Trans-Planckian tail in a theory with a cutoff

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Trans-Planckian frequencies can be mimicked outside a black-hole horizon as a tail of an exponentially large amplitude wave that is mostly hidden behind the horizon. The present proposal requires implementing a final state condition. This condition involves only frequencies below the cutoff scale. It may be interpreted as a condition on the singularity. Despite the introduction of the cutoff, the Hawking radiation is restored for static observers. Freely falling observers see empty space outside the horizon, but are ''heated'' as they cross the horizon. [S0556-2821(97)06504-1]

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I. INTRODUCTION

The standard derivation of the Hawking radiation [1] requires the existence of exponentially high frequency modes in a classical space-time background. Indeed, after a short time (of order $t\sim M\ln M$), the required frequency becomes $\omega\sim 1$ in Planck units. For this reason the standard derivation cannot be trusted after few Hawking photons were already emitted. It appears that in any derivation of Hawking radiation, with no new physical ingredients, a naive short distance cutoff will eliminate the Hawking effect [2]. It is, of course, possible that the origin of the Hawking radiation does depend on the behavior of an ultrahigh trans-Planckian spectrum. In this article, however, we suggest an alternative mechanism for generating Hawking's radiation in a theory with an effective cutoff.

Evidence that a theory with a cutoff may reproduce the Hawking radiation has been recently provided by Unruh's work [3]. Unruh has shown that a natural cutoff still gives rise to the Hawking radiation in the case of sonic black holes [4]. In his approach the cutoff modifies the dispersion relation for sound waves. This, in turn, alters the motion of modes with frequency close to the cutoff scale and gives rise to a new type of trajectories which approach the horizon, but eventually ''reflect'' back to infinity. Further works tried to adapt Unruh's model to real black holes [5–7]. It is not clear that a similar process is indeed realized for real black holes.

In this article we present another possibility. It is shown that even without modifying the ordinary field equations and the ensuing dispersion relations, as in the above proposals, one can still restore the Hawking radiation in a theory with a cutoff. In the present approach the Hawking radiation is generated by an apparent trans-Planckian tail outside the black-hole horizon. The source of this tail is an exponentially large wave that is mostly hidden behind the black-hole horizon. To develop this picture we shall use two new key ingredients: (I) *Ultrahigh frequency modes can be mimicked to arbitrary accuracy in a bounded region even with a finite band spectrum.* The basic idea was discovered by Aharonov *et al.* [8] and was further developed by Berry [9], who coined the term ''superoscillations'' to describe such a behavior. A simple example of a function $F(t)$ which exhibits superoscillations was given in [8]:

$$F(t;N,\omega^*)=\left[\left(\frac{1-\omega^*/\omega_0}{2}\right)e^{it\omega_0/N}+\left(\frac{1+\omega^*/\omega_0}{2}\right)e^{-it\omega_0/N}\right]^N. \tag{1}$$

Here, $N>1$ is an integer, and ω^* and ω_0 being the super and reference frequencies. For small t we expand $\exp(it\omega_0/N)$ and find

$$F(t;N,\omega^*)=\left[e^{-i\omega^*t/N}+\frac{(\omega^{*2}-\omega_0^2)t^2}{2N^2}+O(N^{-3})\right]^N$$
$$=e^{-i\omega^*t}\left[1+\frac{(\omega^{*2}-\omega_0^2)t^2}{2N}+O(N^{-2})\right]\cong e^{-i\omega^*t}. \tag{2}$$

Although the spectrum of Eq. (1) includes only modes with frequencies $\omega\in(-\omega_0,+\omega_0)$, in the time interval $|t|\ll\sqrt{N}/\sqrt{\omega^{*2}-\omega_0^2}\equiv T$, $F(t)$ behaves as a wave with arbitrary large frequency ω^*. The number of superoscillations in this interval is $\sim\sqrt{N}$. Systems that interact with the wave F only during $|t|\ll T$ will not distinguish between F and a pure wave $e^{-i\omega^*t}$ that extends for all times.

