Post-Newtonian effects of gravity on quantum interferometry

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The effects of general relativistic gravity on quantum mechanics are thoroughly investigated. The Schrödinger equation is derived for a nonrelativistic spinless particle and for a spin-1/2 particle, with the general relativistic corrections of Earth's gravity up to the first post-Newtonian order, $O(1/c^2)$. The phase differences due to the post-Newtonian corrections in quantum interferometer experiments are also calculated. [S0556-2821(97)00604-8]

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I. INTRODUCTION

It has been long regarded that the fundamental issues in quantum mechanics may be studied only by theoretical experiments. However, recent progress in technology has made these experiments possible in a laboratory and stimulated new interest in the problem of the interpretation in quantum mechanics. Another interesting and fundamental issue in quantum mechanics is its relation to gravity. In this respect the development of a neutron interferometer [1] has played a decisive role. Although we do not have the fundamental unification between quantum mechanics and gravity, it is by now clear that gravity has interesting and observable effects on quantum interference.

It is expected that the use of an atomic interferometer [2] will improve the accuracy of the measurement. For example the original Colella-Overhauser-Werner (COW) experiment [3] using a neutron interferometer has a sensitivity of $10^{-2}g$ where g is the acceleration of Earth's gravity. It has been argued that the measurement of the rate of change in the Doppler shift of a falling atom may reach a sensitivity of $10^{-12}g$. Moreover, it is becoming possible to use polarized neutrons in interferometer experiments and this will give us direct information on the interaction between gravity and spin.

In this situation it seems useful and desirable to study higher order effects of gravity as well as of spin-gravity coupling on the quantum interferometer experiment. This is the aim of the present paper. We shall derive the Schrödinger equation for a spinless particle and for a Dirac particle with general relativistic corrections of Earth's gravity up to the first post-Newtonian order which is $O(1/c^2)$ higher than Newtonian order. Then the effect of general relativistic gravity on the quantum interferometer is investigated.

The organization of the present paper is as follows. We shall use the Kerr metric as the external gravitational field of Earth. The metric and coordinates are specified in terms of a 3+1 formalism used in Sec. II. Then we shall write down the Schrödinger equation with a general relativistic correction for a nonrelativistic spinless particle and study the phase difference in the quantum interferometer. We then turn our attention to the case of a spin-1/2 particle using the covariant Dirac equation in curved spacetime in Sec. IV. Finally we shall give a brief summary and discussion in Sec. V.

II. SPACETIME AND THE OBSERVER'S FRAME

What we have in mind is the quantum interferometry experiments in the laboratory on Earth. We assume that the external gravitational field of Earth is described by the Kerr metric. The weak field approximation up to the first post-Newtonian order, i.e., up to $O(1/c^2)$, gives

$$\begin{aligned} ds^2 \simeq & \left(c^2 + 2\phi + \frac{2\phi^2}{c^2} + \frac{2GMa}{c^2 r^3}(x'^2 + y'^2) \right) dt^2 \\ & + \frac{4GMa}{c^2 r^3}(x' dy' - y' dx') dt \\ & - \left(1 - \frac{2\phi}{c^2} \right)(dx'^2 + dy'^2 + dz'^2), \end{aligned} \tag{2.1}$$

where M is the mass of Earth, ϕ is the Newtonian gravitational potential, $\phi \equiv -GM/r$ with $r = \sqrt{x'^2 + y'^2 + z'^2}$, and a is the Kerr parameter which corresponds to the angular momentum J of Earth per unit mass. Hereafter, we assume that Earth is a sphere of radius R with uniform density. Then

$$a = \frac{J}{M} = \frac{2}{5} R^2 \omega, \tag{2.2}$$

where ω is the angular velocity of Earth. (The numerical factor 2/5 might be changed by a factor of order unity, if the effects of the density inhomogeneity, deviations from the spherical shape and so on, are included.) The observer's rest frame (t,x,y,z) is fixed on the rotating Earth, and the relation to the asymptotically static coordinates (t,x',y',z') is

$$\begin{aligned} x' &= x\cos\omega t - y\sin\omega t, \\ y' &= x\sin\omega t + y\cos\omega t, \\ z' &= z. \end{aligned} \tag{2.3}$$

