

## Phase space approach to the gravitational arrow of time

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As the universe evolves, it becomes more inhomogeneous due to gravitational clumping. We attempt to find a function that characterizes this behavior and increases monotonically as inhomogeneity increases. We choose  $S = \ln\Omega$  as the candidate “gravitational entropy” function, where  $\Omega$  is the phase-space volume below the Hamiltonian  $H$  of the system under consideration. We perform a direct calculation of  $\Omega$  for transverse electromagnetic waves and gravitational wave, radiation, and density perturbations in an expanding FLRW universe. These calculations are carried out in the linear regime under the assumption that the phases of the oscillators comprising the system are random. Entropy is thus attributed to the lack of knowledge of the exact field configuration. The time dependence of  $H$  leads to a time-dependent  $\Omega$ . We find that  $\Omega$ , and, hence,  $\ln\Omega$  behaves as required. We also carry out calculations for Bianchi type IX cosmological models and find that, even in this homogeneous case, the function can be interpreted sensibly. We compare our results with Penrose’s  $C^2$  hypothesis. Because  $S$  is defined to resemble the fundamental statistical mechanics definition of entropy, we are able to recover the entropy in a variety of familiar circumstances including, evidently, black-hole entropy. The results point to the utility of the relativistic Arnowitt-Deser-Misner (ADM) Hamiltonian formalism in establishing a connection between general relativity and statistical mechanics, although fully nonlinear calculations will need to be performed to remove any doubt. [S0556-2821(97)06604-6]

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### I. INTRODUCTION

It has been recognized for some time that gravity behaves in an “antithermodynamic” fashion. Whereas ordinary thermodynamic systems, a gas for example, tend to become more homogenous with time, gravitating systems tend to become more inhomogeneous with time. The anomalous behavior can be viewed as a manifestation of the long-range nature of the gravitational force, which tends to cause the components of a gravitating system to clump. If we associate an increase in homogeneity with an increase in entropy for thermodynamic systems, then for gravitating systems an increase in entropy will imply an increase in inhomogeneity. The “gravitational arrow of time” points in the direction of increasing inhomogeneity.

There have, apparently, been only a few attempts in the literature to characterize the gravitational arrow of time. The most well known is the suggestion of Penrose [1] that “gravitational entropy” should be measured by  $C^2$ , the square of the Weyl tensor. Penrose hoped that the Weyl tensor would provide a measure of inhomogeneity and increase monotonically in time. This proposal met with some degree of success with a slight modification [2–4]. However, this “entropy” function is not well defined for all spacetimes; for example, conformally flat or vacuum models. Furthermore, Bonnor [5] has found an example in which the gravitational arrow points in the opposite sense when compared to the flow of radiation from a collapsing fluid, throwing doubt on

the entire proposal. There have been several other efforts to define the entropy of the gravitational field from various standpoints (Smolin [6], Hu and Kandrup [7], and Brandenberger *et al.* [8]), but none appear to have established an explicit connection to the Hamiltonian formulation of gravity, and none has addressed the arrow-of-time question.

In this paper we attack the problem of gravitational entropy by a direct approach. The goal is to find a function that behaves like entropy, i.e., that increases monotonically as a gravitating system becomes more inhomogeneous. We therefore choose a function that resembles entropy as much as possible:

$$S = \ln\Omega. \quad (1)$$

Here,  $S$  is gravitational entropy and  $\Omega$  is the volume of phase space for the system. (Unless stated otherwise, throughout the paper we use units in which  $\hbar = c = k = G = 1$ .)

For this choice we have reverted to the fundamental statistical mechanics’ definition of entropy. However, although we will refer to particle models, it is absolutely crucial to realize our goal is to characterize the phase space and entropy of the field itself, *not* of systems of particles.

There are several advantages and disadvantages to the above definition for gravitational entropy. For thermodynamic systems, a direct evaluation of the phase space is extremely difficult, if not impossible. Instead, one chooses the simpler path of evaluating the partition function,  $Z = \sum_i e^{-\beta E_i}$ , from which the entropy is readily derived as  $S = k(\ln Z + \beta \bar{E})$ , where  $\beta \equiv 1/(kT)$  and  $\bar{E}$  is the mean energy.

Here, however, we encounter the first conceptual difficulty in carrying over the procedure to relativity: To evaluate

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the partition function requires knowing the temperature of the system. In general relativity we usually deal with dynamical, not thermodynamic, systems, and a temperature is not well defined. A macroscopic pendulum executing simple harmonic motion, for example, constitutes a dynamical, not a thermodynamic system. Of course, one could assign an effective temperature  $kT \sim mv^2$  where  $v$  is the pendulum's velocity, but the system is nevertheless not in thermal equilibrium and so the concept of a partition function is not obviously useful.

However, a pendulum's motion does define a trajectory in phase space and it can be calculated without recourse to temperature. This is one of the two main reasons for reverting to the statistical mechanics expression for entropy [Eq. (1)]. The other, anticipating the application to cosmology, is that the phase space approach is intimately connected with Hamiltonian dynamics, and a Hamiltonian formalism of relativity [the Arnowitt-Deser-Misner (ADM) formalism [9]] already exists.

Now, normally, one is not interested in the absolute value of the entropy, but in the change of entropy with time. If the above pendulum were given a higher energy, it would include a larger phase space and the logarithm of the area between the two paths would formally resemble an entropy change. However, neglecting dissipative forces, the pendulum does not change its trajectory. Furthermore, the concept of entropy implies a loss of information, i.e., that we do not know the pendulum's location within this band. This requires that the system be ergodic, which is not true in the case of the pendulum.

On the other hand, if one imagines a system with  $N$  independent oscillators and assumes that their trajectories are uncorrelated, in other words, that their phases are random, then one should be able to compute an entropy via Eq. (1).

Thus our approach is basically simple: we calculate the phase space of dynamical systems in relativity, assuming that the trajectories of the components are independent and that consequently each region of phase space is occupied with equal probability. We then derive an entropy via Eq. (1). (The number of degrees of freedom need not be large as long as the system is chaotic, as in the case of Bianchi type IX cosmologies.)

Usually, one computes the entropy as the logarithm of a volume of phase space constrained by an energy  $E$ . Our approach, when applied to cosmological models, forces us to substitute the Hamiltonian  $H$  for  $E$ . This is the most natural and conservative extension of the usual definition but it should be emphasized that  $H$  does not always correspond to the energy. In most of the systems we consider,  $H$  will be time dependent, resulting in an entropy change.

The main advantage of the entire method is its conceptual clarity. The main disadvantage of the procedure is that it is technically cumbersome. However, we have found a number of systems for which the computation is tractable in the classical, perturbative limit. In these limits the entropy function does appear to increase or decrease monotonically when appropriate, and by suitable identification of parameters we recover the entropy familiar from a variety of circumstances including, evidently, black holes. In this way the universality of the phase-space concept is established. Extensions to the nonperturbative and quantum limits need to be carried out.

In Secs. II–IV of the paper we summarize some preliminary calculations necessary for what follows. In Sec. V we apply our method to the electromagnetic field. In Sec. VI and Sec. VII we treat gravitational waves and density perturbations. In Sec. VIII we give a more formal mathematical basis for the results of Sec. VI and Sec. VII. Section IX concerns Bianchi type IX. In Sec. X we compare our method with Penrose's  $C^2$  suggestion. In Sec. XI we argue that our entropy is indeed the Bekenstein-Hawking entropy under appropriate circumstances, and in Sec. XII we discuss further applications of our procedure.

## II. TWO PARTICLE MODEL

We now present a simple model to illustrate the basic approach. This model describes the Newtonian gravitational interaction between two particles, and although it is an extremely idealized particle model, it does highlight several important aspects of our treatment that will remain unchanged in the more complex field models.

Consider two particles, each of mass  $m$  free to slide on a one-dimensional frictionless track of length  $L$  with hard "bumpers" set at the two ends. The Hamiltonian for this system is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1, x_2), \quad (2)$$

with

$$V = - \frac{Gm^2}{\sqrt{(x_1 - x_2)^2 + r_0^2}}. \quad (3)$$

The factor  $r_0$  softens the potential and is introduced to avoid singularities. The system thus represents two objects that can pass through each other, such as colliding galaxies. In this case the Hamiltonian  $H = E_0$ , is the total energy, which remains constant.

The "air-track" model is a closed, isolated system and the available phase space can be computed. For two particles, however, the system cannot be regarded as ergodic and hence an entropy is not really defined. When generalized to  $N \gg 1$  particles and three dimensions the system may be regarded as a microcanonical ensemble and the particles can be assumed to be in random motion and equally likely to be found in any region of phase space. In that case an entropy would be well defined. Unfortunately, the complexity of a  $2N$ -dimensional phase space prohibits analytic solutions; hence, we demonstrate the basic results with just two particles then argue a connection to more general  $N$ -body systems.

For the two-particle case, we can write the phase space below energy  $E_0$  as

$$\begin{aligned} \Omega(E < E_0) = & \int_{-L/2}^{L/2} dx_1 \int_{x_{2\min}}^{x_{2\max}} dx_2 \int_{-\sqrt{2m(E-V)}}^{\sqrt{2m(E-V)}} dp_1 \\ & \times \int_{-\sqrt{2m(E-V)-p_1^2}}^{\sqrt{2m(E-V)-p_1^2}} dp_2. \end{aligned} \quad (4)$$

Note this procedure is similar to evaluating the volume of a four-sphere, although the exact topology and hence volume will depend on the form of  $V$ .

Generally one defines the accessible region of phase space as a shell between  $E_0$  and  $E_0 + \Delta E$ ; thus  $\Omega \equiv \Omega[E_0 < E < (E_0 + \Delta E)]$ . It is, however, easier to evaluate the full volume  $\Omega(E < E_0)$ , a procedure that we will follow throughout the paper. In the limit of a large number of degrees of freedom, the two results are identical, since most of the volume of an  $N$ -object resides infinitesimally near the surface.

The first important step in the phase-space procedure is to find the limits of integration, which are not always obvious. Because of the quadratic form of the momenta in the Hamiltonian (2), the momentum integrals give the volume of a two-sphere and the limits are set simply by requiring the  $p_i^2$  to be positive definite but constrained by the total energy of the system, as in Eq. (4). After evaluating the momentum integrals we have

$$\Omega(E < E_0) = 2\pi m \int_{-L/2}^{L/2} dx_1 \int_{x_{2\min}}^{x_{2\max}} (E - V) dx_2. \quad (5)$$

The lower and upper limits on  $x_2$  are set by restricting our attention to bound systems, such that  $E \leq 0$  and by requiring  $E - V \geq 0$ , which leads to

$$x_2 = x_1 \pm \sqrt{\frac{G^2 m^4}{E^2} - r_0^2}. \quad (6)$$

The entire volume of phase space can then be evaluated analytically and is found to be

$$\Omega(E < E_0) = 4\pi m^3 GL \left[ \sinh^{-1} \sqrt{\frac{V_0^2 - E_0^2}{E_0^2}} - \sqrt{\frac{V_0^2 - E_0^2}{V_0^2}} \right], \quad (7)$$

where  $V_0 = -Gm^2/r_0$  is the minimum potential.