This remarkable feature is derived at the expense of having such functions grow exponentially in other regions. In the example above, for $|t|>T$, we get $F\sim e^N$. Nevertheless, as we shall see, the large amplitudes can be confined to a compact region. In particular, by adapting Berry's integral representation [9], the large amplitudes can be entirely confined to the interior region of a black hole while only a *high frequency tail* remains outside the black-hole horizon. This ''tail'' will be seen by the external observer as the origin of the Hawking radiation. The observer cannot probe the inte-

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rior of the black hole and distinguish between the mimicked tail and a ''truly'' trans-Planckian frequency mode.

If the above function F is viewed as a wave function, then the probability to see a photon coming out from the tail is exponentially small. In order to avoid this, we shall make an additional assumption which is the second basic ingredient: (*II*) *A black hole is described by two conditions: by the ordinary ingoing state and by a final condition.* Under this assumption, the black hole is described in a fashion similar to that of a pre- and postselected system [10–12]. A pre- and postselected ensemble is prepared according to given initial and final conditions. Observations can be then made at some intermediate time between the pre- and postselections, and the probability of the measured results can be expressed as conditional probabilities. However, in our case the final state will be given a more fundamental role. It will not be determined by a postselection done by some fictitious observer in the future, rather it will be conceived as arising from some new fundamental law, which is required by the presence of a singularity in the future.

Such a final condition can be anticipated, for example, in a theory that replaces past or future curvature singularities by smooth initial or final conditions. To some extent, the Hartle-Hawking ansatz for the cosmological wave function [13] can be interpreted as corresponding to initial and final conditions. When the WKB approximation is valid, the Hartle-Hawking wave function is expressed in terms of the action S as

$$\Psi \simeq e^{iS} + e^{-iS}. \tag{3}$$

It is possible to interpret these two terms as two wave functions which travel forward and backward in time, and correspond to conditions in the past or in the future. It is possible that a similar fundamental principle is available also for the case of a black-hole singularity.

Although our assumption *II* above might seem at first radical, we shall show, in Sec. II, that final conditions may be constructed which do not affect low energy observables. Such a final condition will manifest only in very extreme cases. The basic idea, suggested by Aharonov [14], is to implement the condition only on very high frequency modes above some scale ω_h.

In Sec. III we shall construct the special superoscillatory function which mimics a high frequency tail using a bounded spectrum. In Sec. IV we study the response of a stationary detector in a black-hole geometry to a scalar field when a cutoff with respect to Kruskal coordinates was introduced. In Sec. V the two main ingredients, namely, superoscillations and a final condition, are combined for the simple case of an eternal black hole. A cutoff is assumed with respect to the Kruskal coordinates both on the initial Kruskal vacuum state $|O_K\rangle$ and our final state $|f\rangle$. It is shown that these initial and final conditions cause the observer to see Hawking radiation emitted from the black hole. Finally, we conclude with a discussion of our results and remaining difficulties.

II. FINAL CONDITION ON ULTRAHIGH MODES

In this section it is shown how a nontrivial final condition can be imposed without affecting low energy observables. The basic idea [14] is to implement a final condition only on high frequency modes.

Over the last decade, Aharonov and collaborators have elaborated on the two-vector formalism of quantum mechanics. In this formalism one specifies both an initial and a final state and considers measurements done at intermediate time. (For a detailed discussion see Refs. [11,12].) Let the initial and final conditions on a system be that at $t=-\infty$ $(+\infty)$ the field is in the state $|i\rangle$ $(|f\rangle)$. Indeed, one can in ordinary quantum mechanics impose two such conditions. These states are independent, but need to be nonorthogonal.

