

## Canonical quantization of the kink model beyond the static solution

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A new approach to the quantization of the relativistic kink model around the solitonic solution is developed on the grounds of the collective coordinates method. The corresponding effective action is proved to be the action of the nonminimal  $d=1+1$  point particle with curvature. It is shown that upon canonical quantization this action yields the spectrum of the kink solution obtained first with the help of WKB quantization. [S0556-2821(97)01302-7]

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### I. INTRODUCTION

It is well known that the particles spectrum in quantum field theories (QFT's) possessing a topologically nontrivial solution for the corresponding equation of motion may be obtained either by the semiclassical WKB method [1], or with the help of effective actions [2]. The latter describes the low-energy dynamics of a stable time-independent solution (solitons, for instance) plus small quantum oscillations around it. It was established earlier that these actions at the classical level are written down as minimal  $p$ -brane actions, supplemented with some additional nonminimal terms [3]. Depending on curvatures of the relevant world volumes of  $p$  branes, such nonminimal terms may become important when the quantum corrections due to the field fluctuations in a neighborhood of static solutions are not negligible. Upon quantization these actions allow us to reveal the true quantum content of their prototype field theories.

This approach amounts to a nonlinear change of the space-time variables in the action of a given QFT model with spontaneously broken relativistic symmetry [4,5]. As compared with the old one, the new set of space-time variables has another form of transformations under the original space-time symmetry group. In what follows the static solutions of the QFT model under consideration become completely covariant with respect to the relativistic group transformations. To achieve this the special dependence of the aforementioned nonlinear change of the space-time variables on the collective coordinates of the solitons has been assumed. In order to provide a canonical relationship between these variables and the Goldstone excitations of the spontaneously broken space-time symmetries the original set of field variables should be covariantly constrained. Sometimes this is

achieved by putting all the fields with zero expectation values equal to zero. Thus, we get the minimal  $p$ -brane action as the residual effective action for the QFT model we start with. However, as it was mentioned above, this procedure is not suitable for models in which quantum perturbations of fields near the static solutions are rather essential. In these cases one is compelled to be more thorough in treating the excited modes of static solutions.

In this article we discuss the simplest example of such a model, the  $d=1+1$  relativistic model  $\phi^4$ , in which the problem of exciting modes can be resolved both on the classical and quantum levels simultaneously.

The outline of the paper is as follows. In Sec. II we review some well-known results about the field-dependent transformations for the action of the model under consideration. It is shown how this transformation is associated with the spontaneously broken relativistic symmetries of the initial action and how the latter can be recast in the corresponding splitting form.

In Sec. III we have succeeded in showing that the equation of motion obtained from the "splitting" action for the perturbations of field about the kink solution is covariantly separated into an equivalent system of two equations. One of them is solved by the excited modes of the static solution, while the other one defines a world-line trajectory of the effective nonminimal point particle in the weak curvature limit.

The implications of this solution for the construction of the corresponding effective action from the original field action are proposed in Sec. IV. In Sec. V we extend this approach to the quantum level. Here, the canonical quantization of the effective action is carried out for obtaining the quantum states of kink as the spectrum of the underlying point particle with curvature. We finish with some speculations about the possible generalization of our results to more complicated cases.

### II. THE SPLITTING FORM ACTION FOR KINK

To show more precisely how the collective coordinate method works, let us consider the action

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$$S[\phi] = \int d^2x L(\phi, \partial_m \phi), \quad (1)$$

$$L(\phi, \partial_m \phi) = \frac{1}{2} (\partial_m \phi)^2 - \frac{1}{4} g^2 \left[ \phi^2 - \left( \frac{m}{g} \right)^2 \right]^2, \quad (2)$$

where  $\phi(x, t)$  is a dimensionless scalar field;  $m$  and  $g$  are real parameters. It is well known that the corresponding equation of motion [6],

$$\partial^m \partial_m \phi + g^2 \left[ \phi^2 - \left( \frac{m}{g} \right)^2 \right] \phi = 0, \quad (3)$$

leads to the kink solution

$$\phi_c(x) = \frac{m}{g} \operatorname{th} \frac{mx}{\sqrt{2}}, \quad (4)$$

the time-independent solution, which describes the bend of the field at the point  $x=0$ , with the width of order  $m^{-1}$ , and the nontrivial behavior at infinity:

$$\phi_c(+\infty) = -\phi_c(-\infty) = \frac{m}{g}. \quad (5)$$

Using the semiclassical method, it was found in [1] that upon quantization the kink solution (4) yields a heavy quantum-mechanical particle. The modified approach which we follow here shows that the same result may be obtained straightforwardly from the corresponding effective action. The way in which such a nonlinear action would arise from the QFT model [Eqs. (1) and (2)] can be seen by the following considerations, analogous to those which were implemented for the rigid string in [4]. Let us pass to a new set of the basis variables in the action (1):

$$\begin{aligned} x^m &= x^m(s) + e_{(1)}^m(s) \rho \equiv y^m(\sigma_A), \quad A=0,1, \\ \phi(x, t) &= \tilde{\phi}(\sigma_A), \quad \sigma_{A=0} = s, \quad \sigma_{A=1} = \rho, \end{aligned} \quad (6)$$