For the sake of convenience, we use the $3+1$ formalism for representing the spacetime. In the $3+1$ formalism, the four-dimensional metric tensor $g_{\mu\nu}$ is split in the following way:

$$g_{00}=N^2-\gamma_{ij}N^iN^j,\qquad g_{0i}=-\gamma_{ij}N^j\equiv-N_i,\qquad g_{ij}=-\gamma_{ij}, \tag{2.4}$$

where N is the lapse function, N^i is the shift vector, and γ_{ij} is the spatial metric (γ^{ij} is the inverse matrix of γ_{ij}) on the t-constant hypersurface. In this formalism, the slowly rotating, weak gravitational field of Earth in the observer's rest frame is expressed by the quantities

$$N=c\left(1+\frac{\phi}{c^2}+\frac{\phi^2}{2c^4}\right),\qquad N^x=-\left(1-\frac{4GMR^2}{5c^2r^3}\right)\omega y,\qquad N^y=\left(1-\frac{4GMR^2}{5c^2r^3}\right)\omega x,\qquad N^z=0,\qquad \gamma_{ij}=\left(1-\frac{2\phi}{c^2}\right)\delta_{ij}. \tag{2.5}$$

III. CASE OF A NONRELATIVISTIC SPINLESS PARTICLE

In this section, we summarize a general relativistic framework for the quantum mechanics of a spinless particle in the external gravitational field.

A. Classical Hamiltonian

The relativistic Lagrangian for a particle with mass m in the external gravitational field is given by

$$L=-mc\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}=-mc\sqrt{N^2-\gamma_{ij}(N^i+\dot{x}^i)(N^j+\dot{x}^j)}, \tag{3.1}$$

where an overdot denotes d/dt. Using the canonical momentum

$$p_i=\frac{\partial L}{\partial\dot{x}^i}, \tag{3.2}$$

we obtain the following classical Hamiltonian for the particle:

$$H\equiv p_i\dot{x}^i-L-mc^2=N\sqrt{m^2c^2+\gamma^{ij}p_ip_j}-N^ip_i-mc^2, \tag{3.3}$$

where the rest mass energy is subtracted from the conventional definition of the Hamiltonian for later convenience.

In this paper, we restrict ourselves to the case that the motion of the particle is nonrelativistic, $\gamma^{ij}p_ip_j\ll(mc)^2$. Then, the nonrelativistic Hamiltonian up to the order of our interest is

$$H=-N^ip_i+\left(\frac{N}{c}-1\right)mc^2+\frac{N}{c}\left[\frac{\gamma^{ij}p_ip_j}{2m}-\frac{(\gamma^{ij}p_ip_j)^2}{8m^3c^2}\right]+O\left(\frac{1}{c^4}\right). \tag{3.4}$$

B. Quantum Hamiltonian

Once we obtain the classical Hamiltonian, we follow the canonical quantization procedure to have the quantum Hamiltonian. The essential point is to replace the momentum p_i in Eq. (3.4) by the momentum operator $\hat{p}_i$, which satisfies the canonical commutation relation

$$[x^i,\hat{p}_j]=i\hbar\,\delta^i_j \tag{3.5}$$

and is Hermitian:

$$(\hat{p}_i\psi,\varphi)=(\psi,\hat{p}_i\varphi). \tag{3.6}$$

Note that we assume that the wave function is a scalar quantity. Therefore, the inner product which is invariant under the coordinate transformations in the three-dimensional sense is

$$(\psi,\varphi)\equiv\int\psi^*\varphi\sqrt{\gamma}d^3x. \tag{3.7}$$

From the definition of the inner product, we adopt the following expression for the momentum operator:

$$\hat{p}_j=-i\hbar\gamma^{-1/4}\frac{\partial}{\partial x^j}\gamma^{1/4}\equiv\gamma^{-1/4}\overline{p_j}\gamma^{1/4}, \tag{3.8}$$

where $\overline{p_j}$ is the momentum operator in flat space.