Note several aspects of this result. For a *fixed form* of the potential  $V$ , there are only two ways to change the phase space: one must change either  $E_0$  or  $V_0$ . As expected, for a larger  $E_0$ , the particles are free to roam around in a larger region of phase space and thus  $\Omega$  increases. Were  $E_0$  to decrease due to dissipation, the particles would be confined to a smaller volume. This is one example of gravitational clumping. Now, to change  $V_0$ , one must change  $r_0$ , the softening parameter. Decreasing  $r_0$  makes the potential deeper and vice-versa. Thus, imposing a finite  $r_0$  has the effect of excluding a certain region of phase space compared to the usual gravitational potential, in which  $V \rightarrow -\infty$  as  $r \rightarrow 0$ . The dependence of  $\Omega$  on  $E_0$  and  $r_0$  is shown more explicitly in Fig. 1 where we plot phase space trajectories for the two particle system, assuming one particle to be stationary with zero momentum.

The parameter  $E_0$  merely determines whether the system is overall bounded. For the behavior of entropy in  $N$ -body systems with constant total energy  $E_0$ ,  $r_0$  is the relevant parameter. It is  $r_0$  that governs the clumping process within a bounded cluster of particles. In particular,  $r_0$  dictates the extent to which particles can form binaries. This is verified by a number of  $N$ -body simulations which have been per-

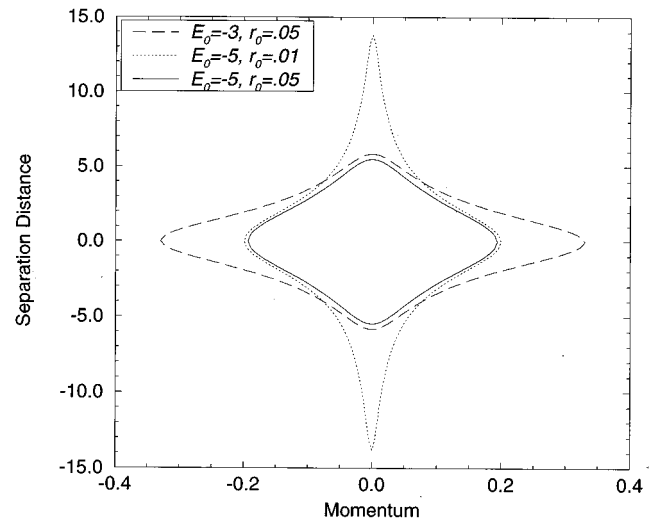


FIG. 1. Phase space trajectories for the two particle model, assuming one particle to be at rest (or equivalently more massive than the other) and  $m_1 = Gm_2 = 1$  with  $m_2 \gg m_1$ . Three different trajectories are displayed: a reference curve of intermediate energy  $E_0$  and softening parameter  $r_0$ , and two other curves varying  $E_0$  and  $r_0$  independently to increase the phase space volume relative to the reference curve.

formed for scenarios ranging from formation of star clusters to clusters of superclusters [10–12]. Invariably, the softening length sets the degree of clumping that is observed: As the softening length  $r_0$  is decreased, particles clump tighter and fall deeper into the central high density cores.

One can relate this behavior to the arrow-of-time question as follows: In  $N$ -body codes the softening length and the mesh discretization scale are equivalent insofar as that, below either, no clumping takes place. (The gravitational force tends to zero and we lose all clumping information.) Hence, by increasing  $r_0$ , the universe becomes effectively more homogeneous, and gravitational entropy decreases. In other words, by changing the mesh size, we change the entropy. This is an example of “coarse graining.”

Furthermore, in  $N$ -body simulations, some particles migrate to a dense central core at the expense of ejecting a few from the system. Thus, the minimum interparticle separation is a decreasing function of time. We would then expect the minimum separation to enter into time-dependent limits of the coordinate integrations. As the separation decreases, the depth of the potential well increases and the phase space also. In terms of the two-particle model, if we associate  $r_0$  with the minimum interparticle separation, then as  $r_0$  decreases, the absolute value between  $x_{2\min}$  and  $x_{2\max}$  in the limits of integration would increase and  $\Omega$  as well (Fig. 1).

To sum up, the “air-track” model is useful in that it exactly illustrates the calculational procedure we will follow, and by reasonable interpretation of  $r_0$  it correctly predicts the behavior of gravitational entropy in  $N$ -body simulations.

### III. HARMONIC OSCILLATOR

Consider a one-dimensional system of  $N$  simple harmonic oscillators with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^N \dot{\phi}_i^2 + V(\phi_i), \quad (8)$$

and potential

$$V(\phi_i) = \frac{\lambda}{2} \sum_{i=1}^N \phi_i^2. \quad (9)$$

With the replacement  $N \rightarrow 3N$  this may be regarded as a three-dimensional system with  $N$  oscillators in each of the  $x, y, z$  directions. The phase space for this system can be evaluated analytically. With the canonical momenta  $\pi_i = \dot{\phi}_i$  we have, for any form of potential,

$$\Omega = \int d\phi_N \cdots \int d\phi_1 \int_{-\ell_N}^{+\ell_N} d\pi_N \cdots \int_{-\ell_1}^{+\ell_1} d\pi_1, \quad (10)$$

with  $\ell_n^2 \equiv 2(H - V) - \sum_{i=n+1}^N \pi_i^2$ . If we further note that  $\ell_n^2 = \ell_{n+1}^2 - \pi_{n+1}^2$ , then each integral is of the form

$$\ell_n^n \int_0^1 x_n^{-1/2} [1 - x_n]^{(n-1)/2} dx_n = \ell_n^n \sqrt{\pi} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \quad (11)$$

where  $x_n = \pi_n^2 / \ell_n^2$ . (This integral is the so-called Beta function [13].)

The final result after  $N$  integrations over the momentum variables yields the volume of an  $N$ -sphere of radius  $\sqrt{H - V}$  and we have, for the remaining coordinate integrations,

$$\Omega = \frac{(2\pi)^{N/2}}{\Gamma\left(\frac{N+2}{2}\right)} \int_{-\ell_N}^{+\ell_N} d\phi_N \cdots \int_{-\ell_1}^{+\ell_1} d\phi_1 (H - V)^{N/2}, \quad (12)$$

where now  $\ell_n^2 = 2H/\lambda - \sum_{i=n+1}^N \phi_i^2$ . In the case of the harmonic oscillator, the  $\phi$ -integrals are identical in form to the  $\pi$ -integrals. Hence, one merely continues the procedure another  $N$  times, and noting that  $\ell_N^2 = 2H/\lambda$ , one gets, finally,

$$\Omega = \frac{(2\pi)^N H^N}{\lambda^{N/2} \Gamma(N+1)}. \quad (13)$$

The same result can be obtained more simply using  $N$ -dimensional spherical coordinates. However, the method outlined here can be applied to a more general class of potentials.

Because Eq. (13) will prove central to much of our analysis, it is worth convincing ourselves that the result is meaningful. We first note that  $\Omega$  decreases as  $\lambda$  increases, in accord with our notion that a stiffer spring constant confines the oscillators to a smaller region of phase space. Note also that  $\lambda \rightarrow 0 \Rightarrow \Omega \rightarrow \infty$ . This behavior is equivalent to that of classical free particles in an infinitely sized box. Indeed, by setting  $V=0$  in Eq. (12), performing each  $\phi$ -integral over the volume of the container and letting  $2^{N/2} \rightarrow (2m)^{N/2}$ , for massive particles, one can recover the usual expression for the entropy of an ideal gas.

As a further check on Eq. (13), we point out that if one makes the identification  $H = \bar{E} = N/\beta$  with  $\beta = 1/kT$ , then  $S = k \ln \Omega$  agrees in the classical limit with Einstein's formula for the entropy of  $N$  harmonic oscillators  $S = kN(1 - \ln \beta \omega)$ , where  $\omega = \sqrt{\lambda}$  [14].

Finally, the usual definition of phase space is the phase space of a shell around  $E_0$ :  $\Omega(\Delta E) = \Omega(E_0 < E < E_0 + \Delta E)$ . This is given by the differential of Eq. (13). From this differential one can derive the partition function

$$Z = \int_0^\infty \Omega(\Delta E) e^{-\beta E} dE = \left(\frac{2\pi}{\beta \omega}\right)^N. \quad (14)$$

Although this formula is not found in texts, if one calculates  $Z$  for  $N$  oscillators with the  $\zeta$ -function approach textbooks apply to free particles [15], one arrives at the same result.

With these checks it appears that Eq. (13) gives a reasonable and meaningful expression for the phase space of  $N$  harmonic oscillators. Perhaps the most important (and useful) feature of Eq. (13) is that it merely considers the amplitudes of the  $\phi_i$ . It ignores the phases. In fact, for its interpretation as phase space, we must assume random phases for the oscillators. Without this assumption, the motion of the system cannot be considered ergodic and the entropy is not defined. Effectively, we are regarding the oscillators as a microcanonical ensemble, in which one does not know the exact energy distribution. However, one could use the definition  $S = -\sum p_i \ln p_i$ , which to a high approximation is equivalent to  $S = \ln \Omega$ , and apply it to other distributions as well.

#### IV. NEAREST NEIGHBOR POTENTIAL

The technique used to derive Eq. (13) can be used without modification for other potentials of the form  $V \sim \phi^\alpha$  for even powers of  $\alpha > 0$ . In addition, as an important application to our analysis, we consider the "nearest-neighbor" potential with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^N \dot{\phi}_i^2 + \frac{\lambda}{2} \sum_{i=1}^N (\phi_i - \phi_{i+1})^2. \quad (15)$$

Note that the product  $\sqrt{\lambda}(\phi_i - \phi_{i+1})$  is a discrete approximation to the gradient  $\partial\phi/\partial x$ ; the spatial scale of the gradient is set by  $1/\sqrt{\lambda}$ .

With the substitution  $\eta_i \equiv \phi_i - \phi_{i+1}$ , the phase space for this nearest neighbor potential can be evaluated in the same way as the harmonic oscillator. After  $N$  integrations over the  $\phi_i$  and  $N-1$  integrations over  $\eta_i$  the result is

$$\Omega = \frac{(2\pi H)^{N-1/2}}{\lambda^{(N-1)/2} \Gamma(N+1/2)}. \quad (16)$$

The lower dimensionality of  $\Omega$  arises from the fact that the last  $\eta$  used in deriving Eq. (16) is  $\eta_{N-1} = \phi_{N-1} - \phi_N$ . The final integration, however, requires one to specify boundary conditions to ensure the dimensionality of momentum space equals that of coordinate space. A natural choice is periodic boundary conditions, such that  $\phi_{N+1} = \phi_1$ , then  $\eta_N = \phi_N - \phi_1$ . The assumption of periodic boundary conditions adds an extra  $(\phi_N - \phi_1)^2$  to the first integral, which can

be handled by ‘‘completing the square’’ and pushing the unwanted terms up to successively higher integrals. However, a rather tedious calculation shows that, surprisingly, the extra terms vanish after  $N-1$  integrations. In other words,  $\mathcal{L}_{N-1}^2 = 2H/\lambda$ . The dimension has not increased. One therefore is still required to specify limits on  $\phi_N$ , which we take to be  $\pm\sqrt{2H/\lambda}$ . Then, for periodic boundary conditions,

$$\Omega = \frac{2}{\sqrt{\pi N}} \frac{1}{\lambda^{N/2}} \frac{(2\pi H)^N}{\Gamma(N+1/2)}. \quad (17)$$

However, we note that one might instead impose ‘‘free-floating’’ boundary conditions such that  $\phi_{N+1} = \phi_N$ , and merely specify that the limits on  $\phi_N$  are  $\pm\sqrt{2H/\lambda}$ . In this case the result is the same as Eq. (17) but without the  $\sqrt{N}$  in the denominator. When logarithms are taken, both results are identical to the harmonic oscillator case except for insignificant numerical factors.

