Radiative transitions of the *D* states of charmonium in potential models

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We derive expressions for the radiative multipole amplitudes and the total decay rates of heavy quarkonia in the transitions $1\,^3D_2 \to 1\,^3P_J + \gamma$ ($J=2,1,0$), $1\,^3D_2 \to n\,\,^1S_0 + \gamma$ ($n=1,2$), $1\,^1D_2 \to 1\,^1P_1 + \gamma$, $1\,^1D_2 \to n\,\,^3S_1 + \gamma$ ($n=1,2$), and $1\,^3D_1 \to 1\,^3P_J + \gamma$ ($J=2,1,0$), in an arbitrary potential model correct to order v^2/c^2. We also numerically evaluate these expressions for charmonium in the nonsingular potential model of Gupta, Johnson, Repko, and Suchyta. [S0556-2821(97)04201-X]

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I. INTRODUCTION

The *D* states of charmonium are quite interesting. Even though they are all above the charm threshold, the $1\,D_2$ and the $1\,^3D_2$ states are expected to have narrow widths since their strong decays to $D+\overline{D}$ are forbidden by parity conservation and their predicted masses are such that their decays into $\overline{D}+D^*$ or $D^*+\overline{D}$ are forbidden by energy conservation. These states can be directly formed in $\overline{p}p$ collisions of the proposed experiments at Fermilab by the E760 group.

Recently one of the authors (K.J.S.) together with Ridener [1,2] has derived model-independent expressions for the angular distributions of the decay products of the *D* states produced as resonances in $\overline{p}p$ collisions. These expressions [1,2] give the angular distribution in terms of the radiative multipole amplitudes. They are independent of any dynamical model and are based only on the general principles of quantum mechanics and symmetry. When these angular distributions are measured experimentally, our expressions [1,2] will enable one to extract the radiative multipole amplitudes from the measured angular distribution and we can then compare them with the predictions of various theoretical models. In this paper we derive expressions for these multipole amplitudes in an arbitrary potential model, correct to order v^2/c^2. We give our expressions in terms of the integrals involving the radial wave functions of the states. We also calculate the total decay rates of these radiative transitions in terms of the multipole amplitudes. We then evaluate our expressions in the nonsingular potential model of Gupta, Johnson, Repko, and Suchyta (GJRS) [3], which has been quite successful in predicting the energy spectrum of charmonium and the $E1$ decay rates in the transitions $\psi' \to \chi_J + \gamma$ and $\chi_J \to \psi + \gamma$ ($J=2,1,0$). The numerical results in the GJRS model are given in Tables I–V. The format of the rest of the paper is as follows. In Sec. II we derive expressions for the multipole amplitudes and the total decay rates in the parity-odd transitions $1\,^3D_2 \to 1\,^3P_J + \gamma$, $1\,^3D_1 \to 1\,^3P_J + \gamma$ ($J=2,1,0$), and $1\,^1D_2 \to 1\,^1P_1 + \gamma$. In Sec. III we consider the corresponding problem for the parity-even radiative transitions $1\,^3D_2 \to n\,\,^1S_0 + \gamma$, and $1\,^1D_2 \to n\,\,^3S_1 + \gamma$ ($n=1,2$). In Sec. IV we numerically evaluate the multipole amplitudes and the total decay rates in the GJRS model [3,4]. Section V gives a summary and concluding remarks. Even though the radiative decays of $1\,^1D_2$ was treated by one of us (K.J.S.) previously [5], we redo the calculations here to correct some errors in the expressions given there [5]. Also, we reevaluate the expressions in the new nonsingular potential model of GJRS where the energy spectrum and the wave functions are calculated in a nonperturbative way using a variational calculation of the full Hamiltonian.

II. MULTIPOLE AMPLITUDES AND THE DECAY RATES IN THE PARITY-ODD TRANSITIONS $1\,^3D_2 \to 1\,^3P_J + \gamma$, $1\,^3D_1 \to 1\,^3P_J + \gamma$ ($J=2,1,0$), AND $1\,^1D_2 \to 1\,^1P_1 + \gamma$

We assume that the heavy quarkonium ($c\overline{c}$ or $b\overline{b}$) to a good approximation can be thought of as the bound state of a quark and an antiquark bound by an arbitrary potential. In the usual model the potential will have a confining piece as well as a perturbative QCD part. In any such model we have shown previously [5–7] that the parity-odd transition amplitude of the one-photon radiative transition $A \to B + \gamma$ could be written [5,6] as

$$T_0(t) = \frac{e_q}{\sqrt{2V\omega}}\,\omega^I_{ABI}\langle B|\hat{\varepsilon}_\alpha \cdot \Bigg\{ \vec{r} - \frac{i}{4mk}\,[(\vec{k}\cdot\vec{r})^2\vec{p} - i(\vec{k}\cdot\vec{r})\vec{k} + 2(\vec{k}\cdot\vec{r})(\vec{S}\times\vec{k})(1+\kappa) - k^2(\vec{S}\times\vec{r}) \times (1+\kappa)]\Bigg\}|A\rangle_I \int_0^t e^{i(\omega-\omega_{AB})t'}dt', \tag{1}$$

where the states $|A\rangle_I$ and $|B\rangle_I$ are eigenstates of the internal Hamiltonian [5–7] with energies E^I_A and E^I_B, respectively. The internal Hamiltonian h is related to the Hamiltonian H_0 of the isolated quarkonium by the relation

$$H_0 = \sqrt{h^2 + P^2}, \tag{2}$$

where $\vec{P}$ is the overall or center-of-mass momentum of the quarkonium and

$$\omega^I_{AB} = E^I_A - E^I_B, \tag{3}$$

TABLE I. Multipole amplitudes and the decay rates in the radiative transitions $1\,^3D_2 \rightarrow 1\,^3P_J + \gamma$ (J=2,1,0) in charmonium using the GJRS model [3]. I_1 and I_2 are the radial integrals. a_1', a_2', and a_3' are the multipole amplitudes. We also give the multipole amplitude relative strengths $|a_2'/a_1'|$ and $|a_3'/a_1'|$.

Decay	Photon energy (MeV)	I_1 (GeV^{-1})	I_2 (GeV^{-1})	a_1' (GeV^{-1})	a_2' (GeV^{-1})	a_3' (GeV^{-1})	$\lvert a_2'/a_1' \rvert$	$\lvert a_3'/a_1' \rvert$	Decay rate (keV)
$1\,^3D_2 \rightarrow \chi_2 + \gamma$	255	2.72	−4.00	1.014 $-i0.007$	$-i0.025$	0.014	0.025	0.014	59.4
$1\,^3D_2 \rightarrow \chi_1 + \gamma$	294	2.44	−3.18	0.844 $-i0.009$	0.029 $-i0.039$	−0.005	0.058	0.006	148
$1\,^3D_2 \rightarrow \chi_0 + \gamma$	383	2.27	−2.74		0.062 $+i0.037$				0.86

$$\vec{r} = \lim_{P\rightarrow 0} (\vec{r}_1 - \vec{r}_2), \tag{4}$$

$$\vec{p} = \lim_{P\rightarrow 0} \vec{p}_1 = \lim_{P\rightarrow 0} (-\vec{p}_2), \tag{5}$$

$$\vec{S} = \vec{s}_1 + \vec{s}_2, \tag{6}$$

where s_1 and s_2 are the spin-$\frac{1}{2}$ operators of the quark and the antiquark. Equation (1) takes into account the full effect of the recoil of the quarkonium after the emission of the photon, to order v^2/c^2. So there is a difference between ω_{AB} and ω_{AB}^I. Here ω_{AB}^I is just the energy difference of Eq. (3), where as the ω_{AB} is the energy of the emitted photon including the recoil effect. When energy conservation holds as it will for large t,

