Global structure of Robinson-Trautman radiative space-times with cosmological constant

Jiří Bičák*
*Department of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
and Max-Planck-Institute for Gravitational Physics, Schlaatzweg 1, Potsdam, D-14473, Germany*

Jiří Podolský†
Department of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

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Robinson-Trautman radiative space-times of Petrov type II with a nonvanishing cosmological constant Λ and mass parameter $m>0$ are studied using analytical methods. They are shown to approach the corresponding spherically symmetric Schwarzschild–de Sitter or Schwarzschild–anti-de Sitter solution at large retarded times. Their global structure is analyzed, and it is demonstrated that the smoothness of the extension of the metrics across the horizon, as compared with the case $\Lambda=0$, can increase for $\Lambda>0$ and decreases for $\Lambda<0$. For the extreme value $9\Lambda m^2=1$, the extension is smooth but nonanalytic. This case appears to be the first example of a smooth but nonanalytic horizon. The models with $\Lambda>0$ exhibit explicitly the cosmic no-hair conjecture under the presence of gravitational waves. [S0556-2821(97)02504-6]

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I. INTRODUCTION

Robinson-Trautman vacuum space-times [1,2] have attracted increased attention in the last decade, in particular, in the works by Lukács *et al.* [3], Schmidt [4], Rendall [5], and, most recently, by Chruściel and Singleton [6–9]. (We refer the reader to the last papers for further references.) In these studies the Robinson-Trautman space-times were shown to exist globally for all positive ''times,'' and to converge asymptotically to a Schwarzschild metric. This global time behavior is true for generic, arbitrarily strong, smooth initial data within the class of the Robinson-Trautman space-times. Interestingly, the extension of these space-times across the ''Schwarzschild-like'' event horizon can only be made with a finite degree of smoothness.

The Robinson-Trautman metrics can easily be generalized to solve the vacuum Einstein equations with a nonvanishing Λ [10]. The results proving the global existence and convergence of the Robinson-Trautman solutions can be taken over from previous studies since Λ does not explicitly enter the basic Robinson-Trautman equation. However, the presence of Λ has a considerable effect on the global structure of the space-times. In our previous work [11], we demonstrated that the Robinson-Trautman space-times of the Petrov type II with $\Lambda>0$ such that $9\Lambda m^2<1$ settle down to the Schwarzschild–de Sitter space-time at large retarded times. They admit a smooth future spacelike infinity and continuation of the metric across the ''Schwarzschild–de Sitter-like'' black-hole horizon can be made with a higher degree of smoothness than those in the corresponding cases with $\Lambda=0$. These space-times may serve as exact models of black-hole formation in nonspherical space-times which are not asymptotically flat. They also represent the only known exact analytic demonstration of the cosmic no-hair conjecture (see, e.g., [12–15]) under the presence of gravitational waves.

The analysis in [11], however, covers only the cases with Λ and m such that $0<9\Lambda m^2<1$, implying the existence of both the black-hole and cosmological horizons. The purpose of this work is to study the ''extreme'' case with $9\Lambda m^2=1$, in which the two horizons coincide, and the cases with $9\Lambda m^2>1$, when the naked singularity arises. We also analyze the global structure of the Robinson-Trautman space-times with $\Lambda<0$, which admit one black-hole horizon.

The formation of an extreme Reissner-Nordström black hole in collapse with small nonspherical perturbations [16,17], as well as motion of particles in extreme black-hole space-times [18], exhibit features qualitatively different from those of generic black holes. Perturbations of extreme black holes seem to be stable with respect to both classical and quantum processes, and there are attempts to interpret them as solitons [19,20]. Extreme black holes with cosmological constant were discussed by Lake and Roeder [21], Mellor and Moss [22,23], Romans [24], Brill and Hayward [25], and others. They were also studied in the context of the Einstein-Yang-Mills-Higgs theory (see, e.g., [26,27] and references therein).

Very recently, Kastor and Traschen [28] have given the solutions with a cosmological constant $\Lambda>0$, containing many extreme black holes. The solutions were used for analytic studies of black-hole collisions and cosmic censorship hypothesis [29]. Horizons of these space-times were analyzed in detail in [29–31].

It is noteworthy that multi-black-hole solutions consisting of the analogues of extremal Reissner-Nordström black holes in asymptotically de Sitter space-time have horizons that are not smooth [29]. In contrast with such black holes in asymptotically flat space-times which have smooth horizons and

*Electronic address: bicak@hp03.troja.mff.cuni.cz
†Electronic address: podolsky@hp03.troja.mff.cuni.cz

are static, the cosmological multi-black-hole solutions are dynamic with gravitational and electromagnetic radiation. The fact that horizons are not smooth is interpreted as due to the existence of radiation which does not have a ''smooth distribution.'' It would then seem natural to interpret the nonsmoothness of the horizons of the Robinson-Trautman black holes in a similar way. On the other hand, one should bear in mind that in five or more dimensions some multi-black-hole solutions to d-dimensional Einstein gravity have horizons that are not smooth although these solutions are static [31]. Their lack of smoothness thus cannot be attributed to the presence of radiation.

In the next section we briefly summarize results of recent studies of the Robinson-Trautman vacuum space-times with $\Lambda=0$, and, in Sec. III, we review results for the Robinson-Trautman space-times with $0<9\Lambda m^2<1$, including their global structure and asymptotic properties at future infinity. Sections IV and V are devoted to analysis of the ''extreme'' case ($9\Lambda m^2=1$) and ''naked-singularity'' cases ($9\Lambda m^2>1$). In Sec. VI the Robinson-Trautman space-times with $\Lambda<0$ are studied. The results are summarized and some general remarks added in Sec. VII.