## V. APPLICATION TO THE ELECTROMAGNETIC FIELD

As a sample problem whose technique will carry over to the gravitational case, we now apply these results to the electromagnetic (EM) field. Because the EM field can be modeled as a collection of harmonic oscillators, we expect the phase space to reflect Eq. (13). To show this is the case we assume a constrained Hamiltonian of the form [16]

$$H = \frac{1}{2} \int (\mathcal{E}^2 + \mathcal{B}^2) d^3x, \quad (18)$$

where  $\mathcal{E}$  and  $\mathcal{B}$  are the electric and magnetic field densities. It is important to remember that in the Hamiltonian formalism, the canonical variables are not the densities but the full field quantities; in this case  $\pi \equiv E$  and  $q \equiv A$ , the vector potential.

To write Eq. (18) in terms of  $E$  and  $A$ , we first discretize  $H$  as follows:

$$H \approx \frac{1}{2} \sum_x^N \sum_y^N \sum_z^N (\mathcal{E}^2 + \mathcal{B}^2) \frac{L_x L_y L_z}{N_x N_y N_z} \Delta N_x \Delta N_y \Delta N_z. \quad (19)$$

Here, the sums are understood to be over the  $x$ ,  $y$ , and  $z$  coordinates covering the enclosed volume  $L_x L_y L_z$ . We have also approximated  $dx$  as  $L\Delta N/N$ , with  $N$  being the number of oscillators in each direction and  $\Delta N = 1$ . Let us further restrict our attention to transverse waves propagating in the  $z$ -direction. Then the  $x$  and  $y$  summations can be easily evaluated with the result

$$H = \frac{1}{2} N^2 \sum_{i=1}^N (\mathcal{E}^2 + \mathcal{B}^2) \frac{L^3}{N^3}, \quad (20)$$

where  $L^3 = L_x L_y L_z$  and  $N^3 = N_x N_y N_z$ .

Now,  $\mathcal{E} \sim \sqrt{e/L^3}$ , if  $e \equiv$  energy. Similarly, if  $\mathcal{A}$  is the potential density, then  $\mathcal{B} \sim \sqrt{e/L^3} \sim \nabla \times \mathcal{A} \sim \mathcal{A}/L$ . Hence  $\mathcal{E}\mathcal{A} \sim e/L^2$ .

The product of the canonical variables,  $EA$  must equal an action  $= eL$  in these units. Consequently,

$$\mathcal{E}\mathcal{A}L^3 = \text{action} = EA, \quad (21)$$

and the proper scaling becomes  $E = \mathcal{E}(L/N)^{3/2}$  and  $B = \mathcal{B}(L/N)^{3/2}$ . The Hamiltonian (20) can then be written as

$$\bar{H} = \frac{1}{2} \sum_{i=1}^N (E^2 + B^2), \quad (22)$$

where  $\bar{H} = H/N^2$ .

To evaluate the phase space below this Hamiltonian, we note that for transverse waves,  $\nabla \equiv \nabla_z$ ;  $E_z = B_z = 0$ ; and  $B = n \times E$ , where  $n$  is the propagation vector. Faraday’s law,  $\nabla \times (E + \dot{A}) = 0$ , implies that  $A_z = 0$  and  $B = -\hat{i}A_{y,z} + \hat{j}A_{x,z}$ . Thus the Hamiltonian can be approximated as

$$\bar{H} = \frac{1}{2} \sum_{i=1}^N [E_x^2(i) + E_y^2(i)] + V(A), \quad (23)$$

with potential

$$V(A) = \frac{\lambda}{2} \sum_{i=1}^N \{[A_y(i) - A_y(i+1)]^2 + [A_x(i) - A_x(i+1)]^2\}, \quad (24)$$

and where  $\lambda$  sets the spatial scale. We see that in this approximation,  $V$  is just given by the nearest-neighbor potential, calculated in Sec. IV, with the phase space given by Eq. (17). In this problem, however, the phase space is  $4N$  dimensions, hence substituting  $2N$  for  $N$  in Eq. (17) yields

$$\Omega_{EM} = \sqrt{\frac{2}{\pi N}} \frac{(2\pi\bar{H})^{2N}}{\lambda^N \Gamma(2N+1/2)}. \quad (25)$$

For the interpretation of  $\Omega$  as phase space we need to assume the phases of the electromagnetic waves are random, which merely means the source is incoherent. This is, in fact, the general case.

A closed solution to the problem requires an evaluation of  $N$ , the number of oscillators. Because we are primarily interested in the time dependence of  $\Omega$  (which is here time independent), it is enough to know that  $N$  is finite; in the quantum limit it will be the number of photons.

To make contact, however, with the usual expression for the entropy of electromagnetic radiation, we imagine transverse waves in a three-dimensional box, assuming each direction is independent. The phase space for that system is obtained by letting  $N \rightarrow 3N$  in the above equation. We assume that  $\bar{H} = 3N\epsilon$ , where  $\epsilon = \omega/(2\pi)$  is the average energy per oscillator and the coupling constant  $\lambda = \omega^2$ . With Stirling’s approximation  $\Gamma(N+1/2) \approx \sqrt{2\pi N} e^{-N} N^N$ , Eq. (25) yields  $S = \ln\Omega \approx 6N(1 - \ln 2)$ . For a photon gas at temperature  $T$ , the energy of most photons is of order  $\omega = \kappa \sim T$ , where  $\kappa$  is the wave vector. In three dimensions, the number of states is proportional to the volume in  $\kappa$  space for a sphere of radius  $|\kappa|$ . The mean number of photons at a temperature  $T$  is thus proportional to  $N \sim \kappa^3 \sim T^3$ , and we recover the usual scaling for the entropy  $S \sim N \sim T^3$ .

We also point out that Eq. (24) gives some insight into the question of coarse graining. The concept of entropy is subjective in the sense that to calculate an entropy requires that an averaging procedure be selected. If one regards the coupling constant in Eq. (24) to be  $\lambda = N^2/L^2$ , where  $L$  is an

arbitrary length scale, then by increasing  $L$ , one increases the volume per oscillator and hence increases the phase space, as can be seen from Eq. (25). Therefore the coarse graining scale evidently appears in these calculations as the coupling constant.

## VI. EXTENSION TO GRAVITATIONAL WAVES

The extension of the previous formalism to gravitational waves is fairly straightforward except for one crucial point, which we discuss below.

We first consider inhomogeneous perturbations of the spatially flat metric

$$ds^2 = a^2(\eta) \{ -d\eta^2 + [\delta_{ij} + h_{ij}(\eta, z)] dx^i dx^j \}, \quad (26)$$

where  $\eta$  is the conformal time,  $a(\eta)$  is the expansion scale factor and the  $h_{ij} \ll \delta_{ij}$  represent gravitational wave perturbations. Their equation of motion can be found by expanding the Einstein action to second order in the perturbation variables  $h_{\mu\nu}$ . The result is

$$I = \frac{1}{64\pi} \int a^2 (\dot{h}^2 - h'^2) d^4x, \quad (27)$$

where  $(\dot{\cdot}) \equiv d/d\eta$  and  $(\prime) \equiv d/dz$ . Equation (27) is the action appropriate for singly polarized gravitational waves in the transverse traceless gauge. The variable  $h$  ( $\equiv h_{xx} = -h_{yy}$ ) represents the single degree of freedom for the + polarization state. (Brandenberger *et al.* [8] have shown that a similar form is achieved even when one considers two polarizations.)

We will find it convenient (particularly when making a connection to density perturbations) to work with a transformed perturbation function  $\phi = ah/\sqrt{32\pi}$ . The Lagrangian density can then be written as

$$\mathcal{L} = \frac{1}{2} \left[ \dot{\phi}^2 - \phi'^2 + \frac{\ddot{a}}{a} \phi^2 \right]. \quad (28)$$

By definition, if  $\phi$  is the canonical coordinate, then  $\pi \equiv \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$ , and the Hamiltonian density  $\mathcal{H} \equiv \pi\dot{\phi} - \mathcal{L}$  is found to be

$$\mathcal{H} = \frac{1}{2} \left[ \dot{\phi}^2 + \phi'^2 - \frac{\ddot{a}}{a} \phi^2 \right]. \quad (29)$$

Following the same procedure used in the EM case, we find, for the Hamiltonian,

$$\bar{H} = \frac{1}{2} \sum_{i=1}^N \left( \bar{\pi}_i^2 + \bar{\phi}_i'^2 - \frac{\ddot{a}}{a} \bar{\phi}_i^2 \right), \quad (30)$$

where  $\bar{\pi} = \pi(L/N)^{3/2}$ ,  $\bar{\phi} = \phi(L/N)^{3/2}$ ,  $\bar{H} = H/N^2$ , and  $H = \int \mathcal{H} d^3x$ . We see that the Hamiltonian contains potentials similar to the others we have considered with one crucial difference: the sign on the last term. In the matter dominated period,  $a \sim \eta^2$  and  $\ddot{a}/a > 0$ , hence the sign on the quadratic term in Eq. (30) is negative and we have a nearest-neighbor potential plus an ‘‘antiharmonic oscillator’’ (or inverted) potential.

The inverted nature of this term is due to the background curvature of spacetime and its rate of expansion. The potential, then, serves as a reflection barrier in an unbounded phase space. Any calculation must therefore include an arbitrary cutoff. We discuss this point in detail in Sec. VIII. There we show that  $\Omega$  can be calculated by the use of hypergeometric functions with a result that is formally similar to that already achieved for the harmonic oscillator and we can continue to use  $\Omega \sim H^N/\lambda^{N/2}$  to compute the time dependence of  $\Omega$ .

To compute  $\Omega(\eta)$  we imagine that  $\Omega$  is constant on each hypersurface of constant time. Thus  $\dot{a}/a$  can be taken as  $\lambda$ , the coupling constant. We then need to compute  $\bar{H}(\eta)$  and  $\lambda(\eta)$ . To find  $\bar{H}(\eta)$  note that the equation of motion for  $h$  resulting from varying the action (27) is

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} - h'' = 0. \quad (31)$$

Because the waves are linear perturbations, they do not interact except through a linear superposition. The time development of each individual component (or a wave of a particular frequency or phase) evolves according to Eq. (31), which can be regarded as representing a family of solutions. That is, we may assume a separable solution

$$\sum_j \left( \ddot{h}_j + \frac{2\dot{a}}{a}\dot{h}_j - h_j'' \right) = \left( \ddot{h} + \frac{2\dot{a}}{a}\dot{h} - h'' \right) \sum_j e^{i\alpha_j} = 0, \quad (32)$$

with arbitrary or random phases  $\alpha_j$ , where here  $j$  is an index over the different waves (not over  $z$ ). This is, in effect, saying that different spatial regions are taken to be oscillating independently of one another, or that the source is incoherent. We therefore assume the perturbations to be random, and that the field variables describe, not a singly polarized wave, but an ensemble of incoherent plane waves. Entropy is thus attributed to the lack of knowledge in the exact field configuration.

