Nonequilibrium perturbation theory for complex scalar fields

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Real-time perturbation theory is formulated for complex scalar fields away from thermal equilibrium in such a way that dissipative effects arising from the absorptive parts of loop diagrams are approximately resummed into the unperturbed propagators. Low order calculations of physical quantities then involve quasiparticle occupation numbers which evolve with the changing state of the field system, in contrast with standard perturbation theory, where these occupation numbers are frozen at their initial values. The evolution equation of the occupation numbers can be cast approximately in the form of a Boltzmann equation. Particular attention is given to the effects of a nonzero chemical potential, and it is found that the thermal masses and decay widths of quasiparticle modes are different for particles and antiparticles. [S0556-2821(97)01604-4]

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I. INTRODUCTION

Many problems in physics require an understanding of the nonequilibrium properties of highly excited states of a quantum field theory. The principal motivation underlying the present work is to understand the nature of phase transitions in the very early universe, in particular those that might give rise to inflation [1–4]. Most discussions of inflation assume that the nonequilibrium state of the relevant system of quantum fields can adequately be characterized by expectation values governed by a classical potential. However, calculations which attempt to go beyond this classical approximation indicate that it may be seriously inaccurate (see, e.g., [5–8]). Nonequilibrium effects are also likely to be important in the context of baryon-number violation at the electroweak scale [9,10] and in the study of quark-gluon plasmas formed in heavy-ion collisions [11], as well as in many aspects of condensed matter physics.

Our own long-term program, set out in [12], has as its goal a complete description within perturbation theory of the dynamics of symmetry-breaking phase transitions. An essential requirement is to formulate perturbation theory in such a way that the evolving state of the nonequilibrium quantum fields is taken adequately into account in the low orders of the perturbative expansion which are likely to be tractable. The standard closed-time-path formalism [13–18], which deals with real-time phenomena in thermal equilibrium, was generalized by Semenoff and Weiss [19] to apply to a real scalar field with a time-dependent action, such as might arise in a Robertson-Walker spacetime. However, the propagators entering their Feynman rules depend on the state of the field essentially through single-particle occupation numbers which are fixed at their initial values and, therefore, do not properly reflect the evolution of the state.

In earlier work [20–22], we have shown, for real scalar fields, how this difficulty may be overcome. By adding a suitable counterterm (which we call the ''dissipative'' counterterm) to the quadratic part of the action, and subtracting it from the interaction, we obtain a lowest-order theory in which the absorptive parts of loop contributions to the two-point functions are partly resummed into the unperturbed propagators. As a result, these propagators contain quasiparticle occupation numbers which evolve with time in the expected way. Interestingly, the form of the counterterm is rather similar to the action encountered by Hu and various collaborators (see, e.g., [23–25] and references given in these papers). These authors use the Feynman-Vernon influence functional technique to obtain the effective action for a quantum field in the presence of environmental variables (which can be thought of as a heat bath) when the latter is integrated out. Our procedure can be thought of as a self-consistent treatment of a field which provides its own heat bath, though none of the modes of this field are integrated out. A similar structure also arises in nonequilibrium formulations of thermo-field dynamics (see, e.g., [26]) but this formalism turns out to be equivalent to a version of quantum-statistical mechanics restricted to Gaussian initial density matrices [27].

In dealing with a realistic model (such as the standard model or a grand unified theory), one naturally meets complex scalar, spinor, and gauge fields and it is essential to extend the nonequilibrium formalism to encompass fields of these kinds. (Real-time perturbation theory for scalar, spinor, and gauge fields in equilibrium is discussed, for example, by Kobes *et al.* [28].) The purpose of the present work is to extend the formalism to complex scalar fields, giving particular attention to the effects of a nonzero chemical potential, which turns out to be nontrivial. In Sec. II we obtain the structure of the matrix of two-point functions in the closed-time-path formalism, which is applied in Sec. III to find the most general permissible form of the dissipative counterterm. The corresponding unperturbed propagators are derived in Sec. IV, where we find that, in the presence of a nonzero chemical potential, not only occupation numbers but also decay widths and thermal masses are different for the quasiparticle modes corresponding to particles and antiparticles. In Sec. V we evaluate the dissipative counterterm at the lowest nontrivial order and find, with suitable approximations, that the time-dependent occupation numbers obey a

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kinetic equation of the Boltzmann type. Finally, our results are summarized in Sec. VI.

II. PROPAGATORS IN THE CLOSED-TIME-PATH FORMALISM

We consider the theory of a massive, complex, self-interacting scalar field $\phi(\mathbf{x},t)$, described by the Lagrangian density

$$\mathcal{L}(\phi,\phi^*)=\dot{\phi}^*\dot{\phi}-\nabla\phi^*\nabla\phi-m^2(t)\phi^*\phi-\frac{\lambda}{4}(\phi^*\phi)^2, \quad (1)$$

where $\dot{\phi}=\partial\phi/\partial t$. The generating functional for the time-ordered Green's functions of the theory may be written as

$$Z[j]=\mathrm{Tr}\left[\rho T\exp\left(i\int d^4x(j\phi^*+j^*\phi)\right)\right], \quad (2)$$

where $\phi(\mathbf{x},t)$ is the Heisenberg-picture field operator, $j(\mathbf{x},t)$ is a complex source, and T denotes time ordering. For an initial state of thermal equilibrium at time t_0, characterized by the temperature $1/\beta$ and chemical potential μ, the density operator is

$$\rho=\frac{e^{-\beta[H(t_0)-\mu N]}}{\mathrm{Tr}[e^{-\beta[H(t_0)-\mu N]}]}, \quad (3)$$

where $H(t)$ is the time-dependent Hamiltonian and N is the particle number operator. It is convenient to generalize this generating functional, using the Schrödinger-picture field operator $\phi_s(\mathbf{x})$ to write

$$Z[j_1,j_2]=\mathrm{Tr}\left[\rho\overline{T}\exp\left(i\int d^4x(H_s+j_2\phi_s^*+j_2^*\phi_s)\right)\times T\exp\left[-i\int d^4x(H_s-j_1\phi_s^*-j_1^*\phi_s)\right]\right] \quad (4)$$

where $\overline{T}$ denotes antitime ordering and $H_s(t)$ is the Hamiltonian expressed in terms of ϕ_s. This Hamiltonian depends explicitly on time through the time-dependent mass $m(t)$. The original generating functional (2) is given by $Z[j]=Z[j,0]$.