where  $x^m(s)$  are the coordinates of the  $d=2$  point particle,  $e_{(1)}^m(s)$  is the unit spacelike vector orthogonal to its world line,  $\tilde{\phi}(\sigma_A)$  is the notation for the field  $\phi(x, t)$  in terms of the new variables. It is clear that it describes an infinite set of the world-line fields emerging in its expansion with respect to the inert coordinate  $\rho$ . It is worth noticing that unlike  $(x, t)$  the new basis variables  $(s, \rho)$  are invariant under Poincaré transformations. The reason for such a behavior of the variables is explained completely by the presence of the coordinates of the point particle  $x^m(s)$  in the mapping (6). A specific feature of these coordinates is that the Poincaré translation symmetries associated with the action (1),

$$\begin{aligned} x^{m'} &= x^m + \alpha^m, \\ \phi'(x', t') &= \phi(x, t), \end{aligned} \quad (7)$$

are realized on them in the form

$$\begin{aligned} x^{m'}(s) &= x^m(s) + \alpha^m, \\ \tilde{\phi}'(\sigma) &= \tilde{\phi}(\sigma), \end{aligned} \quad (8)$$

indicating that the above symmetries are spontaneously broken with  $x^m(s)$  thought of as the corresponding Goldstone fields (owing to the nonhomogeneous part in their transformation laws typical of the Goldstone fields). In addition, one can check that due to the presence of the variables  $e_{(1)}^m(s)$  in Eqs. (6) the Lorentz symmetry of the action (1) is spontaneously broken as well. So, the only unbroken symmetry we are left with in Eqs. (6) is a gauge symmetry under a world-line reparametrization of the point particle:

$$\begin{aligned} x^{m'}(s') &= x^m(s), \\ s' &= s'(s). \end{aligned} \quad (9)$$

The existence of this symmetry guarantees that after gauge fixing,

$$x^0(s) = t, \quad x^1(s) = \tilde{x}(t), \quad (10)$$

we get the set of physical degrees of freedom needed to recover the field content of the original theory. To see this we make the change of variables (6) in action (1). This gives

$$S[x^m, \tilde{\phi}] = \int d^2\sigma \Delta(\sigma) L(\tilde{\phi}, \nabla_m \tilde{\phi}), \quad (11)$$

where we use the notation

$$\Delta(\sigma) = \det \frac{\partial y^m}{\partial \sigma^A} = \sqrt{\dot{x}^2} (1 - \rho k), \quad (12)$$

$$L(\tilde{\phi}, \nabla_m \tilde{\phi}) = \frac{1}{2} \left[ \frac{1}{\Delta^2} (\partial_s \tilde{\phi})^2 - (\partial_\rho \tilde{\phi})^2 \right] - \frac{1}{4} g^2 \left( \tilde{\phi}^2 - \left( \frac{m}{g} \right)^2 \right)^2. \quad (13)$$

Note that action (11) depends on the curvature of the point-particle world line:

$$k = \sqrt{-a^n a_n}, \quad (14)$$

where the acceleration  $a^n$  is defined by

$$a_n = \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \frac{\dot{x}_n}{\sqrt{\dot{x}^2}}. \quad (15)$$

It is rather evident that, in comparison to Eq. (1), action (11) involves one superfluous degree of freedom  $\tilde{x}(t)$ . The latter is nothing but the linear prototype of the Goldstone field  $x(t)$ , arising in the decomposition of  $\phi(x, t)$  around its static solution (4)

$$\phi(x, t) = \phi_c(x) - x(t) \phi'_c(x) + \dots \quad (16)$$

To avoid the doubling of the Goldstone degrees of freedom in action (11) we are forced to relate both  $x(t)$  and  $\tilde{x}(t)$  fields by an equivalence transformation.

### III. ELIMINATING EXCITED MODES

Here, we wish to show that the most correct way to reveal this relation in the framework of the model under consideration is to restrict the set of variables  $[x^m, \tilde{\phi}]$  in action (11) by the covariant condition

$$\left. \frac{\delta S[x^m, \tilde{\phi}]}{\delta(\delta\tilde{\phi}(\sigma))} \right|_{x^m(s)=\text{const}} = 0, \quad (17)$$

where  $\delta\tilde{\phi}(\sigma)$  denote the perturbations of the  $\sigma$  fields  $\tilde{\phi}(\sigma)$  near the kink solution

$$\tilde{\phi}(\sigma) = \phi_c(\rho) + \delta\tilde{\phi}(\sigma). \quad (18)$$

It is worthwhile to emphasize that when varying the variable  $\delta\tilde{\phi}(\sigma)$  in Eq. (17) one should treat both the  $x^m(s)$  and perturbation fields on equal footing as independent variables. Only then can Eq. (17) be considered as the covariant constraint for eliminating  $\delta\tilde{\phi}(\sigma)$  in terms of  $x^m(s)$  and their derivatives. The reason for this is rather simple and belongs to some common issues of the effective actions theory [7]. We wish to briefly concern ourselves with this problem, with emphasis on the peculiarities brought about by our model.