$$k = \omega = \omega_{AB} = E_A - E_B, \tag{7}$$

where E_A and E_B are the energies [5–7] of the quarkonium (including the energy due to the center-of-mass motion) in the states A and B. The symbols e_q, m, and κ represent the charge, the mass, and the anomalous magnetic moment parameter of the quark. We put $\hbar = c = 1$ throughout this paper. Since we use the relativistic center-of-mass variables in the interaction Hamiltonian, the low energy theorem in Compton scattering and the Drell-Hearn-Gerasimov sum rule for composite systems will be satisfied and Eq. (1) takes into account all first order (of order v^2/c^2) relativistic corrections. In Eq. (1) the first term $\vec{r}$ is the dominant term which comes from the commutator term $[\vec{r}_\mu, H_0]\cdot\vec{A}(\vec{r}_\mu, t)$ (μ=1,2) in the interaction Hamiltonian where in the expansion of the vector potential $\vec{A}(\vec{r}_\mu, t)$ we replace $e^{i\vec{k}\cdot\vec{r}_\mu}$ by 1. Recall that the commutator term in the interaction Hamiltonian includes all terms linear in $\vec{A}(\vec{r}_\mu, t)$, obtained from the minimal replacement $\vec{p}_\mu \rightarrow \vec{p}_\mu - (e_\mu/c)\vec{A}(\vec{r}_\mu, t)$. So this includes all relativistic correction terms which are spin independent. This would mean that $\omega_{AB}^I \cdot \vec{r}$ includes most of the relativistic corrections when we substitute for ω_{AB}^I the experimental energy difference, and for $|A\rangle_I$ and $|B\rangle_I$, we use the eigenfunctions of the full Hamiltonian including the relativistic corrections. The second and third terms also come from the commutator term, when the exponential $e^{i\vec{k}\cdot\vec{r}_\mu}$ is expanded to second order in $(\vec{k}\cdot\vec{r}_\mu)$. These are the so-called "finite size corrections." The fourth and fifth terms in Eq. (1) come from the spin-dependent relativistic correction terms in the interaction Hamiltonian.

Let us apply Eq. (1) to calculate the multipole amplitudes and the decay rates of the parity-odd transitions $1\,^3D_2 \rightarrow 1\,^3P_J + \gamma$, $1\,^3D_1 \rightarrow 1\,^3P_J + \gamma$ (J=2,1,0), and $1\,^1D_2 \rightarrow 1\,^1P_1 + \gamma$.

A. $1\,^3D_2 \rightarrow 1\,^3P_J + \gamma$ (J=2,1,0)

Let us assume that the $1\,^3D_2$ state of charmonium is formed at rest in $\overline{pp}$ collisions. Let the $\chi_J(1\,^3P_J)$ and the photon γ be emitted in the $+z$ and $-z$ directions, respectively. The component of the angular momentum of $1\,^3D_2$ in the $+z$ direction is called ν. The helicities of χ_J and γ are called σ and μ, respectively:

$$\mu = \pm 1, \quad \sigma = -J, -J+1, 0, \ldots, +J. \tag{8}$$

TABLE II. Multipole amplitudes and the decay rates in the radiative transitions $1\,^3D_1 \rightarrow 1\,^3P_J + \gamma$ (J=2,1,0) in charmonium using the GJRS model [3]. I_1, I_2, I_3, and I_4 are the radial integrals. a_1', a_2', and a_3' are the multipole amplitudes. We also give the multipole amplitude relative strengths $|a_2'/a_1'|$ and $|a_3'/a_1'|$.

Decay	Photon energy (MeV)	I_1 (GeV^{-1})	I_2 (GeV^{-1})	I_3 (GeV^{-1})	I_4 (GeV^{-1})	a_1' (GeV^{-1})	a_2' (GeV^{-1})	a_3' (GeV^{-1})	$\lvert a_2'/a_1' \rvert$	$\lvert a_3'/a_1' \rvert$	Decay rate (keV)
$1\,^3D_1 \rightarrow \chi_2 + \gamma$	228	−2.19	2.72	2.04	−4.20	0.296	0.019 $-i0.014$	0.014	0.080	0.047	4.64
$1\,^3D_1 \rightarrow \chi_1 + \gamma$	267	−1.84	2.46	1.15	−3.38	−0.964	0.024 $-i0.026$		0.037		79.5
$1\,^3D_1 \rightarrow \chi_0 + \gamma$	357	−1.63	2.29	0.71	−2.94	1.042					225

TABLE III. Multipole amplitudes and the decay rate in the radiative transition $1\,^1D_2\rightarrow 1\,^1P_1+\gamma$ in charmonium using the GJRS model [3]. G_1 and G_2 are the radial integrals. a_1', a_2', and a_3' are the multipole amplitudes. We also give the multipole amplitude relative strengths $|a_2'/a_1'|$ and $|a_3'/a_1'|$.

Decay	Photon energy (MeV)	G_1 (GeV^{-1})	G_2 (GeV^{-1})	a_1' (GeV^{-1})	a_2' (GeV^{-1})	a_3' (GeV^{-1})	$\lvert a_2'/a_1'\rvert$	$\lvert a_3'/a_1'\rvert$	Decay rate (keV)
$1\,^1D_2\rightarrow 1\,^1P_1+\gamma$	281	2.67	−3.64	2.190	$-i0.017$	0.012	0.008	0.005	288

By angular momentum conservation in the $+z$ direction,

$$\nu=\sigma-\mu. \tag{9}$$

The matrix element of Eq. (1) in this particular kinematic configuration is called the helicity amplitude $A'^{J}_{\sigma\mu}=A'^{J}_{\sigma,\sigma-\nu}$ and the transition amplitude of Eq. (1) in this configuration can be written as

$$T_{0,z}(t)=\frac{e_q}{\sqrt{2V\omega}}\,\omega^{I}_{AB}A'^{J}_{\sigma\mu}\int_0^t e^{i(\omega-\omega_{AB})t'}dt'. \tag{10}$$

In general, if the χ_J is moving with momentum $+\vec{p}$ in the (θ,ϕ) direction and γ with momentum $-\vec{p}$, the corresponding transition amplitude can be written as [8,9]

$$T_0(t)=\frac{e_q}{\sqrt{2V\omega}}\,\omega^{I}_{AB}A'^{J}_{\sigma\mu}D^{2*}_{\nu,\sigma-\mu}(\phi,\theta,-\phi)\times\int_0^t e^{i(\omega-\omega_{AB})t'}dt', \tag{11}$$

where $D^{J}_{mm'}(\alpha,\beta,\gamma)$ is the Wigner D^J function [10] defined by the equation

$$\langle Jm|U_R(\alpha,\beta,\gamma)|Jm'\rangle=D^{J}_{mm'}(\alpha,\beta,\gamma), \tag{12}$$

where U_R is the unitary rotation operator and α, β, and γ are the Euler angles of rotation [10]. Equation (11) leads to Eq. (10) since

$$D^{J}_{\nu,\sigma-\mu}(0,0,0)=\delta_{\nu,\sigma-\mu}. \tag{13}$$

The angular momentum helicity amplitude $A'^{J}_{\sigma\mu}$ is independent of all angles, and all the dynamics of the problem is contained in it. Obviously, the expression for $A'^{J}_{\sigma\mu}$ will depend on the dynamical model.