In the following we shall consider measurements at some intermediate time. Given an observable $A=\Sigma a \Pi_a$, where Π_a are projectors to the eigenstates $|a\rangle$, the probability to measure $A=a$ is given by the conditional probability[1]

$$\mathrm{Prob}(a|f,i) = \frac{\mathrm{Prob}(a,f|i)}{\mathrm{Prob}(f|i)} = \frac{|\langle f|a\rangle\langle a|i\rangle|^2}{\Sigma_{a'}|\langle f|a'\rangle\langle a'|i\rangle|^2}. \tag{4}$$

For certain nontrivial final conditions, low energy laboratory experiments will not depend on the final condition. In the example considered here, a final condition is imposed only on the high energy sector, i.e., only for $\omega > \omega_h$, where ω_h is some high energy scale.

To spell out this proposal, let us, for simplicity, consider a free massless scalar field theory in Minkowski space-time, and assume that in a certain rest frame the final state of the field has the form

$$f_{(L,F)}\rangle = \frac{1}{\sqrt{1+\xi^2}}(|L\rangle + \xi|F\rangle), \tag{5}$$

where $|L\rangle$ and $|F\rangle$ are two normalized states in Fock space, and ξ controls the relative probability. The first component $|L\rangle$ denotes a low energy ''laboratory state,'' which contains only particles of low frequency:

$$|L\rangle = \left(1 + \sum_{\omega_k<\omega_h} C_k(L) a^\dagger_{\omega_k} + \sum_{\omega_k,\omega_l<\omega_h} D_{kl}(L) a^\dagger_{\omega_k} a^\dagger_{\omega_l} + \cdots \right)|0_M\rangle. \tag{6}$$

The second term $|F\rangle$ denotes a certain state of particles with frequencies above ω_h:

$$|F\rangle = \left(\sum_{\omega_k>\omega_h} C_k(F) a^\dagger_{\omega_k} + \sum_{\omega_k,\omega_l>\omega_h} D_{kl}(F) a^\dagger_{\omega_k} a^\dagger_{\omega_l} + \cdots \right)|0_M\rangle. \tag{7}$$

We shall demand that the final state always has the form given in Eq. (5), and is *constrained* always to include the same high energy state $|F\rangle$. We shall not constrain the content of the low energy state $|L\rangle$. [In terms of the pre- and postselection terminology, this corresponds to postselection

[1]Clearly, Eq. (4) is different from $|\langle a|i\rangle|^2$, the probability obtained if only the initial state is fixed. The latter is obtained from Eq. (4) by further summing over f.

of an ensemble with fixed F, but arbitrary L in the specific combination of Eq. (5) above.]

Since in the final condition the low energy state $|L\rangle$ is left unspecified, we need to modify Eq. (4) accordingly. The probability to find $A=a$ is now given by further summing over a basis of the subspace, $\mathcal{H}_L=\{|L\rangle\}$, of low energy states:

$$\mathrm{Prob}(a|F,i)=\frac{\Sigma_L P(a,L,F|i)}{\Sigma_{L,a'}P(a',L,F|i)}. \quad (8)$$

Thus,

$$\mathrm{Prob}(a|F,i)=\frac{\Sigma_L|\langle f_{(L,F)}|\Pi_a|i\rangle|^2}{\Sigma_{L,a'}|\langle f_{(L,F)}|\Pi_{a'}|i\rangle|^2}. \quad (9)$$

We call Π_a a low energy ''laboratory'' projector if

$$\xi||\langle L|\Pi_a|F\rangle||<\epsilon, \quad \forall L\in\mathcal{H}_L, \quad (10)$$

where ϵ is some small number. If Eq. (10) is satisfied for every eigenvalue of the operator A, then A will be termed a low energy laboratory observable. If the initial state is taken to be one of the low energy states, i.e., $i\in\mathcal{H}_L$, then for a low energy observable, Eq. (9) reduces to

$$\mathrm{Prob}(a|F,i)=\frac{\Sigma_L|\langle L|\Pi_a|i\rangle+O(\epsilon)|^2}{\Sigma_{L,a'}|\langle L|\Pi'_a|i\rangle+O(\epsilon)|^2}=\frac{\langle i|\Pi_a|i\rangle}{\langle i|i\rangle}+O(\epsilon), \quad (11)$$

the ordinary F-independent expression.