With  $a = a_0 \eta^2$  for the matter dominated period and assuming  $h \sim e^{ikz}$ , Eq. (31) has the solutions

$$h \propto \eta^{-3/2} J_{\pm 3/2}(k\eta) e^{ikz}, \quad (33)$$

where  $J_{\pm 3/2}$  are Bessel functions. To construct the Hamiltonian (30), we then sum over the coordinate  $z$ . Simplified expressions can be obtained by substituting the standard asymptotic ( $k\eta \ll 1$  and  $k\eta \gg 1$ ) forms of  $J_{\pm 3/2}$ . We can then write for the metric perturbations in the limit  $k\eta \ll 1$

$$h = [h_1(k\eta)^{-3} + h_2] e^{ikz}, \quad (34)$$

where  $h_1$  and  $h_2$  are constants and  $\phi = ah \propto (h_1 \eta^{-1} + h_2 \eta^2) e^{ikz}$ . The constants  $h_1$  and  $h_2$  can thus be interpreted as defining the decaying and growing mode solutions, respectively. In this limit spatial gradients are negligible. For  $k\eta \gg 1$ , we have

$$h \propto \sqrt{\frac{2k^3}{\pi}} \left( \frac{1}{k\eta} \right)^2 [\cos(k\eta) + \sin(k\eta)] e^{ikz} \\ \propto (k\eta)^{-2} \times [\text{oscillations}]. \quad (35)$$

and  $\phi \propto \text{constant} \times [\text{oscillations}]$ . We note that  $k\eta \gg 1$  ( $k\eta \ll 1$ ) represents perturbations with wavelengths much shorter (longer) than the Hubble radius (usually referred to as the ‘horizon’).

The Hamiltonian (30), in the limit  $k\eta \gg 1$ , then becomes  $H \propto \pi^2 + k^2 \phi^2$  which is simply the harmonic oscillator Hamiltonian at a fixed time with coupling constant  $k$ .  $H$  therefore oscillates in time at constant amplitude and we have for the phase space

$$\Omega \propto \frac{H^N}{k^{N/2}} \propto \text{constant} \times [\text{oscillations}]. \quad (36)$$

As expected in this approximation, the phase space does not change. Recall that  $H$  is defined on a single time slice. However, assuming incoherency in time as well as in space, one can average  $H$  over several cycles by defining a general four-Hamiltonian

$${}^{(4)}\mathcal{H} = \int d^4x \left( \pi^2 + k^2 \phi^2 - \frac{\ddot{a}}{a} \phi^2 \right). \quad (37)$$

Over intervals of time greater than the dynamical time, this will be a monotonic function and, in the  $k\eta \gg 1$  case,  $\Omega$  will be strictly constant. However, in a nonlinear regime,  $\Omega$  would increase, which is encouraging for the interpretation of  $\ln \Omega$  as entropy.

For  $k\eta \ll 1$ , spatial gradients are, again, negligible and we have  $H \propto \pi^2 - \ddot{a} \phi^2 / a$  and  $\Omega \propto H^N / (\ddot{a}/a)^{N/2}$ . Therefore

$$H \propto \begin{cases} \eta^2, \\ \eta^{-4}, \end{cases} \quad \text{and} \quad \Omega \propto \begin{cases} \eta^{3N} & \text{for growing modes,} \\ \eta^{-3N} & \text{for decaying modes,} \end{cases} \quad (38)$$

with the caveat that we have not yet shown (see Sec. VIII) that for the inverted oscillator this form of  $\Omega$  is justified.

At first these last results strike one as strange because as seen from Eq. (34), for growing modes  $h$  is frozen-in at superhorizon scales. That is, there are no oscillations and the assumption of random phases is not well motivated. In that case, the phase space trajectories are known precisely and  $\Omega = 0$ . One can see this clearly by examining the Hamiltonian in the variable  $h$ . For superhorizon growing modes, this Hamiltonian is zero, and the increase in  $\Omega$  above is entirely due to the expansion of the universe (i.e.,  $\dot{\phi} = \dot{a}h + a\dot{h} = \dot{a}h$ ). Only the  $\dot{a}$  term causes  $\phi$  to increase in amplitude, and hence increases the effective coarse graining scale and  $\Omega$ , in accord with the gravitational arrow of time. (The decaying modes on superhorizon scales also no longer oscillate but damp out monotonically at a rate faster than the universe expands; the associated entropy thus decreases.) We will discuss the significance of the growing and decaying modes further in Sec. VIII B. For now we point out that, certainly for the growing modes,  $\Omega$  is nonzero only if one continues to regard the phases as random. Otherwise, if the phases are assumed known, then the entropy is zero (or constant), in agreement with Brandenberger *et al.* [8]. It would be of interest to establish a more quantitative comparison of our results to Brandenberger *et al.*

These considerations suggest that our definition of entropy is only appropriate for subhorizon scales. This may

actually be an advantage, because in order to ensure that  $\Omega$  is finite, we must ensure that the number of modes  $N$  must also be finite. This requires us to put the system in a box and consider only a finite spatial region. The horizon thus provides a natural upper limit to wavelengths. An absolute lower limit can obviously be chosen as the Planck scale. We will find that similar considerations are necessary for radiation perturbations (below).

## VII. DENSITY PERTURBATIONS

The analysis of the previous section can be repeated for density perturbations in dust- and radiation-filled models. In the longitudinal gauge, the spacetime metric is written as

$$ds^2 = a^2(\eta) \{ -[1 + 2\Phi(\eta, z)] d\eta^2 + [1 - 2\Phi(\eta, z)] \gamma_{ij} dx^i dx^j \}, \quad (39)$$

where

$$\gamma_{ij} = \delta_{ij} \left[ 1 + \frac{\mathcal{K}}{4} (x^2 + y^2 + z^2) \right]^{-2}, \quad (40)$$

$\Phi$  is the gauge invariant gravitational potential, and  $\mathcal{K} = 0, -1, +1$  for flat, open, and closed universes, respectively. Mukhanov *et al.* [17] give the following general equation for adiabatic density perturbations:

$$\ddot{u} - c_s^2 u'' - \frac{\dot{\theta}}{\theta} u = 0, \quad (41)$$

where

$$u = \frac{a\Phi}{\sqrt{4\pi}} \left( 2 \frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right)^{-1/2}, \quad (42)$$

$$\theta = \sqrt{\frac{3}{2}} \frac{\dot{a}}{a^2} \left( 2 \frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right)^{-1/2}, \quad (43)$$

and  $(\cdot) \equiv d/d\eta$ ,  $(\prime) \equiv d/dz$ , and  $c_s^2$  is  $1/3$  for radiation and zero for dust. The corresponding action from which the (ADM) Hamiltonian and equations of motion are derived is given by writing the Einstein action

$$I = - \frac{1}{16\pi G} \int R \sqrt{-g} d^4x - \int e \sqrt{-g} d^4x, \quad (44)$$

where  $e$  is the energy density of matter, in terms of the ADM metric and expanding to second perturbative order.

### A. $\mathcal{K}=0$ , flat universe

#### 1. $c_s^2=1/3$ , radiation

For radiation,  $a \sim \eta$  and  $\dot{\theta}/\theta = 2/\eta^2$ . Assuming the spatial form  $u(z) \sim e^{ikz}$ , the field equation (41) becomes

$$\ddot{u} - \frac{2}{\eta^2} u + \frac{k^2}{3} u = 0. \quad (45)$$

The general solution to Eq. (45) involves Bessel functions similar to Eq (33). The asymptotic superhorizon ( $k\eta \ll 1$ ) solutions are

$$u = (u_1 \eta^2 + u_2 \eta^{-1}) e^{ikz}, \quad (46)$$

representing both growing and decaying modes. As in the case for gravitational waves, these solutions are taken to be a family of functions with random phase angles  $\alpha$ . From now on the presence of these phase angles is understood but we do not write them out explicitly. In addition, we note that the results presented here are independent of the exact form of perturbations, and we could replace  $e^{i(kz)\Sigma_j} e^{i\alpha_j}$  with an arbitrary function of the three spatial coordinates.

For  $k\eta \gg 1$

$$u \propto \left( \cos \frac{k\eta}{\sqrt{3}} + \sin \frac{k\eta}{\sqrt{3}} \right) e^{ikz}, \quad (47)$$

and, as for gravitational waves, perturbations on these subhorizon scales are oscillatory.

The Hamiltonian for the case  $k\eta \ll 1$  is  $H \propto \pi^2 - \ddot{\theta} u^2 / \theta$ , which results in the following evolution

$$H \propto \begin{cases} \eta^2, \\ \eta^{-4}, \end{cases} \quad \text{and} \quad \Omega \propto \begin{cases} \eta^{3N} & \text{for growing modes,} \\ \eta^{-3N} & \text{for decaying modes.} \end{cases} \quad (48)$$

For  $k\eta \gg 1$ , we have  $H \propto \pi^2 + c_s^2 u'^2$  which oscillates at constant amplitude, and therefore  $\Omega$  is constant over sufficiently long intervals of time. Notice that these results for radiation perturbations are identical to those of gravitational waves and the remarks concerning the superhorizon application of our definition of entropy apply here as well.

## 2. $c_s^2 = 0$ , dust

In this case  $a = a_0 \eta^2$  and  $\ddot{\theta}/\theta = 6/\eta^2$ . Then Eq. (41) becomes

$$\ddot{u} - \frac{6}{\eta^2} u = 0, \quad (49)$$

with solution

$$u = (u_1 \eta^3 + u_2 \eta^{-2}) e^{i(kz)}, \quad (50)$$

or equivalently

$$\Phi = (\bar{u}_1 + \bar{u}_2 \eta^{-5}) e^{i(kz)}, \quad (51)$$

where  $u_1, u_2, \bar{u}_1$  and  $\bar{u}_2$  are constants.

Notice the important point that Eq. (49) does not suggest a natural scale (the horizon in particular) for modes to grow or decay as found in the gravitational wave and radiation cases. However, the horizon scale does appear in the expression for the density fluctuations  $\delta\rho/\rho$  [17]

$$\frac{\delta\rho}{\rho} = \frac{1}{6} [ - (k^2 \eta^2 + 12) \bar{u}_1 - (k^2 \eta^2 - 18) \eta^{-5} \bar{u}_2 ] e^{i(kz)}. \quad (52)$$

Thus, in distinction to the previous cases, one should not examine  $\Phi$  to determine whether the modes are frozen-in or not. One should rather examine Eq. (52). We see that on subhorizon scales ( $k\eta \gg 1$ )  $\delta\rho/\rho$  exhibits growing modes, i.e., actually collapse takes place even while  $\Phi$  remains constant. On scales larger than the horizon,  $\delta\rho/\rho$  remains constant for the dominant growing modes. Given that  $\delta\rho/\rho$  is constant on superhorizon scales for growing modes, this once again suggests that our definition of entropy should be restricted to subhorizon regimes, consistent with our earlier results.