Each of the three factors inside the trace in (4) has a path-integral representation with its own integration variable, say $\phi_1(\mathbf{x},t)$ for the time-ordered factor, $\phi_2(\mathbf{x},t)$ for the antitime-ordered factor, and $\phi_3(\mathbf{x},\tau)$ for the density operator, where the real time t runs from the initial time t_0 to some large final time t_f, while the imaginary time τ runs from 0 to β. As usual [19,20], these three fields can be envisaged as living on the three segments of a contour in the complex time plane illustrated in Fig. 1. The effect of a nonzero chemical potential can formally be taken into account by using $H(t_0)$ as the generator of evolution in imaginary time and imposing the boundary conditions

$$\phi_3(\mathbf{x},\beta)=e^{-\beta\mu}\phi_1(\mathbf{x},t_0), \quad \phi_3^*(\mathbf{x},\beta)=e^{\beta\mu}\phi_1^*(\mathbf{x},t_0). \quad (5)$$

On introducing a third source for ϕ_3, the generating functional (4) is then given by the path integral

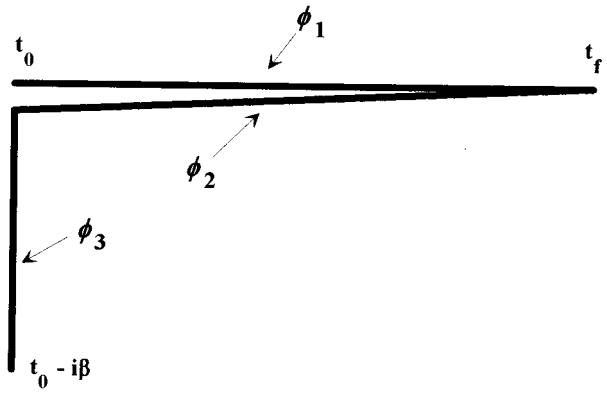

FIG. 1. The closed-time path in the complex time plane.

$$Z[j_1,j_2,j_3]=\mathcal{N}\int[d\phi_1][d\phi_2][d\phi_3]e^{iS(\phi_1,\phi_2,\phi_3)}, \quad (6)$$

where $\mathcal{N}$ is a normalizing constant and

$$S(\phi_1,\phi_2,\phi_3)=\int_{t_0}^{t_f}dt\int d^3x[\mathcal{L}(\phi_1)-\mathcal{L}(\phi_2)+j_1\phi_1^*+j_1^*\phi_1+j_2\phi_2^*+j_2^*\phi_2]+\int_0^\beta d\tau\int d^3x[i\mathcal{L}_E(\phi_3)+j_3\phi_3^*+j_3^*\phi_3]. \quad (7)$$

Here, $\mathcal{L}_E$ denotes the Euclidean version of Eq. (1).

In order to carry out perturbation theory, we wish to split the action into an unperturbed part S_0, an interaction part S_I, and a source term S_j:

$$S(\phi_1,\phi_2,\phi_3)=S_0(\phi_1,\phi_2,\phi_3)+S_I(\phi_1,\phi_2,\phi_3)+S_j(\phi_1,\phi_2,\phi_3). \quad (8)$$

In the standard way, the generating functional (6) can then be rewritten as

$$Z[j_1,j_2,j_3]=\exp[iS_I(-i\delta/\delta j_1,-i\delta/\delta j_2,-i\delta/\delta j_3)]Z_0[j_1,j_2,j_3], \quad (9)$$

$$Z_0[j_1,j_2,j_3]=\exp\left[-\int d^4xd^4yj_a^*(x)g_{ab}(x,y)j_b(y)\right]. \quad (10)$$

The unperturbed propagator $g(x,y)$ satisfies

$$\mathcal{D}_k(x,\vec{\partial}_x)g(x,y)=-i\delta(x-y)=g(x,y)\mathcal{D}(y,-\overleftarrow{\partial}_y), \quad (11)$$

where $\mathcal{D}(x,\partial)$ is the differential operator corresponding to S_0, i.e., $S_0=\int d^4x\phi_a^*(x)\mathcal{D}_{ab}(x,\partial)\phi_b(x)$.

The perturbation theory which results from choosing S_0 to be simply the quadratic part $S^{(2)}$ of S was derived some time ago by Semenoff and Weiss [19] for a theory with somewhat more general time dependence than Eq. (1). It has the disadvantage that coefficients in $g(x,y)$, which correspond roughly to particle occupation numbers, are fixed at their initial values, and do not change so as to reflect the evolving

state of the system. As shown by one of us for the case of a real scalar field [20,21], this situation can be improved by choosing instead

$$S_0(\phi_1,\phi_2,\phi_3)=S^{(2)}(\phi_1,\phi_2,\phi_3)+\int d^4x\,\phi_a^* M_{ab}(x,\partial)\phi_b, \tag{12}$$

where M is a differential operator, to be chosen in such a way that the unperturbed propagator $g(x,y)$ mimics as nearly as possible the dissipative behavior of the full propagator $G(x,y)$. To make perturbation theory tractable, we require the term involving M to be local, so it can involve only the real-time fields ϕ_1 and ϕ_2. Since (as is worth emphasizing) we do not wish to change the overall theory, the interaction term S_I now includes a counterterm $-\phi_a^* M_{ab}\phi_b$ in addition to the original interaction $-(\lambda/4)[(\phi_1^*\phi_1)^2-(\phi_2^*\phi_2)^2]$ and M will be chosen so that this counterterm subtracts some part of the loop contributions to the full propagator. In this way, we partially resum the absorptive parts of these loop contributions, and optimize $g(x,y)$ as an approximation to $G(x,y)$. In particular, we will find that $g(x,y)$ now involves quasiparticle occupation numbers which evolve with time in the expected way.