Let an action  $S = S[x, X]$ , which depends on two sets of variables  $\{x\}$  and  $\{X\}$ , achieve its extremum at the stationary point  $x_c, X_c$ . That is,

$$\left. \frac{\delta S}{\delta x} \right|_{x_c, X_c} = 0, \quad \left. \frac{\delta S}{\delta X} \right|_{x_c, X_c} = 0. \quad (19)$$

The corresponding effective action of the model is obtained by putting

$$S_{\text{eff}}[x] \equiv S[x, X(x)], \quad (20)$$

where the constraint

$$\left. \frac{\delta S}{\delta X} \right|_{x=\text{const}} = 0, \quad \Rightarrow X = X(x) \quad (21)$$

is used for excluding the subset of variables  $\{X\}$  in terms of  $\{x\}$ . Now, any stationary point of the effective action (20), with  $X(x)$  subject to the constraint (21), is simultaneously an extremum of the total action  $S[x, X]$ . Indeed, varying the definition (20) with respect to  $x$ , we find

$$\frac{\delta S_{\text{eff}}[x]}{\delta x} = \frac{\delta S[x, X(x)]}{\delta x} + \frac{\delta S[x, X(x)]}{\delta X(x)} \frac{\delta X(x)}{\delta x}. \quad (22)$$

The second term on the right-hand side (RHS) of Eq. (22) disappears owing to Eq. (19), while the first one vanishes only at the solution  $x = x_c$  extremizing the effective action

$$\frac{\delta S_{\text{eff}}[x]}{\delta x} = 0. \quad (23)$$

In our case the constraint (21) is represented by the manifestly covariant condition (17). Let us analyze the structure of this condition in more detail.

By putting Eq. (18) into action (11) one finds

$$\begin{aligned} S[x^m, \phi_c + \delta\tilde{\phi}] = \int d^2\sigma \left\{ \Delta \left[ L(\phi_c) + \frac{1}{2\Delta^2} (\partial_s \delta\tilde{\phi})^2 \right. \right. \\ \left. \left. - \frac{1}{2} (\partial_\rho \delta\tilde{\phi})^2 - \frac{1}{2} g^2 \left( 3\phi_c^2 - \left(\frac{m}{g}\right)^2 \right) \delta\tilde{\phi}^2 \right. \right. \\ \left. \left. - g^2 \phi_c \delta\tilde{\phi}^3 - \frac{1}{4} g^2 \delta\tilde{\phi}^4 \right] + \phi'_c \delta\tilde{\phi} \partial_\rho \Delta \right\}, \\ \phi'_c \equiv \partial_\rho \phi_c, \end{aligned} \quad (24)$$

$$L(\phi_c) = -\frac{1}{2} \phi_c'^2 - \frac{1}{4} g^2 \left[ \phi_c^2 - \left(\frac{m}{g}\right)^2 \right]^2, \quad (25)$$

where we have used the equation of motion for  $\phi_c$

$$-\phi_c'' + g^2 \phi_c \left[ \phi_c^2 - \left(\frac{m}{g}\right)^2 \right] = 0. \quad (26)$$

From Eq. (24) it follows that the perturbations (18) are governed by the equation

$$\begin{aligned} \left\{ \partial_s \Delta^{-1} \partial_s - \partial_\rho \Delta \partial_\rho + g^2 \Delta \left[ 3\phi_c^2 - \left(\frac{m}{g}\right)^2 \right] \right\} \delta\tilde{\phi} + \phi'_c k \sqrt{x^2} \\ + O(\delta\tilde{\phi}^2) = 0. \end{aligned} \quad (27)$$

After the field rescaling  $\delta\tilde{\phi} \rightarrow (m/g) \delta X$  and variable redefinition  $\rho \rightarrow \epsilon u, \epsilon \equiv \sqrt{2}/m$  Eq. (27) can be rewritten in a more convenient form:

$$\begin{aligned} [\epsilon^2 \partial_s \Delta^{-1} \partial_s - \partial_u \Delta \partial_u + 2\Delta(3X_c^2 - 1)] \delta X + \epsilon X'_c k \sqrt{x^2} \\ + O(\delta X^2) = 0, \\ \Delta = \sqrt{x^2} (1 - \epsilon u k), \end{aligned} \quad (28)$$

$$X_c = \tanh u,$$

where the prime indicates the differentiation by  $u$ . Even in a linear approximation when  $O(\delta X^2) = 0$ , this equation is rather complicated mainly due to its nonlinearities. So, we are not able to solve it unless some additional assumptions are adopted. First, we suppose that the corresponding ansatz is given by

$$\delta X(s, u) = \epsilon k(s) f(u). \quad (29)$$

Note that the zeroth-order solutions of Eq. (29) are omitted owing to the nonhomogeneity term on the left-hand side (LHS), which is of first order in  $\epsilon$ .