By parity invariance [9] of the transition operator, we obtain

$$A'^{J}_{\sigma\mu}=(-1)^{J+1}A'^{J}_{-\sigma-\mu}. \tag{14}$$

We will label the linearly independent helicity amplitudes as

$$A'^{J}_{\sigma}=A'^{J}_{\sigma 1}=(-1)^{J+1}A'^{J}_{-\sigma-1},\quad \sigma=-J,-J+1,\ldots,+J, \tag{15}$$

with the restriction

$$-2\leqslant\nu=\sigma-\mu\leqslant+2\quad\text{and}\quad\mu=\pm1. \tag{16}$$

This would mean that for $J=2$, σ cannot take the value -2. So for $J=2$ there are four linearly independent helicity amplitudes, for $J=1$ three, and for $J=0$ only one. These independent helicity amplitudes are related to the multipole amplitudes a'^{J}_{k} by an orthogonal linear transformation

$$A'^{J}_{\sigma}=\sum_{\mathrm{Max}(k=|2-J|;1)}^{J+2}a'^{J}_{k}\left(\frac{2k+1}{5}\right)^{1/2}\langle k-1;J\sigma|2,\sigma-1\rangle. \tag{17}$$

Since the relation in Eq. (17) is orthogonal,

$$\sum_{\sigma}|A'^{J}_{\sigma}|^2=\sum_{k}|a'^{J}_{k}|^2. \tag{18}$$

Using Eq. (1), we can write an expression for this helicity amplitude in an arbitrary potential model. We get

$$A'^{J}_{\sigma,\mu=\pm1}=\langle J\sigma|\pm\frac{1}{\sqrt{2}}(x\pm iy)-\frac{i}{4mk}k^2z^2\pm\frac{1}{\sqrt{2}}(p_x\pm ip_y)-\frac{2i}{4mk}(-kz)\left(\pm\frac{1}{\sqrt{2}}\right)(\hat{x}\pm i\hat{y})\cdot(\vec{S}\times\vec{k})(1+\kappa)+\frac{i}{4mk}k^2\left(\pm\frac{1}{\sqrt{2}}\right)\times(\hat{x}\pm i\hat{y})\cdot(\vec{S}\times\vec{r})(1+\kappa)|2\nu\rangle. \tag{19}$$

In writing Eq. (19) we assumed that the photon is moving in the $-z$ direction with helicity ±1, so that the momentum and polarization vectors of the photon become

$$\vec{k}=-k\hat{z},\quad \hat{\varepsilon}_{\alpha}=\pm\frac{1}{\sqrt{2}}(\hat{x}\pm i\hat{y})\quad\text{for}\ \mu=\pm1. \tag{20}$$

In order to write the helicity amplitude in Eq. (19) in terms of the multipole amplitudes using Eq. (17), we should express the operators in the matrix element of Eq. (19) in terms of irreducible spherical tensor operator components. For this purpose we define the following highest weight spherical tensor components:

TABLE IV. Multipole amplitudes and the decay rates in the radiative transitions $1\,^1D_2 \to n\,^3S_1+\gamma$ (n=1,2) in charmonium using the GJRS model [3]. J_0, J_1, J_2, J_3, and J_4 are the radial integrals. a_1', a_2', and a_3' are the multipole amplitudes. We also give the multipole amplitude relative strengths $|a_2'/a_1'|$ and $|a_3'/a_1'|$.

Decay	Photon energy (MeV)	J_0 (GeV^{-1})	J_1 (GeV^{-1})	J_2 (GeV^{-1})	J_3 (GeV^{-1})	J_4 (GeV^{-1})	a_1' (GeV^{-1})	a_2' (GeV^{-1})	a_3' (GeV^{-1})	$\|a_2'/a_1'\|$	$\|a_3'/a_1'\|$	Decay rate (keV)
$1\,^1D_2 \to \psi+\gamma$	639	−0.207	0.215	−0.233	−0.193	0.093	i0.004	$-i$0.029	$-i$0.010	6.64	2.41	0.699
$1\,^1D_2 \to \psi'+\gamma$	132	−0.084	−0.027	0.025	−0.162	−0.156	i0.002	i0.000	i0.000	0.016	0.164	0.001

$$T_{33}=x_+^2 p_+ ,\quad T_{22}=x_+(\vec{x}\times\vec{p})_+ ,$$

$$S_{22}=x_+ S_+ ,\quad x_{11}=X_+ ,$$

$$T_{11}=\sqrt{\tfrac{2}{5}}\,r^2 p_+ ,\quad T_{11}'=\sqrt{\tfrac{2}{5}}\,x_+(\vec{x}\cdot\vec{p}),$$

$$S_{11}=S_+ ,\quad S_{11}'=(\vec{x}\times\vec{S})_+ , \tag{21}$$

where

$$A_\pm=\mp\frac{1}{\sqrt{2}}\,(A_x\pm iA_y). \tag{22}$$

The other components of the spherical tensors can be obtained from the commutation relation

$$[J_-,T_{kq}]=\sqrt{k(k+1)+q(1-q)}\,T_{k,q-1}. \tag{23}$$

Using Eqs. (21)–(23), we can write the helicity amplitude of Eq. (19) as

$$A_{\sigma 1}'^{J}=A_\sigma'^{J}=\langle J\sigma|-X_{11}+\frac{ik}{2m\sqrt{10}}\,(T_{11}-\tfrac{1}{2}T_{11}')+\frac{ik}{m}\,T_{21}$$
$$+\frac{ik}{2m}\,\frac{1}{\sqrt{15}}\,T_{31}+\frac{k}{2\sqrt{2}m}\,(1+\kappa)S_{21}$$
$$+\frac{ik}{2m}\,(1+\kappa)S_{11}'|2\nu\rangle. \tag{24}$$

The Wigner-Eckart theorem [10] tells us that

$$\langle J\sigma|R_{k1}|2\nu\rangle=\langle J\sigma|k1;2\nu\rangle\langle J\|R_k\|2\rangle, \tag{25}$$

where R_{k1} stands for X_{11}, T_{11}, T_{11}', T_{21}, T_{31}, S_{21}, or S_{11}', multiplied by the appropriate factors. Using Eq. (25), Eq. (24) becomes

$$A_\sigma'^{J}=\sum_k\,\langle J\sigma|k1;2\nu\rangle\langle J\|R_k\|2\rangle. \tag{26}$$

Comparing Eqs. (26) and (17) through the use of the symmetry relations [10] among Clebsch-Gordon coefficients, we get

$$a_k'^{J}=(-1)^{k+1}\left(\frac{2J+1}{2k+1}\right)^{1/2}\langle J\|R_k\|2\rangle \tag{27}$$

where R_k can be read off from Eq. (24) for various values of k. The explicit expressions for the multipole amplitudes obtained from Eqs. (27) and (24) are given below for the three cases J=2,1,0.

1. $J=2$ ($1\,^3D_2\to\chi_2+\gamma$)

There are four multipole amplitudes in this case, $E1$, $M2$, $E3$, and $M4$ or a_1', a_2', a_3', and a_4', respectively:

$$a_1'=-\sqrt{\tfrac{5}{3}}\Bigg[\langle\chi_2\|X_1\|1\,^3D_2\rangle-\frac{ik}{2\sqrt{10}m}\,\langle\chi_2\|T_1\|1\,^3D_2\rangle$$
$$+\frac{ik}{4\sqrt{10}m}\,\langle\chi_2\|T_1'\|1\,^3D_2\rangle-\frac{ik}{2m}\,(1+\kappa)$$
$$\times\langle\chi_2\|S_1'\|1\,^3D_2\rangle\Bigg],$$

$$a_2'=-\frac{ik}{2m}\left[\langle\chi_2\|T_2\|1\,^3D_2\rangle-\frac{i}{\sqrt{2}}\,(1+\kappa)\langle\chi_2\|S_2\|1\,^3D_2\rangle\right],$$

$$a_3'=\sqrt{\frac{5}{7}}\left(\frac{ik}{2m}\right)\frac{1}{\sqrt{15}}\,\langle\chi_2\|T_3\|1\,^3D_2\rangle,$$

$$a_4'=0. \tag{28}$$

TABLE V. Multipole amplitudes and the decay rates in the radiative transitions $1\,^3D_2\to n\,^1S_0+\gamma$ (n=1,2) in charmonium using the GJRS model [3]. J_1', J_2', J_3', and J_4' are the radial integrals. a_2' is the multipole amplitude.

Decay	Photon energy (MeV)	J_1' (GeV^{-1})	J_2' (GeV^{-1})	J_3' (GeV^{-1})	J_4' (GeV^{-1})	a_2' (GeV^{-1})	Decay rate (keV)
$1\,^3D_2\to\eta_c+\gamma$	716	0.211	−0.227	−0.221	0.078	$-i$0.038	2.21
$1\,^3D_2\to\eta_c'+\gamma$	193	−0.056	0.050	−0.143	−0.159	i0.002	0.001

It is interesting to notice that the M_4 amplitude a'_4 is identically zero in any potential model correct to order v^2/c^2 since in this approximation there is no fourth rank tensor component in the transition operator.