In order to determine the permissible form of M, we first investigate the structure of the full propagator, whose real-time components $(a,b=1,2)$ are given by

$$\begin{aligned}G_{ab}(x,x') &= -\frac{\partial}{\partial j_a^*(x)}\frac{\partial}{\partial j_b(x')}Z[j_1,j_2,j_3]\Big|_{j=0}\\ &=\begin{bmatrix}\langle T[\phi(x,t)\phi^*(x',t')]\rangle_\mu & \langle\phi^*(x',t')\phi(x,t)\rangle_\mu\\ \langle\phi(x,t)\phi^*(x',t')\rangle_\mu & \langle\overline{T}[\phi(x,t)\phi^*(x',t')]\rangle_\mu\end{bmatrix},\end{aligned} \tag{13}$$

where $\langle\cdots\rangle_\mu$ indicates a thermal average in the ensemble with chemical potential μ. Writing

$$\langle\phi(x)\phi^*(x')\rangle_\mu=\mathcal{H}_\mu(x,x')\Theta(t-t')+\mathcal{K}_\mu(x,x')\Theta(t'-t), \tag{14}$$

it is simple to show from the Hermiticity of ρ that $\mathcal{K}_\mu(x,x')=\mathcal{H}_\mu^*(x',x)$ and from the charge conjugation (C) symmetry of Eq. (1) that

$$\langle\phi^*(x')\phi(x)\rangle_\mu=\mathcal{H}^*_{-\mu}(x,x')\Theta(t-t')+\mathcal{H}_{-\mu}(x',x)\Theta(t'-t). \tag{15}$$

For any complex function of μ, we define the $\sharp$-conjugate $f^\sharp(\mu)=f^*(-\mu)$, which is related to charge conjugation. With this notation, the full propagator can be expressed as

$$\begin{aligned}G(x,x')=&\begin{bmatrix}\mathcal{H}(x,x') & \mathcal{H}^\sharp(x,x')\\ \mathcal{H}(x,x') & \mathcal{H}^\sharp(x,x')\end{bmatrix}\Theta(t-t')\\ &+\begin{bmatrix}\mathcal{H}^{*\sharp}(x',x) & \mathcal{H}^{*\sharp}(x',x)\\ \mathcal{H}^*(x',x) & \mathcal{H}^*(x',x)\end{bmatrix}\Theta(t'-t),\end{aligned} \tag{16}$$

where the suffix μ has been suppressed. This structure should be maintained if the scalar field is embedded in a more general C- or CP-invariant theory.

III. THE DISSIPATIVE COUNTERTERM

We now want to construct a differential operator $\mathcal{D}$ and the corresponding counterterm matrix M in such a way that Eq. (11) admits a solution for $g(x,y)$ of the form (16). Since we are dealing with a spatially homogeneous theory, we can take a spatial Fourier transform, obtaining a propagator $g_k(t,t')$, and operator $\mathcal{D}_k(t,\partial_t)$ which obey Eq. (11) in the form

$$\mathcal{D}_k(t,\vec{\partial}_t)g_k(t,t')=-i\delta(t-t')=g_k(t,t')\mathcal{D}_k(t',-\overleftarrow{\partial}_{t'}). \tag{17}$$

In the following, we will usually suppress the suffix k. As in [20], we first construct the operator $\mathcal{D}(\partial_t)$ with constant coefficients appropriate to a temporally homogeneous system, and then allow these coefficients to depend on time. In the temporally homogeneous case, $g(t,t')$ can be expressed in the form of Eq. (16) in terms of a function $h(t-t')$ (which also depends on k and μ) for which we write

$$h(t-t')=u(t-t')+iv(t-t'), \tag{18}$$

$$h^{*\sharp}(t-t')=w(t-t')+iz(t-t'), \tag{19}$$

where u, v, w, and z are real functions. [Note that $f^{*\sharp}(\mu)=f(-\mu)$ for any function $f(\mu)$.] For the temporal Fourier transform

$$\hat{g}(\omega)=\int_{-\infty}^{\infty}dt e^{i\omega t}g(t), \tag{20}$$

we obtain

$$\begin{aligned}\hat{g}_{11}(\omega)=\hat{g}_{22}^*(\omega)&=[A(\omega)-b(\omega)+C(\omega)+d(\omega)]\\ &\quad+i[a(\omega)+B(\omega)-c(\omega)+D(\omega)],\end{aligned} \tag{21}$$

$$\hat{g}_{12}(\omega)=2[C(\omega)+d(\omega)],$$

$$\hat{g}_{21}(\omega)=2[A(\omega)-b(\omega)],$$

where

$$A(\omega)+ia(\omega)=\int_0^\infty dtu(t)[\cos\omega t+i\sin\omega t],$$

$$B(\omega)+ib(\omega)=\int_0^\infty dtv(t)[\cos\omega t+i\sin\omega t],$$

$$C(\omega)+ic(\omega)=\int_0^\infty dtw(t)[\cos\omega t+i\sin\omega t],$$

$$D(\omega)+id(\omega)=\int_0^\infty dtz(t)[\cos\omega t+i\sin\omega t].$$

Note that the change of variables $(\omega,\mu)\to(-\omega,-\mu)$ leads to the interchange $[A(\omega),a(\omega),B(\omega),b(\omega)]\leftrightarrow[C(\omega),-c(\omega),D(\omega),-d(\omega)]$, and consequently,

$$\hat{g}_{ab}(\omega)=\hat{g}^{*\sharp}_{ba}(-\omega), \tag{22}$$

which also follows directly from Eq. (16). Upon Fourier transformation, Eq. (17) becomes an algebraic equation whose solution for $\mathcal{D}(-i\omega)$ is

$$\mathcal{D}(-i\omega)=\frac{-i}{\det|\hat{g}|}\begin{bmatrix}\hat{g}_{22}(\omega) & -\hat{g}_{12}(\omega)\\ -\hat{g}_{21}(\omega) & \hat{g}_{11}(\omega)\end{bmatrix}. \tag{23}$$