For  $\mathcal{K} = 0$  dust we have simply  $H \propto \pi^2 - \ddot{\theta} u^2 / \theta$ . From Eq. (50) and the results from Sec. VIII B for the inverted oscillator potential, we find

$$H \propto \begin{cases} \eta^4, \\ \eta^{-6}, \end{cases} \quad \text{and} \quad \Omega \propto \begin{cases} \eta^{5N} & \text{for growing modes,} \\ \eta^{-5N} & \text{for decaying modes.} \end{cases} \quad (53)$$

We again point out that the fact that  $H$  and  $\Omega$  are time dependent while the conformal metric components (or equivalently  $\Phi$ ) are constant is not a contradiction. The growth of  $H$  and  $\Omega$  is an indication that collapse is taking place on some (subhorizon) scale. Note once more the monotonic behavior of these quantities.

## B. $\mathcal{K} = \pm 1$ , dust-filled open and closed universes

The gauge invariant potential in the case of a dust-filled open universe can be written as [17]

$$\Phi = c_1(z) \frac{2 \sinh^2 \eta - 6 \eta \sinh \eta + 8 \cosh \eta - 8}{(\cosh \eta - 1)^3} + c_2(z) \frac{\sinh \eta}{(\cosh \eta - 1)^3} \quad (54)$$

where  $a \sim (\cosh \eta - 1)$  is the expansion factor,  $c_1 \equiv \bar{u}_1 e^{ikz}$ , and  $c_2 \equiv \bar{u}_2 e^{ikz}$ . Also

$$\frac{\delta\rho}{\rho} = \frac{1}{3} [ (\cosh \eta - 1) \nabla^2 \Phi + 9\Phi - 6c_1 ]. \quad (55)$$

Expanding Eq. (54) in the small time limit  $\eta \ll 1$ , we obtain the flat space solution (51). Eqs. (43) and (42), taken together with the approximate asymptotic solution for the scale factor  $a \sim \eta^2$ , yield the same result for  $H$  and  $\Omega$  as the flat space case (53). In the opposite late time limit,  $\eta \gg 1$ , the hyperbolic functions become exponentials and  $\Phi \propto c_1 e^{-\eta} + c_2 e^{-2\eta}$ . As expected,  $\Phi$  decays in this limit, and Eq. (55), along with the asymptotic form of the scale factor  $a \sim e^\eta$ , shows that the matter density fluctuations do not grow on either super- or subhorizon scales. Equations (43) and (42) yield  $\ddot{\theta}/\theta = \text{constant}$  and  $u \propto c_1 + c_2 e^{-\eta}$ .  $H$  and  $\Omega$  then evolve as

$$H \propto \begin{cases} \text{constant,} \\ e^{-2\eta}, \end{cases} \quad \text{and} \quad \Omega \propto \begin{cases} \text{constant} & \text{for growing modes,} \\ e^{-2N\eta} & \text{for decaying modes.} \end{cases} \quad (56)$$



This is consistent with the fact that  $\delta\rho/\rho = \text{constant}$  for the dominant modes and no collapse takes place on either sub- or superhorizon scales.

The growing modes for the closed model ( $\mathcal{K} = +1$ ) can be obtained by letting  $\eta \rightarrow i\eta$  in Eq. (54). In this case  $0 < \eta < 2\pi$ . For  $\eta \ll 1$ , the closed model gives the same result as the open and flat cases. Equation (55) also holds for the closed model, and so the same consistency among the behaviors of  $H$ ,  $\Omega$ , and  $\delta\rho/\rho$  is found here as well. There is no asymptotic limit  $\eta \gg 1$  in the  $\mathcal{K} = +1$  case. However, by expanding Eq. (54) around  $\eta = \pi$ , we find to lowest order that  $\Phi \propto u \sim \text{const}$ ,  $\dot{\theta}/\theta \sim \text{const}$ , and that therefore  $H$  and  $\Omega$  are constant as well, again consistent with the density perturbations  $\delta\rho/\rho \sim \text{const}$ . In the neighborhood of maximum expansion, then, the model acts like the open case. As recollapse takes place, one finds that toward the singularity  $\eta \sim 2\pi$ ,  $\Phi \propto \epsilon^{-5}$ ,  $u \propto \epsilon^{-2}$ , and  $\dot{\theta}/\theta \propto \epsilon^{-2}$ , where  $\epsilon = 2\pi - \eta$ . These dependences yield  $H \propto \epsilon^{-6}$  and  $\Omega \propto \epsilon^{-5N}$ . Since  $\epsilon$  is decreasing, all these quantities are increasing, as expected;  $\delta\rho/\rho$  grows again as well. Taken with the behavior at  $\eta \sim 0$  and  $\eta \sim \pi$  we see that  $H$  and  $\Omega$  behave monotonically as required.

## VIII. ANTIHARMONIC OSCILLATOR POTENTIAL

### A. Time dependence of $\Omega$

We now consider in detail a Hamiltonian of the form found in Eq. (30) without the gradient term, that is,

$$H = \lambda \sum_{i=1}^N x_i^2 - \xi \sum_{i=1}^N y_i^2. \quad (57)$$

We first justify the expressions found for  $\Omega$  in Sec. VI, then interpret the meanings of  $H < 0$  and  $H > 0$  for the inverted oscillator potential in Sec. VIII B. The inverted nature of the potential in Eq. (57) results in a reflection barrier and a phase space that is unbounded. To compute  $\Omega$ , then, we will need to put in arbitrary cutoffs to the allowable phase space. How this is done will become clearer below.

First consider the case when  $H > 0$ . From Eq. (57) the  $y_i$  then correspond to the canonical coordinates  $\phi_i$  and the  $x_i$  correspond to the previous  $\pi_i$ . One can easily evaluate  $\Omega$  using  $N$ -dimensional spherical coordinates. Integration over the  $y_i$  coordinates yields

$$\Omega_y = \frac{\pi^{N/2}}{\Gamma\left(\frac{N+2}{2}\right)} \left(\frac{\lambda}{\xi}\right)^{N/2} \left[ \sum_{i=1}^N x_i^2 - \frac{H}{\lambda} \right]^{N/2}, \quad (58)$$

and then over the  $x_i$

$$\begin{aligned} \Omega &= \frac{\pi^N N}{\Gamma\left(\frac{N+2}{2}\right)\Gamma\left(\frac{N+2}{2}\right)} \left(\frac{H}{\xi}\right)^{N/2} \int_{R^2 \geq H/\lambda} \left[ \frac{\lambda}{H} R^2 - 1 \right]^{N/2} \\ &\quad \times R^{N-1} dR, \end{aligned} \quad (59)$$

where  $R^2 \equiv \sum_{i=1}^N x_i^2$ .

This integral will be unbounded as  $R^2 \rightarrow \infty$ . We therefore let  $R_{\text{max}}^2 \equiv \text{maximum}(\sum_{i=1}^N x_i^2)$  be the assumed cutoff in

$x$ -space (which here corresponds to momentum space). Further defining  $u_{\text{max}} \equiv R_{\text{max}}^2 \lambda / H$  and  $w = (u - 1)/(u_{\text{max}} - 1)$  the above expression becomes

$$\begin{aligned} \Omega &= \frac{\pi^N H^N}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N+2}{2}\right)} \left(\frac{1}{\xi\lambda}\right)^{N/2} (u_{\text{max}} - 1)^{(N+2)/2} \\ &\quad \times \mathcal{F}\left(\frac{2-N}{2}, \frac{2+N}{2}, \frac{4+N}{2}, 1 - u_{\text{max}}\right), \end{aligned} \quad (60)$$

where  $\mathcal{F}$  is a hypergeometric function

$$\begin{aligned} \mathcal{F}\left(\frac{2-N}{2}, \frac{2+N}{2}, \frac{4+N}{2}, 1 - u_{\text{max}}\right) \\ = \int_0^1 [1 + (u_{\text{max}} - 1)w]^{(N-2)/2} w^{N/2} dw. \end{aligned} \quad (61)$$

The function  $\mathcal{F}$  is absolutely convergent for  $|u_{\text{max}} - 1| \leq 1$ . In that limit, the power series [13] for  $\mathcal{F}$  gives

$$\Omega \approx \frac{2\pi^N}{(N+2)} \frac{H^{(N-2)/2}}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N+2}{2}\right)} \frac{\lambda}{\xi^{N/2}} R_{\text{max}}^{N+2}, \quad (62)$$

which can also be obtained by direct integration of Eq. (60) if one keeps only the  $w^{N/2}$  term in the integrand. Thus to evaluate the time dependence of  $\Omega$  we will consider

$$\Omega \propto \frac{R_{\text{max}}^{N+2}}{k^{N/2}} H^{(N-2)/2} \quad \text{for } R_{\text{max}}^2 = \text{const} \sim \frac{H}{\lambda}, \quad (63)$$

where  $k \equiv 2\xi$  is the ‘‘spring constant.’’ With the definition of  $u_{\text{max}}$  this can be rewritten as

$$\Omega \propto \frac{H^N}{k^{N/2}} \quad \text{for } |u_{\text{max}} - 1| \sim 1, \quad (64)$$

assuming  $u_{\text{max}}$  is constant. That is, which form used depends on which variable is assumed constant. Conceptually, it is easier to visualize the meaning of  $R_{\text{max}}$ , which puts an absolute limit on the allowable momentum of oscillations. We stress that these limits are meant to be constant in time. If, however, we allow  $R_{\text{max}}$  to evolve with  $H$ , then we get the condition  $u_{\text{max}} = \text{constant}$  in time. The parameter  $u_{\text{max}}$  effectively scales the ratio of the allowable kinetic energy to the total energy  $H$ .

We can also derive an analogous scaling in the opposite limit,  $R_{\text{max}}, u_{\text{max}} \rightarrow \infty$ . In this case, Eq. (60) is approximately

$$\Omega \rightarrow \frac{\pi^N}{2\Gamma\left(\frac{N+2}{2}\right)\Gamma\left(\frac{N+2}{2}\right)} \left(\frac{\lambda}{\xi}\right)^{N/2} R_{\text{max}}^{2N} \propto \frac{R_{\text{max}}^{2N}}{k^{N/2}} \propto \frac{H^N u_{\text{max}}^N}{k^{N/2}}. \quad (65)$$

Now we turn to  $H < 0$ . In this case the meanings of the  $x_i$  and  $y_i$  in Eq. (57) are reversed. If we let  $H \equiv |H|$ , then a repetition of the previous analysis gives

$$\Omega \sim \begin{cases} H^N u_{\max}^N / k^{N/2}, & u_{\max} = \text{const}, \\ k H^{(N-2)/2} R_{\max}^{N+2}, & R_{\max}^2 = \text{const} \sim H/k, \\ k^{N/2} R_{\max}^{2N}, & R_{\max}^2 = \text{const} \rightarrow \infty, \end{cases} \quad (66)$$

where the ‘‘spring constant’’ is now  $k \equiv 2\lambda$  and  $R_{\max}$  corresponds to a cutoff in  $\phi$ -space (or places a limit on the amplitude of oscillations).