Using the definitions of the spherical tensor components given by Eq. (21) and constructing the $1\,^1D_{2\nu}$ and $1\,^3P_{J\sigma}$ states in terms of the radial and angular momentum wave functions, we can derive expressions for the reduced matrix elements in terms of the following radial integrals I_1 and I_2:

$$I_1=\int_0^\infty R_{1\,^3P_2}(r)R_{1\,^3D_2}(r)r^3dr,$$

$$I_2=\int_0^\infty R_{1\,^3P_2}(r)\,\frac{dR_{1\,^3D_2}(r)}{dr}\,r^4dr. \tag{29}$$

We obtain

$$\langle\chi_2\|X_1\|1\,^3D_2\rangle=\frac{1}{\sqrt{10}}\,I_1,$$

$$\langle\chi_2\|T_1\|1\,^3D_2\rangle=\frac{i}{5}\,(I_2+3I_1),$$

$$\langle\chi_2\|T'_1\|1\,^3D_2\rangle=\tfrac{1}{5}I_2,$$

$$\langle\chi_2\|S'_1\|1\,^3D_2\rangle=0,$$

$$\langle\chi_2\|T_2\|1\,^3D_2\rangle=-\tfrac{1}{2}\sqrt{\tfrac{7}{5}}(I_1+\tfrac{1}{14}I_2),$$

$$\langle\chi_2\|S_2\|1\,^3D_2\rangle=0,$$

$$\langle\chi_2\|T_3\|1\,^3D_2\rangle=\frac{2i}{7\sqrt{5}}\,(8I_1+3I_2). \tag{30}$$

2. $J=1$ ($1\,^3D_2\rightarrow\chi_1+\gamma$)

There are three multipole amplitudes in this case, namely, the $E1$, $M2$, and $E3$ or a'_1, a'_2, and a'_3:

$$a'_1=-\Bigg[\langle\chi_1\|X_1\|1\,^3D_2\rangle-\frac{ik}{2\sqrt{10}m}\,\langle\chi_1\|T_1\|1\,^3D_2\rangle+\frac{ik}{4\sqrt{10}m}\,\langle\chi_1\|T'_1\|1\,^3D_2\rangle-\frac{ik}{2m}\,(1+\kappa)\times\langle\chi_1\|S'_1\|1\,^3D_2\rangle\Bigg],$$

$$a'_2=-\frac{ik}{2m}\,\sqrt{\frac{3}{5}}\Bigg[\frac{1}{3}\,\langle\chi_1\|T_2\|1\,^3D_2\rangle-\frac{i}{\sqrt{2}}\,(1+\kappa)\times\langle\chi_1\|S_2\|1\,^3D_2\rangle\Bigg],$$

$$a'_3=\frac{ik}{2m}\,\frac{1}{\sqrt{35}}\,\langle\chi_1\|T_3\|1\,^3D_2\rangle. \tag{31}$$

As before, the reduced matrix elements are given in terms of the radial integrals I_1 and I_2, but this time $R_{1\,^3P_2}(r)$ is replaced by $R_{1\,^3P_1}(r)$ in their definitions:

$$\langle\chi_1\|X_1\|1\,^3D_2\rangle=\frac{1}{\sqrt{2}}\,I_1,$$

$$\langle\chi_1\|T_1\|1\,^3D_2\rangle=-\frac{i}{\sqrt{5}}\,(3I_1+I_2),$$

$$\langle\chi_1\|T'_1\|1\,^3D_2\rangle=\frac{1}{\sqrt{5}}\,I_2,$$

$$\langle\chi_1\|S'_1\|1\,^3D_2\rangle=-i\sqrt{2}I_1,$$

$$\langle\chi_1\|T_2\|1\,^3D_2\rangle=\frac{1}{\sqrt{42}}\,[(12-\sqrt{\tfrac{7}{2}})I_1+(\sqrt{\tfrac{2}{7}}-1)I_2],$$

$$\langle\chi_1\|S_2\|1\,^3D_2\rangle=-\frac{1}{\sqrt{3}}\,I_1,$$

$$\langle\chi_1\|T_3\|1\,^3D_2\rangle=-\frac{i}{3}\,\sqrt{\frac{2}{35}}\,(8I_1+3I_2). \tag{32}$$

3. $J=0$ ($1\,^3D_2\rightarrow\chi_0+\gamma$)

There is only one multipole amplitude in this case, namely, the $M2$ or a'_2:

$$a'_2=-\frac{ik}{2m}\,\frac{1}{\sqrt{5}}\Bigg[\frac{1}{3}\,\langle\chi_0\|T_2\|1\,^3D_2\rangle-\frac{i}{\sqrt{2}}\,(1+\kappa)\times\langle\chi_0\|S_2\|1\,^3D_2\rangle\Bigg]. \tag{33}$$

The reduced matrix elements are again given in terms of the corresponding radial integrals I_1 and I_2 of Eqs. (29), where $R_{1\,^3P_2}$ is now replaced by $R_{1\,^3P_0}$:

$$\langle\chi_0\|T_2\|1\,^3D_2\rangle=(\tfrac{2}{3}\sqrt{\tfrac{2}{7}}+\tfrac{17}{21})I_1+(\tfrac{2}{21}-\tfrac{1}{3}\sqrt{\tfrac{2}{7}})I_2. \tag{34}$$

Next we derive the expressions for the decay rates. Using Eq. (11), the probability of transition after time t to the specific final helicity states σ, μ is given by

$$P^J_{\nu,\sigma\mu}(t)=|T_0(t)|^2=\frac{e_q^2}{2V\omega}\,(\omega^I_{AB})^2|A'^J_{\sigma\mu}|^2\times|D^2_{\nu,\sigma-\mu}(\phi,\theta,-\phi)|^2\,\frac{\sin^2[(\omega-\omega_{AB})/2]t}{[(\omega-\omega_{AB})/2]^2}. \tag{35}$$

The transition probability integrated over all the directions of the out-going photon and over a small but finite energy spread of the photon and summed over the final helicity states σ and μ is

$$P^J(t)=\frac{e_q^2}{2V}(\omega_{AB}^I)^2\int\left(\sum_{\sigma\mu}|A'^J_{\sigma\mu}|^2\right)|D^2_{\nu,\sigma-\mu}(\phi,\theta,-\phi)|^2\times\frac{1}{\omega}\frac{\sin^2[(\omega-\omega_{AB})/2]t}{[(\omega-\omega_{AB})/2]^2}\frac{V}{(2\pi)^3}\omega^2 d\omega\, d\Omega. \quad (36)$$

Using the fact that [10]

$$\int|D^J_{mm'}(\phi,\theta,-\phi)|^2 d\Omega=\frac{4\pi}{(2J+1)}, \quad (37)$$

Eq. (36) for large t ($t\gg 1/\omega_{AB}$) becomes

$$P^J(t)=\frac{e_q^2}{(2\pi)^3}(\omega_{AB}^I)^2\frac{4\pi}{5}\left(\sum_{\sigma\mu}|A'^J_{\sigma\mu}|^2\right)(\omega_{AB})\pi t, \quad (38)$$

where

$$\sum_{\sigma\mu}|A'^J_{\sigma\mu}|^2=\sum_{\sigma}|A'^J_{\sigma 1}|^2+\sum_{\sigma}|A'^J_{\sigma -1}|^2=2\sum_{\sigma}|A'^J_{\sigma 1}|^2=2\sum_{\sigma}|A'^J_{\sigma}|^2=2\sum_{k}|a'^J_k|^2. \quad (39)$$

So the transition probability per unit time for large t becomes

$$W^J=\frac{P^J(t)}{t}=\left(\frac{e_q}{e}\right)^2\frac{4}{5}\alpha(\omega_{AB}^I)^2\omega_{AB}\sum_k|a'^J_k|^2, \quad (40)$$

where α is the fine-structure constant:

$$\alpha=\frac{e^2}{4\pi}. \quad (41)$$

In Eq. (40), ω_{AB}^I is the energy difference between the two states and ω_{AB} is the energy of the actual emitted photon which takes into account the recoil energy of the quarkonium. In potential models, a'^J_2 and a'^J_3 are of order k/m or v^2/c^2 compared to a'^J_1 when it is nonzero. So to first order in k/m or v^2/c^2 we only need to keep a'^J_1. So in this approximation

$$W(1\,^3D_2\to 1\,^3P_2+\gamma)=\frac{4}{5}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}|a'_1|^2=\frac{2}{15}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}\times\left[I_1^2+\frac{k}{5m}I_1(I_2+3I_1)\right], \quad (42)$$

correct to first order in k/m. In the same approximation,

$$W(1\,^3D_2\to 1\,^3P_1+\gamma)=\frac{2}{5}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}\times\left[I_1^2-\frac{k}{5m}I_1(I_2+3I_1)-\frac{2k}{m}(1+\kappa)I_1^2\right]. \quad (43)$$