This solution shows how $\mathcal{D}$ can be constructed from the (unknown) functions $A(\omega)\cdots D(\omega)$ which are real, even functions of ω and $a(\omega)\cdots d(\omega)$ which are real and odd. We will choose these eight functions in such a way that $\mathcal{D}$ has certain essential properties. First, we require $\mathcal{D}(\partial_t)$ to be a second-order differential operator. The $\delta(t-t')$ in Eq. (17) arises from the derivatives of $\Theta(t-t')$ and $\Theta(t'-t)$ and, in order that these appear only in the diagonal elements, second derivatives may appear only in the diagonal elements of $\mathcal{D}(\partial_t)$. To maintain the normalization of ϕ as in Eq. (1), we fix the coefficients of ∂_t^2 or of $-\omega^2$, to be ±1. Expanding Eq. (23) in powers of ω, we find that the most general form of $\mathcal{D}$ satisfying these requirements can be expressed in terms of six real coefficients α, β, γ, $\overline{\alpha}$, $\overline{\gamma}$, $\widetilde{\gamma}$ as

$$\mathcal{D}(-i\omega)=\begin{bmatrix}-\omega^2-i(\overline{\gamma}-i\widetilde{\gamma})\omega+\beta-i\alpha & -i(\gamma-\overline{\gamma})\omega+i(\alpha+\overline{\alpha})\\ i(\gamma+\overline{\gamma})\omega+i(\alpha-\overline{\alpha}) & \omega^2-i(\overline{\gamma}+i\widetilde{\gamma})\omega-\beta-i\alpha\end{bmatrix}. \tag{24}$$

Since $\mathcal{D}(-i\omega)$ is equal to $-ig^{-1}(\omega)$, it has the same symmetry [Eq. (22)] as $g(\omega)$, and we may deduce that α, β, and γ are even functions of μ, while $\overline{\alpha}$, $\overline{\gamma}$, and $\widetilde{\gamma}$ are odd. Since these quantities are also real, we have

$$\alpha^\sharp=\alpha,\quad \beta^\sharp=\beta,\gamma^\sharp=\gamma,\quad \overline{\alpha^\sharp}=-\overline{\alpha},$$
$$\overline{\gamma^\sharp}=-\overline{\gamma},\quad \widetilde{\gamma}^\sharp=-\widetilde{\gamma}. \tag{25}$$

In particular, when μ vanishes, we have $\overline{\alpha}=\overline{\gamma}=\widetilde{\gamma}=0$ and $\mathcal{D}$ reduces to the form found in [20] for a real scalar field.

For the nonequilibrium theory, with time-dependent bare mass, $\mathcal{D}(t,\partial_t)$ is given by

$$\mathcal{D}_{11}=\partial_t^2+(\overline{\gamma}-i\widetilde{\gamma})\partial_t+\beta-i\alpha+(\dot{\overline{\gamma}}-i\dot{\widetilde{\gamma}})/2,$$
$$\mathcal{D}_{12}=(\gamma-\overline{\gamma})\partial_t+i(\alpha+\overline{\alpha})+(\dot{\gamma}-\dot{\overline{\gamma}})/2,$$
$$\mathcal{D}_{21}=-(\gamma+\overline{\gamma})\partial_t+i(\alpha-\overline{\alpha})-(\dot{\gamma}+\dot{\overline{\gamma}})/2,$$
$$\mathcal{D}_{22}=-\partial_t^2+(\overline{\gamma}+i\widetilde{\gamma})\partial_t-\beta-i\alpha+(\dot{\overline{\gamma}}+i\dot{\widetilde{\gamma}})/2, \tag{26}$$

where the coefficients are now time dependent, and the derivatives $\dot{\gamma}$, etc. ensure that $\mathcal{D}(t,\partial_t)$ is a symmetrical operator, as required in Eq. (11). That is, $\gamma\partial_t+\dot{\gamma}/2=\gamma^{1/2}\partial_t\gamma^{1/2}$, etc. We emphasize again that the six coefficients in Eq. (26), while undetermined at this stage, do not represent arbitrary modifications of the theory, but rather an optimal choice of the lowest-order theory which is to serve as the basis for perturbation theory. As indicated above, these coefficients will be found self-consistently from a suitable renormalization prescription, and we shall find that they correspond to six properties of a gas of quasiparticles, namely, the thermal masses, occupation numbers, and relaxation rates of particles and antiparticles.