The quantum Hamiltonian is obtained by replacing p_i in Eq. (3.4) with $\hat{p}_i$ and taking the appropriate ordering (note that $\hat{p}_i$ does not in general commute with N and γ^{ij}). The result is

$$H=\frac{\overline{\mathbf{p}^2}}{2m}+m\phi-\boldsymbol{\omega}\cdot\mathbf{L}+\frac{1}{c^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot\mathbf{L}-\frac{\overline{\mathbf{p}^4}}{8m^3}+\frac{m}{2}\phi^2+\frac{3\phi\overline{\mathbf{p}^2}}{2m}\right), \tag{3.9}$$

where $\mathbf{L}=\mathbf{x}\times\overline{\mathbf{p}}$ is the angular momentum in flat space. The Schrödinger equation is then

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi. \tag{3.10}$$

Although the covariant definition of the inner product naturally arises in the general relativistic framework, it is sometimes convenient to adopt the same definition of the inner product as that in flat space, instead of the definition in Eq. (3.7). We can easily convert our result to this case in the following way:

$$\Psi' = \gamma^{1/4}\Psi, \quad H' = \gamma^{1/4}H\gamma^{-1/4}. \tag{3.11}$$

Now the wave function Ψ' is not a scalar under coordinate transformations, but a scalar density. Then, the expectation value of the Hamiltonian, for example, is

$$\langle H'\rangle \equiv \int \Psi'^* H' \Psi' d^3x. \tag{3.12}$$

Note that no factor of $\sqrt{\gamma}$ is now necessary. H' is now Hermite with respect to the inner product in flat space. Under this definition, the Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\Psi' = H'\Psi' = \left[\frac{\overline{\mathbf{p}}^2}{2m} + m\phi - \boldsymbol{\omega}\cdot\mathbf{L} + \frac{1}{c^2}\left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot\mathbf{L} - \frac{\overline{\mathbf{p}}^4}{8m^3} + \frac{m}{2}\phi^2 + \frac{3}{2m}\overline{\mathbf{p}}\cdot\phi\overline{\mathbf{p}}\right)\right]\Psi'. \tag{3.13}$$

The only difference between H and H' is the ordering of $\overline{\mathbf{p}}$ and ϕ in the last term. Each term in Eq. (3.13), including the last correction term, is now Hermitian in the sense, e.g.,

$$\int (\overline{\mathbf{p}}\cdot\phi\overline{\mathbf{p}}\psi')^*\varphi' d^3x = \int \psi'^*\overline{\mathbf{p}}\cdot\phi\overline{\mathbf{p}}\varphi' d^3x. \tag{3.14}$$

C. Phase difference in quantum interferometry

The Hamiltonian in Eq. (3.13) may be divided into parts

$$H' = H_0 + \sum_k \Delta H_k, \tag{3.15}$$

where $H_0 = \overline{\mathbf{p}}^2/(2m)$ is the Hamiltonian for a freely propagating particle in flat space. The Schrödinger equation (3.13) is then formally solved in the following way:

$$\Psi' = \Psi_0 \exp\left(i\sum_k \beta_k\right),$$

$$i\hbar\frac{\partial}{\partial t}\Psi_0 = H_0\Psi_0, \tag{3.16}$$

$$\beta_k = -\frac{1}{\hbar}\int^t \Delta H_k dt.$$

Therefore, for a pair of beams in a quantum interferometer which split into different paths, say, path A and path B, and then recombine, the phase difference at the interference point is

$$\Delta\beta_k = \beta_{k\,(\text{path B})} - \beta_{k\,(\text{path A})} = -\frac{1}{\hbar}\oint \Delta H_k dt. \tag{3.17}$$

Let us evaluate the phase difference due to each correction term in Eq. (3.13) in order.

In the first place, the gravitational potential term $\Delta H_1 = m\phi$ gives the phase shift

$$\Delta\beta_1 = \frac{m^2 g A \lambda}{2\pi\hbar^2}\sin\theta, \tag{3.18}$$

where g is the acceleration of gravity, A is the area inside the interferometry loop, λ is the de Broglie wavelength, and θ is the rotation angle of the interferometer relative to the horizontal plane. The theoretical prediction, Eq. (3.18), was first derived by Overhauser and Colella [4], and it was in good agreement with the observations using a neutron interferometer [3].