Nevertheless, if the initial state does contain states with $\omega>\omega_h$, or when the condition (10) is not satisfied, the full expression (9) must be used, and the probability generally depends on $|F\rangle$.

Although we have seen that expectation values for low energy laboratory observables reduce to the ordinary expression, it is possible that the fluctuations of the field are still sensitive to the condition F. To investigate this question let us consider the case of continuous measurements at intermediate times. In particular, let us consider the interaction of a particle detector with the field. This example will be useful in the following sections as well.

A particle detector [15,16] can be described as a two-level system with an energy gap Ω. The detector is coupled to a scalar field $\phi(x,t)$ via the action

$$S_I=\lambda\int d\tau dx(A+A^\dagger)\phi(x,t)\delta(x-X_D). \quad (12)$$

Here, τ is the proper time in the rest frame of the detector, and $X_D(t)$ is the classical trajectory of the detector. $A,A^\dagger$ act on the two internal states $|\pm\rangle$ according to

$$A^\dagger|-\rangle=|+\rangle, \quad A|+\rangle=|-\rangle,$$
$$A^\dagger|+\rangle=0, \quad A|-\rangle=0. \quad (13)$$

A detection of a particle will be described as a transition from the ground state to the excited state. In the limit of small coupling constant, we shall be interested in obtaining the transition amplitude, computed to the first order in λ.

With the final condition, the transition probability is given by

$$T(+|-,F,i)=\frac{\Sigma_L|\langle +,f_{(L,F)}|U_I|-,i\rangle|^2}{\Sigma_{\pm,L}|\langle \pm,f_{(L,F)}|U_I|-,i\rangle|^2}, \quad (14)$$

where $\Sigma_\pm$ denotes a summation over the final internal states, and $U_I=\exp(-i\lambda\int L_I d\tau)$ is the unitary evolution operator in the interaction picture.

To lowest order in the coupling constant, we get

$$T(+|-,F,i)$$
$$=\lambda^2\frac{\Sigma_L|\langle +,f_{(L,F)}|\int L_I d\tau|-,i\rangle|^2}{\Sigma_L(|\langle f_{(L,F)}|i\rangle|^2+\lambda^2|\langle +,f_{(L,F)}|\int L_I d\tau|-,i\rangle|^2)}$$
$$=\lambda^2\sum_L|\langle +|\otimes(\langle L|+\xi\langle F|)\int L_I d\tau|-,i\rangle|^2+O(\lambda^4), \quad (15)$$

where in passing to the last line we used the identity $\langle F|i\rangle=0$.

The transition probability obtained in Eq. (15) has the ordinary form, except that now it contains the additional component $\lambda^2\Sigma_L|A_F|^2$, where

$$A_F=\xi\int d\tau\langle +,F|L_I|-,i\rangle. \quad (16)$$

When A_F vanishes, Eq. (15) reduces to the ordinary transition probability.

Let us now consider the new amplitude A_F. Using the representation (7) for $|F\rangle$, and neglecting possible multiparticle contributions, we obtain

$$A_F=\xi\sum_{\omega_k}\frac{C_k^*(F)}{\sqrt{4\pi\omega_k}}\int_{-\tau_0}^{\tau_0}d\tau\exp(i\Omega\tau)$$
$$\times\exp\{i\omega_k[t(\tau)-kX_D(\tau)]\}$$
$$\equiv\sum_{\omega_k}A_F(k). \quad (17)$$

For an inertial detector, $t=\tau/\sqrt{1-V_D^2}$, we find that

$$A_F(k)=\xi\frac{C_k^*(F)}{\sqrt{4\pi\omega_k}}\frac{\sin\{[\Omega+(\omega_k-C_Dk)/\sqrt{1-V_D^2}]\tau_0\}}{[\Omega+(\omega_n-V_Dk)/\sqrt{1-V_D^2}]\tau_0}. \quad (18)$$

The last equation reduces to $\delta[\Omega+(\omega_n-V_Dk)/\sqrt{1-V_D^2}]=0$ only when $\tau_0\gg 1/\omega_k>1/\omega_h$, and ξ is finite, say $\xi\sim 1$. This means that as long as the relative amplitude ξ of $|F\rangle$ is not large, the fluctuations are averaged out to zero after a time which is determined by $1/\omega_h$. Intuitively, this seems natural. An interaction on a time scale shorter than $1/\omega_h$ involves energy fluctuations of order $\sim\omega_h$, which in turn depend on the condition F.