At this point we reiterate that our approach is to compute phase space with the assumed cutoffs at each time slice and then allow the system to evolve in time. Since the Hamiltonian is, in general, a function of time, it is reasonable to impose cutoffs that scale with  $H$ . This implies that we should hold  $u_{\max}$  constant, thereby preserving the self-similarity in the energy distribution. This choice of cutoff is also computationally convenient in that it results in a scaling for  $\Omega$  that is similar to the harmonic oscillator case, namely  $\Omega \propto H^N / k^{N/2}$ . However, we have verified that the other cutoff criteria (constant  $R_{\max}^2$ ) gives qualitatively the same behavior

as the constant  $u_{\max}$  case for both growing and decaying modes.

### B. Meaning of $H > 0$ and $H < 0$

In the previous section we considered  $\Omega$  for both  $H < 0$  and  $H > 0$ . We now wish to explore the meaning of positive and negative Hamiltonians in the present context. Classically, one associates negative energy states with bound systems and positive energy states with unbound systems. Here, however, the situation is slightly different.

The equations describing the evolution of dust and superhorizon radiation and gravitational wave (with the identification  $u \equiv \phi = ah$ ) perturbations can be unified into a single differential equation  $\ddot{u} - cu\eta^{-2} = 0$ , where  $c = 6$  for dust and 2 for radiation and gravitational waves. For both cases we can write the solution as  $u = A\eta^{n_1} + B\eta^{n_2}$ , where  $A$  and  $B$  are constants and  $(n_1, n_2) = (3, -2)$  for dust and  $(2, -1)$  for radiation and waves. In short,  $A$  and  $B$  define the growing and decaying modes, respectively. From our previous solutions to the equations of motion on superhorizon scales, the Hamiltonian  $H \sim \dot{u}^2 - cu^2\eta^{-2}$  can be written as

$$H \sim \begin{cases} 3A^2\eta^4 - 2B^2\eta^{-6} - 24AB\eta^{-1} & \text{for dust,} \\ 2A^2\eta^2 - B^2\eta^{-4} - 8AB\eta^{-1} & \text{for radiation and waves.} \end{cases} \quad (67)$$

Now, note that  $A = 0$  results in

$$H \sim \begin{cases} -2B^2\eta^{-6} < 0 & \text{for dust,} \\ -B^2\eta^{-4} < 0 & \text{for radiation and waves.} \end{cases} \quad (68)$$

That is, in both cases,  $H < 0$  corresponds to a decaying mode. Similarly, setting  $B = 0$  leads to

$$H \sim \begin{cases} 3A^2\eta^4 > 0 & \text{for dust,} \\ 2A^2\eta^2 > 0 & \text{for radiation and waves.} \end{cases} \quad (69)$$

In other words,  $H > 0$  corresponds to a growing mode.

It is important now to attach a physical picture to these results because they are, in a sense, opposite from what one intuitively expects from a particle model. In a particle model, one associates  $H < 0$  with bound systems undergoing gravitational collapse. Growing modes, then, correspond to  $H < 0$  and particles moving together.

However, it is crucial to bear in mind that we are considering not a particle model but an oscillator model, where growing modes correspond to increasing amplitudes of oscillation. One therefore can imagine a lattice of points undergoing perturbations that eventually lead to gravitational collapse. As the perturbations grow, the grid points move farther from their initial unperturbed, or ‘‘uniformly’’ arranged positions. For decaying modes, the grid points relax to their homogeneously spaced positions. This is why in Sec. VI,  $\Omega$  grew for growing modes and decreased for decaying modes. In the oscillator picture, then, increasing inhomogeneity automatically gives an increase in phase space and hence gravitational entropy.

We can also make contact with the ‘‘qualitative cosmology’’ approach of Hamiltonian cosmology. The turning points of trajectories with the inverted potential Hamiltonian take place when the momenta are zero and  $H = V$ . Since for the inverted potential  $V < 0$ ,  $H > 0$  necessarily implies  $H > V$ . The motion here is ‘‘unbounded,’’ in the sense that there are no turning points and perturbations continue to grow. For  $H < 0$ , we have  $|H| = \xi \sum_{i=1}^N \phi_i^2$  at the turning points. The potential barrier is thus an  $N$ -sphere of radius  $r = \sqrt{|H|/\xi}$ . However, in general,  $\sum_{i=1}^N \phi_i^2 = |H|/\xi + \lambda \sum_{i=1}^N \pi_i^2/\xi > |H|/\xi$ , so the world point is actually outside this sphere.

For decaying modes, the sphere shrinks in time and the world point attempts to catch up with it. However by comparing  $\dot{u}$  for decaying dust and radiation modes with the time dependence of  $r$  for the potential barrier, one easily shows that the system point can never catch up with the barrier in finite time. The barrier, then, serves as an attractor for the decaying modes but it is never actually reached except in an asymptotic sense. This picture is similar to that of the Bianchi cosmologies, in which the universe is often represented as a point moving in a potential well. We turn to Bianchi type IX cosmologies now.

### IX. BIANCHI TYPE IX COSMOLOGY

For the Bianchi type IX cosmological models, we adopt a metric of the form

$$ds^2 = -dt^2 + e^{2\alpha}(e^{2\beta})_{ij}\sigma^i\sigma^j, \quad (70)$$

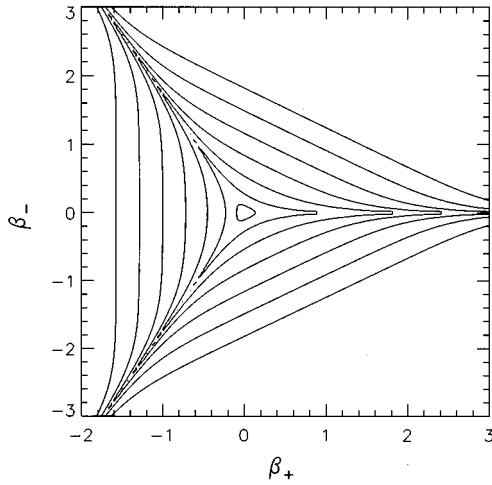


FIG. 2. Contour plot of the Bianchi type IX potential  $V$ . Seven level surfaces are shown at equally spaced decades ranging from  $10^{-1}$  to  $10^5$ . For  $V > 1$ , the potential is open and exhibits a strong triangular symmetry with three narrow channels extending to spatial infinity. For  $V < 1$ , the potential closes and is approximately circular.

where  $a = e^\alpha$  is the mean expansion scale factor,  $\sigma^i$  are the dual one-forms for the rotation group  $SO(3, R)$ , and  $(e^{2\beta})_{ij}$  is an exponential of a  $3 \times 3$  symmetric traceless matrix defining the anisotropy of the spatial hypersurfaces and parametrized as

$$\|\beta\| = \text{diag} \|\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+\|. \quad (71)$$

The Bianchi models are anisotropic but homogeneous cosmologies, so by definition they cannot show the effects of gravitational clumping. Nevertheless, there are three reasons for investigating Type IX. First, it can be conveniently cast into a Hamiltonian form and a phase space can be formally calculated. Second, if one regards anisotropy as the long-wavelength limit of inhomogeneity, we might hope to make contact with our previous results. Finally, it provides a transition to the full ADM formalism, which one will necessarily employ in nonperturbative models.

The ADM Hamiltonian for Bianchi type IX is [18,19]

$$H^2 = p_+^2 + p_-^2 + 36\pi^2 e^{4\alpha} (V(\beta_+, \beta_-) - 1). \quad (72)$$

Hence, Bianchi type IX can be cast into a system of two degrees of freedom (meaning two canonical pairs.) In Eq. (72)  $V$  is Misner's anisotropy potential, which is a function of the canonical coordinates  $\beta_+$  and  $\beta_-$ , the independent components of the metric anisotropy. The precise form of  $V$  is:

$$V = 1 + \frac{1}{3} e^{-8\beta_+} - \frac{4}{3} e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + \frac{2}{3} e^{4\beta_+} (\cosh 4\sqrt{3}\beta_- - 1). \quad (73)$$

This potential (shown in Fig. 2) is symmetric about the  $\beta_-$  axis and has exponentially steep walls. For large isocontours of  $V$  ( $> 1$ ), the potential exhibits a strong triangular symmetry with three narrow channels that extend to infinity. For  $V < 1$ , the potential is closed and asymptotic ( $\beta_\pm \ll 1$ ) iso-

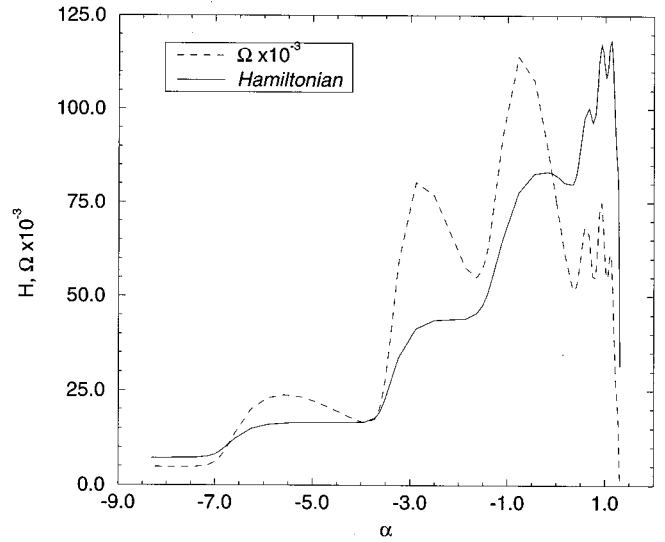


FIG. 3. The Hamiltonian and entropy for Bianchi type IX as a function of  $\alpha$ . The evolution is initialized at  $\alpha = 0$  with the following data:  $\beta_+ = \beta_- = 0$ ,  $\dot{\beta}_+ = 2$ , and  $\dot{\beta}_- = 1$ , and run both forward in time and backward towards the singularity  $\alpha \rightarrow -\infty$ .

contours describe a circle. The motion of the universe point in this potential well is chaotic [20], so we can regard any region of phase space to be filled with equal probability and the concept of an associated entropy is reasonable.