The next contribution is due to the rotation of Earth, $\Delta H_2 = -\boldsymbol{\omega}\cdot\mathbf{L}$. It is a quantum analogue of the Sagnac effect in optical interferometry. The phase difference due to the Sagnac effect is

$$\Delta\beta_2 = \frac{2m}{\hbar}\boldsymbol{\omega}\cdot\mathbf{A}, \tag{3.19}$$

where

$$\mathbf{A} = \frac{1}{2}\oint \mathbf{r}\times d\mathbf{r} \tag{3.20}$$

is the area vector of the interferometry loop. The phase shift is caused by the inertial force, and it does not depend on gravity. The phase difference, Eq. (3.19), was derived first by Page [5] from optical analogy, and later by other authors using various methods [6–8]. The experimental confirmation was made in Ref. [9]. The Sagnac effect was observed also in atomic interferometry [10].

The third term of the phase shift is a general relativistic effect due to the so-called Lense-Thirring effect:

$$\Delta\beta_3 = -\frac{4GMR^2 m}{5c^2\hbar}\boldsymbol{\omega}\cdot\oint\frac{\mathbf{r}\times d\mathbf{r}}{r^3} = \frac{2m}{5\hbar}\frac{r_g}{R}\left[\boldsymbol{\omega} - 3\left(\frac{\mathbf{R}}{R}\cdot\boldsymbol{\omega}\right)\frac{\mathbf{R}}{R}\right]\cdot\mathbf{A}, \tag{3.21}$$

where $\mathbf{R}$ is the position vector of the interferometer from the center of Earth, and $r_g \equiv 2GM/c^2$ is the Schwarzschild radius of Earth. Note that it is very similar to the Biot-Savart law in the classical electromagnetism. The form of the phase difference in Eq. (3.21) was derived and discussed by the authors of Ref. [11], including two of us.

The fourth correction term $\Delta H_4 = \overline{\mathbf{p}}^4/(8m^3c^2)$ in Eq. (3.13) is a purely special relativistic correction to the kinetic energy. It does not depend on the path, and thus does not produce the phase difference in the interferometer experiments.

The fifth contribution $\Delta H_5 = m\phi^2/(2c^2)$ can be regarded as the redshift correction to the potential energy. It produces the phase difference

$$\Delta\beta_5 = -\frac{1}{2}\frac{r_g}{R}\Delta\beta_1. \tag{3.22}$$

Finally, $\Delta H_6 = 3\overline{\mathbf{p}}\cdot\phi\overline{\mathbf{p}}/(2mc^2)$ is the redshift correction to the kinetic energy, which produces

$$\Delta\beta_6 = \frac{3}{2}\left(\frac{\lambda_C}{\lambda}\right)^2\Delta\beta_1, \tag{3.23}$$

where λ_C is the Compton wavelength.

It should be noted that the phase differences $\Delta\beta_5$ and $\Delta\beta_6$, due to the redshift corrections, have the same rotation angle dependence as that of $\Delta\beta_1$. Therefore, as far as the experiments are done only in different rotation angles, these post-Newtonian effects are not separable from the conventional Newtonian effect. On the other hand, the phase shift $\Delta\beta_3$ due to the Lense-Thirring effect depends on the orientation in a different way from that due to the Sagnac effect $\Delta\beta_2$. In particular,

$$\Delta\beta_3 = \frac{1}{5}\frac{r_g}{R}\Delta\beta_2 \quad \text{on the equator} \quad (\mathbf{R}\perp\boldsymbol{\omega}), \tag{3.24}$$

whereas

$$\Delta\beta_3 = -\frac{2}{5}\frac{r_g}{R}\Delta\beta_2 \quad \text{on the North Pole} \quad (\mathbf{R}\parallel\boldsymbol{\omega}). \tag{3.25}$$

Therefore, it is in principle separable from the Newtonian effect, if the experiments are done in different places on Earth. Until now, the Lense-Thirring effect has not yet been observed in any interferometer experiments. This is of course due to the smallness: The phase shift due to the Lense-Thirring effect is $r_g/R \sim 10^{-9}$ times smaller than that due to the Sagnac effect.