II. THE ROBINSON-TRAUTMAN SPACE-TIMES WITH $\Lambda=0$

In the standard form the Robinson-Trautman vacuum metric reads (see [1,2,10])

$$ds^2=-\Phi du^2-2du dr+2r^2P^{-2}d\zeta d\bar{\zeta}, \tag{1}$$

where $P=P(u,\zeta,\bar{\zeta})$, ζ is a complex spatial coordinate, $r\in[0,\infty)$ is the affine parameter along the rays $u=$const, $\zeta=$const, and

$$\Phi=\Delta\ln P-2r(\ln P)_{,u}-\frac{2m}{r}. \tag{2}$$

Here, $\Delta=2P^2\partial^2/\partial\zeta\partial\bar{\zeta}$ and m is a constant related to the Bondi mass of the system. The function P satisfies the Robinson-Trautman equation

$$(\ln P)_{,u}=-\frac{1}{12m}\Delta\Delta(\ln P). \tag{3}$$

This equation can be formulated (see, e.g., [7–9]) by introducing a smooth metric $g^0_{ab}(x^c)$ on a two-dimensional manifold (here we shall concentrate on the physical case S^2) and a u-dependent family of two-metrics $g_{ab}=[f(u,x^c)]^{-2}g^0_{ab}$ which, with respect to the coordinate ζ, takes the form $2P^{-2}d\zeta d\bar{\zeta}$. Writing

$$P=fP_0,\quad P_0=1+\frac{1}{2}\zeta\bar{\zeta}, \tag{4}$$

we find Eq. (3) becomes

$$\frac{\partial f}{\partial u}=-\frac{f}{24m}\Delta_g R, \tag{5}$$

where R is the curvature scalar and Δ_g the Laplacian of the metric g_{ab}. Using R_0 and Δ_0 to denote the curvature scalar and the Laplacian of g^0_{ab}, one has

$$R=f^2(R_0+2\Delta_0\ln f),\quad \Delta_g=f^2\Delta_0. \tag{6}$$

Choosing standard coordinates on the sphere, $\zeta=\sqrt{2}e^{i\varphi}\tan\theta/2$, we obtain

$$2P_0^{-2}d\zeta d\bar{\zeta}=d\theta^2+\sin^2\theta d^2\varphi,\quad \Delta_0\ln P_0=1,\quad R_0=+2. \tag{7}$$

Therefore, the metric (1) with $P=P_0$ is just the Schwarzschild metric

$$\begin{aligned}ds^2&=-\left(1-\frac{2m}{r}\right)du^2-2du dr+r^2(d\theta^2+\sin^2\theta d\varphi^2)\\&=-\left(1-\frac{2m}{r}\right)dt^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2\\&\quad+r^2(d\theta^2+\sin^2\theta d\varphi^2),\end{aligned} \tag{8}$$

where $u=t-r^*$, and $r^*=\int\Phi^{-1}(r)dr=r+2m\ln(r/2m-1)$ is the usual ''tortoise'' coordinate.

The most general analysis of the existence and behavior of solutions of the Robinson-Trautman equation was recently given by Chruściel [7,8] and by Chruściel and Singleton [9] (cf. also [4,6,32]). The main result is that when $f_0\equiv f(u=u_0,x^a)$ is an arbitrary, sufficiently smooth, initial-value function for f, then f satisfying Eqs. (5) and (6) exists for all times $u\geqslant u_0$; an asymptotic expansion of $f(u,x^a)$ for large u has the form

$$\begin{aligned}f&=\sum_{i,j\geqslant0}f_{i,j}u^je^{-2iu/m}\\&=1+f_{1,0}e^{-2u/m}+f_{2,0}e^{-4u/m}+\cdots+f_{14,0}e^{-28u/m}\\&\quad+f_{15,1}ue^{-30u/m}+f_{15,0}e^{-30u/m}+\cdots,\end{aligned} \tag{9}$$

where $f_{i,j}$ are smooth functions on S^2. Therefore, as $u\to+\infty$, *Robinson-Trautman metrics approach exponentially fast a Schwarzschild metric*, $f=1$. (In general, $f\to f_{\text{Schw}}$, where f_{Schw} corresponds to a boosted Schwarzschild solution; performing this boost, we can without loss of generality assume that $f_{\text{Schw}}=1$. The analogous assumption will be made in the cases with $\Lambda\neq0$ in the following.) Some of the functions $f_{i,j}$ may vanish, but Chruściel and Singleton [9] prove that there exist space-times for which $f_{15,1}$ is nonvanishing. This implies a surprising fact that, although there exist extensions through the null hypersurface $\mathcal{H}^+$ given by $u=+\infty$ which are C^{117}, in general the Robinson-Trautman metrics *cannot be extended smoothly*. Also, there exists an infinite number of C^5 extensions through $\mathcal{H}^+$. In particular, we may join the radiative metrics to the Schwarzschild metric so that the Robinson-Trautman space-time ''settles down'' to the Schwarzschild space-time including the interior of the black hole, as shown in Fig. 1. In order to see the smoothness across $\mathcal{H}^+$, one introduces an advanced time coordinate v by $v=u+2r^*=u+2r+4m\ln(r/2m-1)$, and Kruskal-type coordinates $\hat{u},\hat{v}$ by (see, e.g., [32])

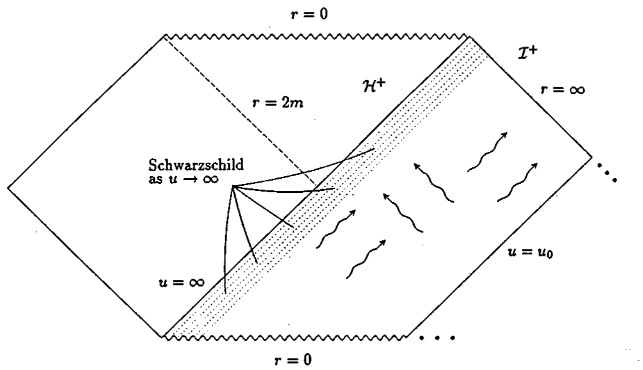

FIG. 1. Starting with arbitrary, smooth initial data at $u=u_0$, the radiative Robinson-Trautman metrics with $\Lambda=0$ converge exponentially fast to a Schwarzschild metric as $u\to\infty$. However, extension beyond the null hypersurface $\mathcal{H}^+$ $(u=+\infty)$ can only be done with a finite degree of smoothness.

$$\hat{u}=-\exp(-u/4m),\quad \hat{v}=\exp(v/4m). \tag{10}$$

The hypersurface $u=+\infty$ now becomes a boundary given by $\hat{u}=0$. The metric (1) becomes

$$ds^2=-\frac{32m^3}{r}\exp(-r/2m)d\hat{u}d\hat{v}-16m^2\hat{\Phi}d\hat{u}^2 + 2r^2P^{-2}d\zeta d\overline{\zeta}, \tag{11}$$

where

$$\hat{\Phi}=e^{u/2m}\left(\frac{1}{2}R-1+\frac{r}{12m}\Delta_g R\right), \tag{12}$$

with R and Δ_g being given by Eq. (6) (for $f=1\Rightarrow\hat{\Phi}=0$ it reduces to the Schwarzschild space-time in standard Kruskal coordinates). In terms of $\hat{u}$, the expansion (9) becomes

$$f=1+f_{1,0}\hat{u}^8+f_{2,0}\hat{u}^{16}+\cdots+f_{14,0}\hat{u}^{112} -4mf_{15,1}(\ln|\hat{u}|)(\hat{u})^{120}+f_{15,0}\hat{u}^{120}+\cdots. \tag{13}$$

Because of the presence of the $\ln|\hat{u}|$ terms, the function f is not smooth at $\hat{u}=0$; indeed, it is C^{119} if $f_{15,1}\neq 0$. The full metric (11) is C^{117} at $\hat{u}=0$, since $\hat{\Phi}$ contains the additional factor $e^{u/2m}\sim 1/\hat{u}^2$.