IV. THE DISSIPATIVE PROPAGATOR

The basic equation (17) for the unperturbed propagator matrix $g(t,t')$ is satisfied if the function $h(t,t')$ obeys the conditions

$$\partial_t[h(t,t')-h^{*\sharp}(t',t)]|_{t'=t}=-i, \tag{27}$$

$$\partial_t[h(t,t')-h^{*}(t',t)]|_{t'=t}=0, \tag{28}$$

$$[\mathcal{D}_{11}(t,\partial_t)+\mathcal{D}_{12}(t,\partial_t)]h(t,t')=0, \tag{29}$$

$$\mathcal{D}_{11}(t,\partial_t)h^{*\sharp}(t',t)+\mathcal{D}_{12}(t,\partial_t)h^{*}(t',t)=0, \tag{30}$$

together with their $\sharp$-conjugates and the continuity conditions

$$h(t,t)=h^*(t,t)=h^\sharp(t,t). \tag{31}$$

We find that the solution of these equations can be written in the form

$$h(t,t')=\frac{\exp\left(-\frac{1}{2}\int_{t'}^{t}[\gamma(\overline{t})-i\widetilde{\gamma}(\overline{t})]d\overline{t}\right)}{4\sqrt{\Omega(t)\Omega(t')}}\Bigg[[1+N(t')]\times\exp\left(-i\int_{t'}^{t}\Omega(\overline{t})d\overline{t}\right)+[-1+N^{\sharp}(t')]\times\exp\left(i\int_{t'}^{t}\Omega(\overline{t})d\overline{t}\right)\Bigg], \tag{32}$$

in terms of the auxiliary functions $\Omega(t)$ and $N(t)$ which are solutions of

$$\frac{1}{2}\frac{\ddot{\Omega}}{\Omega}-\frac{3}{4}\frac{\dot{\Omega}^2}{\Omega^2}+\Omega^2=\beta-\frac{1}{4}(\gamma-i\widetilde{\gamma})^2+i\overline{\alpha}, \tag{33}$$

$$\left[\partial_t+2i\Omega-\frac{\dot{\Omega}}{\Omega}+\gamma\right][(\partial_t+\gamma)N-\overline{\gamma}]=2i\alpha+2i\overline{\alpha}N+i\widetilde{\gamma}(\gamma N-\overline{\gamma}), \tag{34}$$

with the subsidiary conditions

$$\left(\frac{N+N^{\sharp}}{\Omega}\right)^{*}=\left(\frac{N+N^{\sharp}}{\Omega}\right), \tag{35}$$

$$\frac{d}{dt}\left(\frac{N+N^{\sharp}}{\Omega}\right)=-\frac{1}{2}\left(\frac{\dot{\Omega}}{\Omega}+\frac{\dot{\Omega}^{*}}{\Omega^{*}}+2\gamma\right)\left(\frac{N+N^{\sharp}}{\Omega}\right)-i[(N-N^{\sharp})-(N-N^{\sharp})^{*}]. \tag{36}$$

It is straightforward, though tedious, to show that these conditions are preserved by Eq. (34) if they hold at the initial time.

With a nonzero chemical potential, $\widetilde{\gamma}$ and $\overline{\alpha}$ are, in general, nonzero, so the frequency $\Omega(t)$ which satisfies Eq. (33) is complex, though it has the property $\Omega^{\sharp}=\Omega$.

To obtain the initial conditions that apply to Eqs. (33) and (34), we require the full 3×3 matrix of propagators which satisfies an equation of the form (17) with

$$\mathcal{D}_{33}(\partial_\tau)=i(\partial_\tau^2-\omega^2), \tag{37}$$

and $\mathcal{D}_{13}=\mathcal{D}_{23}=\mathcal{D}_{31}=\mathcal{D}_{32}=0$, where $\omega^2=k^2+m^2(t_0)$. The boundary conditions which apply to these nine propagators are set out in Appendix, where results for those involving imaginary times are also given.

For the case of a time-independent mass m^2, we expect a steady state solution, with $\dot{\Omega}=\dot{N}=0$. In this case, we find that the boundary conditions can be satisfied only if three relations hold between α, β, ... The first of these is

$$\widetilde{\gamma}^2=4(\beta-\omega^2). \tag{38}$$

It seems natural to require that the renormalized masses (or, more generally, the k-dependent frequencies) in $\mathcal{D}_{11}$ and $\mathcal{D}_{33}$ should be equal, although this is not obligatory. This means that $\beta=\omega^2=m^2+k^2$, and hence that $\widetilde{\gamma}=0$. We will indeed assume that $\widetilde{\gamma}=0$, since this introduces considerable simplifications, but the consequences of this assumption will need some discussion at a later stage. In this case, the two other relations are

$$\tfrac{1}{2}[\alpha+\overline{\alpha}-(\gamma-\overline{\gamma})\omega]-(\gamma\omega-\overline{\alpha})n^{-}=0, \tag{39}$$

$$\tfrac{1}{2}[\alpha-\overline{\alpha}-(\gamma+\overline{\gamma})\omega]-(\gamma\omega+\overline{\alpha})n^{+}=0, \tag{40}$$

where

$$n^{\pm}=\frac{1}{e^{\beta(\omega\pm\mu)}-1} \tag{41}$$

are the usual occupation numbers for particles and antiparticles in a free Bose-Einstein gas. The coefficient N in the real-time propagators is then given by

$$N=\left(\frac{\Omega+i\gamma/2}{\omega}\right)(1+n^{+}+n^{-})+(n^{-}-n^{+}), \tag{42}$$

and the frequency Ω by

$$\Omega^2=\omega^2-\gamma^2/4+i\overline{\alpha}. \tag{43}$$

Note that when the dissipative counterterm is neglected, we have $\Omega=\omega$ and $\gamma=0$, so that $N=1+2n^{-}$. We then find, as expected, that the coefficient of the positive frequency term in the propagator (32) is $(1+N)=2(1+n^{-})$ while that of the negative frequency term is $(-1+N^{\sharp})=2n^{+}$. Thus, with the dissipative terms present, the real-time propagators describe a gas of quasiparticles, in which quasiparticle modes contain a small admixture of bare antiparticles and quasi-antiparticle modes contain a small admixture of bare particles.