IV. CASE OF A SPIN-1/2 PARTICLE

In this section, we obtain the quantum Hamiltonian with the post-Newtonian corrections of Earth's gravity for a nonrelativistic spin-1/2 particle. We pay attention to the correspondence to the Hamiltonian for a spinless particle derived in the previous section.

A. Covariant Dirac equation in curved spacetime

Here we briefly review the formalism and define the notation used in this section (see also Ref. [14]). Let us start from the generally covariant Dirac equation in curved spacetime:

$$\left[-i\hbar\gamma^\mu\left(\frac{\partial}{\partial x^\mu}-\Gamma_\mu\right)+mc\right]\Psi=0, \tag{4.1}$$

where γ^μ is the covariant Dirac matrix satisfying the relations

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}I_4, \tag{4.2}$$

with the 4×4 unit matrix I_4, and Γ_μ is the spin connection.

It is convenient to introduce the orthogonal tetrad $e^\mu_{(a)}$ defined by

$$g_{\mu\nu}e^\mu_{(a)}e^\nu_{(b)}=\eta_{ab}, \tag{4.3}$$

where $\eta_{ab}=\text{diag}(c^2,-1,-1,-1)$. Then,

$$\gamma^\mu=\gamma^{(a)}e^\mu_{(a)}, \tag{4.4}$$

where $\gamma^{(a)}$ are the constant special-relativistic matrices defined by

$$\gamma^{(a)}\gamma^{(b)}+\gamma^{(b)}\gamma^{(a)}=2\eta^{ab}I_4. \tag{4.5}$$

The spin connection can also be expressed in terms of the tetrad and $\gamma^{(a)}$,

$$\Gamma_\mu=-\tfrac{1}{4}\gamma^{(a)}\gamma^{(b)}g_{\lambda\nu}e^\lambda_{(a)}\nabla_\mu e^\nu_{(b)}. \tag{4.6}$$

Based on the $3+1$ formulation, we choose the tetrad

$$e^\mu_{(0)}=c\left(\frac{1}{N},-\frac{N^i}{N}\right),\quad e^\mu_{(k)}=(0,e^i_{(k)}), \tag{4.7}$$

where the spatial triad $e^i_{(k)}$ is defined by

$$\gamma_{ij}e^i_{(k)}e^j_{(\ell)}=\delta_{k\ell}. \tag{4.8}$$

Consequently, the Dirac equation (4.1) can be rewritten as

$$i\hbar\frac{\partial}{\partial t}\Psi=H_D\Psi\equiv[(N\alpha^{(j)}e^i_{(j)}-N^i)(\overline{p_i}+i\hbar\Gamma_i)+i\hbar\Gamma_0 + Nmc\beta]\Psi, \tag{4.9}$$

where $\alpha^{(j)}$ and β are the constant Dirac matrices. Particularly in the case of the Minkowski spacetime, Eq. (4.9) immediately gives the well-known form

$$i\hbar\frac{\partial}{\partial t}\Psi=(c\boldsymbol{\alpha}\cdot\overline{\mathbf{p}}+mc^2\beta)\Psi. \tag{4.10}$$

The form of Eq. (4.9) is fully valid in general relativistic situation.

B. Nonrelativistic Hamiltonian

As was stated previously, we restrict our consideration to the case that the particle's motion is nonrelativistic. The nonrelativistic Hamiltonian is obtained by applying the Tani-Foldy-Wouthuysen (TFW) [12,13] transformation to H_D. The basic procedure is to perform an appropriate unitary transformation to obtain the ''even'' operator up to the order of our interest:

$$\tilde{H}_D=UH_DU^\dagger=\begin{pmatrix}\tilde{H}_+ & 0\\ 0 & \tilde{H}_-\end{pmatrix}+O\left(\frac{1}{c^4}\right). \tag{4.11}$$