$W(1\,^3D_2\to 1\,^3P_0+\gamma)$ is nonzero only to order k^2/m^2, and it is equal to

$$W(1\,^3D_2\to 1\,^3P_0+\gamma)=\frac{4}{5}\left(\frac{e_q}{e}\right)^2\alpha^2(\omega_{AB}^I)^2\omega_{AB}|a'_2|^2=\frac{4}{25}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}\frac{k^2}{m^2}\left[\tfrac{1}{8}(1+\kappa)^2I_1^2+\left|\tfrac{1}{6}\left(\tfrac{2}{3}\sqrt{\tfrac{2}{7}}+\tfrac{17}{21}\right)I_1+\tfrac{1}{6}\left(\tfrac{2}{21}-\tfrac{1}{3}\sqrt{\tfrac{2}{7}}\right)I_2\right|^2\right]. \quad (44)$$

B. $\psi''(3770)\to 1\,^3P_J+\gamma$

Let us assume that $|\psi''(3770)\rangle$ and $|\psi'(3685)\rangle$ are a linear combination of $|1\,^3D_1\rangle$ and $|2\,^3S_1\rangle$ states. That is,

$$|\psi''(3770)\rangle=a'|1\,^3D_1\rangle-b'|2\,^3S_1\rangle,$$

$$|\psi'(3685)\rangle=a'|2\,^3S_1\rangle+b'|1\,^3D_1\rangle, \quad (45a)$$

where a' and b' are real coefficients with

$$a'^2+b'^2=1. \quad (45b)$$

We should expect $a'\approx 1$ and $|b'|\ll 1$. Using a calculation similar to what we used for $1\,^3D_2\to 1\,^3P_J+\gamma$, we obtain the following results.

1. $\psi''(3770)\to 1\,^3P_2+\gamma$

There are three multiple amplitudes in this case, $E1$, $M2$, and $E3$. We will call them a'_1, a'_2, and a'_3:

$$a'_1=\frac{\sqrt{5}}{3}\left(\frac{a'}{5\sqrt{2}}I'_2+b'I'_1\right)\times\left[1-\frac{k}{20m}\frac{(a'/5\sqrt{2})(I'_4+6I'_2)+b'I'_3}{(a'/5\sqrt{2})I'_2+b'I'_1}\right],$$

$$a'_2=-\frac{ik}{2m}\left[\frac{a'}{10}I'_2+\frac{i}{2}(1+\kappa)\left(\frac{\sqrt{2}a'}{5}I'_2-b'I'_1\right)\right],$$

$$a'_3=\frac{1}{70}\sqrt{\frac{2}{5}}\frac{k}{m}a'(3I'_4+8I'_2). \quad (46)$$

The radial integrals I'_1, I'_2, I'_3, and I'_4 are defined as

$$I_1'=\int_0^\infty r^3 dr R_{2\,^3S_1}(r)R_{1\,^3P_J}(r),$$

$$I_2'=\int_0^\infty r^3 dr R_{1\,^3D_1}(r)R_{1\,^3P_J}(r),$$

$$I_3'=\int_0^\infty r^4 dr\,\frac{dR_{2\,^3S_1}(r)}{dr}\,R_{1\,^3P_J}(r),$$

$$I_4'=\int_0^\infty r^4 dr\,\frac{dR_{1\,^3D_1}(r)}{dr}\,R_{1\,^3P_J}(r). \tag{47}$$

To first order in k/m, the decay rate for the process $\psi''\to 1\,^3P_2+\gamma$ is given by the expression

$$w(\psi''\to\chi_2+\gamma)=\frac{20}{27}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}\left(\frac{a'}{5\sqrt{2}}\,I_2'+b'I_1'\right)^2\times\left[1-\frac{k}{10m}\,\frac{(a'/5\sqrt{2})(I_4'+6I_2')+b'I_3'}{(a'/5\sqrt{2})I_2'+b'I_1'}\right]. \tag{48}$$

2. $\psi''(3770)\to 1\,^3P_1+\gamma$

There are only two multipole amplitudes in this case, namely, $E1$ and $M2$ or a_1' and a_2':

$$a_1'=-\frac{1}{\sqrt{3}}\left(\frac{a'}{\sqrt{2}}\,I_2'-b'I_1'\right)\times\left[1-\frac{k}{20m}\,\frac{(a'/\sqrt{2})(I_4'+6I_2')-b'I_3'}{(a'/\sqrt{2})I_2'-b'I_1'}\right],$$

$$a_2'=-\sqrt{3}\,\frac{ik}{20m}\left[a'I_2'+i(1+\kappa)\left(\frac{2\sqrt{2}}{3}\,a'I_2'-\frac{5}{3}\,b'I_1'\right)\right]. \tag{49}$$

To first order in k/m, the decay rate is given by

$$w(\psi''\to\chi_1+\gamma)=\frac{4}{9}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}\left(\frac{a'}{\sqrt{2}}\,I_2'-b'I_1'\right)^2\times\left[1-\frac{k}{10m}\,\frac{(a'/\sqrt{2})(I_4'+6I_2')-b'I_3'}{(a'/\sqrt{2})I_2'-b'I_1'}\right]. \tag{50}$$

3. $\psi''(3770)\to 1\,^3P_0+\gamma$

There is only one multipole amplitude, namely, the $E1$ amplitude or a'. It is a pure $E1$ transition:

$$a_1'=\tfrac{1}{3}(\sqrt{2}a'I_2'+b'I_1')\left[1-\frac{k}{20m}\,\frac{\sqrt{2}a'(I_4'+6I_2')+b'I_3'}{\sqrt{2}a'I_2'+b'I_1'}\right]. \tag{51}$$

The decay rate to order k/m is given by

$$W(\psi''\to\chi_0+\gamma)=\frac{4}{27}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}(\sqrt{2}a'I_2'+b'I_1')^2\times\left[1-\frac{k}{10m}\,\frac{\sqrt{2}a'(I_4'+6I_2')+b'I_3'}{\sqrt{2}a'I_2'+b'I_1'}\right]. \tag{52}$$

C. $1\,^1D_2\to 1\,^1P_1+\gamma$

From angular momentum and parity conservation, there are three multipole amplitudes in this case, $E1$, $M2$, and $E3$ or a_1', a_2', and a_3'. Applying Eq. (1) to this specific case, we have the following expressions for them:

$$a_1'=\sqrt{\frac{2}{3}}\,G_1\left[1+\frac{k}{20m}\left(10+3\,\frac{G_2}{G_1}\right)\right],$$

$$a_2'=-\frac{i}{20}\,\frac{k}{m}\,G_1,$$

$$a_3'=\frac{1}{35\sqrt{10}}\,\frac{k}{m}\,(3G_2+8G_1). \tag{53}$$

The decay rate to first order in k/m is given by

$$W(1\,^1D_2\to 1\,^1P_1+\gamma)=\frac{8}{15}\left(\frac{e_q}{e}\right)^2\alpha(\omega_{AB}^I)^2\omega_{AB}G_1^2\times\left[1+\frac{k}{m}+\frac{3k}{10m}\,\frac{G_2}{G_1}\right]. \tag{54}$$

The radial integrals G_1 and G_2 are defined as

$$G_1=\int_0^\infty R_{1\,^1P_1}(r)R_{1\,^1D_2}(r)r^3 dr,$$

$$G_2=\int_0^\infty R_{1\,^1P_1}(r)\,\frac{dR_{1\,^1D_2}(r)}{dr}\,r^4 dr. \tag{55}$$