III. THE ROBINSON-TRAUTMAN SPACE-TIMES WITH $0<9\Lambda m^2<1$

When a Robinson-Trautman space-time with $\Lambda=0$ is known, it is straightforward to generalize it to the case of a nonvanishing Λ (cf. [10,11]). The metric still keeps the form (1) with P satisfying the Eq. (3). The only place where Λ enters is through the function Φ. The cosmological Robinson-Trautman metric reads

$$ds^2=-\Phi_\Lambda du^2-2dudr+2r^2P^{-2}d\zeta d\overline{\zeta}, \tag{14}$$

where

$$\Phi_\Lambda=\Delta\ln P-2r(\ln P)_{,u}-\frac{2m}{r}-\frac{\Lambda}{3}r^2. \tag{15}$$

We may still write $P=fP_0$, as in Eq. (4), where P_0 gives Eq. (7) and f satisfies Eqs. (5) and (6). Since Λ does not enter the equation for f, we may take over the results for $\Lambda=0$ described in Sec. II. Therefore, as $u\to\infty$, the metric (14) will now approach the Schwarzschild–de Sitter metric given by $f=1$, corresponding to $\Phi_\Lambda^0=1-2m/r-\Lambda r^2/3$:

$$\begin{aligned} ds^2&=-\left(1-\frac{2m}{r}-\frac{\Lambda}{3}r^2\right)du^2-2dudr + r^2(d\theta^2+\sin^2\theta d\varphi^2)\\ &=-\left(1-\frac{2m}{r}-\frac{\Lambda}{3}r^2\right)dt^2+\left(1-\frac{2m}{r}-\frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\varphi^2). \end{aligned} \tag{16}$$

Again, $u=t-r^*$, but the ''tortoise-type'' coordinate r^* for $0<9\Lambda m^2<1$ is

$$\begin{aligned} r^*&=\int\frac{dr}{\Phi_\Lambda^0(r)}\\ &=\delta_+\ln\frac{|r-r_+|}{r+r_++r_{++}}-\delta_{++}\ln\frac{|r_{++}-r|}{r+r_++r_{++}} + \delta_+\left[\ln\left(\frac{r_{++}}{r_+}\right)-\frac{1}{2}\right], \end{aligned} \tag{17}$$

where

$$\delta_+=\frac{r_+}{1-\Lambda r_+^2},\quad \delta_{++}=-\frac{r_{++}}{1-\Lambda r_{++}^2}. \tag{18}$$

Here, $r_+=(2/\sqrt{\Lambda})\cos(\alpha/3+4\pi/3)$, with $\cos\alpha=-3m\sqrt{\Lambda}$, describes the black-hole horizon, and $r_{++}=(2/\sqrt{\Lambda})\cos(\alpha/3)$ is the cosmological horizon, see, e.g., [11] for more details about dependence of parameters on Λ. (Analytic continuation of the Schwarzschild–de Sitter metric is discussed, for example, in [21] and in [33–36].)

The presence of a cosmological constant does not affect the smoothness of future infinity $\mathcal{I}^+$ in these space-times; however, $\mathcal{I}^+$ becomes spacelike for $\Lambda>0$ in contrast with the cases with $\Lambda=0$ (cf. Fig. 2). Moreover, the presence of Λ has a considerable effect on the smoothness of extensions through $\mathcal{H}^+$ given by $u=+\infty$. The approach of f to its Schwarzschild–de Sitter form $f=1$ is again characterized by the expansion (9) but the transformation to Kruskal-type coordinates is now given by

$$\hat{u}=-\exp(-u/2\delta_+),\quad \hat{v}=\exp(v/2\delta_+), \tag{19}$$

where $v=u+2r^*$, r^* being given by Eq. (17). Hence, instead of Eq. (13), we get the expansion

$$f=1+f_{1,0}(-\hat{u})^{4\delta_+/m}+f_{2,0}(-\hat{u})^{8\delta_+/m} +\cdots+f_{14,0}(-\hat{u})^{56\delta_+/m}$$

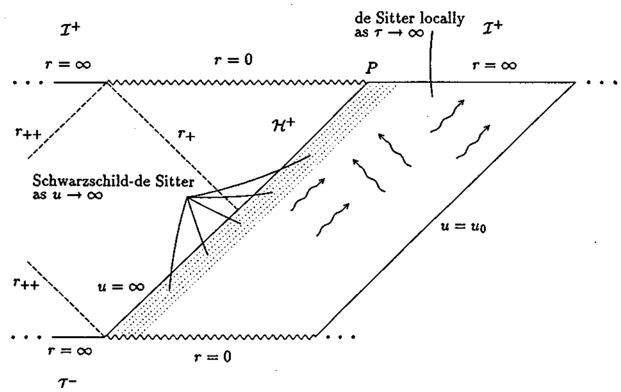

FIG. 2. Starting with initial data at $u=u_0$, the Robinson-Trautman metrics with $0<9\Lambda m^2<1$ converge to a Schwarzschild–de Sitter metric as $u\to\infty$. Although traces of gravitational waves will persist at future infinity $\mathcal{I}^+$, for all geodesic observers the metric will approach the de Sitter metric within their past light cone. The metric at the horizon $\mathcal{H}^+$ has only a finite degree of smoothness, although this can be higher than that in the case with $\Lambda=0$.