The Dirac spinor is also divided into

$$\tilde{\Psi}=\begin{pmatrix}\Phi\\ \chi\end{pmatrix}, \tag{4.12}$$

where Φ and χ are called the ''large'' and ''small'' components, respectively. Using the reduced Hamiltonian $H_+\equiv\tilde{H}_+-mc^2$, the Schrödinger equation for the ''large'' component is

$$i\hbar\frac{\partial}{\partial t}\Phi=H_+\Phi. \tag{4.13}$$

Before applying the TFW transformation, it should be noted that the invariant scalar product in this case is

$$(\psi,\varphi)\equiv\int\psi^\dagger\varphi\sqrt{\gamma}d^3x. \tag{4.14}$$

Therefore, for the sake of convenience to compare with Eq. (3.13), we follow the same discussion as in the previous section and redefine the state vector and the Hamiltonian in the following way:

$$\Psi \to \gamma^{1/4}\Psi, \quad H_D \to \gamma^{1/4} H_D \gamma^{-1/4}. \tag{4.15}$$

Under this definition, we get the Schrödinger equation with post-Newtonian corrections for a spin-1/2 particle:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}\Phi &= H_+ \Phi \\ &= \Bigg[\frac{\overline{\mathbf{p}^2}}{2m} + m\phi - \boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) + \frac{1}{c^2} \\ &\quad \times \left(\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) - \frac{\overline{\mathbf{p}^4}}{8m^3} + \frac{m}{2}\phi^2 + \frac{3}{2m}\overline{\mathbf{p}}\cdot\phi\overline{\mathbf{p}} \right) \\ &\quad + \frac{1}{c^2}\left(\frac{3GM}{2mr^3}\mathbf{L}\cdot\mathbf{S} + \frac{6GMR^2}{5r^5}\mathbf{S}\cdot[\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})] \right) \Bigg] \Phi, \end{aligned} \tag{4.16}$$

where $\mathbf{S}=\hbar\boldsymbol{\sigma}/2$ is the spin of the particle with $\boldsymbol{\sigma}$ the Pauli matrix. The details of the calculations to derive Eq. (4.16) are summarized in the Appendix (see also Ref. [15]).

Comparing Eq. (4.16) with the spinless case Eq. (3.13), we observe the following points. First, setting $\mathbf{S}=0$ in Eq. (4.16) evidently gives the same form as Eq. (3.13), which assures that the canonical quantization procedure employed in the spinless case is fully consistent with the covariant generalization of the Dirac equation. Second, all terms in Eq. (4.16), except the last two terms, are obtained by simply replacing $\mathbf{L}$ in Eq. (3.13) by the total angular momentum $\mathbf{J}\equiv\mathbf{L}+\mathbf{S}$. Finally, the last two terms in Eq. (4.16) are unique in the spin-1/2 case: The former is a spin-orbit coupling term, and the latter represents a coupling between spin and Earth's rotation due to the dragging of inertia.

Before concluding this subsection, we make one comparison with a previous work. Bordé *et al.* [16] derived gravitational effects for atomic interferometry, starting from covariant equations for a two-level spin-1/2 atom interacting with laser fields in a gravitational background. Their result is mostly consistent with ours. But there is one difference. In their paper, all contribution of the spin, except the spin-orbit coupling, appear in the form of the total angular momentum $\mathbf{J} = \mathbf{L}+\mathbf{S}$. However, in our Eq. (4.16), the last term depends only on $\mathbf{S}$.

C. Phase difference in quantum interferometry

The phase differences due to the spin correction terms can also be calculated in the same way as that in the spinless case. Instead of doing the calculation, however, we will mention just a few points. Let us consider the interferometer experiments that the spin is constant along the paths in the interferometer. Then, the term $-\boldsymbol{\omega}\cdot\mathbf{S}$ does not produce the phase difference between the paths. Concerning the remaining terms, the relative orders of magnitude of the other spin correction terms to the orbital angular momentum terms are typically λ/ℓ, where λ is again the de Broglie wavelength and ℓ is a typical size of the interferometer loop. For neutron interferometers of the first generation, the typical values are $\lambda\sim10^{-8}$ cm and $\ell\sim$ 10 cm. Hence, for such interferometers, the effects of the spin corrections are generally 10^{-9} times smaller than those of the orbital angular momentum terms.