For this gas of quasiparticles, the positive- and negative-frequency mode functions are $\exp[-i\Omega_{+}(t-t')]$ and $\exp[i\Omega_{-}(t-t')]$, respectively, where

$$\Omega_{\pm}=[\mathrm{Re}\Omega\mp\widetilde{\gamma}/2]\mp i[\gamma/2\mp\mathrm{Im}\Omega]. \tag{44}$$

We see that the decay rates for particle and antiparticle modes will be different if $\mathrm{Im}\Omega\neq0$, which will, in general, be true if either $\widetilde{\gamma}$ or $\overline{\alpha}$ is nonzero. On the other hand, the thermal masses of quasiparticles and quasi-antiparticles will be different if and only if $\widetilde{\gamma}$ is nonzero. Which of these conditions actually applies will be discussed in the next section.

For the nonequilibrium theory, any solution of Eq. (33) may, in principle, be used for $\Omega(t)$. Clearly, however, it is desirable that our lowest-order theory should approximately retain the characteristics of the steady-state solution in the case where $m^2(t)$ is slowly varying near t_0. We will, therefore, choose a renormalization prescription for which $\beta(t_0)=\omega^2$, and the solution of Eq. (33) which has $\dot{\Omega}(t_0)=0$ and $\Omega(t_0)$ given by Eq. (43). Retaining the assumption that $\widetilde{\gamma}(t_0)=0$, Eq. (33) shows that $\ddot{\Omega}(t_0)=0$ also. We would now like to interpret $N(t)$ in terms of time-dependent occupation numbers $n^{\pm}(t)$. To this end, we define

$$n^{\pm}(t)=\frac{1}{4\Omega(t)}\{[\sqrt{\beta(t)}\pm i\gamma(t)/2][N(t)+N^{\sharp}(t)]\mp\Omega(t)[N(t)-N^{\sharp}(t)]-2\Omega(t)\}, \tag{45}$$

so that

$$N(t)=\left(\frac{\Omega(t)+i\gamma(t)/2}{\sqrt{\beta(t)}}\right)[1+n^{+}(t)+n^{-}(t)] +[n^{-}(t)-n^{+}(t)]. \quad (46)$$

It is easy to see that $n^{+}(t)$ and $n^{-}(t)$ are real and that $n^{+\sharp}(t)=n^{-}(t)$ as required for the occupation numbers. If $n^{\pm}(t_0)$ are taken to be the equilibrium values (41), then the desired initial condition $\dot{N}(t_0)=0$ ensures that Eq. (36) is satisfied.

As in [20], we can now show that with reasonable approximations, the evolution equation (34) and its $\sharp$-conjugate reduce to a pair of Boltzmann-like kinetic equations for $n^{\pm}(t)$. We first rewrite Eq. (34) with $\tilde{\gamma}=0$ as

$$\Omega[\partial_t+2i(\Omega-i\gamma/2)]P=2i(\alpha+\overline{\alpha N}), \quad (47)$$

$$P=\Omega^{-1}[(\partial_t+\gamma)N-\overline{\gamma}], \quad (48)$$

and assume that $\partial_t P\ll(2\Omega-i\gamma)P$, which will be valid if the characteristic relaxation time $1/\gamma$ is much greater than $1/\Omega$. It will turn out, as in [20], that γ is smaller than Ω by a factor of λ^2. Indeed, γ, $\overline{\gamma}$, α, and $\overline{\alpha}$ are all of order λ^2. Consequently, to order λ^2, we can replace N by its lowest-order value $N\approx(1+2n^{-})$ and neglect $i\gamma/2$ in comparison with Ω. With these approximations, Eq. (47) and its $\sharp$-conjugate become

$$\frac{dn^{-}}{dt}=\frac{1}{2\Omega}[\alpha+\overline{\alpha}-(\gamma-\overline{\gamma})\Omega]-\frac{1}{\Omega}(\gamma\Omega-\overline{\alpha})n^{-}, \quad (49)$$

$$\frac{dn^{+}}{dt}=\frac{1}{2\Omega}[\alpha-\overline{\alpha}-(\gamma+\overline{\gamma})\Omega]-\frac{1}{\Omega}(\gamma\Omega+\overline{\alpha})n^{+}. \quad (50)$$

Evidently, the relations (39) and (40) which apply to the steady-state solution are just the conditions for the occupation numbers to be constant. In the next section, we evaluate α, $\overline{\alpha}$, γ, and $\overline{\gamma}$ explicitly, and find that the right-hand sides of Eqs. (49) and (50) have approximately the form of the scattering integrals which appear in the Boltzmann equation.

V. EVALUATION OF THE DISSIPATIVE COUNTERTERM

As explained above, our strategy is to choose the counterterm M so as to optimize the unperturbed propagator $g(x,y)$ as an approximation to the full propagator $G(x,y)$. The relation between these may be expressed by the Dyson-Schwinger equation

$$G_{ab}(x,y)=g_{ab}(x,y)+i\int\int d^4z\,d^4z'\,g_{ac}(x,z) \times\Sigma_{cd}(z,z')G_{db}(z',y) \quad (51)$$

in terms of the self-energy $\Sigma(x,y)$. As explained in detail in [20], we express Σ in terms of the average $\overline{t}$ and difference Δt of its time arguments, and take the Fourier transform with respect to $(\mathbf{x}-\mathbf{y})$ and Δt. The result can be written as

$$\Sigma(\overline{t},\omega)=M(\overline{t},-i\omega)+\tilde{\Sigma}^{(1)}(\overline{t},\omega^2)\omega+\tilde{\Sigma}^{(2)}(\overline{t},\omega^2), \quad (52)$$

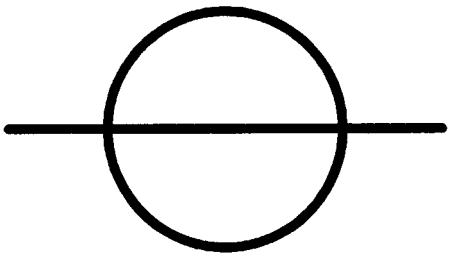

FIG. 2. Lowest-order Feynman diagram contributing to the dissipative counterterm.