Anticipating the discussion in Sec. V, let us also consider the large ξ case. By insisting that Eq. (10) is satisfied, we find that the expectation values (11) are still unmodified. Nevertheless, the fluctuating A_F seen by a particle detector are not negligible. In this case, to average out such fluctuations, we will need times $\tau_0 \gg \xi/\omega_h$. Otherwise, our detector will observe particles which are not present in the initial state $|i\rangle$.

Finally, we note that the above considerations can be easily extended to the case of a final mixed state. The analogue of the state $|f_{(L,F)}\rangle$ is given by the density matrix

$$\rho_f = |L\rangle\langle L| + \rho_F, \tag{19}$$

where ρ_F is constructed from states with frequency $\omega > \omega_h$.

III. ULTRAHIGH FREQUENCY FROM A BOUNDED SPECTRUM

The other key element I of our approach is the use of superoscillatory functions, alluded to in the introduction. These functions, having only a bounded Fourier spectrum, can still mimic an arbitrarily high frequency though in a finite region.

In the Fourier representation of such a function,

$$\Phi(u) = \int_0^1 d\omega C_\omega \exp(i\omega u), \tag{20}$$

the trick is to choose certain coefficients C_ω, such that at a finite interval of u, Φ exhibits rapid oscillations with a frequency $\omega^* \gg 1$.

As superoscillations necessitate large amplitudes at other regions, our purpose is to find a representation in which these large oscillations are confined to a bounded region of u. To construct such a function, we will use a variant of an integral representation for a superoscillatory function that was found by Berry [9]. Consider the function:

$$\Phi_{A,\Delta}(u) = \frac{1}{\sqrt{2\pi\Delta^2}} \int_0^{2\pi} d\alpha \exp\left(\frac{i}{\Delta^2}\cos(\alpha - iA)\right) e^{i\cos\alpha u}, \tag{21}$$

where A and Δ are real parameters. The modes of $\Phi_{A,\Delta}(u)$ are bounded by $|\omega| = |\cos\alpha| < 1$.

The integration above can be analytically performed to yield

$$\Phi_{A,\Delta}(u) = \sqrt{2\pi/\Delta^2} I_0\left(\frac{i}{\Delta^2}\sqrt{1 + 2\cosh(A)\Delta^2 u + \Delta^4 u^2}\right), \tag{22}$$

where I_0 is the zeroth modified Bessel function.

Expanding Eq. (22) around $u=0$, we note that $\Phi_{A,\Delta}(u)$ behaves as

$$\Phi_{A,\Delta}(u) = \exp[i\cosh(A)u]. \tag{23}$$

$\Phi_{A,\Delta}(u)$ ''superoscillates'' with frequency $\omega^* = \cosh A$. This expansion is valid in a region $|u| < [\cosh(A)\Delta^2]^{-1} \equiv \delta u$. Thus, the parameter Δ controls the number n_S of superoscillations: $n_S \sim \delta u/\omega^* = 1/\Delta^2$.

By modifying the two parameters A and Δ, we can control the frequency and number of superoscillations. However, the limits $A \to \infty$ or $\Delta \to 0$ are singular. Outside the region δu, where the function superoscillates, Φ grows exponentially. Φ gets its maximal value at $u = -\cosh A/\Delta^2$, where the amplitude grows to

$$\Phi \sim \exp(\cosh A/\Delta^2). \tag{24}$$

The superoscillations are hence found at the tails of an exponentially high peak.