V. SUMMARY AND DISCUSSION

We have investigated the effect of general relativistic gravity on a quantum interferometer. We have followed the standard canonical quantization procedure to derive the curved space version of Schrödinger equations for spinless as well as spin-1/2 particles with the general relativistic corrections of Earth's gravity up to first post-Newtonian order. In particular we have studied the phase difference in a quantum interferometer due to the post-Newtonian corrections and spin-gravity coupling.

It has been discussed from the viewpoint of the equivalence principle that the effect of gravity is studied by the coordinate transformation of the flat space Schrödinger equation. This kind of discussion is useless for general-relativistic gravity since the coordinate transformation is unable to eliminate some of the general-relativistic gravity. Thus it is very interesting to see that the standard canonical quantization procedure may lead to the correct answer to the quantum phenomena in the presence of general-relativistic gravity. It is hoped that the next generation of quantum interferometer may have enough accuracy to detect the general-relativistic effect discussed in the present paper and answer to this question.

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APPENDIX

Here we summarize the details of the calculations to derive Eq. (4.16).

1. Tetrad

The tetrad is defined in Eqs. (4.7) and (4.8). Using Eq. (2.5), we obtain

$$\begin{aligned} e^0_{(0)} &= 1 - \frac{\phi}{c^2} + \frac{\phi^2}{2c^4}, \\ e^1_{(0)} &= \left[1 - \frac{1}{c^2}\left(\phi + \frac{4MR^2}{5r^3} \right) \right]\omega y, \\ e^2_{(0)} &= -\left[1 - \frac{1}{c^2}\left(\phi + \frac{4MR^2}{5r^3} \right) \right]\omega x, \\ e^3_{(0)} &= 0, \\ e^0_{(i)} &= 0, \\ e^j_{(i)} &= \left(1 + \frac{\phi}{c^2} \right)\delta^j_i . \end{aligned} \tag{A1}$$

2. Spin connection

The spin connection is defined by Eq. (4.6) or, equivalently, by

$$\Gamma_\mu = -\tfrac{1}{8}[\gamma^{(a)},\gamma^{(b)}]g_{\lambda\nu}e^{\lambda}_{(a)}\nabla_\mu e^{\nu}_{(b)}. \tag{A2}$$

We use the standard representation for the Dirac matrices:

$$\gamma^{(0)} = \frac{1}{c}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^{(i)} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{A3}$$

where I is the 2×2 unit matrix, and σ_i are Pauli's spin matrices.

It is convenient to use the 4×4 matrices

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{A4}$$

They satisfy the commutation relations

$$[\rho_1,\rho_2] = 2i\rho_3, \quad [\rho_2,\rho_3] = 2i\rho_1, \quad [\rho_3,\rho_1] = 2i\rho_2. \tag{A5}$$

It is apparent that terms proportional to ρ_1 or ρ_2 are ''odd'' whereas those proportional to ρ_3 are ''even.''

The components of the spin connection are

$$i\hbar\Gamma_0 = -\boldsymbol{\omega}\cdot\mathbf{S} + \frac{1}{2c}\rho_1\boldsymbol{\sigma}\cdot(\overline{\mathbf{p}}\phi) + \frac{1}{c^2}\left\{\frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot\mathbf{S} - \frac{GM}{r^3}\mathbf{S}\cdot[\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})] + \frac{6GMR^2}{5r^5}\mathbf{S}\cdot[\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})]\right\},$$

$$i\hbar\Gamma_1 = -\frac{1}{2c^2}(\phi_{,2}\sigma_3 - \phi_{,3}\sigma_2) + \frac{i}{c^3}\rho_1\frac{3GMR^2}{5r^5}\omega\{-2xy\sigma_1 + (x^2-y^2)\sigma_2 - yz\sigma_3\}, \tag{A6}$$

$$i\hbar\Gamma_2 = -\frac{1}{2c^2}(\phi_{,3}\sigma_1 - \phi_{,1}\sigma_3) + \frac{i}{c^3}\rho_1\frac{3GMR^2}{5r^5}\times\omega\{(x^2-y^2)\sigma_1 + 2xy\sigma_2 + zx\sigma_3\},$$

$$i\hbar\Gamma_3 = -\frac{1}{2c^2}(\phi_{,1}\sigma_2 - \phi_{,2}\sigma_2) + \frac{i}{c^3}\rho_1\frac{3GMR^2}{5r^5}\times\omega(-yz\sigma_1 + zx\sigma_2).$$

Note that the terms of $O(1/c^3)$ are also necessary to calculate the Hamiltonian correctly up to the order of $1/c^2$.