The phase space is formally calculated as we have done with the other cases:

$$\Omega_{IX} = \int d\beta_+ \int d\beta_- \int dp_+ \int dp_-. \quad (74)$$

To facilitate integration, however, we eliminate  $p_-$  in favor of  $H$ . The integral then becomes from Eq. (72)

$$\Omega_{IX} = \int_{H_{\min}}^{H_{\max}} H dH \int_{\beta_{+\min}}^{\beta_{+\max}} d\beta_+ \int_{\beta_{-\min}}^{\beta_{-\max}} d\beta_- \int_{-\ell}^{+\ell} \frac{dp_+}{\sqrt{\ell^2 - p_+^2}}, \quad (75)$$

where  $\ell^2 = H^2 - 36\pi^2 e^{4\alpha} (V - 1)$ . We note that this problem is very similar to the two-particle model of Sec. II, except for the more complicated potential. The integral over  $p_+$  is simply an arcsin and the result after applying the boundary conditions is  $\pi$ . The lower limit on  $H$  is 0, and due to the symmetry of the potential, we can take the  $\beta_-$  limits to be 0 and  $\beta_{-\max}$ , the maximum value of  $\beta_-$ , and double the result. Therefore we are left with

$$\Omega_{IX} = 2\pi \int_0^{H_{\max}} H dH \int_0^{\beta_{-\max}} d\beta_- \int_{\beta_{+\min}}^{\beta_{+\max}} d\beta_+. \quad (76)$$

The remaining integrals are evaluated numerically. To do this requires first determining the limits of integration. As with the two-particle model, we set limits by equating the momenta in Eq. (72) to zero and finding the reflection points. In other words, we demand that  $H^2$  always remain positive:

$$H^2 \geq 36\pi^2 e^{4\alpha} (V - 1) \Rightarrow V \leq \frac{H^2}{36\pi^2 e^{4\alpha}} + 1. \quad (77)$$

For a fixed value of  $H$  and  $\beta_-$ , we march across the potential well varying  $\beta_+$  until this inequality is violated. We then perform interpolations at the two endpoints to find the minimum and maximum values of  $\beta_+$ . Then  $\beta_-$  is incremented and the process is repeated. The limits on  $\beta_-$  are found in a similar manner. For large values of  $\beta_{\pm}$ , we treat the equipotentials as equilateral triangles. For intermediate values of  $\beta_{\pm}$  we take into account the deformation of the contours and follow them part way into the channels (the area here becomes vanishingly small). For a closed potential in which  $\beta_{\pm} \ll 1$ , the area is approximated as a circle of radius  $\sqrt{\beta_+^2 + \beta_-^2}$ . At each time step the integrals in Eq. (76) are evaluated with a 40-point Gaussian quadrature scheme.

To evolve the system in time, we integrate the evolution equations for  $\beta_{\pm}$  and  $\alpha$ ,

$$\ddot{\beta}_+ = -3\dot{\alpha}\dot{\beta}_+ - \frac{1}{8}e^{-2\alpha}\frac{\partial V}{\partial \beta_+}, \quad (78)$$

$$\ddot{\beta}_- = -3\dot{\alpha}\dot{\beta}_- - \frac{1}{8}e^{-2\alpha}\frac{\partial V}{\partial \beta_-}, \quad (79)$$

$$\dot{\alpha} = \left[ \dot{\beta}_+^2 + \dot{\beta}_-^2 - \frac{1}{4}e^{-2\alpha}(1-V) \right]^{1/2}, \quad (80)$$

using a fourth-order Runge-Kutta scheme. The Hamiltonian is then updated at each time step by

$$H = 12\pi\dot{\alpha}e^{3\alpha}. \quad (81)$$

This value of  $H$  is then used for the upper limit of the outer integral in Eq. (76).

We note that the  $\beta$ -integrals basically give the area of the triangular potential. Thus we can estimate the size of the phase space as

$$\Omega_{IX} = 2\pi \int^H H' dH' \int \Delta \approx 2\pi \frac{H^2}{2} \frac{A}{3}, \quad (82)$$

where  $A$  is the area of largest triangle and the factor of  $1/3$  is introduced to approximate the size of an average triangle in the inverted pyramid of the potential well. Estimates performed this way typically agree with the computed results to within a factor of two or better.

Results of the numerical integrations are shown in Fig. 3 where we plot the Hamiltonian and the volume of phase space as a function of  $\alpha$ . We note that the limit  $\alpha \rightarrow -\infty$  corresponds to the ‘‘Big Bang’’ singularity. One of the most striking features is that both the Hamiltonian  $H$  and the phase space  $\Omega$  are seen to oscillate. We now demonstrate that these oscillations are real. From Eq. (82) we have  $\Omega \sim AH^2$ . Then

$$\frac{d\Omega}{d\alpha} \sim 2AH \frac{dH}{d\alpha} + H^2 \frac{dA}{d\alpha}, \quad (83)$$

where from Eq. (72) we have the fundamental equation

$$\frac{dH}{d\alpha} = \frac{72\pi^2}{H} e^{4\alpha}(V-1). \quad (84)$$

Assuming the boundary triangles for the unbounded (open potential) phase space are equilateral, the enclosed area is approximately  $A \sim \sqrt{3}\beta_-^2$ . Also, for large values of  $\beta_-$ , the asymptotic form of  $V$  is from Eq. (73)  $V \sim (1/3)e^{4\sqrt{3}\beta_-}$ . Thus  $A \sim (\ln 3V)^2$ . Now, at the potential wall, Eq. (72) shows that  $V \approx H^2 e^{-4\alpha}/36\pi^2$  and we can write

$$A \sim \left[ \ln \left( \frac{H^2 e^{-4\alpha}}{12\pi^2} \right) \right]^2,$$

$$\frac{dA}{d\alpha} \sim 8 \left( \frac{1}{2H} \frac{dH}{d\alpha} - 1 \right) \ln \left( \frac{H^2 e^{-4\alpha}}{12\pi^2} \right). \quad (85)$$

Analytic approximations for the behavior of  $\Omega$  can be found for two limiting cases: (i) ‘‘Free-particle trajectories.’’ Such trajectories correspond to the plateaus in Fig. 3. In these regions, the universe point is sufficiently far from the potential walls that the potential terms in Eq. (72) can be neglected. The universe point propagates like a free particle with constant ‘‘energy’’  $H$ , so that  $dH/d\alpha \sim 0$ .

(ii) ‘‘Wall collisions.’’ At or near the potential barriers the momenta in  $H$  are negligible so that  $dH/d\alpha \sim 2H$  or  $H \sim e^{2\alpha}$ . In Fig. 3, wall collisions correspond to the places where  $H$  suddenly increases. Here the system is gaining energy from the gravitational field. During wall collisions the area remains approximately constant so  $dA/d\alpha \sim 0$ .

From Eq. (83) we then have for the free and bounce cases, respectively,

$$\frac{d\Omega}{d\alpha} \sim -8H^2(2\ln H - 4\alpha - \ln 12\pi^2), \quad \frac{dH}{d\alpha} \sim 0, \quad (86)$$

and

$$\frac{d\Omega}{d\alpha} \sim H^2, \quad \frac{dH}{d\alpha} \sim H. \quad (87)$$

Equation (86) shows that for large negative  $\alpha$ ,  $d\Omega/d\alpha < 0$ , as observed in Fig. 3. Furthermore, this scales roughly as  $\sim H^2$ , so as  $H$  increases,  $d\Omega/d\alpha$  becomes more negative and the phase space evolves more rapidly, which is also observed.

Note that  $d\Omega/d\alpha$  in Eq. (87) is positive definite, so at wall collisions  $\Omega$  always increases and, for a large enough energy,  $\Omega$  increases more rapidly than  $H$ . Furthermore, the greater the value of  $H$ , the greater the slope, as observed. Together, the competing behaviors in the two limiting cases account for the oscillations observed in the figure.

In terms of finding a monotonic function to call entropy, we reiterate that  $H$  is time dependent but also that this system is inhomogeneous. However, two aspects of Fig. 3 are highly encouraging. For late times ( $\alpha > 0$ ), the phase space—and hence entropy—is seen to increase monotonically in the direction of increasing anisotropy. Furthermore, in the oscillatory regime, along the plateaus, where the model most closely resembles a typical ‘‘closed’’ system ( $E = \text{const}$ ), the entropy is again seen to increase in the direction of increased anisotropy. Both behaviors correspond with the notion that anisotropy represents the long-wavelength limit of inhomogeneity.

In the limit of small anisotropy  $V \approx 8(\beta_+^2 + \beta_-^2) \ll 1$ , Eqs. (78)–(80) become

$$\ddot{\beta}_\pm + 3\frac{\dot{a}}{a}\dot{\beta}_\pm + \frac{2}{a^2}\beta_\pm = 0, \quad (88)$$

$$\dot{\alpha}^2 = \dot{\beta}_+^2 + \dot{\beta}_-^2 - \frac{e^{-2\alpha}}{4}, \quad (89)$$

where we have defined  $a = e^\alpha$  and  $\dot{\alpha} = \dot{a}/a$ . Equation (88) is similar to Eq. (31) for gravitational wave perturbations. However, the type IX solution is further complicated by Eq. (89) which couples the anisotropy to the expansion factor. Nevertheless, at late times, we expect  $\Omega$  to behave similarly in both the Bianchi type IX and gravitational wave cases: increase monotonically with increasing anisotropy or inhomogeneity.

## X. COMPARISON WITH $C^2$

Penrose [1] had suggested that the square of the Weyl tensor  $C^2 \equiv C_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\alpha\beta}$  might act as an arrow of time, increasing monotonically in time as the universe becomes more inhomogeneous. Of course, this presupposes an initial low entropy state at the singularity, in which the matter distribution is homogeneous and the Weyl tensor tends to zero. However, Wainwright and Anderson [2] (see also Goode and Wainwright [3]) have shown that cosmological models which admit an isotropic singularity contradict Penrose's hypothesis. They also noted that the Ricci tensor diverges, but in such a way as to dominate the Weyl tensor. This led them to propose a weakened form of Penrose's hypothesis in which the quantity

$$\frac{C^2}{R^2} \equiv \frac{C_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\alpha\beta}}{R_{\alpha\beta}R^{\alpha\beta}} \quad (90)$$

might be the appropriate indicator. However, subsequent work by Bonnor [5] has thrown even this weakened form into question.

Here we calculate the two variants of Penrose's proposal for cosmological density perturbations in an expanding flat universe. Assuming, for simplicity, the perturbations to be functions only of conformal time  $\eta$  and a single spatial coordinate  $z$ , the spacetime metric is given by Eq. (39) with  $\mathcal{K}=0$ . We find (using MATHTENSOR and MATHEMATICA) to lowest order in the smallness parameter  $\Phi \ll 1$ ,

$$C_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\alpha\beta} = \frac{16(\Phi_{,zz})^2}{3a^4}, \quad (91)$$

and the solution for  $\Phi$  is given by Eq. (51). During the matter dominated regime, the scale factor evolves as  $a \sim \eta^2$ , and we immediately see that Eq. (91) does not produce the right behavior for the growing modes. The Weyl tensor *decreases* with increasing time and inhomogeneity. Because the overall time dependence is monotonic, one might think to correct this by introducing a negative sign: However, then for the decaying modes  $C^2$  is increasing for decreasing inhomogeneity.

Following the suggestion of Wainwright and Anderson [2] we also calculate

$$\frac{C_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\alpha\beta}}{R_{\alpha\beta}R^{\alpha\beta}} = \frac{4a^4(\Phi_{,zz})^2}{9(\dot{a}^4 - a\dot{a}^2\ddot{a} + a^2\ddot{a}^2)} = \frac{\eta^4(\Phi_{,zz})^2}{27}. \quad (92)$$

Equation (92) does have the correct behavior. In fact, it is interesting to note that to this order the time dependence is identical to that found for the corresponding Hamiltonian (53), i.e.,  $\eta^4$  and  $\eta^{-6}$  for the growing and decaying modes, respectively.