$$-2\delta_+ f_{15,1}(\ln|\hat{u}|)(-\hat{u})^{60\delta_+/m}$$
$$+f_{15,0}(-\hat{u})^{60\delta_+/m}+\cdots \tag{20}$$

at $u\to+\infty$, i.e., $\hat{u}\to 0_-$ [cf. Eq. (19)]. The full metric takes the form

$$ds^2=-\frac{4\Lambda\delta_+^2 r_+ e^{1/2}}{3r_{++}r}(r_{++}-r)^{1+\delta_{++}/\delta_+}$$
$$\times(r+r_++r_{++})^{2-\delta_{++}/\delta_+}d\hat{u}d\hat{v}$$
$$-4\delta_+^2\hat{\Phi}_\Lambda d\hat{u}^2+2r^2P^{-2}d\zeta d\bar{\zeta}, \tag{21}$$

where

$$\hat{\Phi}_\Lambda=e^{u/\delta_+}\left(\frac{1}{2}R-1+\frac{r}{12m}\Delta_g R\right), \tag{22}$$

with f being of the form (20) above. We may join the radiative Robinson-Trautman metrics with $\Lambda>0$ to the Schwarzschild–de Sitter metric so that the space-time ''settles down'' to the Schwarzschild–de Sitter black hole including its interior (see Fig. 2). Such an extension across $\hat{u}=0$ will, in general, be C^5 in the case of vanishing Λ. [For example, $\hat{\Phi}$ and all its derivatives vanish for $\hat{u}=0$ in the Schwarzschild case, whereas $\partial_{\hat{u}}^{(6)}\hat{\Phi}\neq 0$ with f given by Eq. (13).] With $\Lambda>0$, much higher smoothness can be obtained. For those values of Λ which imply $4\delta_+/m$ equals an integer, the smoothness is always better than that for $\Lambda=0$. Moreover, the horizon $\mathcal{H}^+$ can be made ''arbitrarily smooth'' by letting Λ approach its extremal value, $\Lambda\to 1/9m^2$ (i.e., $r_+\to 3m$). Then, δ_+ becomes arbitrarily large and the terms $\sim(-\hat{u})^{i\delta_+/m}$, $i=4,8,\ldots$, in Eq. (20) will guarantee arbitrarily high smoothness of the function f at $\hat{u}=0$.

The Robinson-Trautman metrics with $\Lambda>0$ may serve as exact analytic models demonstrating the cosmic no-hair conjecture under the presence of gravitational waves, they all approach de Sitter space-time locally close to $\mathcal{I}^+$, i.e., near $r\to\infty$, u finite (cf. Fig. 2). As discussed in detail in [11], the transformation of the form

$$r=\chi e^{H\tau}-H^{-2}(f_{\infty,u}/f_\infty)+\sum_{n=1}^{\infty}A_n e^{-nH\tau},$$
$$e^{Hu}=H\chi-e^{-H\tau}+\sum_{n=3}^{\infty}B_n e^{-nH\tau},$$
$$\zeta=\eta+\sum_{n=3}^{\infty}C_n e^{-nH\tau}, \tag{23}$$

in which A_n, B_n, C_n are suitable functions of $\chi,\eta,\bar{\eta}$, and $H=\sqrt{\Lambda/3}$ brings the metric (14) into the asymptotic form

$$ds^2=-d\tau^2+e^{2H\tau}[d\chi^2+f_\infty^{-2}\chi^2(d\theta^2+\sin^2\theta d\varphi^2)]$$
$$+\sum_{m=0}^{\infty}e^{-mH\tau}h_{ab}^{(m)}dx^a dx^b, \tag{24}$$

where the coordinates θ, φ are reintroduced by

$$\eta=\sqrt{2}e^{i\varphi}\tan(\theta/2), \quad f_\infty=f|_{\tau\to\infty}=f(u=H^{-1}\ln|H\chi|,\theta,\varphi),$$

and $h_{ab}^{(m)}$ depend on $\{x^a\}=\{\chi,\theta,\varphi\}$ only. It is seen explicitly that for $\tau\to\infty$, the metric (24) does not approach the de Sitter metric globally, the gravitational waves leave ''an imprint'' on $\mathcal{I}^+$ which is demonstrated by the presence of the function f_∞. However, any geodesic observer will see *locally*, inside his past light-cone, space-time approach de Sitter space-time exponentially fast in accordance with the cosmic no-hair conjecture (see [11] for details).

IV. THE ROBINSON-TRAUTMAN SPACE-TIMES WITH $9\Lambda m^2=1$

Above, we summarized the approach to Schwarzschild–de Sitter space-time in the case $0<9\Lambda m^2<1$ characterized by the existence of two distinct horizons r_+ and r_{++}, with $0<2m<r_+<3m<r_{++}$. With Λ approaching its extremal value, $\Lambda\to 1/9m^2$, the black-hole horizon r_+ monotonically increases and the cosmological horizon r_{++} decreases to the common value $3m$. In this section we shall analyze the extreme case $9\Lambda m^2=1$ for which there exists only one ''double'' Killing horizon at $r_e=3m$.

The metric of the Robinson-Trautman space-time is still given by Eqs. (14) and (15), and the corresponding extreme Schwarzschild–de Sitter metric by Eq. (16). However, the ''tortoise-type'' coordinate r^* is now

$$r^*=\frac{9m^2}{r-3m}+2m\ln\left|\frac{r+6m}{r-3m}\right|, \tag{25}$$

where an additive constant was chosen such that $r^*\to 0$ at $r\to\infty$. By introducing the Kruskal-type null coordinates

$$\hat{u}=-\operatorname{arccot}(-u/\delta), \quad \hat{v}=\arctan(v/\delta), \tag{26}$$

where

$$\delta=-m(3-2\ln 2)<0, \tag{27}$$

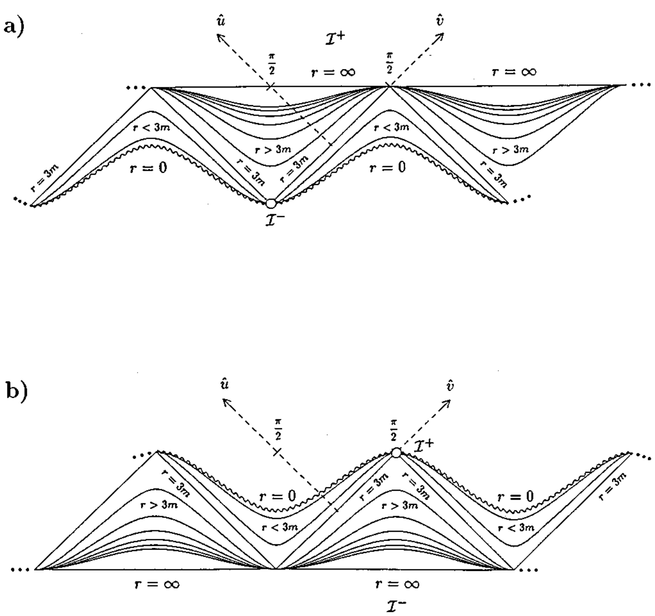

FIG. 3. (a) Conformal diagram of the extreme Schwarzschild–de Sitter space-time with $9\Lambda m^2=1$ and the singularity in the past, corresponding to a white hole. The maximal analytic extension of the geometry is obtained by glueing an infinite number of regions shown in the figure, or joining a finite number of regions via identification of events along two horizons $r=3m$. (b) The time-reversed diagram ($\hat{u}\rightarrow-\hat{u},\hat{v}\rightarrow-\hat{v}$), corresponding to a black hole.