III. MULTIPOLE AMPLITUDES AND THE DECAY RATES IN THE PARITY-EVEN TRANSITIONS $1\,^1D_2\to n\,^3S_1+\gamma$ and $1\,^3D_2\to n\,^1S_0+\gamma$ ($n=1,2$)

Following the treatment given previously [5–7], we can write the parity-even transition amplitude of the one-photon transition of quarkonium between spin-singlet and triplet states as

$$T_e(t)=\frac{1}{\sqrt{2V\omega}}\,{}_I\langle A|\,\frac{e_q}{m}\,(\vec{k}\times\hat{\varepsilon}_\alpha)\cdot\vec{\mathcal{S}}\left[1+\frac{k}{2m}-\frac{p^2}{2m^2}-\frac{1}{8}\,(\vec{k}\cdot\vec{r})^2\right]+ik\,\frac{e_q}{4m^2}\,(\vec{k}\cdot\vec{r})\hat{\varepsilon}_\alpha\cdot(\vec{\mathcal{S}}\times\vec{p})$$
$$-\frac{e_q}{4m^2}\,\frac{1}{r}\,\frac{\partial U^{(0)}}{\partial r}\,(\vec{k}\cdot\vec{r})\hat{\varepsilon}_\alpha\cdot(\vec{\mathcal{S}}\times\vec{r})+\frac{e_q}{2m^3}\,(\hat{\varepsilon}_\alpha\cdot\vec{p})\vec{k}\cdot(\vec{\mathcal{S}}\times\vec{p})|B\rangle_I\int_0^t e^{i(\omega-\omega_{AB})t'}dt', \tag{56}$$

where

$$\vec{\mathcal{S}}=\vec{s}_1-\vec{s}_2 \tag{57}$$

and $\vec{s}_1$ and $\vec{s}_2$ are the spin operators of the quark and antiquark in the center-of-mass frame where the total momentum $\vec{P}$ is zero. In Eq. (56) we have only retained terms proportional to $\vec{\mathcal{S}}=\vec{s}_1-\vec{s}_2$ as they alone can connect the spin singlet and triplet states. The only nonrelativistic term in the transition operator of Eq. (56) is the term involving ''1'' in the first term. Obviously, this term does not contribute between the D and S states since the spatial wave functions are orthogonal. But if there is mixing between $^3S_1$ and $^3D_1$ states, it can contribute. All the other terms in the transition operator are of relative order v^2/c^2. So to the extent we can neglect the mixing between $^3S_1$ and $^3D_1$ states, all the nonvanishing multipole amplitudes $M1$, $E2$, and $M3$ in the transition $1\,^1D_2\rightarrow n\,^3S_1+\gamma$ $(n=1,2)$ are of relative order v^2/c^2. Also, the $E2$ amplitudes in $1\,^3D_2\rightarrow n\,^1S_0+\gamma$ $(n=1,2)$ are also of order v^2/c^2.

A. $1\,^1D_2^{(\nu)}\rightarrow n\,^3S_1^{(\sigma)}+\gamma(\mu)$ $(n=1,2)$

As before, let us assume that the $1\,^1D_2$ state is formed at rest in $\overline{p}\overline{p}$ collisions. Let the $n\,^3S_1$ state (ψ or ψ') and the photon γ be emitted in the $+z$ and $-z$ directions, respectively. The component of the angular momentum of $^1D_2$ in the direction of z is called ν. The helicities if $n\,^3S_1$ and γ are called σ and μ, respectively. The matrix element of the transition amplitude of the process $1\,^1D_2(\nu)\rightarrow n\,^3S_1(\sigma)+\gamma(\mu)$ in Eq. (56) in this kinematical configuration is called $A_{\sigma\mu}$, where $\sigma=+1,0,-1$ and $\mu=\pm1$. Equation (56) in this kinematical configuration can be written as

$$T_{e,z}(t)=\frac{e_q}{\sqrt{2V\omega}}\,A'_{\sigma\mu}\int_0^t e^{i(\omega-\omega_{AB})t'}dt'. \tag{58}$$

If charmonium in the $n\,^3S_1$ state is moving in an arbitrary direction (θ,ϕ) with momentum $\vec{p}(\theta,\phi)$ and γ in the opposite direction with momentum $-\vec{p}$, the transition amplitude in this general case will then become

$$T_e(t)=\frac{e_q}{\sqrt{2V\omega}}\,A'_{\sigma\mu}D^{2*}_{\nu,\sigma-\mu}(\phi,\theta,-\phi)\int_0^t e^{i(\omega-\omega_{AB})t'}dt'. \tag{59}$$

By angular momentum conservation,

$$\nu=\sigma-\mu, \tag{60}$$

where

$$\sigma=+1,0,-1,\quad \mu=\pm1. \tag{61}$$

Because of parity invariance,

$$A'_{\sigma\mu}=A'_{-\sigma,-\mu}. \tag{62}$$

We will call the three independent helicity amplitudes A'_2, A'_1, and A'_0:

$$A'_2=A'_{1-1}=+A'_{-11},$$
$$A'_1=A'_{0-1}=+A'_{01},$$
$$A'_0=A'_{-1-1}=+A'_{11}. \tag{63}$$

These helicity amplitudes are related to $M1$, $E2$, and $M3$ multipole amplitudes a'_1, a'_2, and a'_3 by the orthogonal transformation

$$A'_\nu=\sum_{k=1}^{3}a'_k\left(\frac{2k+1}{3}\right)^{1/2}\langle 1,\nu-1|k-1;2\,\nu\rangle$$
$$=\sum_{k=1}^{3}(-1)^{2+\nu}a'_k\langle k1|1,1-\nu;2\,\nu\rangle. \tag{64}$$

As before, since the relation in Eq. (64) is orthogonal,

$$\sum_{\nu=0}^{2}|A'_\nu|^2=\sum_{k=1}^{3}|a'_k|^2. \tag{65}$$

In any potential model, to order v^2/c^2, we have an explicit expression for the helicity amplitude, which can be obtained from Eq. (56) by putting

$$\vec{k}=-k\hat{z}\quad\text{and}\quad\hat{\varepsilon}_\alpha=\hat{\varepsilon}_{-1}=-\frac{1}{\sqrt{2}}\,(\hat{x}-i\hat{y}), \tag{66}$$

$$A'_\nu = A'_{\nu-1,-1} = \langle 1,\nu-1| \frac{ie_q k}{\sqrt{2}m} (\mathcal{S}_x - i\mathcal{S}_y)\left(1+\frac{k}{2m}-\frac{p^2}{2m^2}-\frac{1}{8}k^2z^2\right) + \frac{ik^2}{\sqrt{2}}\frac{e_q}{4m^2} z[(\vec{\mathcal{S}}\times\vec{p})_x - i(\vec{\mathcal{S}}\times\vec{p})_y]$$
$$-\frac{e_q k}{4\sqrt{2}m^2}\frac{1}{r}\frac{\partial U^{(0)}}{\partial r} z\{(\vec{\mathcal{S}}\times\vec{r})_x - i(\vec{\mathcal{S}}\times\vec{r})_y\} + \frac{e_q k}{2\sqrt{2}m^3}(p_x - ip_y)(\vec{\mathcal{S}}\times\vec{p})_z|2\nu\rangle. \quad (67)$$

In order to write the helicity amplitude in Eq. (67) in terms of the multipole amplitudes using Eq. (64), we will express the operators in the matrix element of Eq. (67) in terms of the irreducible spherical tensor components [10]. For this purpose we define the following highest weight spherical tensor components:

$$\Pi_{22} = p_+(\vec{\mathcal{S}}\times\vec{p})_+, \quad Q_{22} = x_+(\vec{\mathcal{S}}\times\vec{r})_+,$$
$$M_{11} = \left(1+\frac{k}{2m}-\frac{p^2}{2m^2}\right)\mathcal{S}_+, \quad Q_{33} = x_+x_+\mathcal{S}_+, \quad L_{22} = x_+(\vec{\mathcal{S}}\times\vec{p})_+,$$
$$L_{11} = \{(\vec{\mathcal{S}}\times\vec{p})\times\vec{x}\}_+, \quad Q'_{11} = r^2\mathcal{S}_+, \quad Q_{11} = [(\vec{\mathcal{S}}\times\vec{q})\times\vec{q}]_+. \quad (68)$$