Far away from the region of superoscillations, for $u \gg \delta u$, Eq. (22) reduces to a low frequency wave:

$$\Phi_{A,\Delta}(u) = \frac{1}{u}\exp(iu). \tag{25}$$

In Sec. V we will show that these properties of Φ allow finding a state $|F\rangle$ that mimics the trans-Planckian Hawking photons close to the horizon. A particle detector will not distinguish between a ''fake'' tail of superoscillations and ''real'' trans-Planckian model. Before proceeding to this final task, it will be useful to reexamine the interaction of a particle detector with a scalar field near a black hole.

IV. PARTICLE DETECTOR IN KRUSKAL GEOMETRY WITH A CUTOFF

In this section we examine the response of a particle detector in the space-time of an eternal black hole. We shall assume that the initial state is the Unruh vacuum [15], but that modes above a certain frequency are cut off. In this section we use usual, only preselected, quantum-mechanics framework. (We shall defer the discussion of an additional final condition to the next section.)

The geometry of an eternal black hole is described in terms of Kruskal coordinates U,V, that are defined via the relations

$$\begin{aligned} ds^2 &= (1-2M/r)dt^2 - (1-2M/r)^{-1}dr^2 - r^2 d\Omega^2 \\ &= (1-2M/r)du\,dv - r^2 d\Omega^2 \\ &= \frac{\exp(-r/4M)}{r/2M} dU\,dV - r^2 d\Omega^2. \end{aligned} \tag{26}$$

Here,

$$u,v = t \pm r^*, \tag{27}$$

where $r^* = r + 2M\ln(r/2M - 1)$, and

$$U = -4M\exp(-u/4M), \quad V = 4M\exp(v/4M). \tag{28}$$

The Unruh vacuum corresponds to an initial state which reproduces Hawking radiation at $\mathcal{I}^+$. It is imposed by selecting the in-vacuum with respect to the Killing vector ∂_U on the past horizon ($V=0$) as follows. Consider a scalar field in the reduced $1+1$ spherical approximation. By the conformal invariance, we have

$$\phi(U,V) = \phi_R(U) + \phi_L(V). \tag{29}$$

In terms of creation and annihilation operators, we have

$$\phi_R(U)=\int \frac{d\omega}{\sqrt{4\pi\omega}}(e^{-i\omega U}a^R_\omega+\text{H.c.}), \tag{30}$$

and a similar expression of the left-moving part ϕ_L.

Now, for the Unruh vacuum $|0_U\rangle$,

$$a^R_\omega|0_U\rangle=0. \tag{31}$$

Here, we furthermore assume that modes with $\omega>\omega_c$ are cut off.

Let us consider now a static particle detector that is located at a constant radius r and interacts with the cutoff vacuum state defined above. The trajectory (r,t) of the detector can be described in terms of the Kruskal coordinates (U,V) as

$$U_D=-(r/2M)^{1/2}l\exp[(-t+r)/4M],$$
$$V_D=(r/2M)^{1/2}l\exp[(-t-r)/4M], \tag{32}$$

where $l\equiv 4M(1-2M/r)^{1/2}$.

Using the interaction (12), and Eqs. (30) and (32), we shall obtain the transition amplitude from an initial vacuum state, an unexcited detector to a final state with excited detector state, and a one-scalar photon of frequency ω:

$$A(+,\omega|-,0_K)=\lambda\int d\tau\langle +,1_\omega|(A+A^\dagger)\phi(U_D)|-,0_U\rangle, \tag{33}$$

where the time coordinate is related to the proper time by $dt=d\tau/(1-2M/r)^{1/2}$. We find that

$$A(+,\omega|-,0_U)=\lambda^2\int \frac{d\tau}{\sqrt{4\pi\omega_k}}e^{i\Omega\tau}\exp(i\omega_k(r/2M)^{1/2} \times l\exp\{[-t(\tau)+r]/4M\}). \tag{34}$$

The total probability of jumping to an exited state is obtained by summation over the final emitted photon states:

$$\text{Prob}(+)=\sum_{\omega<\omega_c}|A(+,\omega)|^2. \tag{35}$$

By inspecting the transition amplitude equation (34), one finds that the integral is dominated by a stationary point at

$$\omega_{s.p}=-4M\exp\{[t(\tau)-r]/4M\}\left(\frac{2M}{r}\right)^{1/2}. \tag{36}$$

Since the maximal frequency is ω_c, after u time of order $t-r\sim 4M\ln(\omega_C/4M)$, this transition amplitude vanishes.