$v=u+2r^*$, r^* given by Eq. (25), the ''extreme'' Robinson-Trautman metric can be written in the form

$$ds^2=-\frac{\delta^2}{27m^2r}\frac{(r+6m)(r-3m)^2}{\cos^2\hat{v}\,\sin^2\hat{u}}d\hat{u}d\hat{v}-\hat{\Phi}_\Lambda d\hat{u}^2+2r^2P^{-2}d\zeta d\bar{\zeta}, \tag{28}$$

where

$$\hat{\Phi}_\Lambda=\frac{\delta^2}{\sin^4\hat{u}}\left(\frac{1}{2}R-1+\frac{r}{12m}\Delta_g R\right). \tag{29}$$

The asymptotic expansion (9) becomes

$$f=\sum_{i,j\geqslant 0} f_{i,j}\delta^j\cot^j\hat{u}\, e^{-(2i\delta/m)\cot\hat{u}}$$
$$=1+f_{1,0}e^{-(2\delta/m)\cot\hat{u}}+f_{2,0}e^{-(4\delta/m)\cot\hat{u}}+\cdots+f_{14,0}\times e^{-(28\delta/m)\cot\hat{u}}+\delta f_{15,1}\cot\hat{u}\, e^{-(30\delta/m)\cot\hat{u}}+\cdots. \tag{30}$$

In particular, if $f=1$ we get $\hat{\Phi}_\Lambda=0$, $P=P_0$ [see Eqs. (4), (6), and (7)], and the metric (28) describes the spherically symmetric extreme Schwarzschild–de Sitter space-time, see Fig. 3 for its conformal diagram. It is regular on the horizon $r=r_e=3m$ for all finite u and v since

$$\lim_{r\rightarrow 3m}\frac{(r-3m)^2}{\cos^2\hat{v}}=\lim_{r\rightarrow 3m}\frac{(r-3m)^2}{\sin^2\hat{u}}=\frac{(18m^2)^2}{\delta^2}. \tag{31}$$

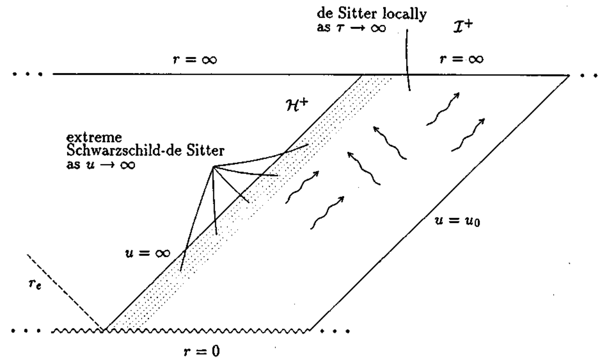

FIG. 4. Starting with initial data at $u=u_0$, the Robinson-Trautman metrics with $9\Lambda m^2=1$ converge to an extreme Schwarzschild–de Sitter space-time as $u\rightarrow\infty$. The extension beyond the horizon $\mathcal{H}^+$ is smooth but not analytic.

As in the previous case, the general Robinson-Trautman space-times with $9\Lambda m^2=1$ approach an extreme Schwarzschild–de Sitter space-time as $u\rightarrow+\infty$, i.e., $\hat{u}\rightarrow 0_-$ [$\hat{u}<0$, cf. Eq. (26)]. Indeed, introducing $g_i=a_i\cot\hat{u}$, where $a_i=-(2i\delta/m)>0$, $i=1,2,3,\ldots$, and $h_j=\exp(g_j)$, $j=1,2,3,\ldots$, the expansion of the function $f-1$ given by Eq. (30) can be written as a linear combination of terms $(g_i)^k h_j$, $k=0,1,2,\ldots$. Clearly, $g_i\rightarrow-\infty$ as $\hat{u}\rightarrow 0_-$, so that $(g_i)^k h_j\rightarrow 0$; this implies $f\rightarrow 1$. All Robinson-Trautman spacetimes (28)-(30) are thus settling down to the extreme Schwarzschild–de Sitter space-time as $u\rightarrow\infty$, i.e., at the null hypersurface $\mathcal{H}^+$ given by $\hat{u}=0_-$ (see Fig. 4). A question again naturally arises, whether one can extend the space-time through $\mathcal{H}^+$ by glueing to it, for example, an extreme Schwarzschild–de Sitter space-time (with $\hat{u}>0$). It is not difficult to see that one can make such an extension and, in contrast with the cases $0\leqslant 9\Lambda m^2<1$, this *extension is smooth*.

First, it can be shown by induction and using the relation $dg_i/d\hat{u}=-(a_i+g_i^2/a_i)$ that the nth derivative, $n=1,2,\ldots$, of $(g_i)^k$ with respect to $\hat{u}$ can be expressed as a polynomial of the $(n+k)$th order in g_i, i.e., $(g_i^k)^{(n)}=\Sigma_{s=0}^{n+k}c_{ks}g_i^s$, where the coefficients c_{ks} are constants. Similarly, $h_j^{(n)}=h_j\Sigma_{s=0}^{2n}d_s g_j^s$, where d_s are constants. Leibnitz's formula then gives $(g_i^k h_j)^{(n)}\rightarrow 0$ as $g_i\rightarrow-\infty$, which implies

$$\lim_{\hat{u}\rightarrow 0_-}f=1,\ \lim_{\hat{u}\rightarrow 0_-}f^{(n)}=0. \tag{32}$$

Moreover, we find

$$\lim_{\hat{u}\rightarrow 0_-}(\hat{\Phi}_\Lambda)^{(n)}=0, \tag{33}$$

since $\sin^{-4}\hat{u}\sim g_i^4\cos^{-4}\hat{u}$, so that $\hat{\Phi}_\Lambda=\{$linear combination of $g_i^{k+4}h_j\}\cos^{-4}\hat{u}$; an arbitrary derivative of the first factor tends to zero as $\hat{u}\rightarrow 0_-$ while derivatives of the second factor remain finite.