The other components of the above eight spherical tensor operators can be obtained by means of Eq. (23). In terms of these spherical tensor components, the helicity amplitude of Eq. (67) takes the form

$$A'_\nu = \langle 1,\nu-1| \frac{ik}{m}M_{1-1} + \frac{ik^3}{40m}Q_{1-1} - \frac{ik^3}{20m}Q'_{1-1} + \frac{k^2}{8\sqrt{2}m^2}L_{1-1} + \frac{ik}{8\sqrt{2}m^2}\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_{1-1} - \frac{ik}{4\sqrt{2}m^3}\Pi_{1-1}|2\nu\rangle$$
$$+\langle 1,\nu-1| \frac{k}{2\sqrt{2}m^3}\Pi_{2-1} - \frac{k}{4\sqrt{2}m^2}\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_{2-1} - \frac{\sqrt{2}k^3}{24m}Q_{2-1} + \frac{ik^2}{4\sqrt{2}m^2}L_{2-1}|2\nu\rangle + \langle 1,\nu-1| \frac{-ik^3}{4\sqrt{15}m}Q_{3-1}|2\nu\rangle. \quad (69)$$

Using the Wigner-Eckart theorem

$$\langle 1,\nu-1|R_{k-1}|2\nu\rangle = \langle 1,\nu-1|k,-1;2\nu\rangle\langle 1\|R_k\|^2\rangle, \quad (70)$$

we can write Eq. (69) as

$$A'_\nu = \sum_{k=1}^{3}\langle 1,\nu-1|k,-1;2\nu\rangle\langle 1\|R_k\|2\rangle, \quad (71)$$

where R_k is the appropriate symbol for each k, which can be read off from Eq. (69). Comparing Eq. (71) with Eq. (64), we obtain

$$a'_k = \left(\frac{3}{2k+1}\right)^{1/2}\langle 1\|R_k\|2\rangle, \quad k=1,2,3. \quad (72)$$

We have the following explicit expressions for the three multipole amplitudes $M1$, $E2$, and $M3$ or a'_1, a'_2, and a'_3:

$$a'_1 = \left\langle 1\middle\| \frac{ik}{m}M_1 + \frac{ik^3}{40m}Q_1 - \frac{ik^3}{20m}Q'_1 + \frac{k^2}{8\sqrt{2}m^2}L_1 + \frac{ik}{8\sqrt{2}m^2}\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_1 - \frac{ik}{4\sqrt{2}m^3}\Pi_1\middle\| 2\right\rangle,$$

$$a'_2 = \sqrt{\frac{3}{5}}\left\langle 1\middle\| \frac{k}{2\sqrt{2}m^3}\Pi_2 - \frac{k}{4\sqrt{2}m^2}\frac{1}{r}\frac{\partial U^{(0)}}{\partial r}Q_2 - \frac{\sqrt{2}k^3}{24m}Q_2 + \frac{ik^2}{4\sqrt{2}m^2}L_2\middle\| 2\right\rangle,$$

$$a'_3 = \sqrt{\frac{3}{7}}\left\langle 1\middle\| -\frac{ik^3}{4\sqrt{15}m}Q_3\middle\| 2\right\rangle. \quad (73)$$

By explicitly calculating the matrix elements of specific spherical tensor components using explicit wave functions for the $1\,{}^1D_{2\nu}$ and $n\,{}^3S_{1\sigma}$ states and then using the Wigner-Eckart theorem, we can evaluate the different reduced matrix elements in terms of integrals involving the radial wave functions of the $1\,{}^1D_2$ and the $n\,{}^3S_1$ states. In this way we obtain

$$a_1'=-\frac{ik}{12m}(J_1+J_2+J_3+J_4-12bJ_0),$$

$$a_2'=-\frac{ik}{2m}\left(\frac{5}{6\sqrt{3}}J_1+\frac{1}{\sqrt{6}}J_2-\frac{1}{\sqrt{10}}J_3+\frac{1}{\sqrt{10}}J_4\right),$$

$$a_3'=-\frac{ik}{6m}J_1, \tag{74}$$

where b is the mixing coefficient between the $n\ {}^3S_1$ and $1\ {}^3D_1$ states in the physical states of charmonium, which are identified as $n\ {}^3S_1$ states. The dimensionless integrals J_i ($i=0,1,\ldots,4$) are given by the expressions

$$J_0=\int_0^\infty R_{1\,{}^3D_1}(r)R_{1\,{}^1D_2}(r)r^2dr,$$

$$J_1=\frac{\sqrt{2}}{10}k^2\int_0^\infty R_{n\,{}^3S_1}(r)R_{1\,{}^1D_2}(r)r^4dr,$$

$$J_2=\frac{k}{2m}\int_0^\infty \frac{dR_{n\,{}^3S_1}(r)}{dr}R_{1\,{}^1D_2}(r)r^3dr,$$

$$J_3=-\frac{1}{m^2}\int_0^\infty\left(\frac{d^2R_{n\,{}^3S_1}(r)}{dr^2}-\frac{1}{r}\frac{dR_{n\,{}^3S_1}(r)}{dr}\right)R_{1\,{}^1D_2}(r)r^2dr,$$

$$J_4=\frac{1}{2m}\int_0^\infty R_{n\,{}^3S_1}(r)\frac{\partial U^{(0)}}{\partial r}R_{1\,{}^1D_2}(r)r^3dr. \tag{75}$$

Since the radial integral J_0 is quite large compared to other J_i, even a small mixing between $n\ {}^3S_1$ and $1\ {}^3D_1$ can lead to a significant contribution from J_0 to the $M1$ amplitude a_1'.

Next, we derive the expression for the decay rate. Using Eq. (59), the probability of transition after time t to the specific final helicity states (σ,μ) is given by the expression

$$P_{\sigma\mu}(t)=|T_e(t)|^2=\frac{e_q^2}{2V\omega}|A'_{\sigma\mu}|^2|D^2_{\nu,\sigma-\mu}(\phi,\theta,-\phi)|^2\times\left|\int_0^t e^{i(\omega-\omega_{AB})t'}dt'\right|^2. \tag{76}$$

The transition probability (after time t) integrated over all the directions of the outgoing photon and over a small but finite energy spread of the photon and summed over the final helicity states σ and μ is

$$P(t)=\frac{e_q^2}{2V}\int\left(\sum_{\sigma\mu}|A'_{\sigma\mu}|^2\right)|D^2_{\nu,\sigma-\mu}(\phi,\theta,-\phi)|^2\frac{1}{\omega}\frac{\sin^2[(\omega-\omega_{AB})/2]t}{[(\omega-\omega_{AB})/2]^2}\frac{V}{(2\pi)^3}\omega^2d\omega\, d\Omega$$
$$=\frac{e_q^2}{2}\left(\frac{4\pi}{5}\right)\frac{1}{(2\pi)^3}\sum_{\sigma\mu}|A'_{\sigma\mu}|^2\int\frac{\sin^2[(\omega-\omega_{AB})/2]t}{[(\omega-\omega_{AB})/2]^2}\omega\, d\omega=e_q^2\left(\frac{4\pi}{10}\right)\frac{1}{(2\pi)^3}2\sum_{\sigma\mu}|A'_{\sigma\mu}|^2\int\frac{\sin^2(xt)}{x^2}(2x+\omega_{AB})dx$$
$$=e_q^2\left(\frac{4\pi}{5}\right)\frac{1}{(2\pi)^3}2\sum_{\nu}|A'_\nu|^2\pi t(\omega_{AB})=\frac{e_q^2}{5\pi}\sum_k|a_k'|^2(\omega_{AB})t. \tag{77}$$

So the decay rate or the transition probability per unit time is given by the expression

$$W(1\ {}^1D_2\to n\ {}^3S_1+\gamma)=\frac{P(t)}{t}=\frac{4}{5}\left(\frac{e_q}{e}\right)^2\alpha\omega_{AB}\sum_k|a_k'|^2, \tag{78}$$

where $\alpha=e^2/4\pi$ is the fine-structure constant. Using Eq. (74), this becomes

$$W(1\ {}^1D_2\to n\ {}^3S_1+\gamma)$$
$$=\frac{1}{90}\left(\frac{e_q}{e}\right)^2\alpha\omega_{AB}\left(\frac{k}{m}\right)^2\left[\frac{1}{2}|J_1+J_2+J_3+J_4-12bJ_0|^2+18\left|\frac{5}{6\sqrt{3}}J_1+\frac{1}{\sqrt{6}}J_2-\frac{1}{\sqrt{10}}J_3+\frac{1}{\sqrt{10}}J_4\right|^2+2|J_1|^2\right]. \tag{79}$$

In Eq. (79), k is the same as ω_{AB}.