Now, we can obtain approximate equations for  $f(u)$  and  $k(s)$  by simply expanding Eq. (29) in Taylor series around  $\epsilon = 0$ . In doing so we need, however, to be sufficiently careful in treating the first term inside the square brackets in Eq. (28). One cannot drop it as the terms of the second-order correction in  $\epsilon$  in the case when the point-particle world-line metric is a quickly varying function. Let us suppose that the term

$$\lim_{\epsilon \rightarrow 0} \epsilon^2 \frac{d}{ds} \frac{1}{\sqrt{\dot{x}^2}} \frac{dk}{ds} \neq 0, \tag{30}$$

is not zero even in the  $\epsilon=0$  limit. Taking into account Eqs. (29) and (30) one finds that to the lowest order in  $\epsilon$  the variables in Eq. (28) are separated and instead of one equation with partial derivatives the equivalent system of two equations for total derivatives arises:

$$\epsilon^2 \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \frac{1}{\sqrt{\dot{x}^2}} \frac{dk}{ds} + ck = 0, \tag{31}$$

$$-f'' + 2(3X_c^2 - 1)f - cf + X'_c = 0. \tag{32}$$

Here,  $c$  is a constant which can be found as a solution of Eq. (32). Both equations of the system [Eqs. (31) and (32)] are very important for the constitution of the relationship between linear and nonlinear parametrizations of the action. First, it appears that in contrast with the linear version of the theory the lowest eigenvalue of the energy is not zero. Indeed, one can see that for  $c=0$ , Eq. (32) can be rewritten as

$$\frac{d}{du} \left[ \frac{1}{\cosh^4 u} \frac{d}{du} (\cosh^2 u f_0) \right] = \frac{1}{\cosh^4 u}. \tag{33}$$

The corresponding solution can be represented in the form

$$f_0(u) = \frac{1}{4} \cosh^2 u - \frac{1}{12} \frac{\sinh^4 u}{\cosh^2 u} + \frac{b}{\cosh^2 u} + \frac{a}{8 \cosh^2 u} \times \left( 3u + 2 \sinh 2u + \frac{1}{4} \sinh 4u \right). \tag{34}$$

It is evident that this solution ceases to have any appropriate behavior for large  $|u|$ . Thus, one cannot provide the stabilization of the theory until this state is assumed to be rejected from the very beginning. It is not difficult to do this, however, in the case  $c \neq 0$ . To this end, we have to go to the shifted function

$$\tilde{f}(u) = f(u) - c^{-1} X'_c(u) \tag{35}$$

and carry out the calculation for  $\tilde{f}(u)$  and  $c$  from the corresponding equation of motion

$$-\tilde{f}'' + 2(3X_c^2 - 1)\tilde{f} - c\tilde{f} = 0. \tag{36}$$

Here, we display the partial solution which is the most suitable for our purposes:

$$\tilde{f}_1(u) = \frac{\sinh u}{\cosh^2 u}, \quad c_1 = 3. \tag{37}$$

It is appropriate to remark that in addition to Eq. (37), there exist other solutions of Eq. (36) [6]. However, we do not consider such solutions here because they represent more excited eigenstates of the energy.

In the remainder of this section we want to discuss the relationship between collective coordinates  $x^m(s)$  and the Goldstone field  $x(t)$ , which arises in the action upon the

spontaneous breakdown of the translation invariance. Let us note first that we can take the decomposition (18) in the same form as in Eq. (16):

$$\tilde{\phi}(\rho, t) = \phi_c(\rho) - g(s) \phi'_c(\rho) + \dots, \tag{38}$$

where  $g(s)$ , as opposed to  $x(t)$ , is a completely covariant degree of freedom. Comparing this expression with Eqs. (6) in the proper-time gauge (10), we find

$$g(t_r) = x(t_r) - \tilde{x}(t_r) + \dots, \tag{39}$$

where  $t_r = t - \rho$  is the ‘‘retentive’’ time and ellipsis stands for terms which are at least second order in the field  $\tilde{x}(t_r)$ . On the other hand, from solutions (29), (35), (37) it follows that  $g(s) = -(2g/3m^3)k(s)$ . Thus, owing to the constraint (17) there is no independent degree of freedom to be related with the zero mode  $\phi'_c(\rho)$ . Instead of this, we have the relation

$$\tilde{x}(t_r) = x(t_r) + \frac{2}{3m^2} \left( \frac{g}{m} \right) k(t_r) + \dots,$$

which establishes the desired equivalence connection between the Goldstone field  $x(t)$  and corresponding collective coordinate.