Therefore, the radiative Robinson-Trautman space-times with $9\Lambda m^2=1$ can be extended *smoothly* through the hori-

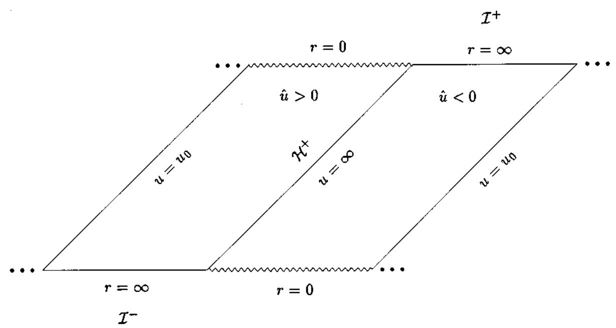

FIG. 5. Another smooth extension of the Robinson-Trautman metric with $9\Lambda m^2=1$ beyond the horizon $\mathcal{H}^+$ can be obtained by glueing two copies of the metric along $u=\infty$ ($\hat{u}=0$). The extreme black-hole space-time illustrated in Fig. 3(b) can also be joined to the Robinson-Trautman space-time along $\mathcal{H}^+$.

zon $\mathcal{H}^+$ to the spherically symmetric extreme Schwarzschild–de Sitter space-time with the same values of Λ and m, $9\Lambda m^2=1$ (see Fig. 4). However, such an extension is not unique. There are other possibilities, the simplest one can be obtained by glueing a copy of the Robinson-Trautman space-time with $9\Lambda m^2=1$ to itself (see Fig. 5). For $\hat{u}>0$ we consider another copy of Eqs. (28)–(30) obtained by the reflection $\hat{u}\rightarrow-\hat{u}$, $\hat{v}\rightarrow-\hat{v}$. [The same reflection connects Figs. 3(a) and 3(b).] Again, since $\lim_{\hat{u}\rightarrow 0_+}(\hat{\Phi}_\Lambda)^{(n)}=0$, the extension across $\hat{u}=0$ is smooth and the space-time can be called an ''extreme'' Robinson-Trautman black hole in the de Sitter universe. Its conformal diagram resembles the diagram in Fig. 2, representing the nonextreme case (cf. [11]). Any timelike geodesic observer falling from the region $\hat{u}<0$ will cross the smooth horizon $\mathcal{H}^+$ and reach the singularity at $r=0$, or escape to ''de Sitter-like'' infinity given by $r=\infty$.

Therefore, the smooth extensions across $\hat{u}=0$ are not unique. Of course, *they are not analytic*. In fact, the functions $\exp(a_i\cot\hat{u})$ in expansion (30) are C^∞ at $\hat{u}=0_-$, but $\hat{u}=0$ is an irremovable singularity.

The behavior of the Robinson-Trautman space-times near future spacelike infinity $\mathcal{I}^+$ (given by $r=\infty$) is similar to the nonextreme case discussed in the previous section. Again, one can perform the transformation (23) converting the metric into the asymptotic form (24) so that those space-times approach the de Sitter metric locally as $\tau\rightarrow\infty$, in correspondence with the cosmic no-hair conjecture.

V. THE ROBINSON-TRAUTMAN SPACE-TIMES WITH $9\Lambda m^2>1$

In this case the corresponding Schwarzschild–de Sitter space-time (16) admits no horizon in the region $r>0$ (cf. [21,36]) so that there is only a naked singularity situated at $r=0$. The metric of the Robinson-Trautman space-time with $9\Lambda m^2>1$ is again given by Eqs. (14) and (15) but now the ''tortoise-type'' coordinate r^* becomes

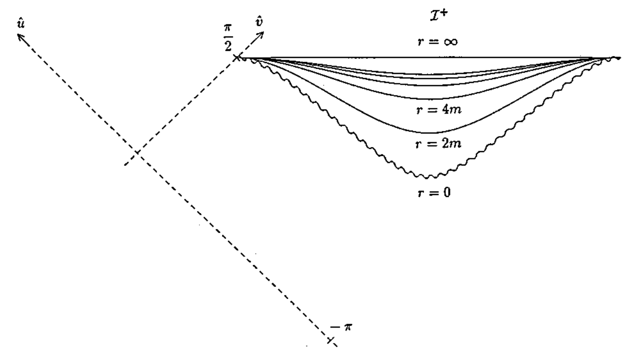

FIG. 6. Conformal diagram of the Schwarzschild–de Sitter space-time with $9\Lambda m^2>1$ describing a spherically symmetric naked singularity in the (asymptotically) de Sitter universe.

$$r^*=-\frac{r_-}{\Lambda r_-^2-1}\left\{\frac{1}{2}\ln\frac{r^2-2r_-r+r_-^2}{r^2+r_-r-6m/\Lambda r_-}+\frac{\frac{r_-}{2}-6m/\Lambda r_-^2}{\sqrt{(3/4)r_-^2-3/\Lambda}}\times\left[\arctan\left(\frac{r+r_-/2}{\sqrt{(3/4)r_-^2-3/\Lambda}}\right)-\frac{\pi}{2}\right]\right\}, \tag{34}$$

where $r_-=-(3m/\Lambda)^{1/3}[(1-C)^{1/3}+(1+C)^{1/3}]<0$ and $C=\sqrt{1-1/(9\Lambda m^2)}$. It can be shown that r^* monotonically decreases from $r^*(r=0)>0$ to $r^*(r=\infty)=0$. The Kruskal-type coordinates are

$$\hat{u}=-\text{arccot}(u/m),\quad \hat{v}=\arctan(-v/m), \tag{35}$$

where $v=u+2r^*$ and r^* is given by Eq. (34). Then, the Robinson-Trautman metric reads

$$ds^2=-\frac{\Lambda m^2}{3r}(r-r_-)\left(r^2+r_-r-\frac{6m}{\Lambda r_-}\right)\frac{d\hat{u}d\hat{v}}{\sin^2\hat{u}\cos^2\hat{v}}-\hat{\Phi}_\Lambda d\hat{u}^2+2r^2P^{-2}d\zeta d\bar{\zeta}, \tag{36}$$

where

$$\hat{\Phi}_\Lambda=\frac{m^2}{\sin^4\hat{u}}\left(\frac{1}{2}R-1+\frac{r}{12m}\Delta_g R\right), \tag{37}$$

and

$$f=\sum_{i,j\geqslant 0}f_{i,j}(-m)^j\cot^j\hat{u}\,e^{2i\cot\hat{u}}=1+f_{1,0}e^{2\cot\hat{u}}+f_{2,0}e^{4\cot\hat{u}}+\cdots+f_{14,0}e^{28\cot\hat{u}}-mf_{15,1}\cot\hat{u}\,e^{30\cot\hat{u}}+\cdots. \tag{38}$$

The metric is regular for all values $r>0$ and, in particular, it describes spherically symmetric Schwarzschild–de Sitter space-time with a naked singularity if $f=1$ (i.e., $\hat{\Phi}_\Lambda=0$); its