B. $1\ {}^3D_2\to n\ {}^1S_0+\gamma$

By angular momentum and parity conservation, there is only one radiative multipole amplitude in this case, namely, the $E2$ amplitude. We will call it a_2':

$$a_2'=-\frac{1}{\sqrt{5}}\frac{ik}{2m}\left[\frac{5}{6}J_1'-\frac{1}{3\sqrt{2}}J_2'-\frac{1}{\sqrt{2}}J_3'+\frac{1}{\sqrt{2}}J_4'\right], \tag{80}$$

where the dimensionless integrals J_i' ($i=1,2,3,4$) can be obtained from Eq. (75) with the replacements $R_{n\,{}^1S_1}\to R_{n\,{}^1S_0}$ and $R_{1\,{}^1D_2}\to R_{1\,{}^3D_2}$. The decay rate is given by the formula

$$W(1\ {}^3D_2\to n\ {}^1S_0+\gamma)$$
$$=\frac{4}{5}\left(\frac{e_q}{e}\right)^2\alpha\omega_{AB}|a_2'|^2$$
$$=\frac{1}{25}\left(\frac{e_q}{e}\right)^2\alpha\omega_{AB}\left(\frac{k}{m}\right)^2\times\left|\frac{5}{6}J_1'-\frac{1}{3\sqrt{2}}J_2'-\frac{1}{\sqrt{2}}J_3'+\frac{1}{\sqrt{2}}J_4'\right|^2. \tag{81}$$

In Eq. (81), k is the same as ω_{AB}, the photon energy.

IV. NUMERICAL EVALUATION OF THE MULTIPOLE AMPLITUDES AND THE DECAY RATES IN THE NONSINGULAR POTENTIAL MODEL OF GUPTA, JOHNSON, REPKO, AND SUCHYTA

The nonsingular potential model of Gupta, Johnson, Repko, and Suchyta (GJRS) has been quite successful in predicting the energy spectrum of charmonium and the $E1$ decay rates of ψ' and the χ_J states. The nonsingular nature of the potential makes it possible to obtain the wave functions by a nonperturbative treatment. This is important for the evaluation of multipole transition rates, since in these calculations the relativistic corrections to the wave functions are usually more significant than that to the transition operators except in the case when the nonrelativistic transition operator is zero. The details of the model and the specific form of the Hamiltonian used is given in Ref. [3]. We solved for the eigenvalue problem of the full Hamiltonian, except for the tensor term, by a variational calculation, with the trial radial wave function as

$$R(r)=\sum_{n=0}^{9} c_n\left(\frac{r}{R}\right)^{n+\ell} e^{-r/R}, \tag{82}$$

where ℓ is the orbital quantum number. The coefficients c_n and the parameter R are variational parameters. The parameter R was determined by satisfying the virial theorem

$$\langle \vec{p}\cdot\vec{\nabla}_p H\rangle=\langle \vec{r}\cdot\vec{\nabla} H\rangle. \tag{83}$$

The coefficients c_n's were determined by minimizing the energy in the appropriate way. We used the same values for the parameters in the Hamiltonian as GJRS [3]. We were able to reproduce more or less their predicted energy spectrum, their values for the $E1$ decay rates of ψ' and $\chi_J(J=2,1,0)$, and the leptonic decay rates of ψ and ψ'. Our predicted results for the multipole amplitudes and the decay rates for the decays $1\,^3D_2\rightarrow 1\,^3P_J+\gamma$ $(J=2,1,0)$, $1\,^3D_2\rightarrow n\,^1S_0+\gamma$ $(n=1,2)$, $1\,^1D_2\rightarrow 1\,^1P_1+\gamma$, $1\,^1D_2\rightarrow n\,^3S_1+\gamma$ $(n=1,2)$, and $1\,^3D_1\rightarrow 1\,^3P_J+\gamma$ $(J=2,1,0)$ are given in Tables I–V, respectively. The predicted energy spectrum, including those of the D states, are given in Table VI. In Tables I–IV we also give the relative strengths of the higher multipole amplitudes compared to the predominant $E1$ amplitude, which is nonzero even in the extreme nonrelativistic limit.

In calculating the photon energies in the tables, we used the experimental values of the masses of the states whenever they are available [12]. When the masses are not known experimentally we used the predicted values in the GJRS model [3].

V. SUMMARY AND CONCLUDING REMARKS

If the potential models are at least approximately right in predicting the energy spectrum of charmonium, the $1\,^1D_2$ and $1\,^3D_2$ states should have a narrow width. If and when they are detected as resonances in $\overline{pp}$ collisions, from their branching ratios and angular distributions of decay products we can measure the decay rates and the multipole amplitudes of the various radiative transitions. In previous works [1,2] one of the authors (K.J.S.) together with Ridener has shown how one can obtain the radiative multipole amplitudes from the experimentally measurable angular distributions. In this paper we have derived expressions for the decay rates and the multipole amplitudes which are valid in any potential model and which are correct to relative order v^2/c^2. In potential models the higher multipole amplitudes are of order v^2/c^2 relative to the $E1$ amplitudes and hence are expected to be smaller. In particular, the $M4$ amplitude in the decay $1\,^3D_2\rightarrow 1\,^3P_2+\gamma$ should vanish in any potential model correct to first order in v^2/c^2. This result is due to the fact that to this order there is no fourth rank tensor component in the transition operator.

TABLE VI. Predicted energy levels of charmonium in the GJRS potential model [3]. Parameters used are $m_c=2.208$ GeV, $\mu=2.58$ GeV, $\alpha_s=0.313$, $A=0.181$ GeV^2, and $B=0.245$. The experimental data are taken from Ref. [12].

State	Predicted energy (MeV)	Experiment data (MeV)
$1\,^3S_1$	3097	3096.88±0.04
$1\,^1S_0$	2987	2978.8±1.9[a]
$2\,^3S_1$	3686	3686.00±0.09
$2\,^1S_0$	3620	3594.0±5.0
$1\,^3P_2$	3554	3556.17±0.13
$1\,^3P_1$	3512	3510.53±0.12
$1\,^3P_0$	3412	3415.1±1.0
$1\,^1P_1$	3527	3526.14±0.24
$1\,^3D_3$	3843	
$1\,^3D_2$	3819	
$1\,^3D_1$	3789	3769.9 ± 2.5
$1\,^1D_2$	3820	

[a]New result announced by the E760 group [13] is 2987.9±3.1 MeV.

We have also numerically evaluated our expressions in the successful nonsingular potential model of Gupta, Johnson, Repko, and Suchyta [3]. Our results are tabulated in Tables I–V. From Tables I–V we find that the most prominent decays of $1\,^3D_2$ and $1\,^1D_2$ are the one-photon transitions into $1\,^3P_J$ and $1\,^1P_1$ states respectively. The transitions into the $1\,^1S_0$ and $1\,^3S_1$ states also have significant rates. Especially the transition of $1\,^3D_2$ into $1\,^1S_0$ is worth noting. It is a pure $E2$ transition which has a rather large rate and hence a significant branching ratio.

We have also corrected some errors in the decay-rate formulas of Refs. [5] and [11]. Our numerical estimate for the transition rate for the decay $1\,^1D_2\rightarrow 1\,^3S_1+\gamma$ is now significantly smaller than the value of about 62 keV we reported in Ref. [5].

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