#### IV. THE EFFECTIVE ACTION FOR KINK WITH QUANTUM CORRECTIONS

Thus, we succeeded in resolving the constraint (17) by the ansatz (29) and (30). Actually, this construction allows us to eliminate  $\delta\tilde{\phi}(\tilde{x})$  in terms of  $x^m(s)$  and its derivatives. As a by-product, the new equation (31) implementing the role of the low-energy dynamical equation of motion for kink in presence of its excited modes was obtained. On the other hand, one can consider the solutions (29) and (37) for elimination of the excited modes of the field  $\tilde{\phi}(\tilde{x})$  from the action. Indeed, putting together Eqs. (29), (31), and (37) and inserting them back into the action (24), we find the following expression for the effective action:

$$S_{\text{eff}} = -\mu \int ds \sqrt{\dot{x}^2} \left( 1 + \frac{1}{3m^2} k^2 \right), \tag{40}$$

where

$$\mu = \frac{2\sqrt{2}}{3} \frac{m^3}{g^2}. \tag{41}$$

Note that in deriving Eq. (40) the  $\rho$  integration in the action (24) (modulo terms of order  $k^3$ ) was performed.

It was shown in Sec. III that the method at hand ensures the consistency of a given action with the action (24). Namely, both of them possess one and the same set of stationary points. To check the correctness of this statement it is sufficient to compare Eq. (31) to the equation of motion derived from the effective action (40):

$$\dot{p}_n = 0,$$

$$p_n = \frac{\partial L_{\text{eff}}}{\partial \dot{x}^n} - \frac{d}{ds} \frac{\partial L_{\text{eff}}}{\partial \ddot{x}^n}. \tag{42}$$

An important property of  $L_{\text{eff}}$  defined from Eq. (40) is that Eqs. (42) admit a very nice representation in terms of the world-line curvature  $k(s)$ . The transition to such a representation is achieved through the implication of the corresponding Frenet equations [8]

$$\begin{aligned}\dot{e}_{(0)}^m &= -k\sqrt{\dot{x}^2}e_{(1)}^m, \\ \dot{e}_{(1)}^m &= -k\sqrt{\dot{x}^2}e_{(0)}^m,\end{aligned}\quad (43)$$

where  $e_{(a)}^m, a=0,1$  are the basis components of the ‘‘moving’’ frame:

$$\begin{aligned}e_{(a)}^m &= \left( e_{(0)}^m = \frac{\dot{x}^m}{\sqrt{\dot{x}^2}}, e_{(1)}^m = -\frac{a^m}{k} \right), \\ e_{(a)}^m e_{m(b)} &= \eta_{(ab)} = \text{diag}(+1, -1), \\ e^{m(a)} e_{(a)}^n &= g^{mn} = \text{diag}(+1, -1).\end{aligned}\quad (44)$$

Now, it is not difficult to show that making a projection of Eq. (42) onto the basis (44) and taking account of Eqs. (43), we obtain the equation of motion in the following compact form

$$\epsilon^2 \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \frac{1}{\sqrt{\dot{x}^2}} \frac{dk}{ds} + \left( 3 - \frac{\epsilon^2}{2} k^2 \right) k = 0. \quad (45)$$

Evidently, Eq. (31) is merely the linearized version of Eq. (45).

## V. QUANTIZATION

So far, our attention has been devoted to the classical subject. It was shown that the quantum field theory action (1) and (2) around the topological defect (4) can be reduced to the effective action (40) which incorporates implicitly the dependence of the theory on zero and nonzero modes. The latter are apparent only from the mass formula (41) which depends inversely on the coupling constant  $g$ . Thus we get, in fact, the new nonperturbative formulation of the theory which may be regarded as fundamental in its own right. In particular, it will be of a great interest to use the action (40) straightforwardly for obtaining the quantum states of kink as the spectrum of point particle with curvature.

We begin with some general comments about the canonical analysis of this system. The Lagrangian (40) depends, apart from the velocity  $\dot{x}_m$ , on the acceleration of the particle (15). This, in accordance with theories with higher derivatives, means that we have to treat the variables  $x_m$  and  $\dot{x}_m$  as canonically independent coordinates [9]. So, the phase space of our model consists of two pairs of the canonical variables:

$$x_m, \quad p_m = \frac{\partial L_{\text{eff}}}{\partial \dot{q}^m} - \dot{\Pi}_m, \quad (46)$$

$$q_m = \dot{x}_m, \quad \Pi_m = \frac{\partial L_{\text{eff}}}{\partial \dot{q}^m}. \quad (47)$$

The explicit forms of the momenta (46) and (47) are given by

$$p^n = -e_{(0)}^n (\mu - \alpha k^2) + \frac{2\alpha e_{(1)}^n}{\sqrt{q^2}} \dot{k}, \quad (48)$$

$$\Pi^n = -\frac{2\alpha e_{(1)}^n}{\sqrt{q^2}} k, \quad (49)$$

where we use the notation

$$k = \frac{\dot{q}_n e_{(1)}^n}{q^2}, \quad \alpha = \frac{\mu}{3m^2}. \quad (50)$$

The components of the Frenet basis  $e_{(a)}^n$  in expressions (48)–(50) are defined in Eq. (44). This allows us to establish the existence of two primary first-class constraints:

$$\Phi_1 = \Pi q \approx 0, \quad (51)$$

$$\Phi_2 = H_{\text{can}} = pq + \Pi \dot{q} - L = pq + \sqrt{q^2} \left( \mu + \frac{1}{4\alpha} q^2 \Pi^2 \right) \approx 0. \quad (52)$$