3. Hamiltonian

Using the above quantities, the Hamiltonian H_D defined in Eq. (4.9) is

$$H_D = \rho_3 mc^2 + c\rho_1\boldsymbol{\sigma}\cdot\overline{\mathbf{p}} + \rho_3 m\phi - \boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) + \frac{1}{c}\rho_1\left[-\frac{1}{2}\boldsymbol{\sigma}\cdot(\overline{\mathbf{p}}\phi) + 2\phi(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})\right] + \frac{1}{c^2}\left[\frac{1}{2}\rho_3 m\phi^2 + \frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) + \frac{6GMR^2}{5r^5}\mathbf{S}\cdot[\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})]\right]. \tag{A7}$$

The redefined Hamiltonian $H'_D \equiv \gamma^{1/4}H_D\gamma^{-1/4}$ is

$$H'_D = \rho_3 mc^2 + c\rho_1\boldsymbol{\sigma}\cdot\overline{\mathbf{p}} + \rho_3 m\phi - \boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) + \frac{1}{c}\rho_1[(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})\phi + \phi(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})] + \frac{1}{c^2}\left[\frac{1}{2}\rho_3 m\phi^2 + \frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) + \frac{6GMR^2}{5r^5}\mathbf{S}\cdot[\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})]\right]. \tag{A8}$$

The difference between H_D and H'_D is the second line on the right-hand side.

4. Tani-Foldy-Wouthuysen (TFW) transformation

It is instructive to divide the TFW transformation into two steps. The first step uses the unitary operator

$$U_1 = \exp\left(i\rho_2\frac{\boldsymbol{\sigma}\cdot\overline{\mathbf{p}}}{2mc}\right), \tag{A9}$$

which is introduced to make the odd term of $O(c)$ vanish. Using the useful formula

$$e^{iS}He^{-iS} = H + i[S,H] + \frac{i^2}{2!}[S,[S,H]] + \frac{i^3}{3!}[S,[S,[S,H]]] + \cdots \tag{A10}$$

and the commutation relation, Eq. (A5), we obtain the transformed Hamiltonian $U_1H'_DU_1^\dagger$ which is even up to $O(c^0)$:

$$U_1H'_DU_1^\dagger = \rho_3 mc^2 + \rho_3\left(\frac{\overline{\mathbf{p}}^2}{2m} + m\phi\right) - \boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) + \frac{1}{c}\rho_1\left[\frac{1}{2}[(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})\phi + \phi(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})] - \frac{1}{3m^2}(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})^3\right] + \frac{1}{c^2}\left[\rho_3\left(\frac{1}{2}m\phi^2 - \frac{\overline{\mathbf{p}}^4}{8m^3} + \frac{3}{2m}\overline{\mathbf{p}}\cdot(\phi\overline{\mathbf{p}}) + \frac{3GM}{2mr^2}\mathbf{L}\cdot\mathbf{S}\right) + \frac{4GMR^2}{5r^3}\boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) + \frac{6GMR^2}{5r^5}\mathbf{S}\cdot[\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})]\right]. \tag{A11}$$

The second step uses

$$U_2 = \exp\left(i\rho_2\frac{3m^2[(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})\phi + \phi(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})] - 2(\boldsymbol{\sigma}\cdot\overline{\mathbf{p}})^3}{12(mc)^3}\right), \tag{A12}$$

which makes the odd terms of $O(1/c)$ vanish. Then, we finally obtain Eq. (4.16).

[1] See, e.g., S. A. Werner and H. Kaiser, in *Quantum Mechanics in Curved Space-Time*, edited by J. Audretsch and V. de Sabbata (Plenum, New York, 1990), and references therein.
[2] See C. S. Adams, M. Sigel, and J. Mlynek, Phys. Rep. **240**, 143 (1994), and references therein.
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