where $\tilde{\Sigma}$ denotes the contribution from loop diagrams, which has been split into parts even and odd in the transform variable ω. In the counterterm contribution, the t dependence is that of $\alpha(\overline{t})\cdots$ while $-i\omega$ replaces ∂_t. While $M(\overline{t},-i\omega)$ is a second-order polynomial in ω, the same is not true of $\tilde{\Sigma}(\overline{t},\omega)$. As in standard renormalization theory, therefore, we choose the counterterm to cancel $\tilde{\Sigma}$ at some reference frequency, which can conveniently be chosen as $\overline{\Omega}=\Omega(\overline{t})$. This gives

$$[\gamma(t)-\overline{\gamma(t)}]=i\Sigma_{12}^{(1)}(t,\overline{\Omega^2}), \quad (53)$$

$$[\alpha(t)+\overline{\alpha(t)}]=i\Sigma_{12}^{(2)}(t,\overline{\Omega^2}), \quad (54)$$

$$[\overline{\gamma(t)}-i\tilde{\gamma}(t)]=i\Sigma_{11}^{(1)}(t,\overline{\Omega^2}), \quad (55)$$

from which α, $\overline{\alpha}$, γ, $\overline{\gamma}$, and $\tilde{\gamma}$ can be deduced by isolating parts which are even and odd under $\sharp$-conjugation. The corresponding equation for β, namely,

$$k^2+m^2(t)-\beta(t)=\mathrm{Re}\Sigma_{11}^{(2)}(t,\overline{\Omega^2}), \quad (56)$$

can be adjusted to meet the requirement $\beta(t_0)=k^2+m_R^2$ where m_R is an appropriate renormalized mass, as discussed in the last section, but the details are not important for our present purposes.

The lowest-order contribution to Σ_{12} is from the graph shown in Fig. 2. An approximate method for evaluating it is described in [20]. Since the whole diagram is proportional to λ^2, we take the lowest-order approximation to its internal propagators, using

$$h(t,t')\approx\frac{1}{2\Omega(\overline{t})}\{[1+n^{-}(\overline{t})]e^{-i\Omega(\overline{t})\Delta t}+n^{+}(\overline{t})e^{i\Omega(\overline{t})\Delta t}\}, \quad (57)$$

where $n^{\pm}(\overline{t})$ and $\Omega(\overline{t})$ are assumed to vary sufficiently slowly that they can be treated as effectively constant. After calculating $\alpha_k(t)$, $\overline{\alpha_k}(t)$, $\gamma_k(t)$, and $\overline{\gamma_k}(t)$ from Eqs. (53)–(55) and substituting the results in Eqs. (49) and (50), we obtain the Boltzmann equation

$$\frac{d}{dt}n_k^-$$

$$\approx \frac{\lambda^2}{32(2\pi)^5}\int d^3k_1 d^3k_2 d^3k_3\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k})$$
$$\times\delta(\Omega_1+\Omega_2-\Omega_3-\Omega_k)[\Omega_1\Omega_2\Omega_3\Omega_k]^{-1}[n_1^+n_2^-(1+n_3^+)$$
$$\times(1+n_k^-)-(1+n_1^+)(1+n_2^-)n_3^+n_k^-+n_1^-n_2^+(1+n_3^+)$$
$$\times(1+n_k^-)-(1+n_1^-)(1+n_2^+)n_3^+n_k^-+n_1^-n_2^-(1+n_3^-)$$
$$\times(1+n_k^-)-(1+n_1^-)(1+n_2^-)n_3^-n_k^-], \tag{58}$$

where Ω_1 denotes $\Omega_{k_1}(t)$, etc. At the order of approximation we are using, $\Omega_k(t)$ is real. The scattering integral on the right correctly describes the rate of production minus the rate of absorption of particles of momentum k due to all two-body processes allowed by charge conservation. It vanishes when the occupation numbers have the Bose-Einstein form

$$n^{\pm}=\frac{1}{e^{\beta(\Omega\pm\overline{\mu})}-1}, \tag{59}$$

where the chemical potential $\overline{\mu}$ need not be the same as the original μ. Of course, the rate of change of n^+ is given by the $\sharp$-conjugate of Eq. (58).

The value of $\widetilde{\gamma}$ obtained from Eq. (55) is not zero although, unlike α, $\overline{\alpha}$, γ, and $\overline{\gamma}$, it involves off-shell processes. As discussed in the last section, this implies that the thermal masses of particles and antiparticles are different when the chemical potential is nonzero, the difference being comparable in magnitude to the decay widths ($\gamma/2\pm\mathrm{Im}\Omega$) of the quasiparticle modes. We are nevertheless entitled to set $\widetilde{\gamma}=0$. This simply means that the difference in thermal masses is treated purely perturbatively, rather than being resummed into the unperturbed propagator. As it happens, the contributions involving $\widetilde{\gamma}$ to both Eqs. (33) and (34) are of order λ^4. Thus, in low order calculations, it would be consistent to retain a nonzero value of $\widetilde{\gamma}$ in Eq. (32), while setting it to zero in the subsidiary calculations needed to determine $\Omega(t)$ and $N(t)$. The approximations needed to obtain the Boltzmann equation in any case entail setting $\widetilde{\gamma}=0$.