One verifies easily that upon substitution of the functions (48) and (49) into Eqs. (51) and (52), these constraints vanish identically with respect to  $q_n$  and  $\dot{q}_n$ . This means that, in fact, the dynamics of our system is constrained on a certain submanifold of the total phase space. For definition of the physical phase space according to Dirac’s gauge-fixing prescription [10] we need to introduce new constraints which enable us to avoid the gauge freedom of the theory generated by the constraints (51) and (52). To this end we shall consider only the ordinary proper-time gauge condition

$$X = \sqrt{q^2} - 1 \approx 0. \quad (53)$$

It is easy to see that  $X$  forms a second-class algebra with the constraint (51):

$$\{\Phi_1, X\} = 1 + X. \quad (54)$$

Therefore, it must be omitted as nondynamical degree of freedom. This result can be achieved if we pass from the Poisson brackets to Dirac ones:

$$\{f, g\}^* = \{f, g\} + \frac{1}{1+X} \{f, \Phi_1\} \{X, g\} - \frac{1}{1+X} \{f, X\} \{\Phi_1, g\}. \quad (55)$$

From Eq. (55) one can observe that the constraints (51) and (53) have zero Dirac’s brackets with everything and, therefore, can be considered as strong equations.

The remaining phase-space coordinates have a transparent physical meaning. Indeed, let us perform the transformation to the new set of canonically conjugated variables:

$$\begin{aligned}\rho &= \sqrt{q^2}, \quad \Pi_\rho = \Pi_{(0)}, \\ v &= \text{arctanh} \frac{p_{(1)}}{p_{(0)}}, \quad \Pi_v = \rho \Pi_{(1)},\end{aligned}\quad (56)$$

$$z^m = x^m - \{x^m, v\} \Pi_v,$$

where  $p_{(a)} = p_n e^n_{(a)}$ ,  $\Pi_{(a)} = \Pi_n e^n_{(a)}$  are the momentum components in the Frenet basis. We would like to note that the above representation of the phase-space variables preserves the structure of the canonical Poisson brackets:

$$\begin{aligned} \{\Pi_\rho, \rho\} &= \{\Pi_v, v\} = 1, \\ \{p^m, z_n\} &= \delta^m_n, \end{aligned} \tag{57}$$

with all others equal to zero. It follows from their definitions that the constraints (51)–(53), when rewritten in terms of new coordinates, have the forms

$$\begin{aligned} \Phi_1 &= \rho \Pi_\rho, \\ \Phi_2 &= \rho \left( -\sqrt{p^2} \cosh v + \mu - \frac{1}{4\alpha} \Pi_v^2 + \frac{1}{4\alpha} \rho^2 \Pi_\rho^2 \right), \\ X &= \rho - 1. \end{aligned} \tag{58}$$

Thus, we see that the physical variables  $\Pi_v, v$  are nothing but the intrinsic angular momentum, i.e., the spin

$$S_{mn} = q_m \Pi_n - q_n \Pi_m = \varepsilon_{mn} \Pi_v, \tag{59}$$

and the corresponding angular variable in this model. The residual first-class constraint  $\Phi_2$  in terms of the remaining coordinates has the form

$$\Phi_2 = -\sqrt{p^2} \cosh v + \mu - \frac{1}{4\alpha} \Pi_v^2. \tag{60}$$

The physical states on the quantum level must satisfy the condition  $\hat{\Phi}_2 |ph\rangle = 0$ . Here, the operator  $\hat{\Phi}_2$  is obtained from Eq. (60) by the normal ordering prescription in the coordinate representation,  $\Pi_v = -i \partial / \partial v$ :

$$\begin{aligned} \hat{\Phi}_2 &= a_\lambda^+ a_\lambda - \xi^2 (1 - \lambda), \\ a_\lambda &= \frac{\partial}{\partial v} + \sqrt{2\lambda} \xi \sinh \frac{v}{2}, \\ \xi &= \frac{2}{\sqrt{3}} \frac{\mu}{m}, \\ \lambda &= \frac{\sqrt{p^2}}{\mu} \equiv \frac{M}{\mu} \geq 0. \end{aligned} \tag{61}$$

Note, that there are other similar modifications of the operator  $\hat{\Phi}_2$  admissible from the classical point of view. Depending on the operators ordering procedure for the product of operators  $a_\lambda, a_\lambda^+$ , they are not equivalent to the expressions (61). We do not find them very satisfactory since in what follows such representations lead to difficulties with the physical interpretation of the system. In particular, the obvious interpretation of the ground state of kink as a state with  $M = \mu$  does not hold.