The generalization to nonflat spacetimes is rather complicated and not qualitatively different from the flat case, so we do not include it here. However, we do compute the Weyl tensor for spacetimes of the form (26), containing singly polarized (+) small amplitude gravitational waves propagating in an expanding universe. In this case

$$\frac{C_{\alpha\beta\gamma\delta}C_{\gamma\delta\alpha\beta}}{R_{\alpha\beta}R^{\alpha\beta}} = \frac{a^4(h_{,zz}^2 - 4h_{,\eta z}^2 + 2h_{,zz}h_{,\eta\eta} + h_{,\eta\eta}^2)}{12(\dot{a}^4 - a\dot{a}^2\ddot{a} + a^2\ddot{a}^2)}. \quad (93)$$

Noting that  $R_{\alpha\beta}R^{\alpha\beta}$ , to zero perturbative order, is the same as for the metric (39), we again find that  $C^2$  alone does not produce the right monotonic behavior, but  $C^2/R^2$  does. To evaluate the latter, we assume the expansion of the universe is governed by density perturbations and that the scale factor evolves as  $a \sim \eta^2$ . For superhorizon scales,  $k\eta \ll 1$ , we may ignore spatial gradients so that only the last term in Eq. (93) survives. Then  $|C^2/R^2| \sim \eta^{-6}$ . In the limit  $k\eta \gg 1$ , only the first term survives and  $|C^2/R^2|$  oscillates at nearly constant amplitude. It is also interesting to note that the subhorizon perturbations evolve similarly to the Hamiltonian (53), i.e., with constant amplitude. The superhorizon evolutions, on the other hand, differ from the Hamiltonian time dependence. Superhorizon perturbations are coupled to the background expansion and, in this case, the expansion is driven by density perturbations. So it is not surprising to find a scaling  $\sim \eta^{-6}$  similar to that of decaying density perturbations.

Finally we present results for the Bianchi type IX metric (70), although due to the complexity of the Weyl tensor, we do not write out  $C^2$  here. Because Bianchi type IX is a vacuum solution with  $R_{\mu\nu}=0$ , we compute only  $|C^2|$ , shown in Fig. 4 using the same initial data as in Fig. 3. For comparison, we also show  $\Omega^4$  (introduced to bring out the structure at the scale of variations in  $|C^2|$ ). Notice that although  $C^2$  oscillates (the kinks evident in  $|C^2|$ , and which correlate with the peaks in  $\Omega$ , are points where  $C^2$  becomes negative), the absolute magnitude diverges exponentially as the singularity is approached. The rate of divergence can be estimated from the "free fall" part of the trajectories during which  $\dot{\alpha} \sim \dot{\beta} \sim e^{-3\alpha}$ , and the dominant terms in the square of the Weyl tensor scale as  $|C^2| \sim e^{-12\alpha}$  for  $\alpha \ll 0$ .

The above results continue to throw doubt on the utility of the  $C^2$  definition of entropy. Indeed, the simple  $C^2$  measure seems to be again ruled out because of its inability to handle both the decaying and growing modes in a sensible fashion.

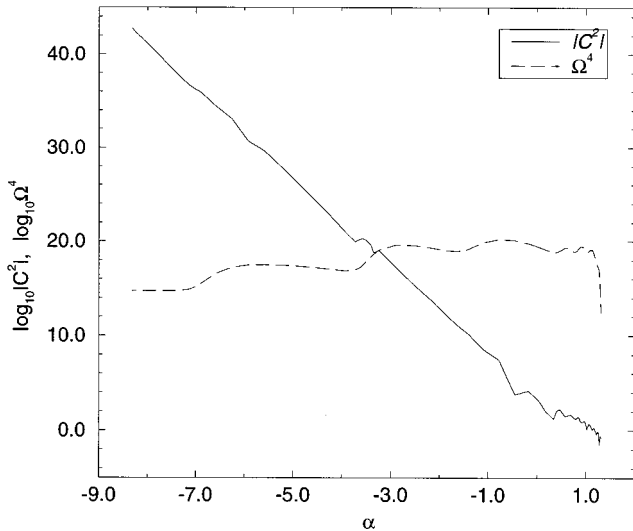


FIG. 4. A comparison plot of the phase space volume (represented by  $\Omega^4$ ) and the magnitude of the Weyl tensor squared  $|C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}|$ . The kinks evident in the  $|C^2|$  curve represent regions where  $C^2 < 0$ , which are correlated with the regions in which  $\Omega$  drops after reaching a peak value, i.e., the wall collisions.

Our results also point to important differences between the phase space and  $C^2$  measures of entropy, as well as several other functions one might consider. As can be seen from above,  $C^2$  is a local quantity, which will vary from point to point. As such it is not a useful measure of the global properties of spacetime, unless some sort of spatial average is introduced. By the same token, even though they are gauge invariant to first order, one can rule out the metric perturbations  $\Phi$  and  $h$  for the density and gravitational wave perturbations. These are also local quantities.

The Hamiltonian in our examples could be considered on its own to be a measure of inhomogeneity since it is summed over the spatial coordinates and has a sensible time dependence. In regard to monotonicity in the time dependence,  $\Omega$  appears to offer no advantages over  $H$  (except perhaps in the case of Bianchi type IX, where we found that along the plateaus of constant  $H$ ,  $\Omega$  increased in the direction of increasing anisotropy). However,  $H$  alone does not provide a statistical description of a system in that it can be changed by the addition of an arbitrary phase.  $\Omega$ , on the other hand is a truly global quantity that expresses the entire allowable dynamical range equally for each of the oscillators in the space-like hypersurfaces. It is not restricted to a particular phase realization, unlike any combination of variables constructed from metric components, which is.  $\Omega$  thus presents the advantage over  $C^2$ ,  $H$  or any single solution to the differential equations in that it is global, shows a sensible time dependence and reduces to familiar entropy under appropriate circumstances. This is apparently true even in one area we have not yet addressed.

## XI. CONNECTION WITH BLACK HOLES

One of the questions one naturally wishes to answer is whether the entropy we have defined results in the well-

known entropy of black holes. To establish the connection would strengthen any claim that the entropy function of this paper is in fact entropy. We now give a Bekenstein-style argument [21] that the logarithm of the phase space does reduce to the entropy of black holes in the appropriate circumstance. The argument resembles the one we gave in Sec. V for the EM field and is also somewhat similar to one found in Zurek and Thorne [22]; we have, however, not seen this demonstration elsewhere.

In Sec. III we showed that the phase space of harmonic oscillators, Eq. (13), gives the classical limit for Einstein's formula and results in a reasonable expression for the entropy of the electromagnetic field. This phase space was, with a slight change in notation,

$$\Omega = \frac{(2\pi)^N H^N}{\omega^N N!}, \quad (94)$$

where  $\omega$  is the angular frequency.

Suppose we wish to construct a black hole out of photons, i.e., quantum oscillators. To do this, we must squeeze the oscillator system to within a Schwarzschild volume and the total energy of the oscillator system should equal  $M$ , the mass of the black hole. The latter condition implies that  $H = M = N\epsilon$ , where  $\epsilon$  is the average energy of an oscillator. We also have  $\epsilon = \nu = 1/\lambda$ , where  $\lambda$  is the wavelength of the photon. The minimum energy per oscillator needed to construct the hole corresponds to the longest allowed wavelength, which should be of order the Schwarzschild diameter, or  $\lambda = 4M$ . Let us, however, parameterize the wavelength as  $\lambda = fM$ . Hence  $\epsilon = 1/(fM)$  and

$$N = fM^2. \quad (95)$$

With the above expression for  $N$ ,  $\omega = 2\pi\epsilon = 2\pi/fM$  and Stirling's formula, we quickly find by taking the logarithm of Eq. (94) that

$$S = \ln\Omega = fM^2. \quad (96)$$

The exact value of  $S$  therefore depends on  $f$ . *A priori*, we expect  $\lambda \sim 4M$  or  $f = 4$ . However, if  $\Delta p \sim 1/\lambda$ , the uncertainty relationship implies that  $\lambda$  may be as large as  $16\pi M$ . In the former case we are a factor of  $2\pi^2$  lower than the Bekenstein-Hawking [23] value of  $8\pi^2 M^2$ , in the latter case a factor of  $\pi/2$  lower. Alternatively, if one chooses  $\lambda = (2T)^{-1}$  where the black-hole temperature  $T^{-1} = 16\pi^2 M = \partial S / \partial M$  [23], one recovers the exact result  $f = 8\pi^2$ .

One might object that we have basically given a dimensional argument. Nevertheless, that the phase space of harmonic oscillators comes so close to the accepted result is striking. With hindsight, the phase-space approach makes clear that  $\ln\Omega \approx N$ , so black-hole entropy must be of order  $M^2$ . A Hamiltonian modified for quantum mechanical systems would, we expect, reproduce the usual result. Note, however, that unlike the cosmological models we have considered, the Hamiltonian here is not the ADM Hamiltonian for the black hole itself. The Hamiltonian for the Schwarzs-

child metric would presumably result in zero entropy since the canonical momenta are zero in the static case. Thus the harmonic oscillator Hamiltonian must be regarded either as perturbations on the background or as the Hamiltonian for the infalling oscillators; this latter corresponds to the usual approach for calculating black-hole entropy. We will explore these matters further and attempt a quantum mechanical calculation in a future paper. As it stands our current result shows that black-hole entropy can be treated profitably as a classical quantity. We also emphasize that, in contrast to the cosmological case, the black-hole Hamiltonian is easily interpreted as the energy and it is constant; the resulting phase space should then be the usual one. The main leap, evidently, in accepting the function we have termed gravitational entropy as genuine entropy lies not in the classical treatment, but in the use of time-dependent Hamiltonians.

## XII. FUTURE WORK

We have evaluated the phase space for a number of models in the perturbative limit under the assumptions that: (1) the phases of the various components can be ignored; (2) that the system can be defined on spacelike hypersurfaces with some prescription for choosing boundaries; (3) the system is constrained by a Hamiltonian on each hypersurface. Under these assumptions  $\ln\Omega$  appears to be a reasonable entropy function in that it increases with increasing inhomogeneity and not otherwise. Because the phase space for the perturbative spacetimes we have considered is computed using gauge invariant functions, entropy as we have defined it is thus also gauge invariant to first order. Moreover, it can be identified with the entropy of more familiar situations. We also point out that the generalized second law of thermodynamics appears to be automatically satisfied. The generalized second law states that the sum of the thermodynamic and gravitational entropies in a closed, isolated system should always increase. Unless for some reason an increase in gravitational entropy actually causes a decrease in thermodynamic entropy, the generalized second law should not be violated. [For some time this was not obvious in the black hole case,

in which the hole *can* decrease the surrounding entropy (at the expense of increasing its surface area). However, since our entropy becomes black hole entropy, this situation is evidently taken care of.] A more detailed investigation of this question may be warranted.

We reiterate that all our calculations have been performed in the classical limit. We will present a quantum calculation for the black hole case in a future paper.

Full, nonperturbative ADM calculations for inhomogeneous model systems would also be desirable. One system to examine is spherically symmetric collapse. However, in this case (as in classical orbital problems) the canonical coordinates and momenta appear to be coupled, making it difficult to perform the integrations. If the system is tractable, it may be possible to get black hole entropy by calculating the phase space available to a collapsing star or dust shells.

These are a few problems we hope to examine in future work. The phase space approach is a generic one, applicable to a wide range of systems, including dust, radiation,  $N$ -body simulations, Newtonian and relativistic problems. Hence, the cases we have mentioned are probably only a small subset of those that can be examined. The more important message is that a consideration of the phase space available to general-relativistic systems appears to open a direct connection to statistical mechanics. This connection is well worth investigating.

## ACKNOWLEDGMENTS

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