In other words, the emission seen by the detector will come to halt very shortly after it started. This corresponds to the usual result that a cutoff will terminate the Hawking radiation.

V. RESTORING THE HAWKING RADIATION

We shall now show that a particular choice for the final condition gives rise to an effective ultrahigh frequency in the vicinity of the horizon and avoids the above ''extinction'' of the Hawking radiation due to the cutoff.

As we have seen in Sec. II, when a final condition on the high frequency modes is imposed, the transition amplitude (15) contains an extra term (16). The contribution of this term to the transition amplitude for a stationary detector located at radius r is

$$A_F=\xi\int d\tau\langle +,F|L_I|-,0_U\rangle$$
$$=\xi\int d\tau e^{i\Omega\tau}\int d\omega_\alpha\frac{C^*_\alpha(F)}{\sqrt{4\pi\omega_\alpha}}\exp(i\omega_\alpha(r/2M)^{1/2} \times l\exp\{[-t(\tau)+r]/4M\}), \tag{37}$$

where the range of integration of ω_α is the band of high frequency modes in $|F\rangle$. In order to replace the contribution of a trans-Planckian frequency ω^* by a superoscillation, we shall require that

$$\langle +,F_{\omega^*,\Delta}|L_I|-,0_U\rangle=\langle +,\omega=\omega^*|L_I|-,0_U\rangle. \tag{38}$$

That is, the superposition of the amplitude given in Eq. (37) gives rise to a single high frequency mode ω^*.

We shall assume that the final state contains only modes with frequencies $\omega\in(\omega_c,\omega_c-\zeta)$, with ω in M_{PL} units. $\zeta<1$ is some pure number that defines the size of the high energy ''band'' *below* the cutoff scale. Using the representation (21) for superoscillatory functions, we can find the coefficients $C_\alpha(F)$ of the single particle states in Eq. (7). (In this article we shall not construct the coefficients of multiparticle terms.) The result is

$$|F_{\omega^*}\rangle=\int_0^{2\pi}d\alpha\sqrt{\frac{\omega_\alpha}{\omega^*}}\left[\exp\left(\frac{i}{\Delta}\cos(\alpha-iA)\right)\right]a^\dagger_{\omega_\alpha}|0_U\rangle, \tag{39}$$

where

$$\omega_\alpha=\omega_c+\zeta(\cos\alpha+1)/2,\quad \omega^*=\cosh A. \tag{40}$$

Outside the black-hole horizon, the effective transition amplitude (37) (with a sufficiently large number of superoscillations $n_S\sim 1/\Delta$), is precisely identical to that obtained without the cutoff. Therefore, we have shown that by a superposition of final states with $\omega\in(\omega_c,\omega_c-\zeta)$, we can mimic a trans-Planckian frequency $\omega^*=\cosh A\gg\omega_c$.

Freely falling detectors are equivalent to inertial detectors in Minkowski space-time. We have seen in Sec. II, that inertial detectors with a small boost factor will not respond to the final condition. In close analogy, freely falling detectors outside the black hole respond only to low frequency modes. Thus, a freely falling detector outside the black hole effectively interacts only with a normal wave and hence sees the space as mostly empty; precisely as in the ordinary picture. Nevertheless, the standard picture fails at the interior of the black hole. Inside the black hole the effective trans-Planckian tail rises sharply to a tremendous amplitude (24) of $\sim\exp(\omega^*/\Delta^2)$.