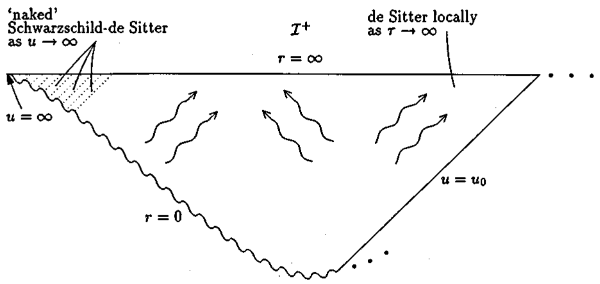

FIG. 7. Starting with smooth initial data at $u=u_0$, the Robinson-Trautman metrics with $9\Lambda m^2>1$ approach a ''naked'' Schwarzschild–de Sitter metric as $u\rightarrow\infty$. No extension of the metric is necessary.

conformal diagram is seen in Fig. 6. Since the expansion (38) is analogous to Eq. (30), we can take over the results (32) and (33), implying that any Robinson-Trautman space-time with $9\Lambda m^2>1$ approaches smoothly the corresponding Schwarzschild–de Sitter space-time as $u\rightarrow\infty$ ($\hat{u}\rightarrow 0_-$). It contains no horizon (contrary to the cases discussed in the previous sections) so that the metrics (36)–(38) need not to be extended past $\hat{u}=0$; it is already geodesically complete for $u>u_0$, as indicated in Fig. 7. Also, it can be put into the asymptotic form (24), again demonstrating explicitly the cosmic ''no-hair'' conjecture under the presence of gravitational waves.

VI. THE ROBINSON-TRAUTMAN SPACE-TIMES WITH Λ<0

We now complete the analysis of the Robinson-Trautman vacuum space-times with Λ with the case $\Lambda<0$. The Schwarzschild–anti-de Sitter metric, which is again a spherically symmetric Robinson-Trautman solution given by Eqs. (14) and (15) with $f=1$ [or (16) with $\Lambda<0$], always admits

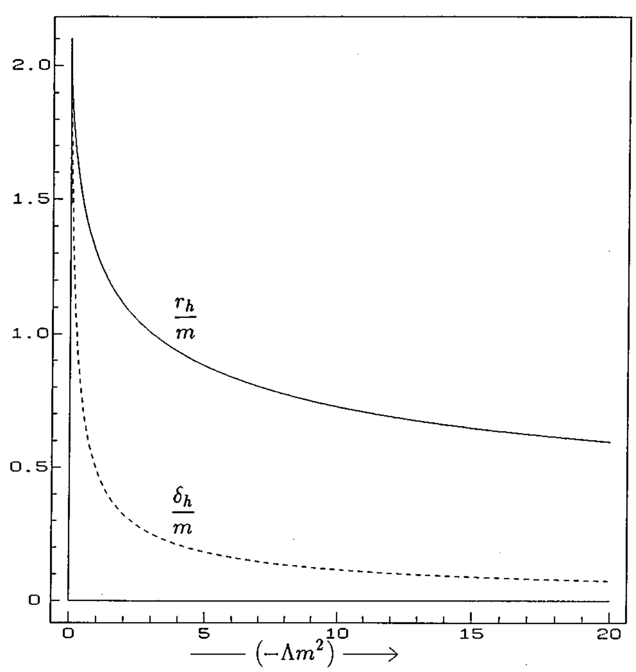

FIG. 8. A plot of the black-hole horizion r_h and the parameter δ_h (dashed line) as a function of $\Lambda<0$ and m.

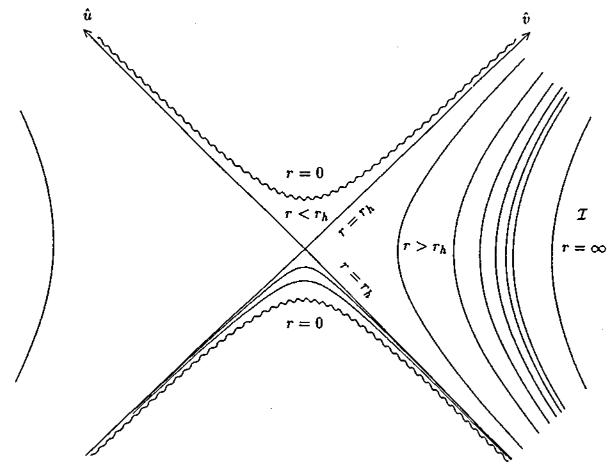

FIG. 9. Conformal diagram of the Schwarzschild–anti-de Sitter space-time with $\Lambda<0$ and $m>0$. Infinity $\mathcal{I}$ is timelike.

a black-hole horizon at $r_h=(-3m/\Lambda)^{1/3}[(C+1)^{1/3}-(C-1)^{1/3}]>0$, where $C=\sqrt{1-1/(9\Lambda m^2)}$. The value of r_h decreases from $r_h=2m$ for $\Lambda=0$ to $r_h\rightarrow 0$ as $\Lambda\rightarrow-\infty$, as seen in Fig. 8 [the expansion of r_h for small $\Lambda<0$ is $r_h=2m+(8/3)m^3\Lambda+O(m^5\Lambda^2)$]. Kruskal-type null coordinates are

$$\hat{u}=-\exp(-u/2\delta_h),\quad \hat{v}=\exp(v/2\delta_h). \tag{39}$$

Here, $v=u+2r^*$, with the ''tortoise-type'' coordinate r^* for $\Lambda<0$ given by

$$r^*=\delta_h\Bigg(\ln|r-r_h|-\frac{1}{2}\ln(r^2+r_hr-\frac{6m}{\Lambda r_h}) + \frac{6m-r_h}{\sqrt{(6m+r_h)(2m-r_h)}}\Bigg\{\arctan\Bigg[\sqrt{\frac{2m-r_h}{6m+r_h}}\Bigg(1+\frac{2r}{r_h}\Bigg)\Bigg] + D\Bigg\}\Bigg), \tag{40}$$

where

$$\delta_h=-\frac{3}{2\Lambda r_h}\frac{2m-r_h}{3m-r_h}, \tag{41}$$

and $D=-m\sqrt{-\Lambda/3}[1+\ln(-4\Lambda m^2/3)]$. Performing the transformation (39), the Robinson-Trautman metric (14) becomes

$$ds^2=\frac{4\Lambda\delta_h^2}{3r}\Bigg(r^2+r_hr-\frac{6m}{\Lambda r_h}\Bigg)^{3/2} \times\exp\Bigg(-\frac{6m-r_h}{\sqrt{(6m+r_h)(2m-r_h)}} \times\Bigg\{\arctan\Bigg[\sqrt{\frac{2m-r_h}{6m+r_h}}\Bigg(1+\frac{2r}{r_h}\Bigg)\Bigg]+D\Bigg\}\Bigg)d\hat{u}d\hat{v} -4\delta_h^2\hat{\Phi}_\Lambda d\hat{u}^2+2r^2P^{-2}d\zeta d\bar{\zeta}, \tag{42}$$