VI. SUMMARY

We have extended to the case of complex scalar fields a formulation of nonequilibrium perturbation theory which partially resums the dissipative effects of loop diagrams into the unperturbed propagator. As a result of this resummation, low order calculations using our modified Feynman rules reflect the evolving state of the nonequilibrium system through terms in the propagator which can roughly be interpreted in terms of quasiparticle occupation numbers. As in earlier work, we find that the evolution equation for these occupation numbers can be cast approximately in the form of a Boltzmann equation. Kinetic equations of a similar form can, of course, be derived by other methods (see, for example [29,30]) if one sets about finding the rate of change of occupation numbers defined in some appropriate manner. In an interacting theory, however, the definition of single-particle modes is generally ambiguous. We, on the other hand, have set about calculating unambiguously defined Green's functions in an optimal manner. For us the quasiparticle modes and their occupation numbers simply provide a convenient way of thinking about quantities which arise naturally in the course of these calculations, and the Boltzmann equation is a low order approximation to a rather more complicated second-order evolution equation.

The novel aspects of the work reported here concern the effects of a nonzero chemical potential. To the extent that the quasiparticle picture is valid, we find that not only the occupation numbers but also the decay widths and thermal masses of the quasiparticles are different for particle and antiparticle modes. In particular, the difference in thermal masses is a two-loop effect [in the $\lambda(\phi^*\phi)^2$ theory], which is not readily apparent in the usual perturbative treatment (see, e.g., [31]), although it should be derivable from a two-loop calculation of the self-energy, even in equilibrium.

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APPENDIX

Boundary conditions on the nine propagators g_{ab} arise from two sources. First, boundary conditions on the fields (5) together with $\phi_1(\mathbf{x},t_f)=\phi_2(\mathbf{x},t_f)$, $\phi_2(\mathbf{x},t_0)=\phi_3(\mathbf{x},0)$, and the complex conjugates of these lead to

$$g_{1a}(t_f,t')=g_{2a}(t_f,t'),\quad g_{a1}(t,t_f)=g_{a2}(t,t_f),$$
$$g_{2a}(t_0,t')=g_{3a}(0,t'),\quad g_{a2}(t,t_0)=g_{a3}(t,0),$$
$$g_{1a}(t_0,t')=e^{\beta\mu}g_{3a}(0,t'),\quad g_{a1}(t,t_0)=e^{-\beta\mu}g_{a3}(t,0), \tag{A1}$$

for ($a=1,2,3$). Second, the evaluation of the Gaussian path integral leading to Eq. (10) is performed, as usual, by completing the square and this involves integrations by parts. The requirement that boundary terms arising from these integrations by parts cancel leads to

$$\partial_t g_{1a}(t,t')|_{t=t_f}=\partial_t g_{2a}(t,t')|_{t=t_f}, \tag{A2}$$

$$\partial_t g_{2a}(t,t')|_{t=t_0}+\frac{1}{2}(\gamma+\overline{\gamma})g_{1a}(t_0,t')-\frac{1}{2}(\overline{\gamma}+i\widetilde{\gamma})g_{2a}(t_0,t')$$
$$=i\partial_\tau g_{3a}(\tau,t')|_{\tau=0}+\frac{\delta}{2}g_{3a}(0,t') \tag{A3}$$

$$\partial_t g_{1a}(t,t')|_{t=t_0}+\frac{1}{2}(\gamma-\overline{\gamma})g_{2a}(t_0,t')+\frac{1}{2}(\overline{\gamma}-i\widetilde{\gamma})g_{1a}(t_0,t')$$
$$=e^{\beta\mu}\left[i\partial_\tau g_{3a}(\tau,t')\Big|_{\tau=\beta}+\frac{\delta}{2}g_{3a}(\beta,t')\right], \tag{A4}$$

for ($a=1,2,3$), together with the $\sharp$-conjugates of these. The coefficient δ corresponds to an additional counterterm which, as explained in detail in [21], enables us to impose

$\dot{N}(t_0)=0$ as befits an initial equilibrium state. As in [21], we find $\delta=-\gamma$. Here, γ, $\overline{\gamma}$, and $\widetilde{\gamma}$ are all evaluated at t_0.

The real-time propagators satisfying these boundary conditions are described in the text. Of those involving imaginary times, three are found to be

$$g_{33}(\tau,\tau')=\frac{1}{2\omega}\{[n^-+\Theta(\tau-\tau')]e^{-\omega(\tau-\tau')} +[n^++\Theta(\tau'-\tau)]e^{\omega(\tau-\tau')}\} \tag{A5}$$

and

$$g_{13}(t,\tau)=g_{23}(t,\tau)=\frac{\exp\left(-\frac{1}{2}\int_{t_0}^{t}[\gamma(\overline{t})-i\widetilde{\gamma}(\overline{t})]d\overline{t}\right)}{4\sqrt{\Omega(t_0)\Omega(t)}} \times[(A_1e^{-\omega\tau}+A_2e^{\omega\tau})e^{-i\int_{t_0}^{t}\Omega(\overline{t})d\overline{t}}+(A_3e^{-\omega\tau} +A_4e^{\omega\tau})e^{i\int_{t_0}^{t}\Omega(\overline{t})d\overline{t}}], \tag{A6}$$

with

$$A_1=(1+n^+)[N(t_0)-1-2n^-]/(1+n^++n^-), \tag{A7}$$

$$A_2=n^-[N(t_0)+1+2n^+]/(1+n^++n^-), \tag{A8}$$

$$A_3=(1+n^+)[N^\sharp(t_0)+1+2n^-]/(1+n^++n^-), \tag{A9}$$

$$A_4=n^-[N^\sharp(t_0)-1-2n^+]/(1+n^++n^-). \tag{A10}$$

The others are given by

$$g_{31}(\tau,t)=g_{32}(\tau,t)=g_{13}^{*\sharp}(t,\tau). \tag{A11}$$

Indeed, the whole propagator matrix has the property

$$g_{ab}^{*\sharp}(t,t')=g_{ba}(t',t), \tag{A12}$$

where t and t' denote real or imaginary times, depending on the indices a and b.

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