Thus, here we shall confine ourselves to the equation of motion

$$[a_\lambda^+ a_\lambda - \xi^2 (1 - \lambda)] \Psi(v) = 0, \tag{62}$$

where the operators  $a_\lambda, a_\lambda^+$  are defined as in Eqs. (61). This equation can be rewritten in the standard form of the Schrödinger equation:

$$\begin{aligned} \left( -\frac{\partial}{\partial v^2} + 2\lambda \xi^2 \sinh^2 \frac{v}{2} - \sqrt{\frac{\lambda}{2}} \xi \cosh \frac{v}{2} \right) \Psi(v) \\ = \xi^2 (1 - \lambda) \Psi(v), \end{aligned} \tag{63}$$

with the potential

$$V(v) = 2\lambda \xi^2 \sinh^2 \frac{v}{2} - \sqrt{\frac{\lambda}{2}} \xi \cosh \frac{v}{2} \tag{64}$$

and the energy  $E = \xi^2 (1 - \lambda)$ . The interesting feature of the potential  $V(v)$  is that it can be written in the form

$$V(v) = G^2(v) - G'(v), \quad G(v) = \sqrt{2\lambda} \xi \cosh \frac{v}{2}, \tag{65}$$

typical to supersymmetric quantum mechanics [14].

One may check that for all the solutions of Eq. (63) with  $\mu \geq m$ , ( $\xi \geq 1$ ) the potential (64) has only one stationary point  $V_{\min} = -\sqrt{\lambda/2} \xi$ . Hence, from the boundary condition  $E \geq V_{\min}$  we get the following constraint on the permissible values of the parameters  $\lambda$  and  $\xi$ :

$$\xi(1 - \lambda) \geq -\sqrt{\lambda/2}. \tag{66}$$

We will come back to this point later on.

Let us now consider the eigenfunction which can be obtained from Eq. (63). First, it is easy to notice that there exists a well-defined ground state solution with the canonical eigenvalue  $M = \mu$  ( $E = 0$ ):

$$a_{\lambda=1} \Psi_{\text{vac}}(v) = 0 \Rightarrow \Psi_{\text{vac}} = C \exp\left(-2\sqrt{2} \xi \cosh \frac{v}{2}\right). \tag{67}$$

In order to construct the wave functions for other physical states it will be convenient to perform the following standard decomposition:

$$\Psi(v) = W(v) \exp\left(-2\sqrt{2\lambda} \xi \cosh \frac{v}{2}\right). \tag{68}$$

When substituting Eq. (68) into Eq. (63), a new equation for the function  $W(v)$  appears:

$$W'' - 2\sqrt{2\lambda} \xi \sinh \frac{v}{2} W' + \xi^2 (1 - \lambda) W = 0. \tag{69}$$

There are only two forms of representation for the function  $W(v)$  compatible with the structure of Eq. (69):

$$W^{(+)} = \sum_{n=0}^{\infty} A_n^{(+)} \cosh \frac{nv}{2}, \tag{70}$$

$$W^{(-)} = \sum_{n=0}^{\infty} A_n^{(-)} \sinh \frac{(n+1)v}{2}. \tag{71}$$

TABLE I. Eigenvalues of  $\lambda^{(-)}$  expressed as functions of  $\xi$ .

$\xi$	3.16	3.87	4.47	5.00	5.48	5.92	6.32	6.71	7.07	10.0	12.0	13.0
$\lambda(\xi)$	1.55	1.425	1.36	1.32	1.289	1.265	1.245	1.23	1.216	1.15	1.144	1.115
$\xi$	14.0	15.0	16.0	17.0	18.0	19.0	20.0	21.0	22.0	23.0	24.0	25.0
$\lambda(\xi)$	1.09	1.047	1.044	1.042	1.040	1.037	1.035	1.033	1.032	1.030	1.029	1.028

The first representation is even and the second one is odd under the transformation  $v \rightarrow -v$ . For definiteness, let us consider the function  $W^{(-)}(v)$ . As a result of the substitution of the decomposition (71) into Eq. (69), we obtain the recursive conditions for the coefficients  $A_n^{(-)}$ :

$$\sum_{n=0}^{\infty} a_{mn}^{(-)} A_n^{(-)} = 0, \quad (72)$$

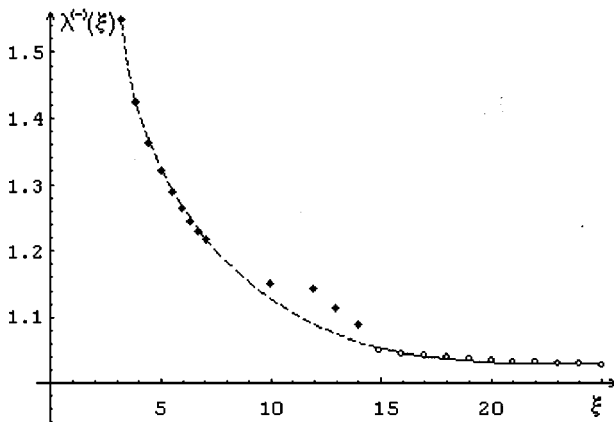
with

$$a_{mn}^{(-)} = -\xi(\lambda/2)^{1/2} m \delta_{n,m-1} + [\xi^2(1-\lambda) + (m+1)^2/4] \delta_{n,m} + \xi(\lambda/2)^{1/2} (m+2) \delta_{n,m+1}. \quad (73)$$