Although the expectation values of low energy observable may remain unchanged, the *fluctuations* become exponentially large. This implies that a probe that couples during a finite time to the field will detect particles with high probability. In the standard picture the black hole is basically empty, and nothing extraordinary occurs when an observer crosses the horizon. In our case, we expect that as the probe crosses the horizon, it will immediately start heating up. We see that in this scenario, while outside the black hole the ordinary predictions are respected, new physics is predicted for the region hidden by the horizon.

So far, we have only demonstrated that a single trans-Planckian mode can be restored. To restore the full transition probability (35), we need to mimic the full trans-Planckian spectrum. Since the different frequencies superpose with no interference, we need to choose a final state which is a density matrix:

$$\rho_f = |L\rangle\langle L| + \int_0^{\omega_{\max}} d\omega^* |F_{\omega^*}\rangle\langle F_{\omega^*}|. \tag{41}$$

The first term represents the nonconditioned final low energy state, which allows the restoring of ordinary low energy physics outside the black hole. The second term represents the final condition on high energy states near the cutoff ω_c and is responsible for the Hawking radiation. Strictly speaking, the state $|F_{\omega^*}\rangle$ becomes ill defined in the limit $\omega^* \to \infty$. Yet, during every finite lifetime, we have a well-defined expression. For example, for an isolated black hole that evaporates according to the standard picture during $t \sim M^3$, we will require in Eq. (41) $\omega_{\max} \sim e^{M^2}$.

VI. DISCUSSION

In this article we have presented a novel picture in which the Hawking radiation can be restored in presence of a cutoff without trans-Planckian modes. To this end, we have introduced two new ingredients: superoscillations which can mimic trans-Planckian modes, and a final condition on the field. The final condition F in Eq. (39) complements the ordinary in-state, and was chosen so as to give rise to superoscillations with respect to static observers. On the other hand, in the reference frame of freely falling observers, F corresponds to a condition on very high frequencies and hence does not affect ''low energy'' observables. As discussed in the introduction, in a more complete theory, F should be derived from some fundamental principle (e.g., that suggested in Ref. [13]). Since our framework still lacks this basic principle, in this work we have not provided a full derivation of Hawking's radiation; rather, we have shown that with a certain condition F, Hawking's radiation can be obtained from superoscillations without invoking trans-Planckian modes of the field. Hawking's radiation is therefore restored independently of the particular nature of the cutoff, i.e., the yet unknown short distance structure of quantum gravity.

The qualitative description of Hawking's effect in this formalism can be summarized as follows. The source of the Hawking radiation is a tail of trans-Planckian oscillations near the horizon, that reaches out from a wave with exponentially high amplitudes hidden inside the black hole. Outside the horizon, we may not notice any significant difference from the ordinary picture; freely falling observers see very little radiation, and static observers see the ordinary Hawking radiation. However, this correspondence with the ordinary picture breaks down inside the black hole. The exponentially high amplitudes inside the black hole imply a ''hot'' black-hole interior, i.e., freely falling observers which cross the horizon will immediately heat up. We therefore anticipate large back-reaction effects for the black-hole interior.

We shall conclude with some comments. The final state of the high energy sector near the cutoff energy seems to be a density matrix. A possible explanation for this could be the following. In our approach gravity is essentially treated semiclassically. If indeed the final state is related to the singularity, we cannot expect it to be expressed as a direct product of a matter state and a semiclassical gravity state. Rather, it should be a highly entangled matter-gravity state. Since outside the black hole the semiclassical approximation is valid, the matter state should result from tracing over gravity states. This procedure may lead to a reduced density matrix such as in Eq. (41).

The formalism presented in this paper is far from being complete, and many important questions remain. For example, can F states also mimic the n-point correlation functions? Or, how does the radiation energy transfer from the black hole to the observer in this picture? We hope that some of the features presented in this approach will turn out useful in understanding the enigma of Hawking radiation.

Note added. After the completion of this work I found out that a concept related to ingredient I was suggested also by Rosu [17].

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