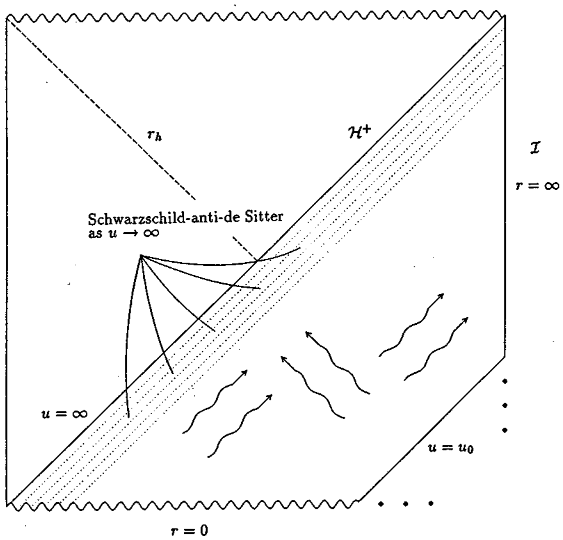

FIG. 10. Starting with smooth initial data at $u=u_0$, the Robinson-Trautman metrics with $\Lambda<0$ converge to a Schwarzschild–anti-de Sitter metric as $u\to\infty$. The metric at the horizon $\mathcal{H}^+$ has only a finite degree of smoothness which is lower than that in the case with $\Lambda=0$.

where

$$\hat{\Phi}_\Lambda=e^{u/\delta_h}\left(\frac{1}{2}R-1+\frac{r}{12m}\Delta_g R\right). \tag{43}$$

If $f=1$, we get $\hat{\Phi}_\Lambda=0$, and the metric reduces to the Schwarzschild–anti-de Sitter metric in Kruskal coordinates; its conformal diagram is indicated in Fig. 9. Letting $\Lambda\to 0$, we obtain back the metric (11). In a general case, the expansion (9) of f in terms of $\hat{u}$ introduced by Eq. (39) becomes

$$\begin{aligned} f=&1+f_{1,0}(-\hat{u})^{4\delta_h/m}+f_{2,0}(-\hat{u})^{8\delta_h/m}\\ &+\cdots+f_{14,0}(-\hat{u})^{56\delta_h/m}-2\delta_h f_{15,1}(\ln|\hat{u}|)(-\hat{u})^{60\delta_h/m}\\ &+f_{15,0}(-\hat{u})^{60\delta_h/m}+\cdots. \end{aligned} \tag{44}$$

Therefore, all radiative Robinson-Trautman metrics with $\Lambda<0$ ''settle down'' to the Schwarzschild–anti-de Sitter metric as $u\to\infty$, or $\hat{u}\to 0_-$ (see Fig. 10). However, the smoothness of the extension of the Robinson-Trautman metric across the horizon $\mathcal{H}^+$ given by $\hat{u}=0$ to the Schwarzschild–anti-de Sitter metric *decreases* with a growing value of $(-\Lambda)$. Indeed, the parameter δ_h given by Eq. (41) monotonically decreases from $\delta_h=2m$ for $\Lambda=0$ to $\delta_h\to 0$ as $\Lambda\to-\infty$ (see Fig. 8). For $(-\Lambda)$ small one gets $7<4\delta_h/m<8$, so that the function f is at least C^7 and the full metric is C^5 (the smoothness of the extension is decreased by two due to the factor $e^{u/\delta_h}\sim 1/\hat{u}^2$ entering $\hat{\Phi}_\Lambda$), as in the case with $\Lambda=0$. For $-\Lambda m^2>4/9$, the black-hole horizon is situated at $r_h<3m/2$ and $4\delta_h/m<3$; the function f is less than C^3 and the metric is not even C^1. If $-\Lambda m^2>3$, then $r_h<m$, $4\delta_h/m<1$, and $df/d\hat{u}$ diverges at $\mathcal{H}^+$.

However, as expected, the presence of a negative cosmological constant does not affect the smoothness of infinity $\mathcal{I}$ (although it changes its character: $\mathcal{I}$ becomes timelike). Introducing a coordinate $l=r^{-1}$ and a conformal factor $\Omega=l$ in Eqs. (14) and (15), one finds (cf. [11])

$$\Omega^2 ds^2=2du\,dl-l^2\Phi_\Lambda du^2+2P^{-2}d\zeta d\bar{\zeta}, \tag{45}$$

where

$$\Phi_\Lambda=\Delta\ln P-2l^{-1}(\ln P)_{,u}-2ml-\frac{\Lambda}{3}l^{-2}. \tag{46}$$

It is easy to see that $l=0$ is a regular timelike hypersurface for arbitrary smooth $P(u,\zeta,\bar{\zeta})$.

VII. CONCLUDING REMARKS

We have shown that all vacuum radiative cosmological Robinson-Trautman space-times of the Petrov type II with $m>0$ settle down to Schwarzschild–de Sitter (if $\Lambda>0$) or Schwarzschild–anti-de Sitter (if $\Lambda<0$) solutions at large retarded times. This is true for ''arbitrary strong'' smooth initial data in the Robinson-Trautman class of metrics. The space-times can then be extended to include the black-hole interiors. As $\Lambda>0$ is increased, the interior of a corresponding Schwarzschild–de Sitter black hole can be joined to an external cosmological Robinson-Trautman space-time across the horizon with an increased degree of smoothness. In the extreme case when $9\Lambda m^2=1$, the extension is C^∞, i.e., smooth, but not analytic. In this sense, the conjecture (2.1) presented for the case $\Lambda=0$ in Ref. [8], that the only ''positive mass Robinson-Trautman space-time which is smoothly extendible through $\mathcal{H}^+$ is (necessarily) the Schwarzschild space-time'' is not true for Robinson-Trautman space-times with a positive cosmological constant. On the other hand, for $\Lambda<0$ the extension to a Schwarzschild–anti-de Sitter black hole has a lower degree of smoothness than those in corresponding cases with $\Lambda=0$.

All space-times with $\Lambda>0$ represent exact explicit models exhibiting the cosmic no-hair conjecture under the presence of gravitational waves. They may serve as test beds in numerical studies of more realistic situations.

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