It is obvious that in accordance with the homogeneity of the system (72) it appears the equation

$$\text{det} a_{mn}^{(-)} = 0 \quad (74)$$

to be satisfied. It gives the relationship between parameters  $\lambda$  and  $\xi$  in accordance with Table I. The roots of this equation are exhibited in Fig. 1, where the eigenvalues of  $\lambda^{(-)}$  are expressed as a function of  $\xi$ . It is evident that only the part of the dots, indicated by the solid line, has been interpreted as the physical ones. The remaining branch, denoted by the dashed line, is not compatible with the condition (66). Therefore, the dot  $\xi = \xi_c \cong 15$  can be regarded as the crucial one. In order to understand better this result it would be useful to remind the reader of our previous lesson. It was mentioned in Sec. III that in accordance with [1], the weak-coupling corrections to the kink solution can be summarized in the mass formula

FIG. 1. The eigenvalues of  $\lambda^{(-)}$  as a function of the parameter  $\chi$ .

$$M = \mu + \frac{1}{2} \sqrt{\frac{3}{2}} m. \quad (75)$$

In our notation [see Eq. (61)], this expression is given by

$$\lambda^{(-)} = 1 + \frac{1}{\sqrt{2}\xi}. \quad (76)$$

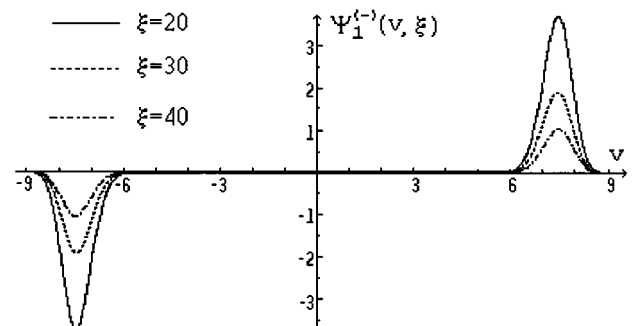
It is of main importance that the last formula is in good agreement with all the numerical data represented in Fig. 1 by the solid line. From the foregoing it follows that the point  $\xi_c$  may be regarded as that in which the weak-coupling perturbative regime of the theory is replaced by its strong-coupling regime and we are left with the solitonic solution only.

Having the spectrum of kink given by the explicit function (76) we can construct eigenfunctions  $\Psi^{(-)}$  straightforwardly by making use of the recursive condition (73). The results of this calculations for  $n$  ranging from 1 to 20 are drawn in Fig. 2. It is worthwhile to remark that the function  $W^{(-)}(v)$  we are dealing with is, in principle, a well-defined and calculable quantity, since the reduction of the series (71) is provided by the recursive condition (73). It turns out that for large  $m$  and the function  $\lambda^{(-)}(\xi)$  defined by Eq. (76), there exists a simple expression for the coefficients  $A_n^{(-)}$  in Eq. (71):

$$\frac{A_m^{(-)}}{A_{m-1}^{(-)}} = \frac{\alpha_m}{1 + \alpha_m^2/1 + \dots},$$

$$\alpha_m = \frac{2\sqrt{2}\xi}{m}. \quad (77)$$

From Eqs. (77) it follows that the calculations of  $W^{(-)}(v)$  are consistently defined.

FIG. 2. The odd wave function  $\Psi_1^{(-)}(v, \xi)$  for the excited kink state with  $\xi = 20, 30, 40$ .

The above considerations are connected to the method of calculation of the odd wave function  $\Psi^{(-)}(v)$  only. In principle, the analogous method allows one to find a formal solution for the even wave function  $\Psi^{(+)}(v)$  also. The latter, however, is not relevant from the physical point of view since the corresponding eigenvalue equation  $\det a_{mn}^{(+)} = 0$  for the function  $\lambda^{(+)}(\xi)$  is in contradiction with the boundary condition (66).

## VI. CONCLUSION

A few comments are in order. In this paper we have constructed a new nonperturbative approach to the problem of quantization of the topologically, nontrivial QFT, models with spontaneously broken relativistic symmetry. Ultimately this approach is based on the observation that the spectrum of the localized field states can be restored from the corresponding effective actions. The method of deriving the effective actions from the field theory is proved to be reduced to the transformation (6) allowing us to eliminate some field variables of the theory in terms of the appropriate collective coordinates. It was shown that in the case of kink solution this yields the action of the nonminimal  $d=1+1$  point particle with curvature. Upon quantization, it describes the

quantum particle with the mass (75) obtained earlier in the framework of the WKB approach [1].

It is clear that the methods developed here can be extended to the cases of field models involving more complicated static solutions. In particular, it would be of great interest to apply this approach to carry out the quantization of the 't Hooft–Polyakov monopole [11] and of the static solution of  $(2+1)$ -dimensional  $SU(2)$  Yang–Mills theory with the Chern–Simons term [12]. We hope also that such an investigation turns out to be useful in the cases of solutions obtained from supersymmetric field theories (see, for example, [13] and references therein).

*Note added in proof.* We are grateful to R. Jackiw who brought our attention to a series of works [15] in which the quantization beyond the static solution was initiated.

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