ell)}{\ell!}+\frac{1}{(\ell+1)!}\right)f^{b\mu c}f_{(\ell)}^{\vec{\sigma}c\gamma}\ A_i^{\mu}(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\sigma}}_{(\ell,p)}(\mathbf{r})\ \partial_jV_j^{\gamma}(\mathbf{r}))\\&+\sum_{\ell=1}\sum_{t=1}\sum_{p=\ell}\ \delta_{p+\ell+t-(N+1)}(-1)^{t-1}\frac{g^{N+1}B(\ell)}{\ell!(t-1)!(t+1)}f_{(t)}^{\vec{\mu}b\lambda}f_{(\ell)}^{\vec{\sigma}\lambda\gamma}\ \mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\ \mathcal{M}^{\vec{\sigma}}_{(\ell,p)}(\mathbf{r})\ \partial_iV_i^{\gamma}(\mathbf{r})\\&+\sum_{\ell=1}\sum_{t=1}\sum_{p=\ell}\delta_{p+\ell+t-N}(-1)^{t}\frac{g^{N+1}B(\ell)}{\ell!(t+1)!}f^{b\mu d}f_{(t)}^{\vec{\nu}d\lambda}f_{(\ell)}^{\vec{\sigma}\lambda\gamma}\ A_i^{\mu}(\mathbf{r})\frac{\partial_i}{\partial^2}(\mathcal{R}^{\vec{\nu}}_{(t)}(\mathbf{r})\ \mathcal{M}^{\vec{\sigma}}_{(\ell,p)}(\mathbf{r})\ \partial_jV_j^{\gamma}(\mathbf{r}))\quad .\end{aligned}\tag{B27}$$

Finally, we combine Eqs. (A19), (B24), (B26), and (B27) to obtain

$$\begin{aligned}\mathsf{A}+\mathsf{B}+\mathsf{C}=&\,i\int d\mathbf{r}'[\partial_i\Pi_i^b(\mathbf{r}),\ \mathcal{A}^{\alpha}_{(N+1)j}(\mathbf{r}')]V_j^{\alpha}(\mathbf{r}')+\delta_Nf^{b\mu\gamma}A_i^{\mu}(\mathbf{r})\ V_i^{\gamma}(\mathbf{r})\\&-\sum_{m=1}\sum_{r=m}\delta_{r+m-N}\frac{B(m)}{m!}f^{b\mu c}f_{(m)}^{\vec{\alpha}c\gamma}\ A_i^{\mu}(\mathbf{r})\ \frac{\partial_i}{\partial^2}(\mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r})\ \partial_jV_j^{\gamma}(\mathbf{r}))\\&+\sum_{m=0}\sum_{t=1}\sum_{r=m}\delta_{r+m+t-(N+1)}(-1)^{t-1}\frac{B(m)}{m!(t-1)!(t+1)}f_{(t)}^{\vec{\mu}b\lambda}f_{(m)}^{\vec{\alpha}\lambda\gamma}\ \mathcal{R}^{\vec{\mu}}_{(t)}(\mathbf{r})\ \mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r})\ \partial_iV_i^{\gamma}(\mathbf{r})\\&+f^{b\mu d}A_i^{\mu}(\mathbf{r})\sum_{m=0}\sum_{t=1}\sum_{r=m}\delta_{r+m+t-N}(-1)^{t}\frac{B(m)}{m!(t+1)!}f_{(t)}^{\vec{\nu}d\lambda}f_{(m)}^{\vec{\alpha}\lambda\gamma}\ \frac{\partial_i}{\partial^2}(\mathcal{R}^{\vec{\nu}}_{(t)}(\mathbf{r})\ \mathcal{M}^{\vec{\alpha}}_{(m,r)}(\mathbf{r})\ \partial_jV_j^{\gamma}(\mathbf{r}))\quad ,\end{aligned}\tag{B28}$$

where we have added a δ_N term that vanishes except for the $N=0$ case; the need for this term in the $N=0$ case has been discussed in Sec. II. The RHS of Eq. (B28) is identical to the LHS of Eq. (2.24) for the value $n=N+1$, and therefore completes the proof of the ''fundamental theorem'' for the construction of Ψ.

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